1. PŘ.

2. PŘ.

1. CV.

- neskouší neodpřednášené důkazy
- 2 neomluvené absence
- 3 zápočtovky – cca. 13.10., ~~10.~~ 24.11., 8.12.
  - z každé 10 bodů (celkem získat aspoň 10)
- 15 - sbírka příkladů

* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$

$$a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$s_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \dots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} \xrightarrow[n\to\infty]{} 1$$

* $\sum_{n=1}^{\infty} \left(\frac{3^n + 2^n}{6^n}\right) = \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{\text{konv.}} + \underbrace{\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n}_{\text{konv.}} = \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} = 1 + \frac{1}{2} = \frac{3}{2}$

* $\sum_{n=1}^{\infty} \frac{1}{(5n-4)(5n+1)} = \frac{1}{5}$

$$\frac{1}{(5n-4)(5n+1)} = \frac{\frac{1}{5}}{1} - \frac{\frac{1}{5}}{5n+1} \qquad s_n = \frac{\frac{1}{5}}{1} - \frac{\frac{1}{5}}{5n+1} = \frac{1}{5} - \frac{1}{5(5n+1)} \xrightarrow[n\to\infty]{} \frac{1}{5}$$

* $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = +\infty$

$$\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$$

$$s_n = -\ln 1 + \ln(n+1) = \ln(n+1) \xrightarrow[n\to\infty]{} +\infty$$

* $\sum_{n=1}^{\infty}\left(\frac{3}{4^{2n-1}}+\frac{2}{4^{2n}}\right) = \sum_{n=1}^{\infty}\frac{14}{4^{2n}} = \sum_{n=1}^{\infty}\frac{14}{(16)^n} = \frac{14}{16}\cdot\frac{1}{1-\frac{1}{16}} = \frac{14}{16}\cdot\frac{16}{15} = \frac{14}{15}$

* $s_n = 1-\frac{1}{2^n},\ n\in\mathbb{N}$

$a_n = s_n - s_{n-1} = \frac{1}{2^{n-1}} - \frac{1}{2^n} = \frac{1}{2^n},\ n\geq 2,\quad a_1 = s_1 = \frac{1}{2}$

$\sum_{n=1}^{\infty}\frac{1}{2^n} = 1$

* DÚ $s_n = \frac{(-1)^n}{n}$

$a_n = s_n - s_{n-1} = \frac{(-1)^n}{n} - \frac{(-1)^{n-1}}{n-1}$

$n\geq 2$

n sudé: $a_n = \frac{1}{n} - \frac{1}{n-1}$

n liché: $a_n = \frac{1}{n-1} - \frac{1}{n}$

$n=1$: $a_1 = s_1 = \frac{-1}{1} = -1$

* $\sum_{n=1}^{\infty}\underbrace{\frac{1}{(n+1)^2}}_{a_n} \leq \sum_{n=1}^{\infty}\underbrace{\frac{1}{n(n+1)}}_{b_n} = 1$

a_n konv.

* $\sum_{n=}^{\infty}\sin\frac{1}{n}$ ... diverguje

$a_n \xrightarrow[n\to\infty]{} 0$ (je splněna nutná podmínka :( )

$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$

* $\sum_{n=1}^{\infty}\frac{n+1}{n^2+1}$ ... diverguje

$\frac{n+1}{n^2+1} \approx \frac{1}{n}$

$\cancel{\lim_{n\to\infty}\frac{\frac{n+1}{n^2+1}}{\frac{1}{n^2+1}} = \lim_{n\to\infty}(n+1) = \infty}$

$\lim_{n\to\infty}\frac{\frac{n+1}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n^2+n}{n^2+1} = 1$

$$* \sum_{n=2}^{\infty} \frac{1}{\ln n} \geq \sum_{n=2}^{\infty} \frac{1}{n} = \text{div} +\infty$$

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = +\infty$$

$$* \sum_{n=1}^{\infty} \frac{\sin\frac{1}{n}}{\sqrt{n^3}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \dots \text{konverguje}$$

konverguje

alternativně: $\frac{\sin\frac{1}{n}}{\sqrt{n^3}} \approx \frac{\frac{1}{n}}{\sqrt{n^3}} = \frac{1}{n^{5/2}}$

$$\lim_{n\to\infty} \frac{\frac{\sin\frac{1}{n}}{\sqrt{n^3}}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

$\boxed{2.\ \text{CV.}}$

$$* \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$a_n = \frac{n^n}{n!}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim \frac{(n+1)^n}{n^n} = \lim \left(1+\frac{1}{n}\right)^n = e > 1$$

