SET-THEORETIC CATEGORY THEORY

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Abstract. The following are expanded notes of a guest lecture given in the course Topics in Category Theory at Masaryk University, 22 May 2018. I am responsible for all content, including any errors, omissions, or little white lies.

The goal of this lecture is to disabuse others of an idea that I held for a disgracefully long time, namely that categorical algebra—categorical model theory, in particular—can and should be done in ZFC, without any untoward dependence on further assumptions about cardinal arithmetic, large cardinals, and so on. I suggest here that, in a sense, one cannot do this; more importantly, I hope to present evidence that one *should* not.

1. The Generalized Continuum Hypothesis, accessibility, and size

We begin with a quick review of infinite cardinal numbers. We all know and love the finite cardinals,

$$0, 1, 2, \dots, n, \dots$$

and all but the most recalcitrant will entertain the idea of a cardinal larger than all such n, namely ω (or $\aleph_0$), the first infinite cardinal. There are infinite cardinals larger than $\aleph_0$—$2^{\aleph_0}$, for example—and we can take $\aleph_1$ to be smallest. And so on:

$$\dots, \aleph_0, \aleph_1, \dots, \aleph_n, \dots$$

And we take $\aleph_\omega = \bigcup_{n<\omega} \aleph_n$, the smallest cardinal larger than all of those. And so on:

$$\dots, \aleph_\omega, \aleph_{\omega+1}, \dots, \aleph_{\omega+n}, \dots$$

It continues in this way, giving us an increasing chain of infinite cardinals $\aleph_\alpha$, indexed by $\alpha \in \mathbf{Ord}$.

We see two phenomena at play here. First, we have a *successor* operation, which takes any cardinal $\aleph_\alpha$ to the least cardinal $\mu > \aleph_\alpha$, i.e.

$$\aleph_\alpha \quad \mapsto \quad \aleph_{\alpha+1}$$

In notational terms, we tend to refer to cardinals by Greek letters (especially λ, κ, μ, and ν) in which case the successor of λ, say, is denoted λ^+.

Definition 1.1. We say that λ is a *successor cardinal* if $\lambda = \kappa^+$ for some κ. Otherwise, λ is a *limit cardinal*.

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Examples 1.2. Clearly, for $1 \leq n < \omega$, the cardinals $\aleph_n$ are successors. In terms of limit cardinals, we have $\aleph_0$, $\aleph_\omega$, and so on.

There is another important division of cardinals, between those that are *regular* and those that are *singular*.

Definition 1.3. For a cardinal κ, we define the *cofinality* of κ, denoted $\mathrm{cf}(\kappa)$, to be the least cardinal λ such that

$$\kappa = \sup_{i<\lambda} \kappa_i$$

for a sequence of cardinals $\kappa_i < \kappa$. We say κ is a *regular cardinal* if $\mathrm{cf}(\kappa) = \kappa$. Otherwise, we say that it is *singular*.

Remark 1.4. One can show that every successor cardinal is regular (hence any singular cardinal is a limit). The limit cardinal $\aleph_0$ is regular, while the uncountable limit cardinal $\aleph_\omega = \sup_{i<\omega} \aleph_i$ is clearly singular ($\mathrm{cf}(\aleph_\omega) = \aleph_0 < \aleph_\omega$). Are there any uncountable regular limit cardinals at all? The existence of such cardinals—*weakly inaccessible cardinals*—is independent of ZFC. These are (very small) large cardinals of the sort that will be discussed in Part II.

Of course, we have another operation on cardinals, namely the exponential:

$$\begin{array}{rcl} X & \mapsto & \mathcal{P}(X) \\ |X| & \mapsto & 2^{|X|} \end{array}$$

Fact 1.5. It is easy to show, in ZFC, that for any λ, $2^\lambda > \lambda$.

You can't say a whole lot more, though, and, in a strong sense, can't say anything about the relationship between the exponential 2^λ and the successor λ^+ without making assumptions that go beyond ZFC. The following hypotheses, all independent of ZFC, are occasionally assumed:

- (CH/Continuum Hypothesis) $2^{\aleph_0} = \aleph_1$.
- (GCH/Generalized Continuum Hypothesis) $2^\lambda = \lambda^+$ for all λ.
- (WGCH/Weak GCH) $2^\lambda < 2^{\lambda^+}$ for all λ.
- (SCH/Singular Cardinal Hypothesis) $2^\lambda = \lambda^+$ for λ any singular *strong limit cardinal*, i.e. singular with $2^\mu < \lambda$ for all $\mu < \lambda$.

