Classification Theory for Accessible Categories

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- Generalizing AMT: Behavior under a large cardinal assumption.

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- Avoiding (odd) syntax: an alternative to Shelah's Presentation Theorem.
- Generalizing AMT: Behavior under a large cardinal assumption.
- Rethinking categorical algebra: a robust, generalized Ehrenfeucht-Mostowski construction.

The fundamental question: given a class of objects axiomatizable in first order logic, how many are there (up to isomorphism) of each cardinality? In particular, in which cardinalities is it *categorical*, i.e. has just one isomorphism class?

Note: here we consider only infinite cardinalities (else *finite model theory...*)

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Route 1: (Baldwin/Lachlan) If T is categorical in an uncountable λ, there is a uniform geometric structure on the uncountable models, and a uniform way of picking out a basis of each...

This led to the development of geometric stability theory...

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Route 2: Calculus of types.

A syntactic type in T is a set of formulas completely describing a potential element of models of T.

Best case scenario: minimum number of types—stability.

Worst case scenario: lots of types, intractable theory.

Morley's proof hinges on delicate balance between realizing and omitting types. Crucial ideas:

- Saturated models: realize all relevant types—"big" models. Think algebraically closed fields...
- Ehrenfeucht-Mostowski models: for linear order I, $EM(I)$ is a "lean" model built along I. This is functorial: if $I \subseteq J$, $EM(I)$ embeds in $EM(J)$. Actual construction and use wildly syntactic.

Many other important ideas, particularly when one generalizes to uncountable theories. Or to more general logics...

There are all kinds of reasons to go beyond first order logic, mostly because it just isn't expressive enough. Possible directions:

- $L_{\kappa\lambda}$: conjunctions, disjunction of fewer than κ formulas, quantification over fewer than λ variables. Special case: $L_{\omega_1\omega}$.
- $L(Q)$: adds counting quantifier Qx, i.e. "there exist uncountable many."

Many of the properties that make classical model theory work disappear. Compactness, especially, only holds for $L_{\kappa\kappa}$ with κ a *compact cardinal*.

Generalize by forgetting logic, retaining essential properties of classes of models.

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- Coherence: Given any $L(\mathcal{K})$-structure embedding $f : M \to N$ and any map $g : N \to N'$ in $\mathcal{M}$, if $gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

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Note

Here we describe $\mathcal{K}$ in terms of properties of the forgetful functor $\mathcal{K} \to \mathbf{Str}(L(\mathcal{K}))$.

Shelah's Conjecture

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This is a very hard problem—we have only partial results.

Having discarded syntax, we need a new notion of type. The best candidate at present: Galois types. In AECs with amalgamation, we define:

Definition

Galois types are equivalence classes of triples (M, a, N)*, where* $M \prec_{\mathcal{K}} N$ *and* $a \in N$*. We say* $(M, a_1, N_1) \sim (M, a_2, N_2)$ *if there are* N^* *and* $h_i : N_i \to N^*$ *so that* $h_1 \restriction M = h_2 \restriction M$ *and* $h_1(a_1) = h_2(a_2)$*.*

Example: If $\mathcal{K}$ consists of models of a complete first order theory, the Galois types over $M \in \mathcal{K}$ correspond to complete syntactic types over M.

We can define Galois-stability and Galois-saturation accordingly:

- $\mathcal{K}$ is λ-Galois-stable if any $M \in \mathcal{K}$ of size λ has only λ types over it: the bare minimum.
- A model M is λ-Galois-saturated if it realizes all types over submodels of size less than λ.

Theorem

A model M is λ-Galois-saturated iff it is injective with respect to the class of morphisms $N \to N'$ with N, N' of size less than λ.

Definition

$\mathcal{K}$ is χ-tame if for any distinct types p, q over any $M \in \mathcal{K}$, there is a submodel N of M of size $\leq \chi$ on which p, q disagree.

This appears in the current best approximation of Shelah's Conjecture:

Theorem (Grossberg/VanDieren)

If $\mathcal{K}$ is χ-tame and λ^+-categorical with $\lambda \geq \chi$, it is categorical in all $\kappa > \chi$.

Earth-shattering recent result due to Will Boney:

Theorem

If there is a proper class of compact cardinals, any $\mathcal{K}$ is χ-tame for some χ.

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Definition

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Example: In **Grp**, the category of groups, an object G is finitely presentable iff G is finitely presented. Same for any finitary algebraic variety.

