ON THE TORSION ON GAUGE-LIKE PROLONGATIONS OF PRINCIPAL BUNDLES

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ABSTRACT. Every fiber product preserving bundle functor $F$ on the category $\mathcal{FM}_m$ defines the gauge-like prolongation $W^F P$ of a principal bundle $P$, that coincides with the $r$-th principal prolongation $W^r P$ in the special case $F = J^r$. For a large class of such functors we introduce the torsion of connections on $W^F P$ and we deduce some its properties analogous to the case of $W^r P$.

The $r$-th principal (or gauge-natural) prolongation $W^r P \to M$ of a principal bundle $P \to M$ is a fundamental structure for both the theory of geometric object fields, [2], [9], and the gauge theories of mathematical physics, [3]. In [5] we introduced the torsion of a connection $\Gamma$ on $W^r P$ to be the covariant exterior differential of the canonical one-form $\theta_r$ of $W^r P$. On the other hand, the Lie algebroid $L(W^r P)$ coincides with the $r$-th jet prolongation $J^r (LP)$ of the Lie algebroid $LP$ of $P$, [11]. In [8], we considered the algebroid form $\gamma: TM \to J^r LP$ of $\Gamma$, we introduced the torsion of $\gamma$ by using the truncated bracket of $J^r LP$ and we deduced that both approaches to the torsion are naturally equivalent.

In the present paper we study a more general setting of this problem. In [1], the authors constructed a principal bundle $W^F P \to M$ for every fiber product preserving bundle functor $F$ on the category $\mathcal{FM}_m$ of fibered manifolds with $m$-dimensional bases and fibered morphisms with local diffeomorphisms as base maps. In the case of the functor $J^r$ of $r$-th jet prolongation, we have $W^{J^r} P = W^r P$, so that $W^F P$ will be called a gauge-like prolongation of $P$. We are going to clarify that geometrically remarkable results concerning torsion appear in the case of a subfunctor $E \subset J^1 \circ F$. In this situation, we can use our results on the generalized $G$-structures on $W^F P$, i.e. the reductions of $W^1(W^F P)$, [4].

2000 Mathematics Subject Classification: 53C05, 53C10, 58A20, 58A32.

Key words and phrases: gauge-like prolongation of principal bundle, Lie algebroid, generalized $G$-structure, connection, torsion.

The author was supported by the Ministry of Education of Czech Republic under the project MSM 0021622409.
In Section 1 we first summarize the basic properties of $WF^P$. Then we construct a canonical map relating the gauge-like prolongation of the iteration of two functors with the iteration of the gauge-like prolongations. In Proposition 1 we determine the Lie algebroid version of this map. In Section 2 we study the reductions $Q$ of $W^1P$ (called generalized $G$-structures in [4]) from our point of view. In Proposition 2 we deduce that both approaches to the torsion on $Q$ are naturally equivalent. Special attention is paid to the additional properties of semiprolongable generalized $G$-structures. Proposition 3 describes a relation between the prolongability of generalized $G$-structures and the existence of torsion-free connections analogous to the case of classical $G$-structures. In Section 3 we specify some fiber product preserving bundle functors on $F\mathcal{M}_m$, the gauge-like prolongations of which are of the form studied in Section 2. We also point out that some further functors can be reduced to this case by using a suitable natural equivalence.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [9]. In particular, we write $P^rM$ for the $r$-th order frame bundle of a manifold $M$ and $G^r_m$ for the $r$-th jet group in dimension $m$. Under a connection we always mean a principal connection.

1. Gauge-like prolongations of principal bundles. We denote by $T^A$ the Weil functor determined by a Weil algebra $A$, [9], [7]. A fundamental result reads that the product preserving bundle functors on the category $\mathcal{M}f$ of all manifolds are in bijection with the Weil functors and the natural transformations $h_M: T^{A_1}M \to T^{A_2}M$ are in bijection with the algebra homomorphisms $h: A_1 \to A_2$, [7]. In the special case of $D^k_r = J^r_0(\mathbb{R}^k, \mathbb{R})$, $T^{D^k_r} = T^r_k$ is the classical functor of $(k, r)$-velocities.