Remark 1.6. As is usually the case with independent axioms, some are deemed plausible, or at least above moral reproach, and some are regarded as unlikely, and assuming them will raise eyebrows. No one likes GCH, particularly, while it's hard to imagine anyone objecting to WGCH. SCH is somewhere on the more plausible side, I suppose.

What could this possibly have to do with anything, let alone category theory? We'll see. First, recall:

Definition 1.7. Let $\mathcal{K}$ be a category.

(1) We say that an object $N \in \mathrm{Ob}(\mathcal{K})$ is *finitely presentable* (or *ω-presentable*) if the functor

$$\mathrm{Hom}_{\mathcal{K}}(N, -) : \mathcal{K} \to \mathbf{Set}$$

preserves directed colimits. That is, if M is a directed colimit, $(\phi_i : M_i \to M)_{i\in I}$, any morphism $f : N \to M$ with N finitely presentable will factor, essentially uniquely, through one of the M_i, i.e. $f = \phi_i \circ f_i$.

(2) A category $\mathcal{K}$ is *locally finitely presentable* (or *locally ω-presentable*) if
  - $\mathcal{K}$ is cocomplete,
  - contains a set (up to isomorphism) of finitely presentable objects, $\mathcal{A}$,
  - and any object in $\mathcal{K}$ is a directed colimit of objects in $\mathcal{A}$.

All kinds of familiar categories are locally finitely presentable (henceforth *lfp*). Here are a few, with a description of the finitely presentable objects in each:

Examples 1.8. (1) **Set**: finite sets.
(2) **Grp**: finitely presented groups, $G = \langle S, R\rangle$ with S and R both finite.
(3) **Rel**(Σ), structures in relational signature Σ: structures $X = \langle |X|, R^X\rangle_{R\in\Sigma}$ with X and $\bigcup_{R\in\Sigma} R^X$ both finite.
(4) **Alg**(Σ), algebras over a functional signature Σ: algebras finitely presented in the sense of **Grp**.
(5) Given a poset P, it is lfp iff it is complete and algebraic, i.e. any element is a directed join of finite elements.

Nonexample 1.9. Consider **Ban**, the category of Banach spaces with linear contractions. The objects here are not even remotely finite, however we think of size: in terms of cardinality, the smallest Banach space is of size $2^{\aleph_0}$. A better notion of size of a space V, though, is its *density character*, $\mathrm{dc}(V)$, the cardinality of the smallest dense subset of V. The small spaces, really, are the separable ones, i.e. those V with $\mathrm{dc}(V) = \aleph_0$. Still not finite...

In this and other cases, we need larger cardinals around.

Definition 1.10. Let λ be a regular cardinal, $\mathcal{K}$ a category.

(1) We say N in $\mathcal{K}$ is *λ-presentable* if the functor

$$\mathrm{Hom}_{\mathcal{K}}(N, -) : \mathcal{K} \to \mathbf{Set}$$

preserves λ-directed colimits.

(2) We say $\mathcal{K}$ is *locally λ-presentable* (or *lλp*) if
  - $\mathcal{K}$ is cocomplete,
  - contains a set (up to isomorphism) of λ-presentable objects, $\mathcal{A}$,
  - and any object in $\mathcal{K}$ is a λ-directed colimit of objects in $\mathcal{A}$.

Fact 1.11. **Ban** is locally $\aleph_1$-presentable, and the $\aleph_1$-presentable objects are precisely the spaces with density character $\aleph_0$. More generally, a space V is λ^+-presentable iff $\mathrm{dc}(V) \leq \lambda$.

Locally λ-presentable categories are still not general enough for many purposes, though.

Fact 1.12. The category **Hilb** of Hilbert spaces and linear contractions, a full subcategory of **Ban**, is not locally λ-presentable for any λ. This is by abstract nonsense, essentially: no locally presentable category is self-dual (see [AR94]). **Hilb**

is, via the map

$$\begin{array}{cccc} (-)^* & : & \mathbf{Hilb} & \to & \mathbf{Hilb}^{op} \\ & & H & \mapsto & H \\ & & (f : H \to H') & \mapsto & (f^* : H' \to H) \end{array}$$

where f^* is the adjoint of f, i.e. the unique map with

$$\langle f(x), y\rangle = \langle x, f^*(y)\rangle.$$

Theorem 1.13 ([MP89] 3.4.2)**.** The category **Hilb** has the following properties:

- **Hilb** has all $\aleph_1$-directed colimits,
- contains a set (up to iso) of $\aleph_1$-presentable objects, $\mathcal{A}$ (the separable spaces),
- and any object in $\mathcal{K}$ is a $\aleph_1$-directed colimits of objects in $\mathcal{A}$.

