Universal classes and locally multipresentable categories

(Joint with Jiří Rosický and Sebastien Vasey)

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We will consider some useful—and under-appreciated—categories, namely

1. Locally μ-multipresentable categories
2. Locally μ-polypresentable categories

We will also look at a few crucial classes from abstract model theory, starting with μ-abstract elementary classes, but focusing on

1. Universal μ-AECs
2. μ-AECs admitting intersections

Punchline

Up to equivalence of categories, these are, respectively, the exact same things.

Recall that a category is locally μ-presentable precisely if and only if it is μ-accessible and contains all (small) limits.

Locally multipresentable and locally polypresentable categories arise by weakening the limit requirement. One sense:

Theorem (Diers '80)

A category is locally μ-multipresentable iff it is μ-accessible and has all connected limits.

Theorem (Lamarche, '88)

A category is locally μ-polypresentable iff it is μ-accessible and has wide pullbacks.

Arguably, the more interesting characterizations are in terms of multicolimits and polycolimits, to which we now turn.

Everything springs from the definition of a multiinitial object, which is much more natural than it sounds:

Example

Let **Fld** *be the category of fields and field homomorphisms. It has an initial object. . . for each individual characteristic: the prime field. That is, there is a family of objects that jointly play the role of the initial object.*

Definition

Let $\mathcal{K}$ be a category. A *multiinitial object* in $\mathcal{K}$ is a set of objects $\mathcal{I}$ such that for any M in $\mathcal{K}$, there is a unique $i \in \mathcal{I}$ with an arrow $i \to M$, and this arrow is unique.

Definition

Let $\mathcal{K}$ be a category, $D : J \to \mathcal{K}$ a diagram. The *multicolimit of D* is the family of initial objects in the category of cocones on D.

Definition

We say that a category is locally μ-multipresentable if it is μ-accessible and has all (small) multicolimits.

Examples

The following categories are locally $\aleph_0$-multipresentable:

1. **Fld**, *the category of fields and homomorphisms.*
2. **Lin**, *the category of linear orders and order-preserving maps.*
3. **pHilb**, *the category of pre-Hilbert (not necessarily complete) spaces and linear orthogonal maps.*

Hilb, *the category of Hilbert spaces and linear orthogonal maps is locally $\aleph_1$-multipresentable.*

Theorem (Adámek/Rosický)

For any category $\mathcal{K}$, the following are equivalent:

1. *$\mathcal{K}$ is locally μ-multipresentable.*
2. *$\mathcal{K}$ is μ-accessible and has all connected limits.*
3. *$\mathcal{K}$ is equivalent to a μ-cone-orthogonality class in a μ-locally presentable category.*

Recall that an object M is orthogonal to a cone $(a_i : A \to A_i)_{i \in I}$ if for any $f : A \to M$ there is a unique $i \in I$ such that f factors through a_i (uniquely).

A μ-cone-orthogonality class consists of objects orthogonal to a family of cones consisting of μ-presentable objects.

This provides the link to universal classes...

A polyinitial object is a slight weakening of the notion of a multiinitial object, which is also easily motivated:

Example

Consider the category ACF of algebraically closed fields and field homomorphisms. In each characteristic, objects admit an embedding of the algebraic closure of the corresponding prime field, but this embedding is not unique, strictly speaking. There is more than one way to embed $\bar{\mathbb{Q}}$ in $\mathbb{C}$, for example, but these embeddings are identical up to unique automorphisms of $\mathbb{C}$ fixing $\mathbb{Q}$.

Definition

Let $\mathcal{K}$ be a category. A *polyinitial object* in $\mathcal{K}$ is a set of objects $\mathcal{I}$ with the property that for any M in $\mathcal{K}$, there is a unique $i \in \mathcal{I}$ with a map $i \to M$. For any two such maps, $f, g : i \to M$, there is an isomorphism $h : M \to M$ such that $hf = g$.

As before, we define the *polycolimit* of a diagram $D : J \to \mathcal{K}$ to be the polyinitial object in the category of cocones on D, and:

Definition

We say that a category $\mathcal{K}$ is *locally μ-polypresentable* if it is μ-accessible and has all polycolimits.

Example

ACF *is locally $\aleph_0$-polypresentable.*

Theorem (Lamarche, '88)

For any category $\mathcal{K}$, the following are equivalent:

1. *$\mathcal{K}$ is locally μ-polypresentable.*
2. *$\mathcal{K}$ is μ-accessible and has wide pullbacks.*

This provides the link to classes admitting intersections.

Recall the definition of a μ-abstract elementary class (or μ-AEC):

Definition (Boney/Grossberg/L/Rosický/Vasey)

An abstract class of structures $\mathcal{K}$ in a μ-ary signature is a μ-AEC if it satisfies the AEC axioms, but with the following modifications:

- $\mathcal{K}$ is only assumed to have μ-directed colimits, and
- there is λ with $\lambda^{<\mu} = \lambda$ such that for any $A \subseteq M \in \mathcal{K}$, there is $A \subseteq N \prec_{\mathcal{K}} M$ and $|UN| \leq |A|^{<\mu} + \lambda$. Define $LST(\mathcal{K})$ to be least such λ.