Definition

A category **C** is finitely accessible (ω-accessible) if

- it has at most a set of finitely presentable objects,
- it is closed under directed colimits, and
- every object is a directed colimit of finitely presentable objects.

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Example: **Ban** lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

This gives a notion of size:

Definition

For any object M in an accessible category $\mathcal{K}$, we define its **presentability rank**, $\pi(M)$ *to be the least cardinal λ such that M is λ-presentable.*

Fact

In well-behaved accessible categories, $\pi(M)$ is always a successor, say $\pi(M) = \lambda^+$. In this case we say λ is the **size** *of M.*

Theorem (L, Beke/Rosický)

If $\mathcal{K}$ is an AEC, $M \in \mathcal{K}$ is of size λ iff $|M| = \lambda$. By DLS, it follows that any AEC $\mathcal{K}$ is $LS(\mathcal{K})^+$-accessible.

In a general accessible category, objects are not structured sets. To define Galois types, we (seem to) need sets and elements around.

We do this via a functor $U : \mathcal{K} \to \mathbf{Sets}$. To ensure good behavior, we insist that this U

- Is faithful: If $f \neq g$ in $\mathcal{K}$, then $U(f) \neq U(g)$.
- Preserves directed colimits: the image of any colimit in $\mathcal{K}$ is the colimit of the corresponding diagram of sets.

You would not lose much in thinking of this U as a *forgetful functor* (or *underlying object functor*), in the usual sense. There are peculiarities, however.

Note

The size of an object M in $\mathcal{K}$ need not correspond to $|U(M)|$. In principle, they could disagree for arbitrarily large M.

This poses little problem for the theory, but one might ask how it can be avoided.

Fact

If U reflects split epimorphisms, it preserves sufficiently large sizes.

We can achieve the same through a stronger, but more familiar condition on U:

Definition

We say $U : \mathcal{K} \to \mathbf{Sets}$ is coherent if, given any set map $f : U(M) \to U(N)$ and $\mathcal{K}$-map $g : N \to N'$, if $U(g) \circ f = U(h)$ for some $h : M \to N'$, then there is $\bar{f} : M \to N$ with $U(\bar{f}) = f$.

Definition

We say that an accessible category with concrete directed colimits, $(\mathcal{K}, U)$, is coherent if U is coherent.

Given an AEC (or accessible category of structures), this encompasses the usual notion. Notice, though, that we need not refer to an ambient category of structures, or signature.

In AECs, the following looms large:

Theorem (Shelah's Presentation Theorem)

For any AEC $\mathcal{K}$ in signature L, there is a signature $L' \supseteq L$, a first order theory T' in L', and a set of T'-types Γ such that

$$\mathcal{K} = \{M \restriction L \mid M \models T', M \textit{ omits } \Gamma\}$$

There are several things to note:

- This result is essential for the computation of Hanf numbers, used in the construction of the EM-functor for AECs.
- The proof makes essential use of coherence.
- The expansion L' and set Γ are ad hoc, unrelated to the actual nature of the AEC.

One can do much better:

Theorem (Makkai/Pare)

If $\mathcal{K}$ is a κ-accessible category of structures in signature L, it is equivalent to the category of models of an infinitary L-sentence σ.

Note

Basic ingredients: a structure A in $\mathbf{Str}(L)$ is in $\mathcal{K}$ if and only if each map $f : C \to A$ with C of size κ in $\mathbf{Str}(L)$ factors through a κ-presentable object D in $\mathcal{K}$. Morphisms out of a structure can be coded via atomic diagrams...

The result is an enormous sentence, but one in the natural signature of $\mathcal{K}$ and which captures the way objects are assembled from smaller ones.

We do not insist that our accessible categories consist of structured sets, but even our weak underlying object functor U is enough to give an analogue:

Theorem (L/Rosický)

For any accessible category with concrete directed colimits, $(\mathcal{K}, U)$*, there is a canonical signature* $\Sigma_{\mathcal{K}}$ *such that* $\mathcal{K}$ *is equivalent to a full subcategory of* $\mathbf{Str}(\Sigma_{\mathcal{K}})$*.*

This signature contains finitary relation symbols corresponding to well-behaved subfunctors of U^n, and function symbols corresponding to natural transformations $U^n \to U$.

So: any category of interest to us has a representation of the form on the previous slide.