We note in passing that, at least in **Hilb**, separability—having a countable dense subset—is equivalent to having a countable orthonormal basis.

Notice the weakening in the theorem above: in the first bullet point, we have only $\aleph_1$-directed colimits, rather than the arbitrary colimits we would have in the locally presentable case. What this means, though, is that **Hilb** is $\aleph_1$-*accessible*.

Definition 1.14. Let λ be a regular cardinal. A category $\mathcal{K}$ is λ-*accessible* if it has the following properties:

- $\mathcal{K}$ has all λ-directed colimits,
- contains a set (up to iso) of λ-presentable objects, $\mathcal{A}$,
- and any object in $\mathcal{K}$ is a λ-directed colimits of objects in $\mathcal{A}$.

Examples 1.15.
(1) Any locally λ-presentable category is λ-accessible.
(2) The category of fields, **Fld**, is finitely accessible (i.e. ω-accessible), but not lfp. To see the latter point, note that **Fld** has no initial object—due to considerations of characteristic—hence is not cocomplete [AR94, 2.3(5)].
(3) The category **Lin** of linear orders and strictly increasing maps is finitely accessible. The full subcategory **Well** of well-orders is $\aleph_1$-accessible, but not finitely accessible [AR94, 2.3(8)].
(4) A poset P, considered as a category, is finitely accessible iff it is a Scott domain.

The change in the colimit assumption has large ramifications, and makes working in accessible categories a bit harder.

Example 1.16. One can characterize locally finitely presentable categories as those categories that are cocomplete and contain a dense set of finitely presentable objects; one cannot characterize finitely accessible categories as those that are closed under directed colimits and contain a dense set of finitely presentable objects. Consider the poset of subsets of ω that are either singletons or infinite, ordered by inclusion (see [AR94, 2.3(4)]).

Index of Accessibility

A much more important difference is the following: any locally λ-presentable category is locally μ-presentable for all $\mu > \lambda$. In the accessible world, this is a much more delicate question.

Definition 1.17. Given regular cardinals $\mu > \lambda$, we say that μ is *sharply larger* than λ, denoted $\mu \triangleright \lambda$, if the following equivalent conditions hold:

(1) Every λ-accessible category is μ-accessible.
(2) For any set X with $|X| < \mu$, the set $\mathcal{P}_\lambda(X)$ consisting of all subsets of X of size λ has a cofinal subset of cardinality less than μ.
(3) In any λ-directed poset, any subset of size less than μ is contained in a λ-directed subset of size less than μ.

Examples 1.18 ((Mostly [MP89])). (1) For any (uncountable) regular μ, $\mu \triangleright \omega$.
(2) For any regular μ, $\mu^+ \triangleright \mu$.
(3) For regular $\mu \geq \lambda$, $(2^\mu)^+ \triangleright \lambda$.
(4) If $\mu > 2^\lambda$, then $\mu^+ \triangleright \lambda^+$ iff $\mu^\lambda = \mu$ ([LR17b, 4.11]).
(5) If $\mu \geq \lambda$ and for all cardinals $\alpha < \lambda$ and $\beta < \mu$, $\beta^\alpha < \mu$, then $\mu \triangleright \lambda$.

The last of these is a very useful test—recall, incidentally, that $\beta^\alpha = |\{f : \alpha \to \beta\}|$. Sharp inequality is a delicate thing: one can show that for any regular λ, there are arbitrarily large μ with $\mu \triangleright \lambda$, but probably quite a few gaps between such μ. The *accessibility spectrum* of a category, in short, is complicated and depends heavily on cardinal arithmetic, hence on GCH, SCH, etc.

Under further assumptions, though, things are simpler.

Theorem 1.19 ([BR12]). If $\mathcal{K}$ is a λ-accessible category with directed colimits, it is μ-accessible for all regular $\mu \geq \lambda$, i.e. it is *well-λ-accessible*.

