

# GALOIS TYPES AND TOPOLOGY

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AMS Sectional Meeting 2007

Motivation:

### Question

*In the context of AECs with amalgamation, can one introduce a topological structure on Galois types in such a way that familiar model-theoretic properties have topological analogues?*

The answer is yes. Tameness translates particularly nicely. Also:

### Question

*Does the topological structure provide a mere translation into topological language, or does it allow the development of nontrivial new machinery?*

Preliminary results suggest that the latter disjunct may be true.

For an arbitrary AEC  $\mathcal{K}$  with amalgamation and model  $M \in \mathcal{K}$ , we will:

- ▶ Define a topology (really, a spectrum of topologies) on  $\text{ga-S}(M)$ , and explore the connection between properties of these spaces and properties of the class  $\mathcal{K}$ .
- ▶ Define, for sufficiently tame  $\mathcal{K}$ , a closely related notion of rank (really, a spectrum of ranks) on  $\text{ga-S}(M)$ , and examine the information it encodes about  $\mathcal{K}$ .

Let  $\mathcal{K}$  be an AEC with monster model  $\mathfrak{C}$ . Let  $\lambda \geq \text{LS}(\mathcal{K})$  and  $M \in \mathcal{K}$ .

### Definition ( $X_M^\lambda$ )

For each  $N \prec_{\mathcal{K}} M$  of size less than or equal to  $\lambda$  and type  $p \in \text{ga-S}(N)$ , let

$$U_{p,N} = \{q \in \text{ga-S}(M) : q \upharpoonright N = p\}$$

The sets  $U_{p,N}$  form a basis for a topology on  $\text{ga-S}(M)$ . We denote by  $X_M^\lambda$  the set  $\text{ga-S}(M)$  endowed with this topology.

### Note

The  $U_{p,N}$  are, in fact, clopen. Types over small submodels play a role analogous to that of formulas in topologizing spaces of syntactic types.

## Remark

The assignment  $(M, \lambda) \mapsto X_M^\lambda$  is functorial in both arguments:

- ▶ For any  $\mu > \lambda$ , the set-theoretic identity map  $\text{Id}_{\mu, \lambda} : X_M^\mu \rightarrow X_M^\lambda$  is continuous.
- ▶ For  $M, M' \in \mathcal{K}$  and  $f : M \rightarrow M'$  a  $\mathcal{K}$ -embedding, the induced map from  $\text{ga-S}(M')$  to  $\text{ga-S}(M)$  is a continuous surjection from  $X_{M'}^\lambda$  to  $X_M^\lambda$ .

So for each  $M \in \mathcal{K}$ , we obtain a well-behaved spectrum of spaces, with topological properties passing up and down the line.

A first result linking model theory and topology:

### Theorem (Tameness As Separation Principle)

*The AEC  $\mathcal{K}$  is  $(\kappa, \lambda)$ -tame iff for all  $M \in \mathcal{K}_\kappa$ ,  $X_M^\lambda$  is Hausdorff.*

One can show that for any AEC  $\mathcal{K}$  and  $M \in \mathcal{K}$ , the space  $X_M^\lambda$  is Hausdorff iff it is  $T_1$  iff it is  $T_0$ , yielding the more precise statement

### Theorem

*The AEC  $\mathcal{K}$  is  $(\kappa, \lambda)$ -tame iff for all  $M \in \mathcal{K}_\kappa$ ,  $X_M^\lambda$  is Hausdorff ( $T_1$ ,  $T_0$ ).*

We obtain an analogous result for the less parametrized notion of  $\chi$ -tameness:

### Theorem

*The AEC  $\mathcal{K}$  is  $\chi$ -tame iff for every  $M \in \mathcal{K}$ ,  $X_M^\chi$  is Hausdorff ( $T_1$ ,  $T_0$ ).*

Since each space  $X_M^\lambda$  has a basis of clopens (i.e. the  $U_{p,N}$ ) we have a bit more:

### Proposition

*$\mathcal{K}$  is  $(\kappa, \lambda)$ -tame iff  $X_M^\lambda$  is totally disconnected for every  $M \in \mathcal{K}_\kappa$ .*

And similarly for  $\chi$ -tameness.

