Comenius University Talk: 18/12/2012

Thank you very much. Naturally I'm delighted to have the opportunity to speak here. My plan is to give a broad introduction to classical model theory – what it is, what it's good for. If there's time, I'll introduce abstract model theory, which is what I do in real life.

Slogan: Model theory = universal algebra + logic

Idea: Given interesting class of structures
- Isolate basic vocabulary needed to describe
- Determine (if poss.) the theory = set of sentences in vocab satisfied by precisely objects of interest

Basics:

Def: A vocabulary/signature is a set of const, fun, rel symbols

$$\Sigma = \mathcal{C} \cup \mathcal{F} \cup \mathcal{R}$$

Gps: $\Sigma_G = \{e, m(-,-), i(-)\}$

Def: A structure X for Σ is an underlying set $|X|$, interpretations of basic symbols:

$$c \in \mathcal{C} \longrightarrow c^X \in |X|$$
$$f \in \mathcal{F} \longrightarrow f^X : |X|^k \to |X|$$
$$R \in \mathcal{R} \longrightarrow R^X \subseteq |X|^k$$

Gps Σ_G-struct: $\langle \mathbb{Z}, 0, +, - \rangle$

$\langle \mathbb{Z}, 3, +, - \rangle$

or $+(\bar{x}, \bar{y}) := x + y - 1$, etc.

Language from Σ: $L_{\omega,\omega}(\Sigma)$. Built from Σ, ctbly many var symbols, logical symbols $\wedge, \vee, \neg, \rightarrow, \forall, \exists$

Nonexample: element not of finite order

$$\exists x \left(\bigwedge_{n \in \mathbb{N}} x^n \neq e \right)$$

← ctble not of conjunction (infinitary)

Semantics: Given Σ-str X, Σ-form φ, say X sats φ or believes φ, den $X \models \varphi$, if interps of symbols in φ actually sat the relationship it describes.

Ex: G gp. $G \models \forall x [m(x, e) = x]$

$G \not\models \forall x [m(x,y) = m(y,x)]$ not nec.

Want to use sentences to pick out structures w/ des properties — theories.

Say X model of theory T if $X \models \varphi$ for all $\varphi \in T$.

Ex: $T_{Gp} = \{$ axioms of gps $\} \cup \{$ logical consequences $\}$

Note: this theory is incomplete — certain sentences not det abelian/nonabelian

If class given as $Mod(T)$, T f.o. complete, elementary class

Ex: 1. T_{inf}

2. T_{DLO}

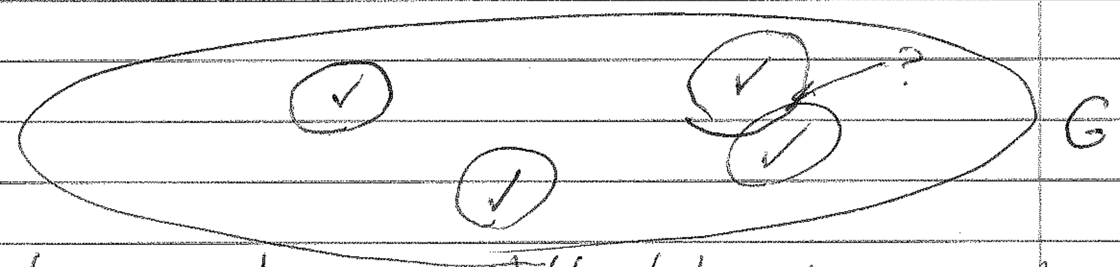
3. T_{ACF_p}

4. T_{RCF}

Applications: Compactness Thm (Gödel): Let Γ be set of sentences. If any finite subset $\Gamma' \subseteq \Gamma$ has model $M' \models \Gamma'$, then there is a model satisfying <u>all of</u> Γ!

Surprising, def property of f.o. Gives surp local to global results.

Ex: De Bruijn/Erdős. Let G be infinite graph. If any f. subgraph can be colored by k colors, then G can be colored by k colors.



f colorings may be incompatible, but cptness guarantees patch.