Proof. Take $\mu > \lambda$. As $\mathcal{K}$ has all λ-directed colimits, it clearly has all μ-directed colimits as well. So it suffices to show that any $M \in \mathrm{Ob}(\mathcal{K})$ is a μ-directed colimit of μ-presentables. Certainly M is a λ-directed colimit of λ-presentables, say

$$(\phi_i : M_i \to M)_{i \in I}$$

Let $\hat{I}$ be the poset of all directed subsets of I of cardinality less than μ. It follows from the fact that $\mu \triangleright \omega$ (Ex. 1.18(1)) that $\hat{I}$ is μ-directed. For each $X \in \hat{I}$, let

$$M_X = \mathrm{colim}_{i \in X} M_i.$$

Lemma 1.20 ([AR94] 1.16). A colimit of a μ-small diagram of μ-presentable objects is μ-presentable.

Note that any λ-presentable object is μ-presentable, and we have arranged that $|X| < \mu$—meaning that the diagrams are *μ-small.* Hence each M_X is μ-presentable. Finally, notice that

$$M = \mathrm{colim}_{X \in \hat{I}} M_X,$$

a μ-directed colimit of μ-presentables, as desired. □

From a certain perspective, we can get a weaker form of well-accessibility much more easily with a set theoretic assumption, namely GCH. In this way, we can also drop the assumption of directed colimits...

Theorem 1.21 (Me, right now? Folk wisdom?). Assume GCH. If $\mathcal{K}$ is a λ-accessible category, it is μ^+-accessible for all μ with $cf(\mu) > \lambda$.

Proof. Consider Example 1.18(4). Say $\mu > 2^\lambda = \lambda^+$, and $cf(\mu) > \lambda$. Then

$$\mu^\lambda \;=\; \begin{cases} \lambda^+ & \mu \leq \lambda^+ \\ \mu^+ & \mathrm{cf}(\mu) \leq \lambda \leq \mu \\ \mu & \lambda < \mathrm{cf}(\mu) \end{cases} \;=\; \begin{cases} \lambda^+ & \mu \leq \lambda^+ \\ \mu^+ & \emptyset \\ \mu & \lambda < \mathrm{cf}(\mu) \end{cases}$$

where the latter reduction uses the fact that $\mathrm{cf}(\mu) > \lambda$. Of course, we are assuming that $\mu > \lambda^+$, so the only possibility is

$$\mu^\lambda = \mu.$$

So, in particular, $\mu^+ \triangleright \lambda^+$. Since $\lambda^+ \triangleright \lambda$, by Example 1.18(2), we also have $\mu^+ \triangleright \lambda$, and $\mathcal{K}$ is μ^+-accessible. The result follows. □

One can almost certainly improve upon the result above—perhaps by replacing GCH with SCH.

Internal Sizes

Fact 1.22. In any accessible category $\mathcal{K}$, each $M \in \mathrm{Ob}(\mathcal{K})$ is λ-presentable for some (regular) cardinal λ.

Definition 1.23. The *presentability rank* of an object M, which we here denote by $\pi_{\mathcal{K}}(M)$, is the least λ such that M is λ-presentable.

In examples, as we've already seen to some extent, there is a small disconnect between presentability ranks and the intuitive notion of size in a category, with the former typically the successor of the latter.

Examples 1.24.

(1) **Grp**:

$$\begin{aligned} \pi_{\mathbf{Grp}}(G) = \aleph_0 &\quad \text{iff} \quad G \text{ finitely presented} \\ \pi_{\mathbf{Grp}}(G) = \lambda^+ &\quad \text{iff} \quad G\ \lambda\text{-presented} \end{aligned}$$

(2) **Set**:

$$\begin{aligned} \pi_{\mathbf{Set}}(X) = \aleph_0 &\quad \text{iff} \quad X \text{ finite} \\ \pi_{\mathbf{Set}}(X) = \lambda^+ &\quad \text{iff} \quad |X| = \lambda \end{aligned}$$

(3) **Hilb**:

$$\begin{aligned} \pi_{\mathbf{Hilb}}(\mathcal{H}) = \aleph_0 &\quad \text{iff} \quad \mathcal{H} \text{ separable} \\ \pi_{\mathbf{Hilb}}(\mathcal{H}) = \lambda^+ &\quad \text{iff} \quad \mathrm{dc}(\mathcal{H}) = \lambda \end{aligned}$$

In (3) above, we can use the cardinality of an orthonormal basis in place of density character with no change—so this notion of size is doubly appropriate.

Question: Is $\pi_{\mathcal{K}}(M)$ always a successor, at least for sufficiently large ranks?

Answer: It's complicated. There is no known counterexample in the accessible case, i.e. an object M in an accessible category $\mathcal{K}$ such that $\pi_{\mathcal{K}}(M)$ is a (regular) limit cardinal (that is, a weakly inaccessible cardinal). Under additional assumptions on $\mathcal{K}$, though, we can guarantee that the answer is in the affirmative.