řada diverguje

$$* \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1 \quad \text{řada konverguje}$$

$$* \sum_{n=1}^{\infty} \frac{n!}{2^n+1} \text{ diverguje}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)!}{2^{n+1}+1}}{\frac{n!}{2^n+1}} = \lim \frac{(2^n+1)(n+1)}{2^n(2+\frac{1}{2^n})} = \lim \frac{(1+\frac{1}{2^n})(n+1)}{(2+\frac{1}{2^n})} = \infty$$

* $$\sum_{n=1}^{\infty} \left(\frac{n+2}{2n-1}\right)^n$$

$$\lim \sqrt[n]{\left(\frac{n+2}{2n-1}\right)^n} = \lim \frac{n+2}{2n-1} = \frac{1}{2} < 1 \ldots$$ řada konverguje

* $$\sum_{n=1}^{\infty} \left(\operatorname{arctg} \frac{1}{n}\right)^n$$

$$\lim \operatorname{arctg} \frac{1}{n} \cancel{= \frac{\pi}{2} > 1} \ldots$$ ~~řada diverguje~~

$$= 0 < 1 \ldots$$ řada konverguje

* $$\sum_{n=1}^{\infty} \frac{1}{(\ln(n+1))^{n+1}}$$

$$\lim \sqrt[n]{\frac{1}{(\ln(n+1))^{n+1}}} = \lim \frac{1}{(\ln(n+1))^{1+\frac{1}{n}}} = 0 < 1 \ldots$$ řada konverguje

* $$\sum_{n=1}^{\infty} \frac{1}{n^n}$$

$$\lim \frac{1}{n} = 0 < 1 \ldots$$ řada konverguje

lze to ukázat i $\frac{1}{n^n} \leq \frac{1}{n^2}$

* $$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \ldots$$

$$a_n = \begin{cases} \frac{1}{2^n}, & n = 2k-1 \\ \frac{1}{3^n}, & n = 2k \end{cases} \qquad \sqrt[n]{a_n} = \begin{cases} \frac{1}{2}, & n = 2k-1 \\ \frac{1}{3}, & n = 2k \end{cases}$$

$\lim \sqrt[n]{a_n}$ neexistuje

$$\sqrt[n]{a_n} \leq \frac{1}{2} < 1 \ldots$$ řada konverguje

* $$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{n^2}{(n+1)^2} = 1 \ldots$$ nelze rozhodnout

$$\lim n\left(1 - \frac{a_{n+1}}{a_n}\right) = \lim \left(n - \frac{n^3}{(n+1)^2}\right) = \lim \frac{n(2n+1)}{(n+1)^2} = 2 > 1 \ldots$$ řada konverguje

* $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim \frac{a_{n+1}}{a_n} = \lim \frac{n}{n+1} = 1$ ... nelze rozhodnout

$\lim n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim\left(n - \frac{n^2}{n+1}\right) = \lim \frac{n}{n+1} = 1$ ... nelze rozhodnout

třeba rozhodnout jinak

* $\sum_{n=1}^{\infty} \frac{1}{3^{1+\frac{1}{2}+\dots+\frac{1}{n}}}$

$\lim n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim n\left(1-\frac{1}{3^{\frac{1}{n+1}}}\right) = \lim\left(n - \frac{n}{\sqrt[n+1]{3}}\right) = \lim\left(\frac{n\sqrt[n+1]{3}-n}{\sqrt[n+1]{3}}\right) \cancel{= \infty \dots \text{diverguje}} =$

$\cancel{= \left|\frac{0}{1}\right| = 0 \dots \text{konverguje}} = \lim n\left(1-\frac{1}{\sqrt[n+1]{3}}\right) = \lim|\infty\cdot 0| = \lim_{n\to\infty}\frac{1-\frac{1}{\sqrt[n+1]{3}}}{\frac{1}{n}} =$

$= \lim_{n\to\infty} \frac{3^{-\frac{1}{n+1}}\cdot \ln 3 \cdot \left(-\frac{1}{(n+1)^2}\right)}{-\frac{1}{n^2}} = \lim\left(3^{-\frac{1}{n+1}}\cdot\ln 3\cdot\frac{n^2}{(n+1)^2}\right) = \ln 3 > 1$

řada ~~diverguje~~ konverguje

* $\sum_{n=1}^{\infty} \left(\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\right)^2$

$\lim n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim n\left(1-\frac{(2n+1)^2}{(2n+2)^2}\right) \cancel{= \frac{1}{4}\lim n\left(\frac{n^2+2n+1-4n^2}{(2n+4)^2}\right)} =$