In [10], the authors deduced that the fiber product preserving bundle functors on $F\mathcal{M}_m$ of base order $r$ are in bijection with the triples $(A, H, t)$ of a Weil algebra $A$, a group homomorphism $H: G^r_m \to \text{Aut } A$ and an equivariant algebra homomorphism $t: D^r_m \to A$, where $\text{Aut } A$ is the group of all algebra automorphisms of $A$ and we take into account $\text{Aut } (D^r_m) = G^r_m$. One constructs the functor $F = (A, H, t)$ as follows. For every manifold $N$, we have an induced action $H_N$ of $G^r_m$ on $T^A N$,

$$H_N(g, z) = H(g)_N(z), \quad g \in G^r_m, \quad z \in T^A N,$$

where $H(g)_N: T^A N \to T^A N$ is the map determined by $H(g): A \to A$. The value of $F$ on the product fibered manifold $M \times N \to M$, dim $M =$
$m$, is the associated bundle

$F(M \times N) = P^r M[T^A N, H_N]$. 

Every local diffeomorphism $f: M \to \bar{M}$ and every map $\varphi: N \to \bar{N}$ determine the product $\mathcal{FM}_m$-morphism $f \times \varphi: M \times N \to \bar{M} \times \bar{N}$. By naturality, the map $T^A \varphi: T^A N \to T^A \bar{N}$ is $G_m^r$-equivariant. Further, $P^r f: P^r M \to P^r \bar{M}$ is a principal bundle morphism. Then we define $F(f \times \varphi): F(M \times N) \to F(\bar{M} \times \bar{N})$ to be the morphism of associated bundles

$F(f \times \varphi) = (P^r f, T^A \varphi): P^r M[T^A N, H_N] \to P^r \bar{M}[T^A \bar{N}, H_{\bar{N}}]$.

For an arbitrary fibered manifold $p: Y \to M$, $FY$ is the subbundle of $P^r M[T^A Y, H_Y]$ of all elements $\{u, Z\}$, $u \in P^r M \subset T_m^r M$, $Z \in T^A Y$ satisfying

$t_M(u) = T^A p(Z)$,

where $t_M: T_m^r M \to T^A M$ is the map induced by $t: \mathcal{D}_m^r \to A$. Since $t$ is equivariant, (3) is independent of the choice of the representatives $u$ and $Z$ of the equivalence class $\{u, Z\} \in FY$. For another fibered manifold $\bar{p}: \bar{Y} \to \bar{M}$ and an $\mathcal{FM}_m$-morphism $f: Y \to \bar{Y}$ over $f: M \to \bar{M}$, $(P^r f, T^A f)$ maps $FY$ into $F\bar{Y}$. Then one defines $Ff$ to be its restriction and corestriction. In the case of $J^r$, we have $A = \mathcal{D}_m^r$, $H = \text{id}_{G_m^r}$, $t = \text{id}_{\mathcal{D}_m^r}$, and (3) expresses, in fact, the classical relation

$J^r Y = \{X \in J^r (M, Y); p_*(X) = j_x f \text{id}_M \}$,

where $x$ is the source of $X = j_x f$ and $p_*(X) = j_x (p \circ f) \in J^r (M, M)$.

Consider a principal bundle $P(M, G)$. First we recall $W^r P = P^r M \times_M J^r P$, [9]. This is a principal bundle over $M$, whose structure group is the group semidirect product $W_m^r G = G_m^r \rtimes T_m^r G$ with the composition

$(g_1, C_1)(g_2, C_2) = (g_1 \circ g_2, (C_1 \circ g_2) \bullet C_2)$,

where $\bullet$ denotes the induced group composition in $T_m^r G$. In [1], the authors defined

$W^F P = P^r M \times_M FP$.

