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Refree's report on the Ph.D. thesis
Unification modulo associativity and idempotency
submitted by **Ondřej Klíma**

Solving equations is, was and will always be in the centre of interest of several branches of mathematics. In our studies we meet them first as solving systems of linear equations over the real numbers. In algebra we arrive very soon to the problems of solving equations or target equations over finitely presented algebraic structures, for example finite rings, groups, monoids. Surprisingly enough similar questions rise from model theory, automata theory or communicational complexity. They call it program-satisfyability, unification or identity checking problems.

The topic of the Ph.D. thesis is the examination of the complexity of these problems over classes of monoids and semigroups.

The basic problem of the dissertation is the EQN_M problem. Given the polynomials (in the universal algebraic sense) $(p_1, q_1, p_2, q_2, \ldots, p_k, q_k)$ over a fixed algebraic structure, M. Decide whether or not the system of equations $p_i = q_i$ has a solution, that is whether or not you can evaluate the variables on such a way that all the above equations hold. If the structure is infinite then it is not clear that the problem is decidable or not. Indeed, there exist finitely presented groups where the question is undecidable.

The EQN problem over a finite signature is called unification problem. Unification problem plays an important role over existentially closed structures in classical model theory. If additional conditions hold over the structure then we are talking about E-unification. E-unification is the EQN problem for the countably generated free algebra in a variety of algebras.

The topic of the thesis is the examination of the complexity of AI and AIU unification problems and some related E-unification problems. In algebraic language it means EQN_M for bands, that is idmepotent semigroups and monoids.

The 5th and last chapter is the most attractive from the point of view of an algebraist. In this chapter Mr. Klíma investigates the complexity of the T-EQN, 1T-EQN and 1-EQN problems for individual monoids (not for the free structure of a variety). The main results are Proposition 5.2.1, Proposition 5.2.3 and Proposition 5.3. In the first two theorems he proves that the solvability of target equations for a band can be decided in polynomial time if the band is regular and NP-complete otherwise. The construction of Proposition 5.3.6 shows an example of a left normal band with NP-complete EQN problem. There is no doubt that the results of this chapter can be published in a high standard mathematical journal. His expectation about the hopelessness of the characterization of the EQN problem into polnomial and NP-complete cases was verified by Larose and others, who proved that for the class LRN the EQN prblem is equivalent to the famed CSP problem.

Chapter 4 discusses the EQN problem for the free algebras in proper varieties of bands. This is called the V-Unification problem. The chapter is an extension of paper [31] that is to appear in IJAC the leading computational algebra journal. Ondřej Kíma presents a beautiful dichotomy for varieties of bands for all three problems discussed in the thesis. Namely, the identities $xyz = xz$, $xyzx = xzyx$, $xyxzx = xyzx$ distinguish the polynomial time and NP-complete cases for the General Unification Problem, the Unification Problem and the Matching problem, respectively. The methods are fancy generalizations of the ones used in Chapter 3.

According to the author, Chapter 3 contains the most important results of the thesis. The main tool of the chapter is an algorithm that decides whether or not the simple AIU-unification problem has a solution. This implies that the simple AIU-unification problem can be solved in polynomial time. On the other hand, the AIU-unification and the general AIU-unification problems are NP-complete. (Theorems 3.4.1 and 3.4.3).

Chapter 2. contains a Term Rewriting System for bands. The concept of TRS is paradigamatic for the study of computational procedures. They play a crucial role in mathematical logic with resoect to formalizing the notion of computability. TRS are attractive because of their simple syntax that facilitates a satisfactory mathematical analysis. The TRS of Chapter 2. gives an easy way for evaluating terms in a free idempotent monoid proving that the matching problems are in NP.

The first chapter contains the introduction and the preliminaries. In sections 1.1 and 1.2 the author lists all notions from both algebra an unfication theory that will be later needed for the discussion. In this chapter I feel the impatiance of the young researcher who wants to get to the results as fast as possible. This section is very condensed. Although everything is listed here that is required to understand the latter concepts, the missing explanations and examples do not make the reader's job easier. We should not forget that we are in a multidisciplinary subject, hence one cannot assume all the background and knowledge of Mr. Klíma from an average reader, who is very likely aware of just 2-3 topics of the approximatley 6 areas that the thesis touches.

The thesis is written in LATEX in English. As the author is Czech and I am Hungarian, I do not feel my job to correct or evaluate the nonstandard usage or grammatical mistakes. Very likely my review contains a few of those as well. He could have probably written a gramatically better thesis in Czech but in that case I would not have been able to read it.

The proofs are well written (most of them are already published in refereed journals) the theorems are clearly stated. The abstract is a very pleasant work itself. Mr. Klíma shows a mastery of the material he works with. He is aware of the current flow of research in the subject(s) he is working in. He is using the references appropriately. The thesis is divided into five individual chapters, but it is not a list of five different results. It is a complete work where the chapters follow one another in a logical order.

Although I was not given any instructions how to referee the thesis of Ondřej Klíma, what the standard of a Ph.D. thesis is at Masaryk University, I think that this work meets any standards for a Ph.D. thesis and I strongly recommend the Ph.D. degree to be awarded to Ondřej Klíma.

Sincerely:

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