CATEGORIES IN ABSTRACT MODEL THEORY

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Infinity Conference
CRM Barcelona

July 21, 2011

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- Does the shift in perspective yield model-theoretic dividends?

Time permitting, we also examine an alternative category-theoretic framework for abstract model theory.

For the purposes of this talk, abstract model theory is the research program focused on sniffing out the fragment of classification theory that is common to naturally occurring logics: first order, $L_{\omega_1\omega}$, $L(Q)$, $L_{\omega_1\omega}(Q)$, etc.

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Strategy: abandon syntax and logic-dependent structure entirely, and simply work with abstract classes of structures equipped with a strong substructure relation that retains certain essential properties of elementary embedding.

Hence abstract elementary classes—which can (and perhaps should) be regarded as the category-theoretic hulls of elementary classes.

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- $\prec_{\mathcal{K}}$ is a partial order.
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

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Side note: when considering Galois types, saturation, and stability in the sequel, we will assume AP. We therefore have a large Galois-saturated, strongly model homogeneous model $\mathfrak{C}$—the monster model—and may identify Galois types with orbits in $\mathfrak{C}$.

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- (Beke/Rosický; L) Accessible categories and AECs.

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Equivalently, N is finitely presentable if for any directed diagram $D : (I, \leq) \to \mathbf{C}$ with colimit cocone $(\phi_i : D(i) \to M)_{i \in I})$, any map $f : N \to M$ factors through one of the cocone maps: $f = \phi_i \circ g$ for some $i \in I$ and $g : N \to D(i)$.

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Example: In **Grp**, the category of groups, an object G is finitely presentable iff G is finitely presented. Same for any finitary algebraic variety.

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This is our analogue of cardinality, and allows a straightforward translation of categoricity questions into the context of abstract categories...

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- it has at most a set of λ-presentables
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Example: **Hilb**, the category of Hilbert spaces with linear contractions, lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

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Theorem (L, Beke/Rosický)

As a category, an AEC $\mathcal{K}$ is μ-accessible for all regular $\mu \geq LS(\mathcal{K})^+$, and, for $\lambda \geq LS(\mathcal{K})$, a model $M \in \mathcal{K}$ has $\pi(M) = \lambda^+$ if and only if $|M| = \lambda$. Moreover, $\mathcal{K}$ is closed under directed colimits.

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With a few more clauses, we can completely axiomatize subcategories of categories of structures that are essentially AECs. Rosický considers categories of this form, defines a number of category-theoretic analogues of notions from model theory. Most notably: weak κ-stability.

Definition

A morphism $f : M \to N$ in a category **C** *is said to be κ-pure if for any commutative square*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ u \uparrow & & \uparrow v \\ C & \xrightarrow{g} & D \end{array}$$

in which C and D are κ-presentable, there is a morphism $h : D \to M$ such that $h \circ g = u$.

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In an EC, an elementary embedding $M \to N$ is κ-pure iff M is κ-saturated relative to N. In an AEC, a strong embedding $M \to N$ is κ-pure only if M is κ-Galois-saturated relative to N. In particular, an inclusion $M \to \mathfrak{C}$ is κ-pure iff M is κ-Galois-saturated.

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A category **C** *is weakly* κ*-stable if for every* κ^+*-presentable* M *and morphism* $f : M \to N$, f *factors as*

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Things are more complicated in AECs, but the connection is still close...

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Proof: Let $M \in \mathcal{K}_\kappa$, hence κ^+-presentable. The inclusion $M \to \mathfrak{C}$ factors through a κ^+-presentable object M' (i.e. a model $M' \in \mathcal{K}_\kappa$) such that $M' \to \mathfrak{C}$ is κ-pure, whence M' is Galois-saturated.

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In certain contexts, weak κ-stability implies κ-Galois-stability, which is interesting because every accessible category is weakly stable in many cardinalities...

Suppose $\mathcal{K}$ is λ-categorical (no assumption of AP, JEP), C is the unique structure of size λ, and M is its monoid of endomorphisms.

Theorem (R,L)

*If $\mathcal{K}$ is λ-categorical, the sub-AEC $\mathcal{K}_{\geq\lambda}$ consisting of models of size at least λ is equivalent to (M^{op}, λ^+)-**Set**, the full subcategory of M^{op}-**Set** consisting of λ^+-directed colimits of M.*

The equivalence is induced by the composition

$$\mathcal{K}_{\geq\lambda} \xrightarrow{y} \mathbf{Set}^{(\mathcal{K}_{\geq\lambda})^{op}} \xrightarrow{r} \mathbf{Set}^{M^{op}} \longrightarrow M^{op}\text{-}\mathbf{Set}$$

where y is the Yoneda embedding, the second map is restriction, and the final map is the obvious equivalence $\mathbf{Set}^{M^{op}} \xrightarrow{\sim} M^{op}$-**Set**.

