Compactness and powerful images: once and for all

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[weakly compact ... strongly compact]

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Definition

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This gets pretty much everything:

- κ (κ, κ)-compact: weakly compact.
- κ (δ, ∞)-compact: δ-strongly compact.
- κ $(< \kappa, \infty)$-compact: *almost strongly compact*. . .
- κ (κ, ∞)-compact: strongly compact.

Theorem (Boney/Unger)

Let κ satisfy $\mu^{\omega} < \kappa$ for all $\mu < \kappa$. The following are equivalent:

(1) *κ is almost strongly compact.*

(2) *The powerful image of any accessible functor $F : \mathcal{K} \to \mathcal{L}$ below κ is κ-accessible and κ-accessibly embedded in $\mathcal{L}$.*

(3) *Every AEC below κ is $< \kappa$-tame.*

Proof.

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Here κ-accessibility is basically free; κ-accessible embeddability, i.e. closure under κ-directed colimits, follows from the large cardinal assumption. □

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(2) $\Rightarrow$ (3): Lieberman and Rosický.

Here $< \kappa$-tameness is reformulated as κ-accessibility of the powerful image of a particular forgetful functor. □

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(2) $\Rightarrow$ (3): Lieberman and Rosický.

(3) $\Rightarrow$ (1): Boney and Unger.

They make use of a combinatorial construction which produces results for, e.g. (δ, θ)-compact cardinals. □

A careful reworking of these arguments gives:

Theorem (Boney/L)

Let δ be an inaccessible cardinal, and θ a κ-closed strong limit cardinal. The following are equivalent:

1. *κ is logically $(\delta, <\theta)$-strong compact.*
2. *If $F : \mathcal{K} \to \mathcal{L}$ is below δ and preserves $\mu_{\mathcal{L}}$-presentable objects, then the powerful image of F is κ-accessible and closed in $\mathcal{L}$ under θ^+-small κ-directed colimits of κ-presentables.*
3. *Any AEC $\mathcal{K}$ with $LS(\mathcal{K}) < \delta$ is $(<\kappa, <\theta)$-tame.*

We will not focus too much on the details—κ-closed, $\mu_{\mathcal{L}}$, etc.—and instead try to get the flavor of the correspondence.

We will prove the following:

Theorem

If κ is (δ, θ)-compact, the powerful image of any (suitably size-preserving) accessible functor $F : \mathcal{K} \to \mathcal{L}$ below δ has powerful image closed under θ^+-small κ-directed colimits of κ-presentable objects.

Proof.

Three steps, almost exactly as in Brooke-Taylor/Rosický:
Step 1: Realize the powerful image of F as the full image of

$$H : \mathcal{P} \to \mathcal{L}$$

where $\mathcal{P}$ is the category of monos $L \to FK$, $K \in \mathcal{K}$. Everything is still nicely accessible... □

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Three steps, almost exactly as in Brooke-Taylor/Rosický:

Step 2: Embed $\mathcal{P}$ and $\mathcal{L}$ into categories of structures, and realize $\mathcal{P}$ as the category of models of an infinitary theory T in some $L_{\delta\delta}(\Sigma)$—in this way, one realizes the full image of F, ultimately, as

$$\mathbf{Red}_{\mathcal{L}}(T)$$

the category of reducts of models of T to the signature of $\mathcal{L}$. □

Crucially, careful reading reveals that $\mathbf{Red}_{\mathcal{L}}(T)$ is κ-preaccessible.

Step 3: We wish to show that $\mathbf{Red}_{\mathcal{L}}(T)$ is closed under θ^+-small κ-directed colimits of κ-presentable objects—take such a diagram $\langle M_i : i \in I \rangle$ in $\mathbf{Red}_{\mathcal{L}}(T)$, and consider its colimit M in $\mathbf{Str}(\mathcal{L})$. It suffices to show there is a mono $M \to N$, $N \in \mathbf{Red}_{\mathcal{L}}(T)$. M is θ^+-presentable, so $|U(M)| = \theta$. Take

$$T_M : \text{atomic/negated atomic diagram of } M, \text{ names } c_m \text{ for } m \in M$$

It suffices to exhibit a model $N \models T \cup T_M$.
Any $\Gamma \subseteq T$ of size $< \kappa$ is included in T plus a piece $T_{M,\Gamma}$ of T_M involving $< \kappa$ of the c_m. We may take M_i with $U(M_i)$ containing their realizations, c_m^M, by κ-directedness: so $M_i \models T \cup T_{M,\Gamma}$. By (δ, θ)-compactness, we are done.

We work in an AEC $\mathcal{K}$, although an arbitrary concrete accessible category will do.

Definition

A *Galois type* over $M \in \mathcal{K}$ is an equivalence class of pairs (f, a), $f : M \to N$ and $a \in UN$. We say $(f_0, a_0) \sim (f_1, a_1)$ if there is an object N and morphisms $g_i : N_i \to N$ such that the following diagram commutes

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

with $U(g_0)(a_0) = U(g_1)(a_1)$.

Assuming the amalgamation property, which we do, this is in fact an equivalence relation.

Definition

We say that Galois types in $\mathcal{K}$ are $(<\kappa,<\theta)$-*tame* if for every $M\in\mathcal{K}_{<\theta}$ and types $p\neq q$ over M, there is $M_0\prec_{\mathcal{K}}M$ of size less than κ such that $p\restriction M_0\neq q\restriction M_0$.

We turn this into a question about powerful images in precisely the same way as in L/Rosický.

- $\mathcal{L}_1$: Category of diagrams witnessing equivalence of pairs:

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & N \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

  with selected elements $a_i\in UN_i$, $U(g_0)(a_0)=U(g_1)(a_1)$.

- $\mathcal{L}_2$: Category of pairs:

$$\begin{array}{ccc} N_0 & & \\ {\scriptstyle f_0}\uparrow & & \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

  with selected elements $a_i \in UN_i$.
- Let $F : \mathcal{L}_1 \to \mathcal{L}_2$ be the obvious forgetful functor.

Facts

1. *The functor F is accessible. If $\mathcal{K}$ is below κ, so is F.*
2. *The powerful image of F consists of precisely the equivalent pairs of (representatives of) types.*

With these facts, there is essentially nothing to do.

In L/Rosický, it is shown that κ-accessibility of the powerful image of F implies $< \kappa$-tameness. We proceed similarly, but assuming only closure under θ-small κ-directed colimits of κ-presentable objects.

Proof.

Suppose M is the κ-directed colimit of θ-presentable objects

$$M = \mathrm{colim}_{i \in I} M_i$$

with colimit maps $\phi_i : M_i \to M$. Suppose types (f_j, a_j), $j = 0, 1$, are equivalent over any M_i, i.e.

$$N_0 \underset{f_0 \phi_i}{\longleftarrow} M_i \underset{f_1 \phi_i}{\longrightarrow} N_1$$

is always in the powerful image of F. Since the original pair is the colimit of this system—of the permitted form—and the powerful image is closed under such colimits, they are equivalent as well. □

We have nearly proven:

Theorem (Boney/L)

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3. *Any AEC $\mathcal{K}$ with $LS(\mathcal{K}) < \delta$ is $(< \kappa, < \theta)$-tame.*

The final link, $(3) \Rightarrow (1)$, is by Boney/Unger, whence cometh all the technical conditions in the preamble...