Toward a new model theory

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What is classical model theory?

Model theory is an area of mathematical logic that seeks to use the tools of logic to solve concrete mathematical problems. Given a class of interesting objects, we:

- isolate the basic vocabulary needed to describe them, and
- identify the rules (expressed in this vocabulary) that characterize precisely the objects of interest.

Based on the size and complexity of this set of rules—and a little bit of first-order logic—we can often draw new and surprising conclusions...

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  $+ : (x, y) \mapsto x + y$

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  Binary function symbol $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
  $$\times : (x, y) \mapsto x \cdot y$$

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  Binary relation $\leq$ on $\mathbb{R} \times \mathbb{R}$
  $\leq (x, y)$ if and only if $x \leq y$

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Another example we'll return to:

- Natural numbers: $\langle 0, S \rangle$

What other objects are we used to talking about using the vocabulary of ordered reals? Among others: $\mathbb{Z}$, the integers. If we take $\mathbb{R}$ and $\mathbb{Z}$ with the standard interpretations of symbols in $\langle 0, 1, +, -, \leq \rangle$, though, there are serious differences between the two...

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To make this distinction clear, we need a precise and unambiguous language.

Given a vocabulary $\mathcal{V}$, we form a *first-order language* through finite combinations of symbols from $\mathcal{V}$, the equals sign, parentheses, variable symbols, and an array of logical symbols:

$\wedge$	"and"
$\vee$	"or"
$\neg$	"not"
$\rightarrow$	"implies"
$\forall x$	"for all x"
$\exists x$	"there exists an x"

Examples:

$$\neg\exists z(0 < z \wedge z < 1) \qquad (\phi_1)$$

$$\forall x \forall y[(x < y) \rightarrow \exists z(x < z \wedge z < y)] \qquad (\phi_2)$$

To ensure that we consider only objects that behave like $\mathbb{R}$, then, we should restrict our attention to those that satisfy the density condition.

In fact, to eliminate as much bad behavior as possible, we consider

$$\mathsf{Th}(\langle \mathbb{R}, 0, 1, +, -, \leq \rangle),$$

the *theory of* $\mathbb{R}$, the set of **all** first-order sentences in this vocabulary that are true in $\mathbb{R}$.

If we restrict to objects with interpretations of 0, 1, $+$, $-$, and $\leq$ satisfying all of these sentences, we've gone a long way toward characterizing $\mathbb{R}$.

In fact, we might refer to this theory as T_{RCF}, as what it really captures are the *real closed fields*, i.e. those densely ordered fields satisfying the following added conditions:

- $\forall x[x > 0 \to \exists y(y^2 = x)]$
- $\forall a_0 a_1 a_2 a_3 \exists x[a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0)]$

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Thinking back to $\mathbb{N}$, let T_S be the set of all first order sentences it satisfies in the vocabulary $\langle 0, S \rangle$. Chief among them:

- $\forall x[x > 0 \rightarrow \exists y(Sy = x)]$
- $\neg \exists x[Sx = 0]$

One of the most fundamental (and awesome) properties of first order logic: it's "compact."

Theorem (Compactness Theorem)

Version 1: Let Γ *be an infinite set of first order sentences. If* Γ *is inconsistent, then there is a finite set of sentences* $\Gamma' \subset \Gamma$ *that is itself inconsistent.*

Version 2: Let Γ *be an infinite set of first order sentences. If for any finite* $\Gamma' \subset \Gamma$ *there is an object* $X_{\Gamma'}$ *obeying all of the sentences in* Γ'*, then there is a single object that obeys the entire infinite list* Γ*.*

Nonstandard reals (Robinson, 1966)

To the basic vocabulary of the ordered reals, $\langle 0, 1, +, \times, \leq \rangle$, we add a new constant symbol α. Let Γ be the set of sentences

$$T_{RCF} \cup \{\alpha < 1, \alpha < 1/2, \alpha < 1/3, \dots\}$$

Any finite subset Γ' of Γ will consist of sentences from T (rules concerning the behavior of the reals) and finitely many of the α-sentences, say $\alpha < 1, \dots, \alpha < 1/k$.

Is there an object that behaves like the reals and contains an element $\alpha < 1/k$? Of course: the reals! Take $\alpha = 1/(2k)$.

Since every finite subset of the rules contained in Γ can be satisfied, there is an object $\mathfrak{R}$ that obeys all of them simultaneously: both the rules guaranteeing $\mathbb{R}$-like behavior, and those ensuring the existence of an infinitesimal.