$= \lim n\left(\frac{4n^2+8n+4-4n^2-4n-1}{(2n+2)^2}\right) = \lim n\frac{4n+3}{(2n+2)^2} = 1$ ... nelze rozhodnout

nelimitní verze Raabe $n\left(1-\frac{a_{n+1}}{a_n}\right) \geq q > 1$ ... konv, $n\left(1-\frac{a_{n+1}}{a_n}\right) \leq 1$ ... div

$\frac{4n^2+3n}{4n^2+8n+4} \leq 1$ ... diverguje

* $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$

$f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}} \geq 0$

$f'(x) = \frac{e^{-\sqrt{x}}\cdot\left(-\frac{1}{2}\cdot x^{-\frac{1}{2}}\right)\cdot\sqrt{x} - e^{-\sqrt{x}}\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}}{x} =$

$\cancel{= -\frac{1}{2}e^{-\sqrt{x}} < 0 \dots f \text{ je nerostoucí}}$

f je nerostoucí

$= \frac{-e^{-\sqrt{x}}\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}\left(\sqrt{x}+1\right)}{x} < 0$ pro $x \in [1,\infty)$

$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = [-2e^{\sqrt{x}}$

$$\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \left| \begin{array}{l} t=\sqrt{x} \\ dt = \frac{1}{2}x^{-\frac{1}{2}}dx \end{array} \right| = 2\int_1^\infty e^{-t}\,dt = 2\left[-e^{-t}\right]_1^\infty = 2e^{-1} = \frac{2}{e} \dots \text{konvergje}$$

tedy i řada konvergje

$\boxed{6.\ \text{PŘ.}}$

* $$\sum_{n=0}^\infty \frac{3}{5^n} = 3\cdot\frac{1}{1-\frac{1}{5}} = \frac{15}{4}$$

* $$\sum_{n=1}^\infty \frac{2n-1}{2^n} = \sum_{n=1}^\infty \frac{2n}{2^n} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = 2\sum_{n=1}^\infty \frac{n}{2^n} - 1 = 3$$

$$s_n = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots + \frac{n}{2^n}$$

$$\frac{s_n}{2} = \frac{1}{2^2} + \frac{2}{2^3} + \dots + \frac{n}{2^{n+1}}$$

$$\frac{s_n}{2} = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} - \frac{n}{2^{n+1}}.$$

$$s_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^{n-1}} - \frac{n}{2^{n}} \xrightarrow[n\to\infty]{} 2$$

* $$\sum_{n=2}^\infty \frac{1}{n^2-1} = \frac{3}{4}$$

$$s_n = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \dots + \frac{1}{n-1} - \frac{1}{n+1}\right) =$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) \xrightarrow[n\to\infty]{} \frac{1}{2}\cdot\frac{3}{2} = \frac{3}{4}$$

* $$\lim_{n\to\infty}\left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2}\right) = \lim_{n\to\infty} \frac{1}{n^2}(1 + \dots + (n-1)) = \lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2}$$

* $$\ln x + \ln\sqrt{x} + \ln\sqrt[4]{x} + \dots = 2$$

$$1\ln x + \frac{1}{2}\ln x + \frac{1}{4}\ln x + \dots = 2$$

$$2\ln x = 2$$

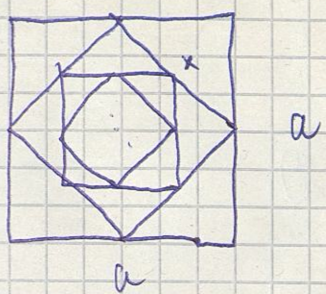
$$\ln x = 1$$

$$x = e$$

* $\sum_{n=1}^{\infty} \left(\sqrt{n} - 2\sqrt{n+1} + \sqrt{n+2}\right) = 1-\sqrt{2}$

$$s_n = \sqrt{1} - \sqrt{2} - \sqrt{n+1} + \sqrt{n+2} \xrightarrow{n\to\infty} 1-\sqrt{2}$$

*



určete součty obsahů a obvodů

$$S = a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \dots = 2a^2$$

$$O = 4a + \frac{4a}{\sqrt{2}} + \frac{4a}{2} + \dots = \frac{4a}{1-\frac{1}{\sqrt{2}}}$$

$$x = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2} = \frac{a}{\sqrt{2}}$$

* $\sum \frac{1}{n(n+1)} \le \sum \frac{1}{n^2} < \infty$ ... konv

* $\sum \frac{n^2}{n^3 - 4n - 5} \ge \sum \frac{n^2}{n^3} = \sum \frac{1}{n} = \infty$ ... diverguje

* $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

$$\sum_{n=3}^{\infty} \frac{\ln n}{n} \ge \sum_{n=3}^{\infty} \frac{1}{n} = \infty \text{ ... diverguje}$$

