CATEGORIES IN ABSTRACT MODEL THEORY

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- Does the shift in perspective yield model-theoretic dividends?

Time permitting, we also examine an alternative category-theoretic framework for abstract model theory.
For the purposes of this talk, abstract model theory is the research program focused on sniffing out the fragment of classification theory that is common to naturally occurring logics: first order, $L_{\omega_1 \omega}$, $L(Q)$, $L_{\omega_1 \omega}(Q)$, etc.
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Strategy: abandon syntax and logic-dependent structure entirely, and simply work with abstract classes of structures equipped with a strong substructure relation that retains certain essential properties of elementary embedding.

Hence abstract elementary classes—which can (and perhaps should) be regarded as the category-theoretic hulls of elementary classes.
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- $\prec_{\mathcal{K}}$ is a partial order.
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\text{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \text{LS}(\mathcal{K})$. 
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A strong embedding $f : M \to N$ is an isomorphism from $M$ to a strong submodel of $N$, $f : M \cong M' \prec_{\mathcal{K}} N$. 
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Side note: when considering Galois types, saturation, and stability in the sequel, we will assume AP. We therefore have a large Galois-saturated, strongly model homogeneous model $\mathcal{C}$—the monster model—and may identify Galois types with orbits in $\mathcal{C}$. 
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- (Beke/Rosický; L) Accessible categories and AECs.
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Equivalently, $N$ is finitely presentable if for any directed diagram $D : (I, \leq) \to \mathbb{C}$ with colimit cocone $(\phi_i : D(i) \to M)_{i \in I}$, any map $f : N \to M$ factors through one of the cocone maps: $f = \phi_i \circ g$ for some $i \in I$ and $g : N \to D(i)$. 

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Example: In $\mathbf{Grp}$, the category of groups, an object $G$ is finitely presentable iff $G$ is finitely presented. Same for any finitary algebraic variety.
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This is our analogue of cardinality, and allows a straightforward translation of categoricity questions into the context of abstract categories...
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Example: $\text{Hilb}$, the category of Hilbert spaces with linear contractions, lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.
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**Theorem (L, Beke/Rosický)**

As a category, an AEC $\mathcal{K}$ is $\mu$-accessible for all regular $\mu \geq \text{LS}(\mathcal{K})^+$, and, for $\lambda \geq \text{LS}(\mathcal{K})$, a model $M \in \mathcal{K}$ has $\pi(M) = \lambda^+$ if and only if $|M| = \lambda$. Moreover, $\mathcal{K}$ is closed under directed colimits.
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With a few more clauses, we can completely axiomatize subcategories of categories of structures that are essentially AECs. Rosický considers categories of this form, defines a number of category-theoretic analogues of notions from model theory. Most notably: weak $\kappa$-stability.
Definition

A morphism \( f : M \to N \) in a category \( \mathcal{C} \) is said to be \( \kappa \)-pure if for any commutative square

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\begin{array}{ccc}
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A category $\mathbf{C}$ is weakly $\kappa$-stable if for every $\kappa^+$-presentable $M$ and morphism $f : M \to N$, $f$ factors as

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Things are more complicated in AECs, but the connection is still close...
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**Proposition (L)**

*If every \( M \in \mathcal{K}_\kappa \) has at most \( \kappa \) many strong extensions of size \( \kappa \), \( \mathcal{K} \) is weakly \( \kappa \)-stable.*
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**Proof:** Let $M \in \mathcal{K}_\kappa$, hence $\kappa^+$-presentable. The inclusion $M \to \mathfrak{C}$ factors through a $\kappa^+$-presentable object $M'$ (i.e. a model $M' \in \mathcal{K}_\kappa$) such that $M' \to \mathfrak{C}$ is $\kappa$-pure, whence $M'$ is Galois-saturated.
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In certain contexts, weak $\kappa$-stability implies $\kappa$-Galois-stability, which is interesting because every accessible category is weakly stable in many cardinalities...
Suppose $\mathcal{K}$ is $\lambda$-categorical (no assumption of AP, JEP), $C$ is the unique structure of size $\lambda$, and $M$ is its monoid of endomorphisms.

**Theorem (R,L)**

If $\mathcal{K}$ is $\lambda$-categorical, the sub-AEC $\mathcal{K}_{\geq \lambda}$ consisting of models of size at least $\lambda$ is equivalent to $(M^{\text{op}}, \lambda^+)-\text{Set}$, the full subcategory of $M^{\text{op}}\text{-Set}$ consisting of $\lambda^+$-directed colimits of $M$.