Analogously to (5), one constructs the group semidirect product

$W^A_H G = G_m^r \rtimes_H T^A G$

with the composition

$(g_1, C_1)(g_2, C_2) = (g_1 \circ g_2, H(g_2^{-1})_G(C_1) \bullet C_2)$,

where $\bullet$ denotes the induced group composition in $T^A G$. In the case of $W_m^r G$, $H(g_2^{-1})_G(C_1) = C_1 \circ g_2$, so that (7) generalizes (5). Then
Let $W^F P$ be a principal bundle over $M$ with structure group $W^1 G$ with respect to the following action, [1]. The right action of $G$ on $P$ can be interpreted as an $\mathcal{FM}_m$-morphism

$$\varrho: P \times_M (M \times G) \to P.$$ 

Applying $F$, we obtain

$$F\varrho: FP \times_M P^r M[T^A G, H_G] \to FP.$$ 

For $(g, X) \in G^r_m \times T^A G$ and $(u, Y) \in P^s M \times_M FP$, one defines

$$(u, Y)(g, X) = (u \circ g, F\varrho(Y, \{u \circ g, X\})).$$ 

In the case of $W^r P$, (8) coincides with the right action of $W^r_m G$ on $W^r P$ described in [9].

We remark that a basic geometric property of $W^F P$ is that for every bundle $D \to M$ associated to $P$, $FD \to M$ is a bundle associated to $W^F P$, [1].

Further, $F$ determines a natural transformation $\tilde{t}_Y: J^r Y \to FY$. Every element $X \in J^r Y$ is of the form $j^l_s$. We interpret the local section $s$ of $Y$ as a local $\mathcal{FM}_m$-morphism $\tilde{s}$ of the trivial fibered manifold $id_M: M \to M$ into $Y$ and we set

$$\tilde{t}_Y(X) = (F\tilde{s})(x) \in FY.$$ 

In the product case, $J^r (M \times N) = P^r M[T^r_m N], F(M \times N) = P^r M[T^A N]$ and $\tilde{t}_{M \times N}$ is of the form $\tilde{t}_{M \times N} = (id_{P^r M}, t_N)$ with $t_N: T^r_m N \to T^A N$.

In particular, we have $\tilde{t}_{TM}: J^r TM \to FT M$. Write $q: LP \to TM$ for the anchor map, so that $Fq: FLP \to FTM$. In [6], we deduced that the Lie algebroid of $W^F P$ is

$$L(W^F P) = J^r TM \times_{FT M} FLP.$$ 

In the special case $F = J^r$, we reobtain

$$L(W^r P) = J^r TM \times_{J^r TM} J^r LP = J^r LP.$$ 

Consider two such functors $F_1$ and $F_2$ of base orders $r$ and $s$. Then the base order of $F_2 \circ F_1$ is $r + s$. Using $(\tilde{t}_2)_{P^{r} M}: J^s P^r M \to F_2 P^r M$, we construct a map

$$W^{F_2 \circ F_1} P \to W^{F_2} (W^{F_1} P)$$ 

as follows. The classical inclusion $P^{r+s} M \hookrightarrow W^s (P^r M) = P^s M \times_M J^s P^r M$ is described in [9]. So we have

$$W^{F_2 \circ F_1} P = P^{r+s} M \times_M F_1 F_2 F_1 P \hookrightarrow P^s M \times_M J^s P^r M \times_M F_2 F_1 P$$

$$\to P^s M \times_M F_2 P^r M \times_M F_2 F_1 P$$

$$\to P^s M \times_M F_2 (W^{F_1} P) = W^{F_2} (W^{F_1} P).$$ 

According to [1], this is a principal bundle morphism. If \( t_2 \) is injective, (11) is an injection.