* $\sum_{n=2}^{\infty} \frac{1}{\sqrt[n]{\ln n}}$

$$1 \le \ln n \le n \quad (n \ge 3)$$

$$1 \le \sqrt[n]{\ln n} \le \sqrt[n]{n} \Rightarrow \lim \sqrt[n]{\ln n} = 1 \Rightarrow \text{diverguje}$$

* $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $\quad f(x) = \frac{1}{x \cdot \ln x} \quad (x \ge 2)$

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx = \left[\ln \ln x\right]_2^{\infty} = \infty \text{ ... diverguje}$$

* $\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^n}$ ... konverguje

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \lim \frac{n^n}{2^n \cdot n!} = \lim \frac{2(n+1)\cdot n^n}{(n+1)^{n+1}} = 2\lim \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$$

* $\lim_{n\to\infty} \frac{(n!)^2}{(2n)!}$

$(2n)! = 2^n \cdot n! \cdot (2n-1)!!$ $\qquad \binom{2n}{n} \qquad \frac{1}{\binom{2n}{n}}$

indukcí:

$n=1 \Rightarrow 2 = 2\cdot 1\cdot 1$

$n=2 \Rightarrow 4! = 2^3 \cdot 2 \cdot 3$

$(2n+2)! = (2n+2)(2n+1)(2n)! = (2n+2)(2n+1)2^n n!(2n-1)!! = 2^{n+1}\cdot(n+1)!\cdot(2n+1)!!$

$$\lim_{n\to\infty} \frac{(n!)}{2^n(2n-1)!!} = \lim_{n\to\infty} \frac{1}{2^n}\cdot\left(\frac{1}{1}\cdot\frac{2}{3}\cdot\frac{3}{5}\cdots\frac{n}{2n-1}\right) \leq \lim \frac{1}{2^n} = 0$$

* $\sum_{n=1}^{\infty} (\arccos\frac{1}{n})^{n^2}$ ... diverguje

$$\lim \sqrt[n]{a_n} = \lim (\arccos\tfrac{1}{n})^n = \text{"}\left(\frac{\pi}{2}\right)^{\infty}\text{"} = \infty$$

* $\sum \operatorname{arctg}^n \frac{1}{n}$ ... konverguje

$\lim \operatorname{arctg}\frac{1}{n} = 0 < 1$ ... konverguje

* $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n-1}$ $\qquad \frac{1}{3n-1} \xrightarrow{n\to\infty} 0$ ... konverguje

* $\sum \frac{(-1)^n n}{5n-2}$ ... osciluje $\qquad \frac{n}{5n-2} \xrightarrow{n\to\infty} \frac{1}{5} \neq 0$

* $\sum (-1)^n \frac{1}{(2n+3)^3}$ ... konv. absolutně

rozhodněte o abs. konvergenci

$$\sum \frac{1}{(2n+3)^3} \leq \sum \frac{1}{n^3} < \infty$$

* $\sum \frac{(-1)^n}{\ln^n(2n+4)}$ ... abs. konv.

$$\lim \frac{1}{\ln(2n+4)} = 0 < 1$$

* $\sum (-1)^{n+1} \cdot \frac{\left(\frac{n+1}{n}\right)^{n^2}}{3^n}$ ... abs. konv

$$\lim \sqrt[n]{|a_n|} = \lim \frac{\left(\frac{n+1}{n}\right)^n}{3} = \frac{e}{3} < 1$$

* $\sum \frac{(-2)^n}{(5+(-1)^{n+1})^n}$ ... abs. konv.

$$\sum \frac{2^n}{(5+(-1)^{n+1})^n} \leq \sum \frac{2^n}{4^n} = \sum \frac{1}{2^n} = 1 \text{ .. konv.}$$

* $\sum_{n=1}^{\infty} \frac{x^n}{n}$ pro která $x \in \mathbb{R}$ máme abs. konvergenci ; pro $x=0$ máme ✓

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{|x|}{\frac{n+1}{n}} = |x|$$

$|x| < 1$ ... abs. konv

$x = 1 \Rightarrow$ diverguje

$x = -1 \Rightarrow$ konverguje neabsolutně

odpověď: $x \in (-1, 1)$

* $\sum_{n=1}^{\infty} \frac{2^n}{n^2} \cdot x^n$ určete poloměr a obor konv.

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{2 \cdot n^2}{(n+1)^2} = 2 \Rightarrow R = \frac{1}{2}$$

$$\left.\begin{aligned} x = -\tfrac{1}{2}: &\ \sum \frac{(-1)^n}{n^2} \\ x = \tfrac{1}{2}: &\ \sum \frac{1}{n^2} \end{aligned}\right\} \text{abs. konv.}$$

tedy $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

* $\sum_{n=1}^{\infty} \frac{(n+6)x^n}{n!}$.