The equivalence is induced by the composition

$$
\mathcal{K}_{\geq \lambda} \xrightarrow{y} \text{Set}(\mathcal{K}_{\geq \lambda})^{\text{op}} \xrightarrow{r} \text{Set}^{M^{\text{op}}} \longrightarrow M^{\text{op}}\text{-Set}
$$

where $y$ is the Yoneda embedding, the second map is restriction, and the final map is the obvious equivalence $\text{Set}^{M^{\text{op}}} \xrightarrow{\sim} M^{\text{op}}\text{-Set}$. 
The assignment is:

\[ N \in \mathcal{K}_{\geq \lambda} \mapsto \text{Hom}_\mathcal{K}(C, N) \]

where \( M = \text{Hom}_\mathcal{K}(C, C) \) acts by precomposition.

That this gives the desired equivalence is an exercise in definitions.

This amounts to an astonishing transformation of a very abstract entity—an AEC—into a category of relatively simple algebraic objects. How useful this might be is less clear...
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- Model-theoretic methods and notions are developed within an abstract institution, hence are ”institution-independent” and pass to all of the particular logics falling under this umbrella. This covers an awful lot of logics: FOL, $L_{\omega}$, HOL, IPL, MFOL, MPL, temporal and behavioral logics, and so on.
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- $\text{Sen}^\mathcal{I}$ is a functor from $\text{Sig}^\mathcal{I}$ to $\text{Set}$ that assigns
  (a) to each $\Sigma$ in $\text{Sig}^\mathcal{I}$ a set of “sentences” $\text{Sen}^\mathcal{I}(\Sigma)$, and
  (b) to each signature map $\phi : \Sigma \to \Sigma'$ a “translation” $\text{Sen}^\mathcal{I}(\phi) : \text{Sen}^\mathcal{I}(\Sigma) \to \text{Sen}^\mathcal{I}(\Sigma')$. 

$\mathit{Mod}^\mathcal{I}$ is a functor from $(\mathbf{Sig}^\mathcal{I})^{\text{op}}$ to $\mathbf{CAT}$ that assigns
(a) to each $\Sigma$ a category of “$\Sigma$-models,” $\mathit{Mod}^\mathcal{I}(\Sigma)$, and
(b) to each signature map $\phi : \Sigma \rightarrow \Sigma'$ a functor
$\mathit{Mod}^\mathcal{I}(\phi) : \mathit{Mod}^\mathcal{I}(\Sigma') \rightarrow \mathit{Mod}^\mathcal{I}(\Sigma)$. 
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**Motivation:** if $\phi : \Sigma \to \Sigma'$ is an inclusion of signatures, and
$\text{Mod}^\mathcal{I}(\Sigma)$ and $\text{Mod}^\mathcal{I}(\Sigma')$ the corresponding categories of
structures, we have the reduct

$$\mid_\Sigma : \text{Mod}^\mathcal{I}(\Sigma') \to \text{Mod}^\mathcal{I}(\Sigma)$$
Mod\textsuperscript{I} is a functor from (Sig\textsuperscript{I})\textsuperscript{op} to CAT that assigns
(a) to each Σ a category of “Σ-models,” Mod\textsuperscript{I}(Σ), and
(b) to each signature map φ : Σ → Σ′ a functor
Mod\textsuperscript{I}(φ) : Mod\textsuperscript{I}(Σ′) → Mod\textsuperscript{I}(Σ).

Motivation: if φ : Σ → Σ′ is an inclusion of signatures, and
Mod\textsuperscript{I}(Σ) and Mod\textsuperscript{I}(Σ′) the corresponding categories of
structures, we have the reduct

\[ \mid_Σ : Mod\textsuperscript{I}(Σ′) \to Mod\textsuperscript{I}(Σ) \]

This is the template for Mod\textsuperscript{I}(φ)…
for each $\Sigma$ in $\text{Sig}^I$,

$$\models^I_\Sigma \subseteq |\text{Mod}^I(\Sigma)| \times \text{Sen}^I(\Sigma)$$

is a relation, “$\Sigma$-satisfaction,” which ensures the ingredients behave as in any concrete logic: for each map $\phi : \Sigma \to \Sigma'$,

$$M' \models^I_\Sigma, \text{Sen}^I(\phi)(s) \text{ iff } \text{Mod}^I(\phi)(M') \models^I_\Sigma s$$

for any sentence $s \in \text{Sen}^I(\Sigma)$ and $M' \in |\text{Mod}^I(\Sigma')|$. 
A fundamental difference with this approach is that we are not tied to a fixed signature, and can pass freely to reducts and expansions, making available many essential tricks from classical model theory:
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Michael Lieberman University of Pennsylvania
Categories in abstract model theory
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- ...

See, in particular, recent work of Diaconescu, who has a (mostly) institution-independent analog of the Keisler-Shelah theorem from first order: any two elementarily equivalent models have isomorphic ultrapowers.
This seems promising...
Accessible Categories and AECs:

- Lieberman, Michael. Category-theoretic aspects of AECs. To appear in APAL.
Further Reading II

AEC Context:


