A category-theoretic characterization of almost measurable cardinals

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We work in a familiar interval in the large cardinal hierarchy:

[weakly compact . . . strongly compact]

Strongly compact: Any κ-satisfiable $L_{\kappa\kappa}$-theory T is satisfiable.

Weakly compact: As above, provided $|T| \leq \kappa$.

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Definition (e.g. [BM14])

A cardinal κ is *μ-strongly compact* if any κ-complete filter on a set of size κ can be extended to a μ-complete ultrafilter. We say κ is *almost strongly compact* if it is μ-strongly compact for all $\mu < \kappa$.

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In terms of consistency strength, there isn't much daylight between almost strongly compact and strongly compact. We won't dwell on this, but μ-strong compactness may not offer the best grading.

Theorem (Boney/Unger)

Let κ satisfy $\mu^\omega < \kappa$ for all $\mu < \kappa$. The following are equivalent:

(1) *κ is almost strongly compact.*

(2) *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below κ is κ-accessible and accessibly embedded in $\mathcal{L}$.*

(3) *Every AEC below κ is $< \kappa$-tame.*

Proof.

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Here κ-preaccessibility is basically free; that κ-directed colimits exist and are computed as in $\mathcal{L}$ follows from the large cardinal assumption. □

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Here $< \kappa$-tameness is reformulated as κ-accessibility of the powerful image of a particular forgetful functor. □

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(3) $\Rightarrow$ (1): Boney and Unger, [BU17].

They make use of a combinatorial construction which produces results for a broad range of other large cardinals... □

We say κ is *measurable* if the following equivalent conditions hold:

1. For any $L_{\kappa\kappa}$-theory T, if T is the union of an increasing chain of satisfiable theories, T is satisfiable ([CK12]/[Bon]).
2. There is a nonprincipal κ-complete ultrafilter on κ.

Definition ([BU17])

We say that a cardinal κ is *μ-measurable* if there is a *uniform* μ-complete ultrafilter on κ, i.e. one in which all sets are of size κ. We say κ is *almost measurable* if it is μ-measurable for all $\mu < \kappa$.

Facts

1. *Any almost measurable cardinal is measurable or a regular limit of measurables.*
2. *Any almost measurable cardinal is strongly inaccessible, and sharply greater than any smaller cardinal.*

The emerging picture is of measurability as a kind of chain completeness/compactness condition. We further this with:

Theorem ([Lie18])

Let κ be a strong limit cardinal. The following are equivalent:

1. *κ is almost measurable.*
2. *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below κ is κ-preaccessible and closed under colimits of κ-chains in $\mathcal{L}$.*
3. *Every AEC $\mathcal{K}$ below κ is $< \kappa$-local: if $M \in \mathcal{K}$ is a union of an increasing κ-chain $\langle M_i | i \in \kappa \rangle$ and types p, q over M satisfy $p \restriction M_i = q \restriction M_i$ for all i, $p = q$.*

The equivalence of (1) and (3) is (at least implicitly) in [BU17]. We focus on (1)⇒(2) and (2)⇒(3), which repurpose arguments of [BTR16], [LR14].

We will prove the following:

Theorem

If κ is almost measurable, then the powerful image of any (suitably size-preserving) accessible functor $F : \mathcal{K} \to \mathcal{L}$ below κ has powerful image closed under colimits of κ-chains.

Proof.

Three steps, almost exactly as in [BTR16]:
Step 1: Realize the powerful image of F as the full image of

$$H : \mathcal{P} \to \mathcal{L}$$

where $\mathcal{P}$ is the category of monos $L \to FK$, $K \in \mathcal{K}$. Everything is still nicely accessible... □

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Three steps, almost exactly as in [BTR16]:
Step 2: Embed $\mathcal{P}$ and $\mathcal{L}$ into categories of structures, and realize $\mathcal{P}$ as the category of models of an infinitary theory T—in this way, one realizes the full image of F, ultimately, as

$$\mathbf{Red}_{\mathcal{L}}(T)$$

the category of reducts of models of T to the signature of $\mathcal{L}$. □

Crucially, careful reading reveals that $\mathbf{Red}_{\mathcal{L}}(T)$ is κ-preaccessible.