To construct the corresponding Lie algebroid homomorphism, we start with a general formula. Consider two principal bundles \( P_1 \to M_1 \) and \( P_2 \to M_2 \) over \( m \)-manifolds and a \( \mathcal{PB} \)-morphism \( f: P_1 \to P_2 \) over a local diffeomorphism \( f: M_1 \to M_2 \). Write \( Lf: LP_1 \to LP_2 \) for the induced algebroid morphism. Using trivializations, one finds easily that the algebroid morphism \( LWf: LWfP_1 \to LWfP_2 \) is

\[
J^*f \times_{FT} FLP: J^*TM_1 \times_{FTM_1} FLP_1 \to J^*TM_2 \times_{FTM_2} FLP_2.
\]

Further, the algebroid form of the injection \( P^{r+} M \hookrightarrow W^r P^r M \) is

\[
J^*TM \hookrightarrow J^*TM \times_{J^*TM} J^*J^*TM,
\]

where we consider both the jet projection \( \pi_s^{r+s}: J^{r+s}TM \to J^sTM \) and the canonical injection \( J^{r+s}TM \hookrightarrow J^sJ^rTM \). According to [1], (\( \hat{f}_2 \))_{P^rM}: J^sP^rM \to F_2P^rM induces a principal bundle morphism

\[
W^sP^rM = P^sM \times_M J^sP^rM \to P^sM \times_M F_2P^rM = W^{F_2P^rM}.
\]

One verifies easily that its algebroid form

\[
(14) \quad J^*TM \times_{J^*TM} J^*J^*TM \to J^*TM \times_{F_2TM} F_2J^*TM
\]

is determined by \( (\hat{f}_2)_{J^*TM}: J^*J^*TM \to F_2J^*TM \). So we have

\[
L(W^{F_2F_1}P) = J^{r+s}TM \times_{F_2TM} F_2F_1LP
\]

\[
\hookrightarrow J^*J^*TM \times_{F_2TM} F_2F_1LP
\]

\[
\to (J^*TM \times_{F_2TM} F_2J^*TM) \times_{F_2TM} F_2F_1LP
\]

\[
= J^sTM \times_{F_2TM} F_2(LW^{F_1}P) = L(W^{F_2}(W^{F_1}P)).
\]

Using (12)–(15), we deduce

**Proposition 1.** The algebroid homomorphism

\[
L(W^{F_2F_1}P) \to L(W^{F_2}(W^{F_1}P))
\]

**corresponding to (11) is the composition of all arrows in (15).**

**2. Reductions of** \( W^1P \). First we summarize our results from [8] on the torsion of connections on \( W^rP \) in the case \( r = 1 \). We consider the principal bundle \( W^1P = P^1M \times_M J^1P \) with structure group \( W^1mG = G^1m \times T^1mG \) and the canonical one-form \( \theta_1: TW^1P \to \mathbb{R}^m \times g \). For a connection \( \Gamma \) on \( W^1P \), the torsion is defined to be the covariant exterior differential \( D\Gamma \theta_1 \). This can be interpreted as a map

\[
\{D\Gamma \theta_1\}: W^1P \to (\mathbb{R}^m \times g) \otimes \Lambda^2\mathbb{R}^m*.
\]
On the other hand, we have $L(W^1P) = J^1(LP)$. Since the bracket $[, ]$ of $LP$ is a first order differential operator, it induces the so-called truncated bracket $[\cdot, \cdot]_1: J^1LP \times_M J^1LP \to LP$.

[8]. If we pass to the algebroid form $\gamma: TM \to J^1LP$ of $\Gamma$, we introduce $\tau\gamma: TM \times_M TM \to LP$ by

$$ \tau\gamma(Z_1, Z_2) = [[\gamma Z_1, \gamma Z_2]_1, \quad (Z_1, Z_2) \in TM \times_M TM. $$

This is a section of $LP \otimes \Lambda^2 T^*M$, what is a fiber bundle associated to $W^1P$ with standard fiber $(\mathbb{R}^m \times g) \otimes \Lambda^2 \mathbb{R}^{m*}$. So the frame form of $\tau\gamma$ is a map

$$ \{\tau\gamma\}: W^1P \to (\mathbb{R}^m \times g) \otimes \Lambda^2 \mathbb{R}^{m*}. $$