## Remark

*One can show that the  $X_M^\lambda$  are uniform spaces. So we may consider the behavior of nets (Cauchy, convergent) in  $X_M^\lambda$ .*

Preliminary results include:

- ▶ If  $\mathcal{K}$  is  $(\chi, \leq \lambda)$ -tame and for all  $M \in \mathcal{K}_\lambda$  there exists  $\mu$  with  $\chi \leq \mu < cf(\kappa)$  such that  $X_M^\mu$  is complete, then  $\mathcal{K}$  is  $(\kappa, \lambda)$ -compact.
- ▶ If for every  $M \in \mathcal{K}_\lambda$  there exists a  $\mu < cf(\kappa)$  such that  $X_M^\mu$  is  $T_0$ ,  $\mathcal{K}$  is  $(\kappa, \lambda)$ -local.

## Fact

*For any  $\mathcal{K}$ ,  $M \in \mathcal{K}$ , and  $\lambda \geq LS(\mathcal{K})$ ,  $X_M^\lambda$  is completely regular.*

Naturally, compactness is too much to hope for. In particular,

### Proposition

*Let  $\mathcal{K}$  be an arbitrary AEC with monster model,  $M \in \mathcal{K}$ , and  $\lambda \geq LS(\mathcal{K})$ . Then  $X_M^\lambda$  is not compact.*

We might hope for some weaker form of compactness, but this proves incompatible with our desire for tameness. The critical complication results from the following:

### Fact

*For any  $\mathcal{K}$  and  $M \in \mathcal{K}$ , the intersection of any  $\lambda$  many open sets in  $X_M^\lambda$  is open.*

Because the spaces  $X_M^\lambda$  are completely regular, they are  $p_\lambda$ -spaces. If  $\mathcal{K}$  is sufficiently tame to guarantee Hausdorffness:

### Proposition

*Let  $\mathcal{K}$  be  $(\kappa, \lambda)$ -tame. Then for any  $M \in \mathcal{K}_\kappa$  and any  $\mu \geq \lambda$ , the space  $X_M^\mu$  is not countably compact.*

Consolation:  $(\kappa, \lambda)$ -compactness implies a very restricted (but compactness-flavored) intersection property:

### Theorem (Topological Compactness, $(\kappa, \lambda)$ -Compactness)

*Suppose  $\mathcal{K}$  is  $(\kappa, \lambda)$ -compact, and let  $M \in \mathcal{K}_\lambda$ . Given any family of sets  $U_{p_i, N_i} \subseteq X_M^\mu$ ,  $i < \kappa$ , where the  $N_i$  form a continuous  $\prec_{\mathcal{K}}$ -increasing sequence with union  $M$ , if the family has the finite intersection property,  $\bigcap_{i < \kappa} U_{p_i, N_i}$  is nonempty.*

Other important consequences of sufficient tameness:

### Proposition

Let  $\mathcal{K}$  be  $(\kappa, \lambda)$ -tame. Then for any  $M \in \mathcal{K}_\kappa$  and any  $\mu \geq \lambda$ ,

- ▶ If  $q \in X_M^\mu$  has a neighborhood of size less than or equal to  $\mu$ , it is an isolated point.
- ▶ A type  $q \in X_M^\mu$  is an accumulation point of  $S \subseteq X_M^\mu$  only if every neighborhood of  $q$  contains more than  $\mu$  elements of  $S$ .

The latter will be worth remembering in the discussion of ranks.

More on isolated points:

### Note

*A type of the form  $ga\text{-tp}(a/M)$  with  $a \in M$  is always isolated in  $X_M^\lambda$ . There is no reason to think that these are the only isolated types in  $X_M^\lambda$ .*

### Theorem

*If  $M$  is  $\lambda^+$ -saturated, isolated points are dense in  $X_M^\lambda$ .*

There is a partial converse, if not a full one:

### Theorem

*If  $\{ga\text{-tp}(a/M) : a \in M\}$  is a dense subset of  $X_M^\lambda$ ,  $M$  is  $\lambda^+$ -saturated.*

The moment we introduce a topology, we have a notion of rank:

### Definition ( $CB^\lambda$ )

For any  $M \in \mathcal{K}$  and  $q \in \text{ga-}S(M)$ , we define  $CB^\lambda[q]$  to be the Cantor-Bendixson rank of  $q$  in  $X_M^\lambda$ .

As one would hope,

### Theorem

If  $CB^\lambda$  is ordinal-valued on  $\text{ga-}S(M)$ , isolated types are dense in  $X_M^\lambda$ .

More interestingly, we may define a related, slightly Morley-like rank.

Assume  $\mathcal{K}$  is  $\chi$ -tame.