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This represents a kind of vindication of Leibniz's vision of infinitesimals, although it was a very long time (almost 300 years) coming...

Starting from T_S, add a new constant c_0, and consider the set of sentences

$$\Gamma = T_S \cup \{c_0 \neq 0, c_0 \neq S0, c_0 \neq SS0, c_0 \neq SSS0, \ldots\}$$

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Explosion! Also need to worry about theories encoding the natural numbers...

T_{RCF} admits quantifier elimination (q.e.): given any formula $\phi(x)$, no matter how complex, there is an equivalent formula $\psi(x)$ without any quantifiers. That is,

$$\mathcal{R} \models \forall x[\phi(x) = \psi(x)]$$

in any model $\mathcal{R}$.

Notes

- *The quantifier-free formulas are Boolean combinations of polynomial equalities and inequalities.*
- *In real algebraic geometry, the semialgebraic sets are those defined by Boolean combinations of polynomial equalities and inequalities.*

Theorem

The projection of any semialgebraic set is semialgebraic.

Proof: All projection does is introduce an existential quantifier, which we can eliminate!

What about algebraically closed fields of fixed characteristic? The first order theory, T_{ACF_p}, also admits q.e. (Tarski, 1951).

Notes

- *The quantifier-free formulas are Boolean combinations of polynomial equalities and inequalities.*
- *In algebraic geometry, the constructible sets are those defined by Boolean combinations of polynomial equalities and inequalities.*

By the same token,

Theorem

The projection of any constructible set is constructible.

So a little bit of (admittedly old) model theory gives us, essentially, Chevalley's Theorem.

Lots more to say (T_{RCF} is o-minimal, T_{ACF_p} is *model complete*), but no time. So, what could possibly be the downside of first order model theory?

We've already noticed that first order logic can't help us distinguish between standard and nonstandard copies of $\mathbb{R}$ and $\mathbb{N}$, which is troubling. Here's something much worse:

Suppose we want to move on to complex analysis. We would need to expand our vocabulary:

$$\langle 0, 1, +, -, \times, \exp(\ )\rangle$$

But if we interpret exp( ) as we must, we can define a copy of the integers in any model, using the formula:

$$\exp(x) = 1$$

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This prevents the explosion, but at a cost...

This disjunction goes beyond first order: it involves countably infinitely many formulas, not finitely many. That is, we have passed from first order logic to

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($< \omega_1$ conjuncts/disjuncts, quantification over $< \omega$ variables)

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This is a fact of life—when the natural numbers are giving you trouble, you pass to $L_{\omega_1\omega}$. In other cases, more flexibility is needed:

$$L_{\kappa\lambda}$$

$(< \kappa$ conjuncts/disjuncts, quantification over $< \lambda$ variables)

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Solution: abstract the essential properties that are common to classes of models of as many of these logics as possible, then forget about syntax entirely...

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Example: $\mathcal{K}$ is the abelian groups, $\prec_{\mathcal{K}}$ is pure subgroup.

These turn up in surprising places, and in surprisingly significant ways:

- Abelian groups (Baldwin/Calvert/Goodrick/...)
- Ext-orthogonality classes of modules (Baldwin/Eklof/Trlifaj)

One can do a lot of almost-classical model theory here.

Model-theorist's consensus: AECs strike the optimal balance between generality and structure.

There's a very large body of existing knowledge not so far away, though. With a little generalization, you find yourself in the realm of accessible categories (Lieberman, Beke/Rosický).

It is possible (Lieberman/Rosický) to completely nail down what AECs are in category theoretic terms: they consist of an accessible category $\mathcal{K}$ with directed colimits, i.e. an abstract category $\mathcal{K}$ such that

- $\mathcal{K}$ is closed under directed colimits, and
- objects in $\mathcal{K}$ can be approximated from below by small subobjects,

together with a faithful, iso-full functor $U : \mathcal{K} \to \mathbf{Sets}$ that preserves directed colimits, and is "coherent."

In a forthcoming paper, we show that several significant recent results on AECs hold even if the coherence assumption is dropped, often yielding cleaner arguments. So there's something for the model theorists.

As we develop more and more of the tools of AECs in this context, there is also the very real prospect of learning more about the structure of accessible categories themselves—results that should filter more readily into the world of categorical algebra...