Step 3: We wish to show that $\mathbf{Red}_{\mathcal{L}}(T)$ is closed under colimits of κ-chains—take a κ-chain $\langle M_i : i < \kappa \rangle$ in $\mathbf{Red}_{\mathcal{L}}(T)$, and consider its colimit M in $\mathbf{Str}(\mathcal{L})$. It suffices to show there is a mono $M \to N$, $N \in \mathbf{Red}_{\mathcal{L}}(T)$. Take

$$T_M : \text{atomic/negated atomic diagram of } M, \text{ names } c_m \text{ for } m \in M$$

It suffices to exhibit a model $N \models T \cup T_M$.
Take a uniform, sufficiently-complete ultrafilter $\mathcal{U}$ on the index set κ, and set

$$N = \prod_{\mathcal{U}} M_i.$$

One can expand the M_i to interpret the c_m so that ϕ is in T_M just in case ϕ holds on a $\mathcal{U}$-large set of M_i. By Łoś, then, N works.

We work in an AEC $\mathcal{K}$, although an arbitrary concrete accessible category will do.

Definition

A *Galois type* over $M \in \mathcal{K}$ is an equivalence class of pairs (f, a), $f : M \to N$ and $a \in UN$. We say $(f_0, a_0) \sim (f_1, a_1)$ if there is an object N and morphisms $g_i : N_i \to N$ such that the following diagram commutes

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

with $U(g_0)(a_0) = U(g_1)(a_1)$.

Assuming the amalgamation property, which we do, this is in fact an equivalence relation.

Definition

We say that Galois types in $\mathcal{K}$ are κ-*local* if for any object M, any continuous κ-chain

$$M_0 \to M_1 \to \cdots \to M_i \to \dots$$

with colimit M and colimit coprojections $(\phi_i : M_i \to M)$, and any pair $(f_0 : M \to N_0, a_0)$ and $(f_1 : M \to N_1, a_1)$, if

$$(\phi_i f_0, a_0) \sim (\phi_i f_1, a_1)$$

for all $i < \kappa$, then

$$(f_0, a_0) \sim (f_1, a_1).$$

We turn this into a question about powerful images in precisely the same way as in [LR14].

1. $\mathcal{L}_1$: Category of diagrams witnessing equivalence of pairs:

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

   with selected elements $a_i \in UN_i$, $U(g_0)(a_0) = U(g_1)(a_1)$.

2. $\mathcal{L}_2$: Category of pairs:

$$\begin{array}{ccc} N_0 & & \\ {\scriptstyle f_0}\uparrow & & \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

   with selected elements $a_i \in UN_i$.

3. Let $F : \mathcal{L}_1 \to \mathcal{L}_2$ be the obvious forgetful functor.

In [LR14], it is shown that κ-accessibility of the powerful image of F implies $< \kappa$-tameness. We proceed similarly, but assuming only closure under colimits of κ-chains.

Proposition

If κ is such that any accessible functor below κ has powerful image closed under colimits of κ-chains, every AEC below κ is κ-local.

We note:

Facts

1. *The functor F is accessible. If $\mathcal{K}$ is below κ, so is F.*
2. *The powerful image of F consists of precisely the equivalent pairs of (representatives of) types.*

With these facts, there is essentially nothing to do.

Proof.

Suppose M is the colimit of a κ-chain

$$M_0 \to M_1 \to \cdots \to M_i \to \dots$$

with colimit maps $\phi_i : M_i \to M$. Suppose types (f_j, a_j), $j = 0, 1$, are equivalent over any M_i, i.e.

$$\begin{array}{ccc} N_0 & & \\ {\scriptstyle f_0\phi_i}\uparrow & & \\ M_i & \underset{f_1\phi_i}{\longrightarrow} & N_1 \end{array}$$

is always in the powerful image of F. Since the original pair is the colimit of this κ-chain, and the powerful image is closed under such colimits, they are equivalent as well. □

That completes the promised proof of:

Theorem ([Lie18])

Let κ be a strong limit cardinal. The following are equivalent:

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(3) *Every AEC $\mathcal{K}$ below κ is $< \kappa$-local.*

This points in many new directions:

1. Some technical details (omitted here) may be simplified if we restrict to nice accessible categories...
2. This analysis of [BTR16] provides an engine for generating more equivalences...

References I

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