In [8], we deduced

$$ \{D\theta_1\} = \frac{1}{2} \{\tau\gamma\}. $$

Remark. The coordinate formula for $\tau\gamma$ can be found in [8]. We find remarkable that this formula implies directly the following assertion. If $\Gamma_1$ and $\Gamma_2$ are two torsion-free connections on $W^1P$ over the same connection on $P$, then every connection of the pencil $t\Gamma_1 + (1 - t)\Gamma_2$, $t \in \mathbb{R}$, is also torsion-free.

Consider a reduction $Q \subset W^1P$ to a subgroup $H \subset W^1mG$. In [4], $Q$ is said to be a generalized $G$-structure. Write $\theta_Q: TQ \to \mathbb{R}^m \times g$ or $[\cdot, \cdot]_Q: LQ \times_M LQ \to LP$ for the restriction of $\theta_1$ or $[, ]_1$, respectively, and $i_Q: Q \to W^1P$ for the injection. A connection $\Gamma$ on $Q$ is canonically extended into a connection $\bar{\Gamma}$ on $W^1P$. Clearly, we have

$$ D\Gamma \theta_Q = i^*_Q(D\bar{\Gamma} \theta_1). $$

Further, for the algebroid form $\gamma: TM \to LQ$ of $\Gamma$, we define

$$ \tau\gamma(Z_1, Z_2) = [[\gamma Z_1, \gamma Z_2]_Q, \quad (Z_1, Z_2) \in TM \times_M TM. $$

Analogously to (19), we have its frame form

$$ \{\tau\gamma\}: Q \to (\mathbb{R}^m \times g) \otimes \Lambda^2 \mathbb{R}^{m*} $$
satisfying

$$ \{\tau\gamma\} = \{\tau\bar{\gamma}\} \circ i_Q. $$

Then (20)–(22) imply
Proposition 2. Both approaches to the torsion of connections on $Q$ are related by

\[ \{D_t \theta_Q\} = \frac{1}{2} \{\tau_\gamma\} : Q \to (\mathbb{R}^m \times \mathfrak{g}) \otimes \Lambda^2 \mathbb{R}^{m*}. \]

Further we need the basic facts concerning the prolongation of generalized $G$-structures. We define the second nonholonomic prolongation $\tilde{W}^2P = W^1(W^1P)$. By [4], $W^2P = \tilde{P}^2M \times_M \tilde{J}^2P$, where $\tilde{P}^2M$ is the second nonholonomic frame bundle of $M$ and $\tilde{J}^2P = J^1(J^1P)$. The second semiholonomic prolongation $\tilde{W}^2P \subset \tilde{W}^2P$ can be defined as $\tilde{P}^2M \times_M \tilde{J}^2P$. We have $W^1Q \subset \tilde{W}^2P$ and $\beta : W^1Q \to Q$ is always surjective. According to [4], $Q$ is called semiprolongable or prolongable, if the restriction of $\beta$ to $W^1Q \cap \tilde{W}^2P$ or $W^1Q \cap W^2P$ is also surjective, respectively. Write $Q_0 = \beta Q$ and $H_0 = \beta H$. In [4], we deduced

Lemma. $Q$ is semiprolongable, if and only if $Q \subset W^1(Q_0)$ or, equivalently, the values of $\theta_Q$ lie in $\mathbb{R}^m \times \mathfrak{h}_0$.

Thus, if $Q$ is semiprolongable, then (23) holds with the additional property that the values lie in $\mathbb{R}^m \times \mathfrak{h}_0$.