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n+7}{(n+1)!} \cdot \frac{n!}{n+6} = 0 \Rightarrow R = \infty \Rightarrow x \in \mathbb{R}$

* $\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n+\sqrt{n}}$

$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{n+\sqrt{n}}{n+1+\sqrt{n+1}} = 1 \Rightarrow R = \frac{1}{1} = 1$

$x=-3$: $\sum \frac{1}{n+\sqrt{n}} \geq \sum \frac{1}{n+n} = \frac{1}{2} \cdot \sum \frac{1}{n} = \infty$ ... div

$x=-1$: $\sum \frac{(-1)^n}{n+\sqrt{n}}$ ... konv. neabs. ale Leibnize

$x \in (-3, -1]$

* $\sum_{n=1}^{\infty} 2^{n/2} \cdot x^n = \sum (\sqrt{2})^n x^n$

$\lim \sqrt[n]{|a_n|} = \lim \sqrt{2} = \sqrt{2} \Rightarrow R = \frac{1}{\sqrt{2}}$

$x = -\frac{1}{\sqrt{2}}$: $\sum (-1)^n$ ... osciluje

$x = \frac{1}{\sqrt{2}}$: $\sum 1$ ... div.

$x \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

* $|x|<1$, ukažte $\sum_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3} = \frac{1}{2} \operatorname{arctg} x + \frac{1}{4} \ln \frac{1+x}{1-x}$

$\left(\sum \frac{x^{4n-3}}{4n-3}\right)' = \sum x^{4n-4} = \sum_{k=0}^{\infty} (x^4)^k = \frac{1}{1-x^4}$

$\int \frac{1}{1-x^4} dx = \int \left(\frac{1/2}{1+x^2} + \frac{1/4}{1-x} + \frac{1/4}{1+x}\right) dx = \frac{1}{2} \operatorname{arctg} x + \frac{1}{4} \cdot \ln \frac{1+x}{1-x} + C$

$x=0 \Rightarrow 0 = \frac{1}{2} \operatorname{arctg} 0 + \frac{1}{4} \ln 1 + C \Rightarrow C=0$

* $\sum_{n=0}^{\infty} \frac{n^2+5n+6}{3^n}$

využijme $\sum_{n=0}^{\infty} (n^2+5n+6)x^n$ $\qquad (n^2+5n+6)=(n+2)(n+3)$

$x\in(-1,1)$: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

$$\sum_{n=0}^{\infty} x^{n+3} = \frac{x^3}{1-x}$$

$$\sum_{n=0}^{\infty} (n+3)x^{n+2} = \frac{3x^2-2x^3}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} (n+2)(n+3)x^{n+1} = \frac{2x^3-6x^2+6x}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} (n+2)(n+3)x^{n} = \frac{2x^2-6x+6}{(1-x)^3}$$

$x=\frac{1}{3}$: $\sum \frac{n^2+5n+6}{3^n} = \dots = \frac{57}{4}$

$\boxed{\text{3.CV.}}$

* $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ ... konverguje

$f: \langle 2,\infty) \to \mathbb{R}$

$f(x) = \frac{1}{x\ln^2 x} \geq 0$ $\qquad f'(x) = -\frac{1}{(x\cdot\ln x)^2}\cdot(\ln^2 x + 2\ln x) \leq 0$ ✓

$$\int_2^{\infty} \frac{1}{x\ln^2 x}dx = \left| \begin{matrix} t=\ln x \\ dt = \frac{dx}{x} \end{matrix} \right| = \int_{\ln 2}^{\infty} \frac{1}{t^2}dt = \left[-\frac{1}{t}\right]_{\ln 2}^{\infty} = \frac{1}{\ln 2}$$

* $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$, $\alpha\in\mathbb{R}$

$f: [1,\infty) \to \mathbb{R}$

$f(x) = \frac{1}{x^{\alpha}} > 0$ $\qquad f'(x) = -\frac{\alpha}{x^{\alpha+1}} < 0$ pro $\alpha>0$

$$\int_1^{\infty} x^{-\alpha}dx \overset{\alpha=1}{=} [\ln x]_1^{\infty}$$

$$\overset{\alpha\neq1}{=} \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]^{\infty} = \begin{cases} \infty & 0<\alpha<1 \\ \frac{1}{1-\alpha} & \alpha>1 \end{cases}$$

$\alpha=0$, zřejmě div.