Theorem 1.25 ([BR12] 4.2)**.** If $\mathcal{K}$ is λ-accessible with all directed colimits (as opposed to merely the λ-directed ones), then for any $M \in \mathrm{Ob}(\mathcal{K})$ with $\pi_{\mathcal{K}}(M) > \lambda$, $\pi_{\mathcal{K}}(M)$ is a successor.

Proof. Let $\pi_{\mathcal{K}}(M) = \mu > \lambda$. By the proof of Theorem 1.19 above, we can express M as a μ-directed colimit of objects M_X which are themselves directed colimits of λ-presentables, where the diagrams are of size $\nu_X < \mu$. Since M is μ-presentable, the identity map $M \to M$ factors through one of the M_X, i.e. M is a retract of some M_X. If $\nu_X < \lambda$, M_X is λ-presentable, hence so is M. This contradicts our initial assumption. So we must have $\nu_X \geq \lambda$, in which case M_X is ν_X^+-presentable, and similarly for K. So $\mu = \nu_X^+$. □

Or, set theory.

Theorem 1.26 ([BR12] 2.3(5))**.** Assume GCH. If $\mathcal{K}$ is λ-accessible, then for any $M \in \mathrm{Ob}(\mathcal{K})$ with $\pi_{\mathcal{K}}(M) > \lambda$, $\pi_{\mathcal{K}}(M)$ is a successor.

Proof. Say $\mathcal{K}$ is λ-accessible, and $\pi_{\mathcal{K}}(M) = \mu > \lambda$. By definition, μ is regular. Suppose that μ is not a successor. Then it is a limit cardinal, meaning that $\kappa^+ < \mu$ for any $\kappa < \mu$. We are assuming GCH, of course, which means that in fact

$$\kappa < \mu \Longrightarrow 2^\kappa < \mu.$$

That is, μ is a *strong limit* cardinal (and, since regular, a *strongly inaccessible* cardinal, the existence of which is independent of ZFC...). In any case, this means that for any cardinals $\alpha < \lambda$ and $\beta < \mu$,

$$\beta^\alpha \leq \beta^\beta = 2^\beta < \mu.$$

Here we have used:

Fact 1.27. For any infinite cardinal β, $\beta^\beta = 2^\beta$.

So, by 1.18(5), $\mu \triangleright \lambda$. We need a hefty technical lemma, which, now that I think of it, may already have been used implicitly in one of the proofs above:

Lemma 1.28 ([MP89] 2.3.11)**.** Let $\mathcal{K}$ be a λ-accessible category, $\mu \triangleright \lambda$. Then $M \in \mathrm{Ob}(\mathcal{K})$ is μ-presentable iff it is a λ-directed colimit of λ-presentable objects over a diagram of size less than μ.

Say the diagram is of size $\nu < \mu$. Then $\mathcal{K}$ is ν^+-presentable, but $\nu^+ < \mu$. This contradicts our assumption on the rank of M. □

Proposition/Definition 1.29. Let $\mathcal{K}$ be a λ-accessible category. If one of the following conditions holds

(1) GCH
(2) $\mathcal{K}$ has directed colimits

then for all M with $\pi_{\mathcal{K}}(M) > \lambda$, $\pi_{\mathcal{K}}(M) = \mu^+$ for some μ. In this case, we define the *internal size of M in $\mathcal{K}$*, denoted $|M|_{\mathcal{K}}$, to be μ.

Remark 1.30.
(1) We can get away with SCH, rather than GCH ([LRV]).
(2) We can make sense of the internal size even in cases where $\pi_{\mathcal{K}}(M)$ is not a successor, stipulating that if $\pi_{\mathcal{K}}(M)$ is a limit then $|M|_{\mathcal{K}} = \pi_{\mathcal{K}}(M)$. In any kind of reasonable category, though, it almost always is...

If you go back to the examples considered earlier, say **Ban** or those in 1.24, you will see that the internal size corresponds exactly to the intuitive notion of size in each category, at least for sufficiently large objects.

2. Large Cardinals

Here we consider a few large cardinal assumptions—all, of course, independent of ZFC—and the effects they have on the behavior of accessible categories. First, an overview:

Examples 2.1. (1) **V=L, the axiom of constructibility:** Under $V = L$, the only sets that are permitted to exist are those formed under very tight conditions—really only those that, being nicely definable, *have* to exist. This rules out all kinds of strange behavior, makes the set-theoretic universe very predictable (consider, in particular, the host of combinatorial diamond principles [Rin11]), and forces us to accept that there are no unicorns; which is to say, no large cardinals.