Immensely powerful, used in constr of
- nonstandard reals (infinitesimals)
- nonstandard nats/ints (elts at infinity)

Categoricity: T ctble, complete. How many models size λ, λ inf card. When is unique model in λ — λ-categorical

T_{inf}: cat in all inf λ Totally categorical

T_{ACF_0}: cat in $\lambda \geq \aleph_1$ (Steinitz)

ctbly cat?

$\overline{\mathbb{Q}}, \overline{\mathbb{Q}(t_1)}, \overline{\mathbb{Q}(t_1, t_2)}, \ldots$

trans

in fact, class by transc degree $\longrightarrow$ ctbly many

Uncountably categorical

T_{DLO}: Cantor: $\mathbb{Q}$ only ctble model (sat, b&f)

many unctble models

Countably categorical

Morley: For ctble complete T, only possibilities

[Shelah: gen to uncountable T ...]

Proofs gave rise to almost all of the essential machinery of f.o. model theory, massive dividends in and out of MT.

Definable sets: Given $M \models T$, say $X \subseteq |M|^n$ is defined by $\varphi(\bar{x})$ if

$$X = \{ \bar{m} \in |M| \mid M \models \varphi(\bar{m}) \}$$

Say X definable if is such a formula.

How complex are these defining formulas? Sometimes very simple.

Quantifier elimination: T admits q.e. if any def subset of $M \models T$ can be def by formula w/o quantifiers

Ex: T_{ACF_p} any set def by polynomial eqns, ineqns

$\hookrightarrow$ q.e. equiv to statement that constr ~~...~~ closed under proj: Chevalley's Thm.

Ax-Kochen $\longrightarrow$ number theory

O-minimality: T ~~...~~ th of ordered structures.
T O-minimal, if every df set ~~finite union of intervals~~ ~~and~~ df by q.f. involving only order,

$$a = x, \quad a \leq x, \quad x \leq b$$

Equiv. Any definable set is f union of intervals, points.

Ex: T_{RCF} O-minimal. Def sets in $\langle \mathbb{R}, +, \cdot, 0, 1 \rangle$ f unions pts, intervals.

Wilkie: Can add e^x:

$$\langle \mathbb{R}, +, \cdot, 0, 1, e^x \rangle \quad \text{0-minimal}$$

Miller, Starchenko, et al: expansions by other (restr) analytic fns

Strongly minimal: Set def iff actually finite or cofinite
Very, very strong.

T_{ACF_p} s. minimal.

In s.min, any unctble model controlled by set w/ well-def notion of independence, basis, dimension

→ Hrushovski, Geometric Mordell-Lang

Special Case: Zilber/Hrushovski: If T totally cat, models are ess
1. discrete sets
2. v.s. over fixed div ring
3. alg closed fields of fixed char

Problem: Many properties tend non-first-order

Exercise: Connectedness cannot be ax by f.o. sentence
Sketch: By contr, compactness. Suppose Φ sentence that works.

s.t. $T_{Graph} \cup \{\Phi\} \cup \{L_i(s,t)\}_{i \in \mathbb{N}}$
$L_i(s,t)$: no path length i b/w s and t
finitely satisfiable.

Others: I Archimedean rngs (chain cond)
II Banach spaces (completeness, no direct lims)
III $\mathbb{C}$ w/ exp

Last one: $\langle \mathbb{C}, +, \cdot, 0, 1, \exp \rangle$ can define $\mathbb{Z}$ as ker of $\exp : \mathbb{C} \to \mathbb{C}^*$.

Zilber: solve prob by pinning down $\mathbb{Z}$ from start:

$$\forall x \left(\exp(x) = 1 \longrightarrow \bigvee_{n \in \mathbb{Z}} x = 2n\pi \right)$$

ctble disj, $L_{\omega_1,\omega}$

infinitary

$L_{\kappa\lambda}$ $L(Q)$

$<\kappa$ conj/disj $Q x \varphi(x)$: exist uncountably many x

$<\lambda$ quants

$L_{\kappa\lambda}(Q)$

Each of these logics has own peculiar idiosyncratic properties, (almost never compact), so

1. F.O. results don't generalize nicely
2. results on MT of one don't transfer to others

How to generalize?

Shelah: drop logic, consider abstract classes of structures that have just a few essential props of classes of models

Abstract Elementary Classes