$\alpha\leq0$, pak zřejmě řada diverguje (dle nutné podmínky) | takže celkem řada konv. $\Leftrightarrow \alpha>1$

div. $\Leftrightarrow \alpha\leq1$.

$* \sum_{n=1}^{\infty} \frac{2}{3+n^2}$ ... konvergují

$f: [1,\infty) \to \mathbb{R}$

$f(x) = \frac{2}{3+x^2} > 0 \qquad f'(x) = -\frac{2}{(3+x^2)^2} \cdot 2x < 0 \ \checkmark$

$$\int_1^{\infty} \frac{2}{3+x^2}\,dx = \frac{2}{3}\int_1^{\infty} \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx = \left| \begin{array}{l} \frac{x}{\sqrt{3}} = s \\ dx = \sqrt{3}\,ds \end{array} \right| = \frac{2\sqrt{3}}{3}\int_{\frac{1}{\sqrt{3}}}^{\infty} \frac{1}{1+s^2}\,ds =$$

$$= \frac{2\sqrt{3}}{3}\left[\operatorname{arctg} x\right]_{\frac{1}{\sqrt{3}}}^{\infty} = \frac{2\sqrt{3}}{3}\cdot\frac{\pi}{2} - \frac{2\sqrt{3}}{3}\cdot\frac{\pi}{6} < \infty$$

*DÚ vyšetřete konv. $\sum_{n=1}^{\infty} \frac{\ln n}{n^{\alpha}}$, $\alpha \in \mathbb{R}$

$* \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ .. konv.

$\frac{1}{n} \geq 0$, $\frac{1}{n}$ je nerostoucí $\checkmark$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$* \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(2n+1)}$ ... konv.

$\frac{1}{\ln(2n+1)} > 0 \qquad \frac{1}{\ln(2n+1)}$ je nerostoucí $\checkmark$

$\lim_{n\to\infty} \frac{1}{\ln(2n+1)} = 0$

$* \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2^n}$ ... konv.

$\frac{(n+1)^2}{2^{n+1}} \leq \frac{n^2}{2^n} \iff (n+1)^2 \leq 2n^2$

$2n+1 \leq n^2$ pro $n \geq 3$

$\frac{n^2}{2^n} \geq 0$ a $\frac{n^2}{2^n}$ je nerostoucí $\checkmark$ pro $n \geq 3$

$\lim_{n\to\infty} \frac{n^2}{2^n} = 0$

$* \sum_{n=1}^{\infty} \frac{\sin n}{6^n}$ ... abs. konv.

$\sum_{n=1}^{\infty} \frac{|\sin n|}{6^n} \leq \sum_{n=1}^{\infty} \frac{1}{6^n} < \infty$ ✓

* $\sum_{n=1}^{\infty} (1)^{\frac{n(n+1)}{2}} \cdot \frac{1}{4^n}$ ... abs. konv. $\quad \sum_{n=1}^{\infty} \frac{1}{4^n} < \infty$

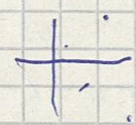
* $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n^2}}{n!}$ ... diverguje

$$\frac{a_{n+1}}{a_n} = \frac{2^{(n+1)^2}}{(n+1)!} \cdot \frac{n!}{2^{n^2}} = \frac{2^{2n+1}}{n+1} > 1 \quad \text{pre } n \geq 1$$

$\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$ ... diverguje

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$$

$\lim_{n\to\infty} \frac{2^{n^2}}{n!} (-1)^{n-1}$ osciluje



* $\sum_{n=1}^{\infty} (-1)^n \cdot \left(\frac{2n+1}{3n+1}\right)^n$ ... abs. konv.

$\sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+1}\right)^n$ ... konv. $\quad \lim_{n\to\infty} \frac{2n+1}{3n+1} = \frac{2}{3} < 1$

* $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3 \cdot 2^n} \cdot x^n$, $x \in \mathbb{R}$ ... konverguje pre $[-2,2)$

$\sum_{n=1}^{\infty} \frac{n^2+1}{n^3 \cdot 2^n} \cdot |x|^n$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^2+1}{(n+1)^3 \cdot 2^{n+1}} \cdot \frac{n^3 \cdot 2^n \cdot |x|}{n^2+1} = \lim_{n\to\infty} \frac{n^3((n+1)^2+1)\cdot |x|}{2(n+1)^3(n^2+1)} = \frac{|x|}{2}$$

$|x| < 2$ ... abs. konv.

$|x| > 2$ ... div.

