On Language Inequalities $XK \subseteq LX$

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Outline:

1. Proving regularity of maximal solutions
   - by constructing finite recognizers.
   - using well quasi-orders.
2. Systems with inequalities $XK \subseteq LX$.
3. Proof of non-regularity of the solution.

finite alphabet $A = \{a, b, \dots\}$

A^* …the monoid of finite words over A

finite set of variables $\mathcal{V} = \{X_1, \dots, X_n\}$

Explicit polynomial equations:

$X_i = P_i, \quad i = 1, \dots, n$

$P_i \subseteq (A \cup \mathcal{V})^*$ finite

components of smallest solutions …context-free languages

Example: $X = \{c, aXb\} \implies X = \{a^k c b^k \mid k \in \mathbb{N}_0\}$

Example: $X \cup Y = A^*$

every language is a component of a minimal solution

Constructing finite recognizers

$P_j \subseteq L_j$

$P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary, $L_j \subseteq A^*$ regular

Conway 1971

maximal solutions:

- regular
- for given L_j: finitely many, computable
- for context-free P_j: algorithmically regular

Example: $X \cdot Y \subseteq L$

$\sim$ ... congruence of A^* of finite index recognizing L,

i.e. $u \sim v \implies (u \in L \iff v \in L)$

Every solution contained in a solution recognized by $\sim$:

$M \cdot N \subseteq L \implies (M \sim) \cdot (N \sim) \subseteq L$

$K_0 \cup K_1 X_1 \cup \cdots \cup K_n X_n \subseteq L_0 \cup L_1 X_1 \cup \cdots \cup L_n X_n$

K_i arbitrary, L_i regular

MK 2005

largest solutions:

- regular
- for given L_i: finitely many, computable
- for context-free K_i: algorithmically regular

Well quasi-orders

Wqo $\leq$ on A^* contains neither  nor $\bullet\;\bullet\;\bullet\cdots$

Monotone: $u \leq v \;\&\; \tilde{u} \leq \tilde{v} \implies u\tilde{u} \leq v\tilde{v}$

Example: (scattered) subword partial order

Theorem: Ehrenfeucht, Haussler, Rozenberg 1983

$L \subseteq A^*$ is regular $\iff$ L is upward closed with respect to a monotone well quasi-order on A^*

Example:

Congruence of finite index is a monotone wqo.

upward closed = recognized by the congruence

Applying well quasi-orders to inequalities

Method:

Construct a wqo on A^* such that every solution is contained in an upward closed solution.

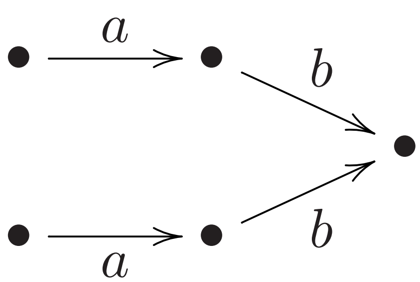
Restriction on constants:

$P_j \subseteq Q_j$

$P_j \subseteq (A \cup \mathcal{V})^*$ arbitrary

$Q_j \ldots$ regular expression over variables and languages recognizable by finite simple semigroups

L recognizable by a finite simple semigroup $\iff$ minimal automaton of L does not contain



MK 2004: all maximal solutions regular

Inequalities $XK \subseteq LX$

has largest solution:

$$M_i K \subseteq L M_i \implies (\bigcup M_i)K \subseteq L(\bigcup M_i)$$

K arbitrary, L regular

MK 2004

largest solution:

- regular
- for context-free K: algorithmically recursive

Rules of the game

$XK \subseteq LX$

position: $w \in A^*$

attacker: $u \in K$, $w \longrightarrow wu$

defender: $v \in L$, $wu = v\tilde{w}$, $wu \longrightarrow \tilde{w}$

largest solution = all winning positions for the defender

Example: $L = \{a, ab, abcde, bc, c, cd, da\}$, $w = abcd$,

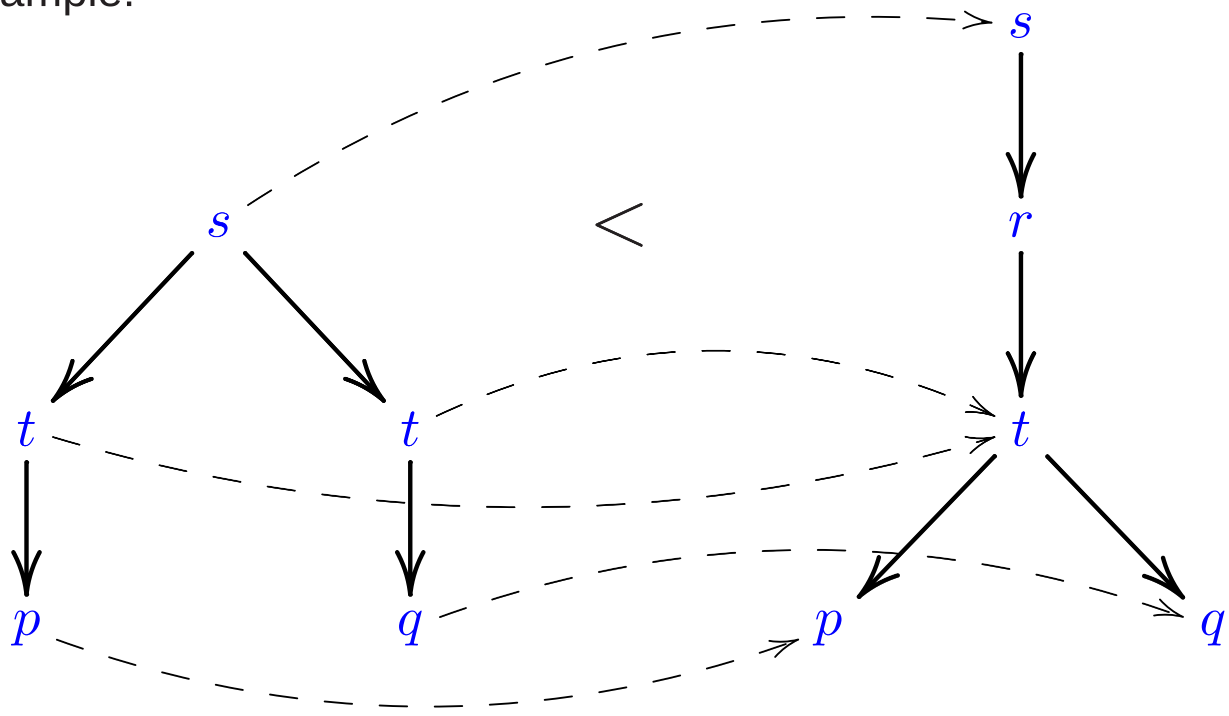
$\sim$ congruence of finite index recognizing L

$$\begin{tikzcd}
 & & & abcd \sim & (\,) \arrow[l, dotted] \arrow[dll] \arrow[dr] & & & \\
 & bcd \sim & (a) \arrow[l, dotted] \arrow[dl] & & & (ab) \arrow[r, dotted] \arrow[dll] \arrow[dr] & cd \sim & \\
d \sim & (a, bc) \arrow[l, dotted] & d \sim & (ab, c) \arrow[l, dotted] & & & (ab, cd) \arrow[r, dotted] & 1
\end{tikzcd}$$

Quasi-ordering of labelled trees

$w \leq v$ ... defender's winning strategies for w can be used also for v

Example:



Largest solution is upward closed with respect to $\leq$

Kruskal 1960: $\leq$ is a wqo

Systems with inequalities $XK \subseteq LX$

Theorem 1:

There exists a finite language K and star-free languages L, M such that the largest solution of the system

$$XK \subseteq LX, \quad X \subseteq M$$

is not recursively enumerable.

Theorem 2:

There exist finite languages K, P and star-free languages L, R such that the largest solution of the system

$$XK \subseteq LX, \quad XP \subseteq RX$$

is not recursively enumerable.

Systems $XK \subseteq LX, PX \subseteq XR$

MK 2005:

There exists a finite language L such that the largest solution of $XL = LX$ is not recursively enumerable.

System $XK \subseteq LX$, $X \subseteq M$

largest solution S non-regular

M ... admissible positions of the game

Alphabet:

$$B = \{a, b, c\}, \quad \hat{B} = \{\hat{a}, \hat{b}, \hat{c}\}, \quad A = B \cup \hat{B}$$

Testing equality of two counters:

simultaneous decrementation and zero test

Configurations: $b\hat{b}(a\hat{a})^{m} c\hat{c}(a\hat{a})^{n}$

m, n ... values of the counters

Languages:

$$\begin{aligned}
K &= \{a\hat{a}, b\hat{b}, c\hat{c}\} \\
L &= \{a\hat{a}, b\hat{b}a\hat{a}, c\hat{c}a\hat{a}, b\hat{b}c\hat{c}b\} \cup \hat{B}B \\
&\quad \cup cA^*a \cup bA^*a \cup A^+bA^*b \cup A^+cA^*c \\
M &= (B\hat{B})^+ \cup \hat{B}
\end{aligned}$$

$$m < n \implies b\hat{b}(a\hat{a})^m c\hat{c}(a\hat{a})^n \notin S$$

By induction on m:

Basis: By contradiction:

$$b\hat{b}c\hat{c}(a\hat{a})^n \in S \implies b\hat{b}c\hat{c}(a\hat{a})^n \cdot b\hat{b} \in SK \subseteq LS \subseteq LM$$

Induction step:

Hypothesis: $b\hat{b}(a\hat{a})^{m-1}c\hat{c}(a\hat{a})^{n-1} \notin S$

By contradiction:

$$\begin{array}{c}
b\hat{b}(a\hat{a})^m c\hat{c}(a\hat{a})^n \in S \\
\downarrow \\
\underline{\underline{\underline{\underline{b\hat{b}}a}\hat{a}}(a\hat{a})^{m-1}c\hat{c}(a\hat{a})^n} \cdot b\hat{b} \\
\downarrow \\
(a\hat{a})^{m-1}c\hat{c}(a\hat{a})^n b\hat{b} \in S \\
\downarrow \ m-1 \text{ times} \\
\underline{(a\hat{a})^{m-1}}c\hat{c}(a\hat{a})^n b\hat{b} \cdot (a\hat{a})^{m-1} \\
\downarrow \ m-1 \text{ times} \\
c\hat{c}(a\hat{a})^n b\hat{b}(a\hat{a})^{m-1} \in S \\
\downarrow \\
\vdots \\
\downarrow \\
b\hat{b}(a\hat{a})^{m-1}c\hat{c}(a\hat{a})^{n-1} \in S
\end{array}$$

$$b\hat{b}(a\hat{a})^n c\hat{c}(a\hat{a})^n \in S$$

$$u_{n,k} = (a\hat{a})^k b\hat{b}(a\hat{a})^n c\hat{c}(a\hat{a})^{n-k}$$
$$v_{n,k} = (a\hat{a})^k c\hat{c}(a\hat{a})^{n+1} b\hat{b}(a\hat{a})^{n-k}$$

$$\begin{aligned} u_{n,0} &\longrightarrow v_{n-1,n-1} \longrightarrow \cdots \longrightarrow v_{n-1,0} \longrightarrow \\ &\longrightarrow u_{n-1,n-1} \longrightarrow \cdots \longrightarrow u_{n-1,0} \end{aligned}$$

$N = \hat{B} \cup \{u_{n,k}, v_{n,k} \mid 0 \leq k \leq n\}$ is a solution

$NK \subseteq LN$: $\quad \hat{B}K \subseteq \hat{B}B \cdot \hat{B} \subseteq LN$

$$k > 0 \implies u_{n,k} \cdot a\hat{a} = a\hat{a} \cdot u_{n,k-1} \in LN$$
$$u_{n,0} \cdot a\hat{a} \in bA^*a \cdot \hat{B} \subseteq LN$$

$$n > 0 \implies u_{n,0} \cdot b\hat{b} = b\hat{b}a\hat{a} \cdot v_{n-1,n-1} \in LN$$
$$u_{0,0} \cdot b\hat{b} = b\hat{b}c\hat{c}b \cdot \hat{b} \in LN$$
$$k > 0 \implies u_{n,k} \cdot b\hat{b} \in A^+bA^*b \cdot \hat{B} \subseteq LN$$

$$u_{n,k} \cdot c\hat{c} \in A^+cA^*c \cdot \hat{B} \subseteq LN$$

Together:

1. $m < n \implies b\hat{b}(a\hat{a})^m c\hat{c}(a\hat{a})^n \notin S$
2. $b\hat{b}(a\hat{a})^n c\hat{c}(a\hat{a})^n \in S$

Pumping lemma $\implies$ S is not regular.

System $XK \subseteq LX$, $XP \subseteq RX$

$\tilde{A} = A \cup \{d\}$

$N \subseteq \tilde{A}^*$ such that $\hat{a} \in N$

N is a solution of $XK \subseteq LX$, $X \subseteq M$ $\iff$
N is a solution of $XK \subseteq LX$, $Xd\hat{a} \subseteq MdX$

Proof of $\Longleftarrow$:

$NK^n \subseteq L^nN$ for arbitrarily large $n \implies N \subseteq A^*$

$Nd\hat{a} \subseteq MdN \implies N \subseteq M$

Open questions

1. Regularity of solutions of similar systems of inequalities, for instance:

   $KXL \subseteq MX$
   $XK \subseteq LX, \; XK \subseteq MX$
   $KX \subseteq LX, \; XM \subseteq XN$

2. Existence of algorithms for computing largest solutions.