# Tameness, compactness, and cocompleteness

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CLMPST, ČVUT Prague
August 8, 2019

We consider a family of equivalences between the following notions, spanning model theory, category theory and set theory.

1. Tameness of AECs
2. Cocompleteness of images of accessible functors
3. Large cardinals

In particular, we will consider category-theoretic characterizations, namely instances of (2), of large cardinals in the range from weakly to strongly compact.

Very much in keeping with John's view of contemporary model theory, it here provides an indispensable linkage between set- and category-theoretic notions.

We begin with a very concrete example, which provided the first connection of the sort we consider here.

### Remark

*We denote by* **Ab** *the category of abelian groups and homomorphisms. This category is finitely accessible, with finitely presentable objects precisely the finitely presented ones. Very nice.*

Consider the subcategory of free abelian groups, **FrAb**. How nice is it? In particular:

1. Is it accessible?
2. Is it closed under sufficiently nice colimits in **Ab**, i.e. is it *accessibly embedded* in **Ab**?

We focus on the second question. As you may know, the answer is highly dependent on set theory...

### Theorem (Eklof/Mekkler, Shelah)

*Assuming $V = L$, for any uncountable regular $\kappa$ there is an abelian group of size $\kappa$ that is $<\kappa$-free—all of its $<\kappa$-presented subgroups are free—but not free.*

Thus, in particular, **FrAb** is not closed under $\kappa$-directed colimits in **Ab**, hence not $\kappa$-accessibly embedded.

### Theorem (Eklof/Mekkler)

*If $\kappa$ is a strongly compact cardinal, any $<\kappa$-free abelian group is free: that is,* **FrAb** *is closed under $\kappa$-directed colimits in* **Ab**.

In particular, then, **FrAb** is $\kappa$-accessibly embedded in **Ab**, for $\kappa$ strongly compact.

We can frame this question more generally:

### Remark

*The category* **FrAb** *is the image of the free functor*

$$F : \mathbf{Sets} \to \mathbf{Ab}$$

*which is a (finitely) accessible functor.*

*Moreover,* **FrAb** *is the* powerful image *of* $F$*—the closure of the image under* **Ab***-subobjects—as it is itself closed in this way: any subgroup of a free abelian group is free abelian.*

So one might ask, is the powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ $\kappa$-accessibly embedded in $\mathcal{L}$ for some $\kappa$?

Obviously, the answer depends on set theory...

## Theorem (Makkai/Paré)

*If there is a proper class of strongly compact cardinals, the powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ is accessibly embedded in $\mathcal{L}$.*

Implicit in this is that given an individual strongly compact cardinal $\kappa$, the conclusion should hold for functors $F$ "below $\kappa$."
A careful analysis of the proof by Brooke-Taylor/Rosický yielded a number of improvements:

1. A detailed analysis of "below $\kappa$."
2. Elimination of use of *sketches*, in favor of, e.g. ultrafilter arguments.
3. Weakening from $\kappa$ strongly compact to $\kappa$ $L_{\mu,\omega}$-compact, i.e. any $< \mu$-satisfiable theory in $L_{\kappa,\kappa}$ is satisfiable.

In particular, they obtain:

### Theorem

*Let $\kappa$ be $L_{\mu,\omega}$-compact. The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\mu$ is $\kappa$-accessibly embedded in $\mathcal{L}$.*

### Note

*Note that almost strongly compact cardinals are precisely those $\kappa$ that are $L_{\mu,\omega}$-compact for all $\mu < \kappa$. Globally, these are the same.*

As we will see later, Brooke-Taylor/Rosický provides a first example of the ways that parametrizations of strong compactness map onto closure conditions on powerful images.

We recall the definition of tameness, and sketch its derivation from cocompleteness of powerful images...

In a concrete accessible category $\mathcal{K}$—an AEC, say—with amalgamation, Galois types over $M \in \mathcal{K}$ are equivalence classes of pairs $(f, a)$, $f : M \to N$ and $a \in UN$, where $(f, a) \equiv (f', a')$ just in case there are $g$ and $g'$ so that

$$\begin{array}{ccc} N & \xrightarrow{g} & N \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g'} \\ M & \xrightarrow[f']{} & N' \end{array}$$

commutes and $U(g)(a) = U(g')(a')$.

Tameness ensures that this equivalence is determined by the restrictions to small subobjects of $M$.

## Definition

*We say that $\mathcal{K}$ is $<\kappa$-tame if for any $M \in \mathcal{K}$, and $(f, a)$, $(f', a')$ over $M$, whenever $(fh, a) \equiv (f'h, a')$ for all $h : K \to M$, $K$ $\kappa$-presentable, then $(f, a) \equiv (f', a')$.*

## Note

*In case $\mathcal{K}$ is an AEC, this ends up being the standard definition: an object $K$ is $\kappa$-presentable just in case $|UK| < \kappa$.*

One can begin to see, perhaps, how this can be captured via cocompleteness—intuitively, at least, $(f, a)$ and $(f', a')$ can be built as a $\kappa$-directed colimit of restrictions $(fh, a)$, $(f'h, a')$ with $\kappa$-presentable domain.

If the subcategory of equivalent pairs is $\kappa$-accessibly embedded, this should give $<\kappa$-tameness.

- $\mathcal{L}_1$: Category of diagrams witnessing equivalence of pairs:

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

  with selected elements $a_i \in UN_i$, $U(g_0)(a_0) = U(g_1)(a_1)$.

- $\mathcal{L}_2$: Category of pairs:

$$\begin{array}{ccc} N_0 & & \\ {\scriptstyle f_0}\uparrow & & \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

  with selected elements $a_i \in UN_i$.

- Let $F : \mathcal{L}_1 \to \mathcal{L}_2$ be the obvious forgetful functor.