(2) **Weakly inaccessible cardinals:** A weakly inaccessible cardinal, as mentioned in Remark 1.4, is an uncountable regular limit cardinal. Their existence is independent of ZFC, but they are too small to be of much use or interest.

(3) **Strongly inaccessible cardinals:** A strongly inaccessible cardinal, as mentioned in the proof of Theorem 1.26, is an uncountable regular strong limit cardinal. These are still quite small by the standards of those who truck in large cardinals, and turn up in very surprising places—the existence of a few such cardinals may be implicit in essentially all informal mathematical reasoning. They are typically needed, roughly speaking, if you want a universe (if, say, you want to avoid any talk of sets versus classes), or want a single very large object containing all those you'd like to study. So, for example, it's needed for:
   - Existence of Grothendieck universes. This means much of arithmetic and algebraic geometry involves at least an implicit assumption of strongly inaccessible cardinals—this includes the patched version of Wiles' proof of Fermat's Last Theorem.
   - Existence of monster models, i.e. large saturated models in which all other (acceptably small) models embed. Folk wisdom suggests that strong inaccessibles are not actually needed, but this is, to my knowledge, not properly written down anywhere.

(4) **Strongly compact cardinals:** Recall that finitary first order logic, $L_{\omega\omega}$, is *compact*: For any first order theory T—i.e. set of sentences in a fixed finitary signature L—if every finite $\Gamma \subseteq T$ is consistent, then T is consistent. In light of the Completeness Theorem, this means that if every finite subset of T has a model, T itself has a model. This is a wildly powerful tool—a magic trick, really. But what happens if we need more general logics, say infinitary logics

$$L_{\kappa\kappa}$$

Here the first subscript indicates that we allow conjunctions and disjunctions over any set of less than κ formulas (not just the finite ones), and the second indicates that we allow quantification over tuples of fewer than κ variables (not just the finite ones). These logics can be genuinely necessary (see [Bal05]). One of many ways of understanding strongly compact cardinals—and the one I find most compelling—is that they are precisely the κ for which $L_{\kappa\kappa}$ is compact in a sense analogous to finitary first order.

Definition 2.2. An uncountable cardinal κ is *strongly compact* if the following equivalent conditions are satisfied:

(a) If T is an $L_{\kappa\kappa}$-theory and if for all $\Gamma \subseteq T$, $|\Gamma| < \kappa$, Γ is consistent, then T is consistent.

(b) Any κ-complete filter extends to a κ-complete ultrafilter.

The ultrafilter characterization of this and other large cardinals tends to be the one that is stressed most heavily in the literature.

(5) There is a vast menagerie of other large cardinals on offer, including weakly compacts, almost strongly compacts, μ-strongly compacts, supercompacts, huge cardinals, Woodin cardinals, and so on. Vopěnka's Principle—essentially a large cardinal hypothesis—is deeply connected with category theory: see Appendix A in [AR94]. Almost and μ-strongly compact cardinals are reasonably well-established in the theory of accessible categories [BTR16]. Measurable cardinals should, morally speaking, be more thoroughly explored in relation to category theory—see the coda of this lecture.

Remark 2.3. In each case, one can assume that there is one cardinal of a particular type, boundedly many cardinals of that type, or arbitrarily large cardinals of that type (which is to say, a "proper class" of cardinals of that type).

You could be forgiven for thinking that, notwithstanding the above comments about uses of strongly inaccessible cardinals, none of the above could possibly be of interest in the context of real, proletarian math...

Example 2.4. Consider the category **FrAb** of free abelian groups, as a full subcategory of the category **Ab** of Abelian groups and group homomorphisms,

$$\mathbf{FrAb} \hookrightarrow \mathbf{Ab}$$

The category **Ab** is finitely accessible, and as nice as you please. What about **FrAb**?

Answer: It's complicated...

Theorem 2.5 ([EM77], building on [She74])**.** Assuming $V = L$, for any regular cardinal κ, there is an indecomposable group that is κ-free, i.e. any of its κ-generated subgroup is free.

In particular,

Corollary 2.6. Assuming $V = L$, **FrAb** is not λ-accessible for any λ.

This feels a little wrong, to be honest. If we want the world to make sense, we need to go the other way: at least one strongly compact cardinal will do the trick.

Theorem 2.7 ([EM90])**.** If there is a strongly compact cardinal κ, **FrAb** is κ-accessible.

