Bootstrapping structural properties, via accessible images

(Joint with Jiří Rosický)

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We consider the interplay between three areas. Template:

Theorem (Boney/Unger, '16)

The following are equivalent:

1. *There is a proper class of almost strongly compact cardinals.*
2. *The powerful image of any accessible functor is accessible.*
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History: Makkai/Paré, '89; Boney, '14; L/Rosický, '16; Brooke-Taylor/Rosický, '16; Boney/Unger, '16.

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We will focus on the machinery provided by (2), and its consequences in abstract model theory.

In particular, we will consider the way in which (2) allows us to push structural properties from small objects in an abstract class to the class as a whole.

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A general λ-accessible category need not be closed under arbitrary directed colimits. . .

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"Accessible" means "λ-accessible for some λ."

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Corollary

Assuming $V = L$, $\mathcal{F}$ is not accessible.

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Theorem (Eklof/Mekler, '90)

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Notes

- *The free abelian group functor* $F : \textbf{Sets} \to \textbf{Ab}$ *is accessible, and* $\mathcal{F}$ *is its image.*

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- *The free abelian group functor F :* **Sets** $\rightarrow$ **Ab** *is accessible, and $\mathcal{F}$ is its image.*
- *$\mathcal{F}$ is closed under subobjects, hence the* powerful image *of F.*

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Let $\mathcal{L}$ be well-λ-accessible, such that there exists a $\mu_{\mathcal{L}}$-strongly compact cardinal κ. The powerful image of any λ-accessible functor to $\mathcal{L}$ that preserves $\mu_{\mathcal{L}}$-presentable objects is κ-accessible.

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Note

We assume *well-λ-accessibility*, purely to avoid technicalities in the exposition...

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For our purposes, $\mathcal{A}$ and $\mathcal{A}'$ will be finite categories, and we can identify the categories of diagrams with $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$. The forgetful functor

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If the (powerful) image of F is accessible, completability of an $\mathcal{A}$-diagram is determined by its small sub-$\mathcal{A}$-diagrams, hence we can bootstrap full completability from completability in the small...

Definition

We say that an accessible category $\mathcal{K}$ has the $< \kappa$-*JEP* if for any κ-presentable $M_0, M_1 \in \mathcal{K}$, there are $f_i : M_i \to N$ for $i = 1, 2$. We say that $\mathcal{K}$ has the *JEP* if this holds for arbitrary $M_0, M_1 \in \mathcal{K}$.

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Let $F_J : \mathcal{K}^{\mathcal{A}'} \to \mathcal{K}^{\mathcal{A}}$ be the forgetful functor that retains only the outermost objects.

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Notes

1. *$\mathcal{K}$ has the $< \kappa$-JEP just in case* $\mathbf{Pres}_\kappa(\mathcal{K})^{\mathcal{A}}$ *is contained in the image of F_J.*

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2. *$\mathcal{K}$ has the JEP just in case F_J is surjective.*
3. *The image of F_J is closed under subobjects, hence powerful.*
4. *As colimits are computed componentwise in $\mathcal{K}^{\mathcal{A}}$, $\mathcal{K}^{\mathcal{A}'}$, F_J preserves everything. Hence F_J is as accessible as $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$ are.*

Proposition

If $\mathcal{K}$ is well-λ-accessible, so are $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$. In either case, the λ-presentables are precisely the diagrams in which all objects are λ-presentable.

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So F_J satisfies the conditions of B-T/R, and thus we are ready to prove the main theorem.

Theorem

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Proof.

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Consider a pair $(M_0, M_1) \in \mathcal{K}$. Since $\mathcal{K}^{\mathcal{A}}$ is λ-accessible, it is also κ-accessible, meaning that (M_0, M_1) is a κ-directed colimit of pairs of κ-presentables. If $\mathcal{K}$ has the $< \kappa$-JEP, all pairs of κ-presentables are in the image of F_J. As the image of F_J is κ-accessible, it is closed under κ-directed colimits. That is, (M_0, M_1) is in the image of F_J. $\square$

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Thus every pair of objects in $\mathcal{K}$ is jointly embeddable. $\square$

If we drop the assumption of well-accessibility, we pick up an extra condition: $\kappa \trianglerighteq \lambda$.

Baldwin/Boney give a syntax-heavy argument for an analogue of this result for AECs.

The theorem above is a two-fold improvement, encompassing metric AECs and generalizations, and offering a tighter characterization of the compactness required of κ.

Definition

We say that $\mathcal{K}$ has the $< \kappa$-*AP* if for all cospans $M_0 \xleftarrow{f_0} M \xrightarrow{f_1} M_1$, there are $g_i : M_i \to N$ such that

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Let $F_A : \mathcal{K}^{\mathcal{A}'} \to \mathcal{K}^{\mathcal{A}}$ be the obvious forgetful functor.

In fact, the argument runs exactly as before, giving:

Theorem

Let $\mathcal{K}$ be well-λ-accessible. If κ is a $\mu_{\mathcal{K}}$-strongly compact cardinal, then if $\mathcal{K}$ has the $< \kappa$-AP, it has the AP.

This again generalizes the analogue for AECs in Baldwin/Boney.

With more care, we can obtain similar results even for the *disjoint* and *nearly disjoint* analogues of JEP and AP.

μ-strongly compact	P. images of accessible functors are accessible (Makkai/Paré; Brooke-Taylor/Rosický)

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A great deal can be said under GCH and certain weakenings, but that is a subject for another day...