## Facts

1. *The functor F is accessible. If $\mathcal{K}$ is below $\kappa$, so is F.*
2. *The powerful image of F consists of precisely the equivalent pairs of types.*

With these facts, it is more or less trivial to prove:

## Theorem (L/Rosický)

*If the powerful image of any accessible functor below $\kappa$ is $\kappa$-accessibly embedded, then any AEC below $\kappa$ is $<\kappa$-tame.*

In particular, if $\kappa$ is (almost) strongly compact [respectively, $L_{\mu,\omega}$-compact], any AEC below $\kappa$ [respectively, $\mu$] is $<\kappa$-tame.

These results are also, more or less, in Boney's paper on tameness under strongly compacts.

Proof.

Suppose $M$ is the $\kappa$-directed colimit of $\kappa$-presentable objects

$$M = \mathrm{colim}_{i \in I} M_i$$

with colimit maps $\phi_i : M_i \to M$. Suppose types $(f_j, a_j)$, $j = 0, 1$, are equivalent over any $M_i$, i.e.

$$N_0 \xleftarrow[f_0 \phi_i]{} M_i \xrightarrow[f_1 \phi_i]{} N_1$$

is always in the powerful image of $F$. Since the original pair is the colimit of this system—of the permitted form—and the powerful image is closed under such colimits, they are equivalent as well. □

## Theorem

*Each of the following implies the next:*

(1) $\kappa$ *is almost strongly compact.*

(2) *The powerful image of any accessible functor* $F : \mathcal{K} \to \mathcal{L}$ *below* $\kappa$ *is* $\kappa$*-accessible and* $\kappa$*-accessibly embedded in* $\mathcal{L}$*.*

(3) *Every AEC below* $\kappa$ *is* $< \kappa$*-tame.*

This holds for the $L_{\mu,\omega}$-compact case as well, with the obvious modifications.

The truly difficult part, (3) $\Rightarrow$ (1), is due to Boney/Unger. This involves a delicate combinatorial construction of AEC whose nice properties—flavors of tameness—are determined by the large cardinal character of a given $\kappa$.

## Theorem (Boney/Unger)

*Let $\kappa$ satisfy $\mu^\omega < \kappa$ for all $\mu < \kappa$. The following are equivalent:*

(1) *$\kappa$ is almost strongly compact.*

(2) *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\kappa$ is $\kappa$-accessible and $\kappa$-accessibly embedded in $\mathcal{L}$.*

(3) *Every AEC below $\kappa$ is $< \kappa$-tame.*

The construction from Boney/Unger gives a great deal more: parametrizations of large cardinals in the range from weakly to strongly compact map to weakenings of tameness, measurable cardinals to *locality* of Galois types.

Can the proofs of the other implications be suitably parametrized, to give level-by-level equivalences? Yes...

We recall, perhaps belatedly, a few flavors of compactness.

- Strongly compact: $\kappa$ uncountable and in $L_{\kappa\kappa}$, if a theory $T$ is $< \kappa$-satisfiable, $T$ is satisfiable.
- Weakly compact: $\kappa$ inaccessible and in $L_{\kappa\kappa}$, if a theory $T$ with $|T| \leq \kappa$ is $< \kappa$-satisfiable, $T$ is satisfiable.

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## Definition

*We say that a cardinal $\kappa$ is $(\delta, \theta)$-compact, $\delta \leq \kappa \leq \theta$, if whenever a theory $T$ of size $\theta$ in $L_{\delta\delta}$ is $< \kappa$-satisfiable, it is satisfiable.*

$(1)_{\delta,\theta}$ $\kappa$ is $(\delta, < \theta)$-compact.

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We need weakenings of the equivalent conditions of the Boney/Unger theorem:

## Definition

*We say that $\mathcal{K}$ is $<\kappa$-tame if for any $M \in \mathcal{K}$, and $(f, a)$, $(f', a')$ over $M$, whenever $(fh, a) \equiv (f'h, a')$ for all $h : K \to M$, $K$ $\kappa$-presentable, then $(f, a) \equiv (f', a')$.*

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So we get a parametrizations:

$(3)_\theta$ Any AEC below $\kappa$ is $(<\kappa, <\theta)$-tame.
$(3)_{\delta,\theta}$ Any AEC below $\delta$ is $(<\kappa, <\theta)$-tame.

We can play a similar game with the closure condition:

(2) The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\kappa$ is $\kappa$-accessibly embedded.

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$(2)_\theta$ The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\kappa$ is closed under $\theta$-small $\kappa$-directed colimits in $\mathcal{L}$.

By $\theta$-*small*, we mean that the diagram (objects and morphisms) is of cardinality less than $\theta$.

$(2)_{\delta,\theta}$ The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\delta$ is closed under $\theta$-small $\kappa$-directed colimits in $\mathcal{L}$.

In each case, we are parametrizing in the most obvious and natural way. But are the parametrized versions equivalent, maybe subject to further assumptions?

## Theorem (L/Boney)

*Let $\delta$ be an inaccessible cardinal, and $\theta$ a $\kappa$-closed strong limit cardinal. The following are equivalent:*

$(1)_{\delta,\theta}$ *$\kappa$ is $(\delta, < \theta)$-compact.*

$(2)_{\delta,\theta}$ *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below $\delta$ is closed under $\theta$-small $\kappa$-directed colimits in $\mathcal{L}$.*

$(3)_{\delta,\theta}$ *Any AEC below $\delta$ is $(<\kappa, <\theta)$-tame.*

The proof results, essentially, from a careful reading of Brooke-Taylor/Rosický, and L/Rosický, and a plundering of the treasure chest of Boney/Unger.

Technical, but—at least in the forthcoming draft—makes clear the fundamental indistinguishability, level-by-level, of these notions...