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits, amalgamation and joint embedding. A Galois type is an equivalence class of pairs (f, a), where $f : M \to N$ and $a \in U(N)$. Pairs (f_0, a_0) and (f_1, a_1) are equivalent if there are morphisms $h_0 : N_0 \to N$ and $h_1 : N_1 \to N$ such that $h_0 f_0 = h_1 f_1$ and $U(h_0)(a_0) = U(h_1)(a_1)$.

If $\mathcal{K}$ has the amalgamation property (which is purely diagrammatic), this is an equivalence relation.

No surprises there: this is a straightforward generalization of the definition for AECs.

In an AEC, Galois types are said to be tame if they are determined by restriction to small submodels of their domains. The situation here is the same:

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits and κ regular. We say that $\mathcal{K}$ is κ-tame if for any non-equivalent types (f, a) and (g, b) over common domain M, there is a morphism $h : X \to M$ with X κ-presentable such that the types (fh, a) and (gh, b) are not equivalent.

$\mathcal{K}$ is called tame if it is κ-tame for some regular cardinal κ.

As mentioned, Boney has shown that if there is a proper class of strongly compact cardinals—henceforth (C)—every AEC is tame.

Theorem (L/R)

Assuming (C), any accessible category with concrete directed colimits is tame.

Proof (Idea): Consider the following categories of configurations:

- $\mathcal{L}_2 : (f_0, f_1, a_0, a_1)$, with $f_i : M \to N_i$, $a_i \in U(N_i)$.
- $\mathcal{L}_1 : (f_0, f_1, a_0, a_1, h_0, h_1)$, with the h_i witnessing equivalence.

Let $G : \mathcal{L}_1 \to \mathcal{L}_2$ be the forgetful functor. Both categories are accessible, as is G. By an old result of Makkai and Paré [assuming (C), the *powerful image* of any accessible functor is accessible], the image of G is χ-accessible for some χ.

If you believe that $G(\mathcal{L}_1)$, the subcategory consisting of equivalent pairs, is χ-accessible, the rest is easy:

Consider (f_0, f_1, a_0, a_1), where $(f_0 u, a_0)$ and $(f_1 u, a_1)$ are equivalent for all $u : X \to M$ with X χ-presentable. Then $(f_0 u, f_1 u, a_0, a_1)$ belongs to $G(\mathcal{L}_1)$ for all such u, and since (f_0, f_1, a_0, a_1) is their χ-directed colimit, it belongs to $G(\mathcal{L}_1)$ as well. That is, (f_0, a_0) and (f_1, a_1) are equivalent.

Thus $\mathcal{K}$ is χ-tame. $\square$

Notice that this proof made no use of coherence—the apparent connection between tameness and large cardinals goes well beyond AECs.

As a last illustration, we consider the appearance of EM-models in accessible categories with concrete directed colimits.

Theorem (Beke/Rosický)

Any large accessible category with directed colimits $\mathcal{K}$ whose morphisms are monomorphisms admits a faithful functor $E : \mathbf{Lin} \to \mathcal{K}$, where **Lin** *is the category of linear orders and order embeddings.*

Specifically, E assigns to each linear order I an object $E(I)$ in $\mathcal{K}$, and to each order embedding $\sigma : I \to J$ a $\mathcal{K}$-map $E(\sigma) : E(I) \to E(J)$. This looks, formally, like the EM-functors from more familiar contexts—but does it retain the same useful properties?

Proposition

*E : **Lin** $\to \mathcal{K}$ preserves sufficiently large sizes.*

That is, for sufficiently large I, the size of $E(I)$ in $\mathcal{K}$ will be precisely $|I|$. Moreover,

Corollary

If $(\mathcal{K}, U)$ is coherent (weaker: U preserves split epimorphisms), $|UE(I)| = |I|$ for sufficiently large I.

In the paper that introduced the use of EM-models in AECs, Baldwin gave an argument for Galois stability below a categoricity cardinal. It can be replicated in this context...

Theorem

Let $(\mathcal{K}, U)$ be coherent (weaker: U preserves split epimorphisms), with amalgamation and joint embedding. If $\mathcal{K}$ is λ-categorical, it is μ-Galois-stable for all $\mu < \lambda$.

As in the proof for AECs, the EM-functor plays an essential role. In the former, Shelah's Presentation Theorem must be invoked, thereby requiring coherence, followed by a delicate syntactic argument involving models $E(I)$.

Our analysis reveals that all of this can be dispensed with: remarkably, functoriality of E seems to be enough. In fact, one can do a lot more. . .