Remark 2.8. Naturally, if **FrAb** is κ-accessible for some ludicrously large cardinal κ, it may also be λ-accessible for some smaller, more reasonable λ. The important thing is that it *is* accessible.

That accessibility of **FrAb** is so heavily dependent on set theory is, in itself, a massively interesting fact. But it also points to a more general question:

Remark 2.9. Consider the free abelian group functor

$$\begin{array}{ccc} \mathbf{Set} & \stackrel{F}{\to} & \mathbf{Ab} \\ X & \mapsto & FX \end{array}$$

where FX is the free group on the elements of X, and the image of any **Set**-map $g : X \to Y$ is the map $Fg : FX \to FY$ determined, in the obvious way, by the action of g on generators. Notice that the image of F is precisely **FrAb**. In fact, given that any subgroup of a free abelian group is free abelian, **FrAb** is the *powerful image* of F—the closure of the image of F under subobjects in **Ab**.

It is worth noting, too, that **Set** and **Ab** are both finitely presentable, and that (as one can readily verify) F preserves directed colimits.

Definition 2.10. Given a functor $F : \mathcal{K} \to \mathcal{L}$, we say that F is λ-accessible if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible, and F preserves λ-directed colimits.

So the free abelian functor $F : \mathbf{Set} \to \mathbf{Ab}$ is $\aleph_0$-accessible.

Question: In the spirit of Theorem 2.7, given an accessible functor $F : \mathcal{K} \to \mathcal{L}$, if there is a (sufficiently large) strongly compact cardinal, is the powerful image of F necessarily accessible?

Answer: yes, definitely.

Theorem 2.11 ([MP89] 5.5.1, essentially)**.** If $F : \mathcal{K} \to \mathcal{L}$ is λ-accessible, and κ a sufficiently large strongly compact cardinal, then the powerful image of F is κ-accessible.

Remark 2.12. In [MP89], this is stated for an arbitrary accessible functor, assuming a proper class of strongly compact cardinals. The more precise version above (together with a specification of "sufficiently large") can be found in [BTR16], but essentially results from a careful analysis of the proof in [MP89].

Or, more in the spirit of [MP89], we have the less parametrized:

Theorem 2.13. If there exists a proper class of strongly compact cardinals, then for any accessible functor $F : \mathcal{K} \to \mathcal{L}$, the powerful image of F is accessible.

In either case, these theorems are like a goddamn magic trick. Beyond ensuring that categories like **FrAb** that should be accessible are accessible, the accessibility of accessible images is a surprisingly powerful tool for pushing structural properties that hold on the small objects of a category to the category at large.

A few such structural properties:

Definition 2.14. Let $\mathcal{K}$ be a category. We say that

(1) $\mathcal{K}$ has the *joint embedding property* (or *JEP*) if for any $M_0, M_1 \in \mathrm{Ob}(\mathcal{K})$, they admit a joint embedding,

$$M_0 \to N \leftarrow M_1$$

We say that $\mathcal{K}$ has the $< \kappa$-*JEP* if the condition above holds for κ-presentable M_0, M_1.

(2) $\mathcal{K}$ has the *amalgamation property* (or *AP*) if every span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to a commutative square

$$\begin{array}{ccc} M_1 & \longrightarrow & N \\ \uparrow & & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array}$$

We say that $\mathcal{K}$ has the $<\kappa$-*AP* if the condition above holds for spans of κ-presentable objects.

Theorem 2.15 ([LR17a] 3.5)**.** Let $\mathcal{K}$ be λ-accessible, and κ a sufficiently large strongly compact cardinal. If $\mathcal{K}$ has the $<\kappa$-JEP, it has the JEP.

Proof. Consider the following categories:

(1) $\mathcal{L}_1$: The category of cospans in $\mathcal{K}$, i.e. the diagram category $\mathcal{K}^{\bullet\leftarrow\bullet\rightarrow\bullet}$. A morphism between cospans $M_1 \leftarrow M_0 \rightarrow M_2$ and $M_1' \leftarrow M_0' \rightarrow M_2'$ is a triple $f_i : M_i \rightarrow M_i'$, $i = 0, 1, 2$, such that the following diagram commutes:

$$\begin{array}{ccccc} M_1 & \longleftarrow & M_0 & \longrightarrow & M_2 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_0}\downarrow & & {\scriptstyle f_2}\downarrow \\ M_1' & \longleftarrow & M_0' & \longrightarrow & M_2' \end{array}$$

(2) $\mathcal{L}_2$: The category of pairs in $\mathcal{K}$, i.e the diagram category $\mathcal{K}^{\bullet\bullet}$. A morphism between pairs (M_0, M_1) and (M_0', M_1') is, of course, just a pair of maps $f_i : M_i \rightarrow M_i'$, $i = 0, 1$.

