Algebraic Characterization of the Finite Power Property

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Outline

Decidability of the finite power property for regular languages.

Classical solutions.

Appropriate syntactic semigroup.

Generalization to rational languages in monoids defined by confluent regular systems of deletions.

finite alphabet $A = \{a, b, \ldots\}$

A^* ... the monoid of finite words over A with concatenation

Regular languages:

$L \subseteq A^*$ definable by a finite automaton

Equivalently, recognizable by a finite semigroup $\mathfrak{S}$:

homomorphism $\sigma\colon A^* \to \mathfrak{S}$

$\sigma(u) = \sigma(v) \implies (u \in L \iff v \in L)$

Operations on languages:

concatenation $KL = \{\, uv \mid u \in K,\ v \in L \,\}$

iteration $L^+ = \bigcup_{n=1}^{\infty} L^n$ (subsemigroup of A^* generated by L)

$L^* = L^+ \cup \{\varepsilon\}$, $\quad \varepsilon$ ... empty word

Finite power property (FPP):

$L^+ = L \cup L^2 \cup \cdots \cup L^n$ for some positive integer n

Brzozowski 1966:

Does a given regular L have the FPP?

Solved independently by Hashiguchi 1979 and Simon 1978.

Classical Solutions

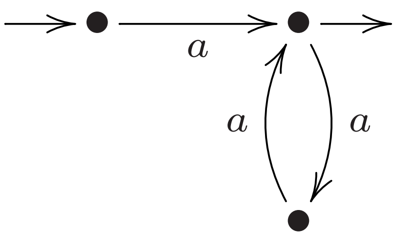
Simon:

automata with weights in the semiring $(\mathbb{N} \cup \{0, \infty\}, \min, +)$

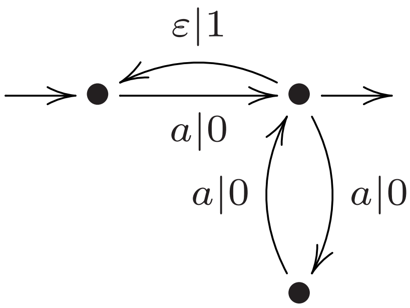
Example:

$L = a\{a^2\}^*$ has the FPP: $\quad L^+ = \{a\}^+ = L \cup L^2$

automaton for L:



automaton for L^+ with weights:



Hashiguchi: direct combinatorial argument on automata

based on the pigeon hole principle

Let us uncover the algebraic background of Hashiguchi's argument.

First steps: Kirsten 2002

Some Basics on Structure of Finite Semigroups

$\mathfrak{S}$ finite semigroup

Quasi-order $\leq_{\mathcal{J}_\mathfrak{S}}$ on $\mathfrak{S}$:

$$s \leq_{\mathcal{J}_\mathfrak{S}} t \iff \exists x, y \in \mathfrak{S} \cup \{1\} \colon s = x \cdot t \cdot y$$

Green relation $\mathcal{J}_\mathfrak{S}$: equivalence relation associated with $\leq_{\mathcal{J}_\mathfrak{S}}$

$s \; \mathcal{J}_\mathfrak{S} \; t \iff$ generate the same ideal of $\mathfrak{S}$

$\leq_{\mathcal{J}_\mathfrak{S}}$ determines a partial order of $\mathcal{J}$-classes

$s \in \mathfrak{S}$ idempotent: $s \cdot s = s$

$\mathcal{J}$-class J regular: contains an idempotent

J regular $\iff \exists s, t \in J \colon s \cdot t \in J$

Example of a Syntactic Monoid

$A = \{a, b\}$

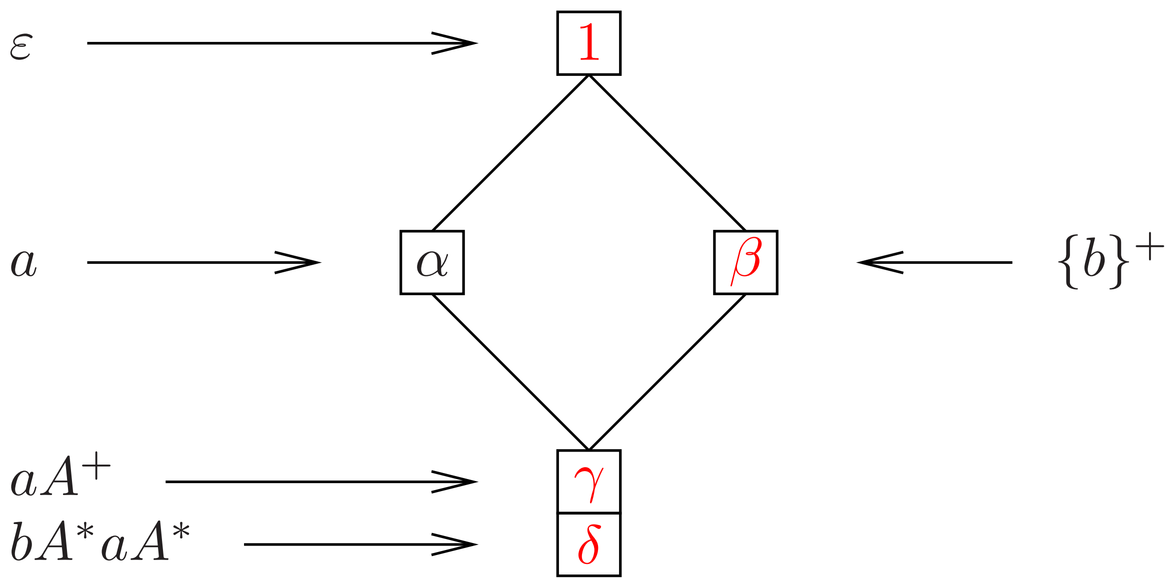
$K = \{a\} \cup bA^*aA^*$

does not have the FPP ($a^+ \subseteq K^+$)