$x = 2$: $\sum \frac{n^2+1}{n^3}$ ... diverguje

$\frac{n^2+1}{n^3} \approx \frac{1}{n} \quad \lim \frac{\frac{n^2+1}{n^3}}{\frac{1}{n}} = \lim \frac{n^2+1}{n^2} = 1$

$x = -2$: $\sum \frac{n^2+1}{n^3} \cdot (-1)^n$ ... relatívne konv. $\quad \lim \frac{n^2+1}{n^3} = 0$

$\boxed{4.\ \text{CV.}}$

- odhady súčtu

$\sum_{n=1}^{\infty} a_n = s \qquad s_n = \sum_{k=1}^{n} a_k, \ \lim_{n\to\infty} s_n = s \qquad R_n = s - s_n$

$\lim_{n\to\infty} R_n = 0$ (odhady vychádzajú z kritérií konv.)

- např. pro podílové kritérium

$$\left|\frac{a_{n+1}}{a_n}\right| \leq q < 1$$

$$|R_n| \leq |a_n| \cdot \frac{q}{1-q}$$

* $\sum_{n=1}^{\infty} \frac{1}{n 2^n}$, chyba $\leq 10^{-2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n 2^n}} = \frac{1}{2} \cdot \frac{n}{n+1} < \overset{=q}{\frac{1}{2}} < 1$$

$$|R_n| \leq |a_n| \cdot \frac{q}{1-q} = \frac{1}{n \cdot 2^n} \cdot \frac{1/2}{1-1/2} = \frac{1}{n \cdot 2^n} \leq 10^{-2}$$

$$n \cdot 2^n \geq 100$$

$$n \geq 5$$

$$s \approx \sum_{n=1}^{5} \frac{1}{n \cdot 2^n} = \dots$$

- pro int. kritérium

$f(x) \geq 0$, $\searrow$ na $[1, \infty)$, $f(n) = |a_n|$

$\int_1^{\infty} f(x)\,dx$ . . . $|R_n| \leq \int_n^{\infty} f(x)\,dx$

* $\sum_{n=1}^{\infty} \frac{1}{n^4}$, chyba $\leq 10^{-2}$

$f(x) = \frac{1}{x^4}$, $f(x) \geq 0$, $\searrow$ na $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3}\right]_1^{\infty} = \frac{1}{3} \dots \text{konv}$$

$$|R_n| \leq \int_n^{\infty} \frac{1}{x^4}\,dx = \left[\frac{-1}{3x^3}\right]_n^{\infty} = \frac{1}{3n^3} \leq 10^{-2}$$

$$3n^3 \geq 100$$

$$n \geq 4$$

$$s \approx \sum_{n=1}^{4} \frac{1}{n^4} = \dots$$

– pro alt. řadu $\sum_{n=1}^{\infty} (-1)^{n-1} a_n \mid a_n \geq a_1 \longrightarrow$

Leibniz konv. $\Longleftrightarrow \lim_{n\to\infty} a_n = 0$

$|R_n| \leq a_{n+1}$

* $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{2^n}{n!}$, chyba $\leq 10^{-3}$

$\frac{2^n}{n!} > 0$, $\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \frac{2}{n+1} \leq 1, \forall n \in \mathbb{N}$

$\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$0 \xleftarrow[n\to\infty]{} 0 < \frac{2^n}{n!} = \frac{2\cdot 2 \cdots 2}{1\cdot 2\cdot 3 \cdots n} \leq \frac{2\cdot 2\cdot 2 \cdots 2}{1\cdot 2\cdot 3\cdot 3 \cdots 3} = 2 \cdot \left(\frac{2}{3}\right)^{n-2}$ pro $n \geq 3$

$\xrightarrow[n\to\infty]{} 0$

$\lim_{n\to\infty} \frac{2^n}{n!} = 0$

$|R_n| \leq a_{n+1} = \frac{2^{n+1}}{(n+1)!} \leq 10^{-3}$

$\frac{(n+1)!}{2^{n+1}} \geq 1000$

$n \geq 9$

$s \approx \sum_{n=1}^{9} (-1)^{n-1} \cdot \frac{2^n}{n!}$

* $\sum_{n=1}^{\infty} \frac{1}{n^3}$, chyba $\leq 10^{-2}$

$f(x) = \frac{1}{x^3}$, $f(x) \geq 0$, $\searrow$ na $[1, \infty)$

$\int_1^{\infty} \frac{1}{x^3}\,dx = \left[\frac{-1}{2x^2}\right]_1^{\infty} = \frac{1}{2}$ ... konv

$\int_n^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2n^2}$ $|R_n| \leq \frac{1}{2n^2} \leq 10^{-2}$