### Definition ( $\text{RM}^\lambda$ )

For  $\lambda \geq \chi$ , we define  $\text{RM}^\lambda$  by the following induction: for any  $q \in \text{ga-S}(M)$  with  $|M| \leq \lambda$ ,

- ▶  $\text{RM}^\lambda[q] \geq 0$ .
- ▶  $\text{RM}^\lambda[q] \geq \alpha$  for limit  $\alpha$  if  $\text{RM}^\lambda[q] \geq \beta$  for all  $\beta < \alpha$ .
- ▶  $\text{RM}^\lambda[q] \geq \alpha + 1$  if there exists a structure  $M' \succ_{\mathcal{K}} M$  such that  $q$  has strictly more than  $\lambda$  many extensions to types  $q'$  over  $M'$  with  $\text{RM}^\lambda[q'] \geq \alpha$ .

For types  $q$  over  $M$  of arbitrary size, we define

$$\text{RM}^\lambda[q] = \min\{\text{RM}^\lambda[q \upharpoonright N] : N \prec_{\mathcal{K}} M, |N| \leq \lambda\}.$$

The ranks  $\text{RM}^\lambda$  satisfy:

- ▶ (Monotonicity) If  $M \prec_{\mathcal{K}} M'$  and  $q \in \text{ga-S}(M')$ ,  $\text{RM}^\lambda[q] \leq \text{RM}^\lambda[q \upharpoonright M]$ .
- ▶ (Invariance) Let  $f \in \text{Aut}(\mathfrak{C})$ ,  $q \in \text{ga-S}(M)$ , and  $M' = f[M]$ . Then the type  $f[q] \in \text{ga-S}(M')$  satisfies  $\text{RM}^\lambda[f[q]] = \text{RM}^\lambda[q]$ .
- ▶ Whenever  $\lambda \leq \mu$ ,  $\text{RM}^\mu[q] \leq \text{RM}^\lambda[q]$ .
- ▶ For any  $q$  (say  $q \in \text{ga-S}(M)$ ),  $\text{CB}^\lambda[q] \leq \text{RM}^\lambda[q]$ .

### Definition ( $\lambda$ -t.t.)

We say that  $\mathcal{K}$  (or perhaps  $\mathfrak{C}$ ) is  $\lambda$ -totally transcendental if for every  $M \in \mathcal{K}$  and  $q \in \text{ga-S}(M)$ ,  $\text{RM}^\lambda[q]$  is an ordinal.

### Note

*A sample consequence: If  $\mathcal{K}$  is  $\lambda$ -t.t., isolated points are dense in  $X_M^\mu$  for any  $\mu \geq \lambda$ .*

There is no guarantee that types have unique extensions of the same  $RM^\lambda$  rank. However,

### Lemma

*Let  $M \prec_{\mathcal{K}} M'$ ,  $q \in \text{ga-S}(M)$  with an ordinal  $RM^\lambda$ -rank. There are at most  $\lambda$  extensions  $q' \in \text{ga-S}(M')$  with  $RM^\lambda[q'] = RM^\lambda[q]$ .*

Also,

### Proposition (Quasi-unique Extension)

*Let  $M \prec_{\mathcal{K}} M'$ ,  $q \in \text{ga-S}(M)$ , and say that  $RM^\lambda[q] = \alpha$ . Given any rank  $\alpha$  extension  $q'$  of  $q$  to a type over  $M'$ , there is an intermediate structure  $M''$ ,  $M \prec_{\mathcal{K}} M'' \prec_{\mathcal{K}} M'$ ,  $|M''| \leq |M| + \lambda$ , and a rank  $\alpha$  extension  $p \in \text{ga-S}(M'')$  of  $q$  with  $q' \in \text{ga-S}(M')$  as its unique rank  $\alpha$  extension.*

It would be nice if one could do vaguely classical stability theory using these ranks. There is some hope that this will be the case:

### Theorem

*If  $\mathcal{K}$  is  $\lambda$ -stable where  $\lambda$  satisfies  $\lambda^{\aleph_0} > \lambda$ , then  $\mathcal{K}$  is  $\lambda$ -t.t.*

This result is promising (there are totally transcendental AECs!), and warrants further exploration. In particular, one might hope that added compactness might allow us to weaken the condition on  $\lambda$  in the antecedent. A pressing question remains unanswered, however.

## Question

*Do these ranks provide new traction in producing stability spectrum results?*

At present, I can prove  $\lambda$ -total transcendentality implies stability in cardinalities  $\kappa$  satisfying  $\kappa^\lambda = \kappa$ , but only if one also assumes  $\lambda$ -stability—this leads to transfer results that are weaker than, e.g., the partial stability spectrum of Grossberg and VanDieren’s “Galois Stability for Tame Abstract Elementary Classes.” However, it seems certain that my first attempts can be improved upon.