$L = \{b\}^+ \cup aA^+$

has the FPP ($L^+ = L \cup \{b\}^+aA^+ = L \cup L^2$)

K and L recognized by the monoid:



The Appropriate Semigroup

$L \subseteq A^+$ regular

homomorphism $\sigma\colon A^* \to \mathfrak{S}$ recognizing L, L^+ and $\{\varepsilon\}$

Define a mapping $\tau\colon A^* \to \wp(\mathfrak{S}^3)$

$$\tau(w) = \{\, (\sigma(x), \sigma(y), \sigma(z)) \mid x, y, z \in A^*,\ w = xyz \,\}$$

Kernel of τ is a congruence of $A^* \implies \tau(A^*)$ is a monoid.

$\mathfrak{T} = \tau(L^+)$ subsemigroup of $\tau(A^*)$.

Algebraic Characterization of the FPP

Theorem: The following conditions are equivalent:

1. L has the FPP.
2. $\forall w \in L^+ \exists n \in \mathbb{N}\colon w^n \in L \cup L^2 \cup \cdots \cup L^n$.
3. Every regular $\mathcal{J}$-class of $\mathfrak{T}$ contains some element of $\tau(L)$.
4. $w \in L^+$, $\mathcal{J}$-class of $\tau(w)$ in $\mathfrak{T}$ regular $\Longrightarrow$
   $\exists y \in L, x, z \in L^* : w = xyz \,\&\, \sigma(y) \;\mathcal{J}_{\mathfrak{S}}\; \sigma(w)$.
5. $L^+ = L \cup L^2 \cup \cdots \cup L^{(j+1)^h}$.

   j ... maximal size of a $\mathcal{J}$-class of $\mathfrak{S}$

   h ... length of the longest chain of $\mathcal{J}$-classes in $\mathfrak{T}$

Proof:

2 $\Longrightarrow$ 3: direct calculation for $w \in L^+$ with $\tau(w)$ idempotent
(common refinement of two decompositions $w^n \in L^m$, $m \leq n$)

4 $\Longrightarrow$ 5: induction with respect to $\mathcal{J}$-classes of $\mathfrak{T}$
(based on length of words; maximality of decompositions)

Monoids Defined by Confluent Deletions

$R \subseteq A^+$ regular

$\mathcal{R} = \{\, w \to \varepsilon \mid w \in R \,\}$ confluent rewriting system

$\mathrm{norm}(w)$ ... normal form of $w \in A^*$ with respect to $\mathcal{R}$

$\mathfrak{G} = (\mathrm{norm}(A^*), \cdot) \qquad u \cdot v = \mathrm{norm}(uv)$

Rational languages in $\mathfrak{G}$: $\mathrm{norm}(L)$, where L is regular in A^*

Example: Free group over $A = \{a, b, \ldots\}$:

Take a disjoint copy $A' = \{a', b', \ldots\}$.

$R = \{\, xx', x'x \mid x \in A \,\} \subseteq (A \cup A')^*$

$L = \{\varepsilon\}$ corresponds to "Dyck language" with symmetric brackets

d'Alessandro and Sakarovitch 2003:

The FPP for rational languages in free groups is decidable.

(involved reduction to boundedness of distance automata)

A Generalization

Theorem: The FPP is uniformly decidable for rational languages in finitely generated monoids defined by a confluent regular system of deletions.

Rational monoids:

$\beta\colon A^+ \longrightarrow A^+$ rational function, $\quad \beta \circ \beta = \beta$

$\mathfrak{M} = (\beta(A^+), \cdot) \qquad u \cdot v = \beta(uv)$

- regular languages behave as in A^*
- can be algorithmically manipulated

The characterization of the FPP holds for monoids $\mathfrak{M}$ satisfying:

1. Well defined length of elements: $\quad \ell\colon \mathfrak{M} \setminus \{0\} \to \mathbb{N}_0$
   $x \cdot y \neq 0 \implies \ell(x \cdot y) = \ell(x) + \ell(y)$
2. Each two decompositions $x \cdot y = z \cdot t \neq 0$ have a common refinement.
3. $\{0\}$ and $\{1\}$ are regular.

Proof of the Generalization

We construct for each regular language $L \subseteq \mathrm{norm}(A^*)$
a different rational monoid satisfying the previous conditions.

homomorphism $\sigma\colon A^* \longrightarrow \mathfrak{S}$ recognizing L, $\mathrm{norm}(A^*)$ and $\{\varepsilon\}$

$$\mathfrak{M} = \big((\mathfrak{S} \times \mathrm{norm}(A^*) \times \mathfrak{S}) \cup \{1, 0\}, \cdot\big)$$

$$(p, u, q) \cdot (r, v, s) = \begin{cases} (p, uv, s) & \text{if } uv \in \mathrm{norm}(A^*) \\ & \quad \text{and } \varepsilon \in \mathrm{norm}(\sigma^{-1}(q) L^* \sigma^{-1}(r)), \\ 0 & \text{otherwise.} \end{cases}$$

$$K = \{\, (\sigma(x), y, \sigma(z)) \mid x, y, z \in A^*,\ y \neq \varepsilon,\ xyz \in L \,\}$$

L has the FPP in $\mathfrak{S}$ $\iff$ K has the FPP in $\mathfrak{M}$

Conclusion

Known positive results on the FPP can be obtained by a transparent algebraic construction.

Open questions

1. Application to star height and related problems?

2. The FPP for recognizable relations
   $\bigcup_{i=1}^{n} K_i \times L_i$, where K_i and L_i regular