For every $X \in W^1P$, $X = (u,U)$, we have $U \circ u \in T^1_mP$. Hence $X$ is identified with an $m$-dimensional subspace $\lambda(X) \subset TP$. So every $Z \in \tilde{W}^2P$ is identified with $\lambda(Z) \subset TW^1P$. According to Proposition 5 of [4], $Z \in \tilde{W}^2P$ satisfies

\[ Z \in W^2P \quad \text{if and only if \,} d\theta_1 \mid \lambda(Z) = 0. \]

This implies an assertion analogous to the classical theory of $G$-structures. Write $\pi : Q \to M$, $\pi_1 : Q \to P^1M$ and $\pi_2 : W^1Q \to J^1Q$ for the canonical projections.

Proposition 3. Let $Q$ be semiprolongable. If $Q$ admits a torsion-free connection, then $Q$ is prolongable. Conversely, if $Q$ is prolongable, then for every $x \in M$ there exists a neighbourhood $U$ and a torsion-free connection on $\pi^{-1}(U)$.

Proof. Let $\Gamma : Q \to J^1Q$ be a torsion-free connection. By (24), the rule

\[ X \mapsto (\pi_1(X), \Gamma(X)) , \quad X \in Q \]

is a section $Q \to W^1Q \cap W^2P$, so that $Q$ is prolongable. Conversely, let $\Sigma : Q \to W^1Q \cap W^2P$ be a section. For every section $\varrho : U \to Q$, the map $\pi_2 \circ \Sigma \circ \varrho : U \to J^1Q$ is canonically extended into a connection on $\pi^{-1}(U)$. This connection is torsion-free due to the fact that $\theta_1$ is a pseudo-tensorial form, [9, p.155].
3. Torsions on certain gauge-like prolongations. If $E$ is a fiber product preserving bundle functor on $\mathcal{FM}_m$ satisfying $E \subset J^1 \circ F$, then $W^E P$ is a reduction of $W^1(W^F P)$ to a subgroup $K \subset W^1(W^A G)$. Write $E_0 = \beta E \subset F$ and $K_0 = \beta K \subset W^A_0 G$. In general, we have $\theta_{EP}: T(EP) \to \mathbb{R}^m \times \mathfrak{w}^G_0$. If $EP$ is semiprolongable, then the values of $\theta_{EP}$ lie in $\mathbb{R}^m \times \mathfrak{k}_0$. The simplest case of such situation is $\beta E = F$.

We are going to present some examples.

Example 1. We start with the functor $\tilde{J}^r$ of $r$-th nonholonomic jet prolongation of fibered manifolds, $\tilde{J}^r Y = J^1(\tilde{J}^{r-1} Y)$. More generally, our approach can be applied to an arbitrary functor $S$ of $r$-th jet prolongation of fibered manifolds. In accordance with [7], this means a fiber product preserving bundle functor on $\mathcal{FM}_m$ satisfying $J^r \subset S \subset \tilde{J}^r$. In particular, $J^r$ can be reduced to this situation by means of the canonical inclusion $J^r \subset J^1 \circ J^{r-1}$. A further well known example is the $r$-th semiholonomic prolongation $\bar{J}^r \subset J^1 \circ \bar{J}^{r-1}$. Clearly, the semiprolongability condition is satisfied in the last two cases.

Example 2. Another example of our type is the composition $J^r \circ F$ for arbitrary $F$. More generally, we can consider every composition $S \circ F$ with $S$ from Example 1.

Some further functors can be studies in this way by using a suitable natural equivalence.

Example 3. In [1] it is deduced that for every fiber product preserving bundle functor $E$ on $\mathcal{FM}_m$ and every vertical Weil functor $V^B$ there exists a canonical natural equivalence

$$\varpi: V^B \circ E \approx E \circ V^B.$$ (25)

If $E \subset J^1 \circ F$, then $\varpi(V^B \circ E) \subset J^1 \circ (F \circ V^B)$. Hence we have the situation of Section 2. In addition, one deduces that $\beta E = F$ implies $\beta(\varpi(V^B \circ E)) = F \circ V^B$ by using the standard Weil algebra manipulations from [1].

References