The assignment is:

$$N \in \mathcal{K}_{\geq\lambda} \quad \mapsto \quad \mathrm{Hom}_{\mathcal{K}}(C, N)$$

where $M = \mathrm{Hom}_{\mathcal{K}}(C, C)$ acts by precomposition.

That this gives the desired equivalence is an exercise in definitions.

This amounts to an astonishing transformation of a very abstract entity—an AEC—into a category of relatively simple algebraic objects. How useful this might be is less clear...

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This covers an awful lot of logics: FOL, $L_{\infty\omega}$, HOL, IPL, MFOL, MPL, temporal and behavioral logics, and so on.

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- $\mathbf{Sig}^{\mathcal{I}}$ is a category whose objects are called "signatures."
- $Sen^{\mathcal{I}}$ is a functor from $\mathbf{Sig}^{\mathcal{I}}$ to **Set** that assigns

  (a) to each Σ in $\mathbf{Sig}^{\mathcal{I}}$ a set of "sentences" $Sen^{\mathcal{I}}(\Sigma)$, and

  (b) to each signature map $\phi : \Sigma \to \Sigma'$ a "translation" $Sen^{\mathcal{I}}(\phi) : Sen^{\mathcal{I}}(\Sigma) \to Sen^{\mathcal{I}}(\Sigma')$.

- $Mod^{\mathcal{I}}$ is a functor from $(\mathbf{Sig}^{\mathcal{I}})^{\mathsf{op}}$ to $\mathbb{CAT}$ that assigns

  (a) to each Σ a category of "Σ-models," $Mod^{\mathcal{I}}(\Sigma)$, and

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Motivation: if $\phi : \Sigma \to \Sigma'$ is an inclusion of signatures, and $Mod^{\mathcal{I}}(\Sigma)$ and $Mod^{\mathcal{I}}(\Sigma')$ the corresponding categories of structures, we have the reduct

$$- \restriction_{\Sigma} : Mod^{\mathcal{I}}(\Sigma') \to Mod^{\mathcal{I}}(\Sigma)$$

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  (a) to each Σ a category of "Σ-models," $Mod^{\mathcal{I}}(\Sigma)$, and

  (b) to each signature map $\phi : \Sigma \to \Sigma'$ a functor $Mod^{\mathcal{I}}(\phi) : Mod^{\mathcal{I}}(\Sigma') \to Mod^{\mathcal{I}}(\Sigma)$.

Motivation: if $\phi : \Sigma \to \Sigma'$ is an inclusion of signatures, and $Mod^{\mathcal{I}}(\Sigma)$ and $Mod^{\mathcal{I}}(\Sigma')$ the corresponding categories of structures, we have the reduct

$$- \restriction_{\Sigma} : Mod^{\mathcal{I}}(\Sigma') \to Mod^{\mathcal{I}}(\Sigma)$$

This is the template for $Mod^{\mathcal{I}}(\phi)$...

- for each Σ in $\mathbf{Sig}^{\mathcal{I}}$,

$$\models^{\mathcal{I}}_{\Sigma} \subseteq |Mod^{\mathcal{I}}(\Sigma)| \times Sen^{\mathcal{I}}(\Sigma)$$

is a relation, "Σ-satisfaction," which ensures the ingredients behave as in any concrete logic: for each map $\phi : \Sigma \to \Sigma'$,

$$M' \models^{\mathcal{I}}_{\Sigma'} Sen^{\mathcal{I}}(\phi)(s) \text{ iff } Mod^{\mathcal{I}}(\phi)(M') \models^{\mathcal{I}}_{\Sigma} s$$

for any sentence $s \in Sen^{\mathcal{I}}(\Sigma)$ and $M' \in |Mod^{\mathcal{I}}(\Sigma')|$.

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See, in particular, recent work of Diaconescu, who has a (mostly) institution-independent analog of the Keisler-Shelah theorem from first order: any two elementarily equivalent models have isomorphic ultrapowers.

This seems promising...

Further Reading

Accessible Categories and AECs:

- Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*. No. 189 in London Math. Soc. Lecture Notes, 1994.
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Further Reading II

AEC Context:

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