Note: AECs are $\aleph_0$-AECs; metric AECs are $\aleph_1$-AECs.

Examples

More broadly, μ-AECs encompass: μ-saturated models in an AEC with amalgamation, μ-complete Boolean algebras, models of $L_{\lambda\mu}(Q^\chi)$-sentences, and so on.

Universal(ly axiomatizable) classes have a long pedigree in model theory. We use a nonsyntactic formulation:

Definition (L/Rosický/Vasey)

Let $\mathcal{K}$ be a class of structures in a fixed μ-ary signature $L = L(\mathcal{K})$. We say that $\mathcal{K}$ is a *μ-universal class* if it is closed under isomorphism, L-substructures, and unions of μ-directed systems of L-structures inclusions.

These have the expected syntactic characterization: Tarski has shown for finite, finitary signatures that any ($\aleph_0$-)universal class is axiomatized by universal sentences in $L_{\omega\omega}$. This generalizes.

Theorem (Vasey)

For $\mathcal{K}$ a class of structures in a μ-ary signature L, TFAE:

1. *$\mathcal{K}$ is a μ-universal class.*
2. *$\mathcal{K}$ is the class of models of a universal $L_{\infty\mu}$-theory.*

Examples

1. *Fields form an $\aleph_0$-universal class, and are $L_{\omega\omega}$*
2. *Locally finite groups are $\aleph_0$-universal, but not $L_{\omega\omega}$.*

We note that any μ-universal class (with substructure) is a μ-AEC: we call this, briefly, a *universal μ-AEC*.

Fact

Equivalently, a μ-AEC $(\mathcal{K}, \prec_{\mathcal{K}})$ is universal iff for any $N \subseteq M$, $M \in \mathcal{K}$, then $N \in \mathcal{K}$ and $N \prec_{\mathcal{K}} M$.

Definition

We say that a μ-AEC $(\mathcal{K}, \prec_{\mathcal{K}})$ *admits intersections* if for every $M \in \mathcal{K}$ and $A \subseteq |M|$,

$$\mathrm{cl}^M_{\mathcal{K}}(A) = \bigcap \{N \in \mathcal{K} \mid N \prec_{\mathcal{K}} M, A \subseteq UN\}$$

is the universe of a strong submodel of M.

Examples

1. *The ($\aleph_0$-)AECs $({}^{\perp}N, \prec_N)$ of Baldwin/Eklof/Trlifaj admit intersections, just in case ${}^{\perp}N$ is cotilting.*
2. *Any universal μ-AEC admits intersections.*

There is an intermediate case, incidentally:

Definition

We say that a μ-AEC $(\mathcal{K}, \prec_{\mathcal{K}})$ is pseudouniversal if it admits intersections and for any $M \in \mathcal{K}$, $A \subseteq M$, and embeddings f, g on M, if $f \upharpoonright A = g \upharpoonright A$ then $f \upharpoonright \mathrm{cl}^M_{\mathcal{K}}(A) = g \upharpoonright \mathrm{cl}^M_{\mathcal{K}}(A)$.

As you might expect, any universal μ-AEC is pseudouniversal.

In fact, any μ-pseudouniversal class admits a functorial expansion to a μ-universal class—in particular, this induces an equivalence of categories. So: a distinction without a difference.

We recall that μ-AECs are, up to equivalence of categories, just accessible categories with all morphisms mono. To be precise,

Theorem (BGLRV)

1. *Any μ-AEC $\mathcal{K}$ is an $LST(\mathcal{K})^+$-accessible category with all morphisms mono (or μ-accessible, if it admits intersections).*
2. *Any μ-accessible category with all morphisms mono is (equivalent to) a μ-AEC.*

Having recalled this, it is essentially trivial to see:

Theorem (LRV)

Let $\mathcal{K}$ be a category, μ regular. Then TFAE:

1. *$\mathcal{K}$ is a μ-polypresentable category with all morphisms mono.*
2. *$\mathcal{K}$ is equivalent to a μ-AEC admitting intersections.*

Matching locally μ-multipresentable categories and universal μ-AECs is less trivial.

Idea

Working with syntactic diagrams of objects, μ-cone-orthogonality can be coded by sentences of (roughly) the form

$$\forall \bar{x}[\phi(\bar{x}) \to \exists ! \bar{y} \left(\bigvee_{i \in I} \psi_i(\bar{x}, \bar{y}) \right)]$$

where ϕ, ψ are conjunctions of atomic formulas. Skolemize.

Theorem (LRV)

Let $\mathcal{K}$ be a category, μ regular. TFAE:

1. *$\mathcal{K}$ is locally μ-multipresentable with all morphisms mono.*
2. *$\mathcal{K}$ is the class of models of a universal $L_{\infty,\mu}$-theory.*
3. *$\mathcal{K}$ is a universal μ-AEC.*