We have an obvious forgetful functor $F : \mathcal{L}_1 \rightarrow \mathcal{L}_2$, which forgets the central element of any cospan, and the central $\mathcal{K}$-map of any $\mathcal{L}_1$-morphism. Clearly, F preserves λ-directed colimits.

Notice that the image of F consists precisely of the pairs (M_0, M_1) that admit a joint embedding. This image is already powerful—clearly, if (N_0, N_1) is a subobject of (M_0, M_1) in the image, N_i is a subobject of M_i, $i = 0, 1$, and the cospan witnessing joint embeddability of M_0 and M_1 will also serve as witness for (N_0, N_1).

Lemma 2.16 ([LR17a] 3.4)**.** If $\mathcal{K}$ is λ-accessible, then so are $\mathcal{L}_1$ and $\mathcal{L}_2$. Moreover, a diagram in $\mathcal{L}_2$ is μ-presentable, $\mu \geq \lambda$, just in case all of the objects in the diagram are μ-presentable in $\mathcal{K}$.

Now, suppose that $\mathcal{K}$ has the $<\kappa$-JEP, i.e. that the (powerful) image of F contains all pairs of κ-presentables.

Consider an arbitrary pair (M_0, M_1) in $\mathcal{L}_2$. The cardinal κ, being sufficiently large, is, in particular, sharply larger than λ, meaning that $\mathcal{L}_2$ is κ-accessible. Hence (M_0, M_1) can be realised as a κ-directed colimit of κ-presentable pairs,

$$(M_0, M_1) = \mathrm{colim}_{i \in I}(M_0^i, M_1^i).$$

By the lemma, M_0^i and M_1^i are κ-presentable, meaning that (M_0^i, M_1^i) is in the image of F. As the image is κ-accessible, by Theorem 2.11, it is closed under κ-directed colimits—so (M_0, M_1) is in the image.

That is, any pair in $\mathcal{K}$ is jointly embeddable: $\mathcal{K}$ has the JEP. $\square$

Theorem 2.17 ([LR17a] 3.6)**.** Let $\mathcal{K}$ be λ-accessible, and κ a sufficiently large strongly compact cardinal. If $\mathcal{K}$ has the $< \kappa$-AP, it has the AP.

Proof. Exercise. Morally speaking, the proof is the same as that for Theorem 2.15. □

In fact, Theorem 2.11 seems to be an engine for generating results of this nature, as well as more delicate properties, e.g. tameness in abstract model theory.

3. Coda

I have not yet mentioned *measurable cardinals*. I have, in fact, been saving them as a treat for the end. Although a good deal is known about measurables vis a vis accessible categories, there is still lots to explore. Something, say, for an ambitious youngster to think about.

Definition 3.1. We say that a cardinal κ is *measurable* if it satisfies the following equivalent conditions:

(1) There exists a κ-additive $\{0, 1\}$-valued measure on $\mathcal{P}(\kappa)$ (Don't worry about this one).
(2) There is a nonprincipal κ-complete ultrafilter on κ.
(3) If a theory T in $L_{\kappa\kappa}$ is the union of an increasing chain of satisfiable theories, then T is satisfiable.

The last condition, resurrected from obscurity in [Bon], tells us that we have a kind of chain compactness holding at measurable cardinals. This is bound to be of use in connection with accessible categories—although we have not discussed it here, any accessible category can be represented syntactically in infinitary logic...

In terms of largeness, measurables lie between strongly inaccessible and strongly compact cardinals. Their existence, therefore, can be expected to have interesting consequences. Likewise, their nonexistence...

Example 3.2 ([AR94] A.19)**.** If there are only boundedly many measurable cardinals (in particular, if there are *no* measurables), then there is an accessible category that is not co-well-powered. That is, it contains an object with a proper class of quotients.

Put another way, if every accessible category is co-well-powered, there is a proper class of measurable cardinals.

Theorem 3.3 ([MP89] 6.2.2, 6.3.8)**.** If there is a proper class of strongly compact cardinals, every accessible category is co-well-powered.

It is an open question whether the large cardinal assumption in this theorem can be weakened, say, to a proper class of measurable cardinals. Wouldn't that be nice...

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