$n^2 \geq 50$

$n \geq 8$ $s \approx \sum_{n=1}^{8} \frac{1}{n^3}$

* $\sum_{n=1}^{\infty} \frac{n}{5^n}$, chyba $\leq 10^{-4}$

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{5^{n+1}} \cdot \frac{5^n}{n} = \frac{n+1}{5n} \leq \overset{=q}{\frac{1}{4}} < 1 \quad \text{pro } n \geq 4$$

$$|R_n| \leq |a_n| \cdot \frac{q}{1-q} = \frac{n}{5^n} \cdot \frac{q}{1-q} = \frac{n}{5^n} \cdot \frac{1/4}{1-1/4} = \frac{n}{3 \cdot 5^n} \leq 10^{-4}$$

$$3 \cdot 5^n \geq \cancel{10000n}\ 10000n$$

$$\cancel{n \geq 6} \qquad n \geq 7$$

$$s \approx \sum_{n=1}^{7} \frac{n}{5^n}$$

* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} \leq \sum_{n=1}^{\infty} \frac{1}{n^4}$ ... konv. chyba $\leq 10^{-4}$

$$\sum_{n=1}^{\infty} a_n , \sum_{n=1}^{\infty} b_n , b_n \geq 0$$

$|a_n| \leq b_n$ pro $\forall n$

$|R_n| \leq r_n$ $\qquad r_n$ ... zbytek u b_n

$$\frac{1}{n^4} = b_n$$

$$|r_n| \leq \int_n^{\infty} \frac{1}{x^4}\,dx = \left[-\frac{1}{3x^3} \right]_n^{\infty} = \frac{1}{3n^3}$$

$$|R_n| \leq |r_n| \leq \frac{1}{3n^3} \leq 10^{-4}$$

$$3n^3 \geq 10000$$

$$n \geq 15$$

$$s \approx \sum_{n=1}^{15} \frac{1}{n(n+1)(n+2)(n+3)}$$

* $\sum_{n=1}^{\infty} \frac{1}{n^2+2n-3}$, chyba $\leq 0,03$

$\cancel{n}\ \frac{1}{n^2+2n-3} \leq \frac{1}{n^2}$ pro $n \geq 2$ $\qquad$ máme konv

$$|R_n| \le |r_n| \le \int_n^{\infty} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_n^{\infty} = \frac{1}{n} \le 0{,}03 = \frac{3}{100}$$

$$3n \geq 100$$

$$n \geq 34$$

$$s \approx \sum_{n=1}^{\cancel{\infty}\,34} \frac{1}{n^2+2n-3}$$

- součiny nebudou na první zápočtovce

- Dirichlet = po elkách
- Cauchy = po diagonálách

* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{q^n}$, $q > 1$

najděte Cauchyho součin sám se sebou

$\left|\frac{1}{q}\right| < 1$ . . . řada konverguje (absolutně) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{q^n} = \frac{1}{q} \cdot \frac{1}{1-\frac{1}{q}} = \frac{1}{q-1}$

$$\left(\sum_{n=1}^{\infty} \frac{1}{q^n}\right)^2 = \cancel{\left(\frac{1}{q}\cdot\frac{1}{q}\right) + \left(\frac{1}{q^2}\cdot\frac{1}{q} + \frac{1}{q}\cdot\frac{1}{q^2}\right) + \left(\frac{1}{q^3}\cdot\frac{1}{q} + \frac{1}{q^2}\cdot\frac{1}{q^2}\right)}$$

$$= \left(\frac{1}{q-1}\right)^2$$

$$\left(\sum_{n=1}^{\infty} \frac{1}{q^n}\right)^2 = \left(\frac{1}{q}\cdot\frac{1}{q}\right) + \left(\frac{1}{q^2}\cdot\frac{1}{q} + \frac{1}{q}\cdot\frac{1}{q^2}\right) + \dots = \frac{1}{q^2} + \frac{2}{q^3} + \frac{3}{q^4} + \dots = \sum_{n=1}^{\infty} \frac{n}{q^{n+1}}$$

tedy $\displaystyle\sum_{n=1}^{\infty} \frac{n}{q^{n+1}} = \left(\frac{1}{q-1}\right)^2$

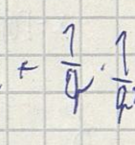
$\boxed{5.\text{CV.}}$

- $\lim_{n\to\infty} f_n(x) = f(x)$ . . . limitní fce.

- bodová konv. na I: $\forall x \in I\ \forall \varepsilon > 0\ \exists n(\varepsilon, x) \in \mathbb{N}\ \forall n \geq n(\varepsilon, x): |f_n(x) - f(x)| < \varepsilon$
- stejnoměrná konv. na I: $\forall \varepsilon > 0\ \exists n(\varepsilon) \in \mathbb{N},\ \forall n \geq n(\varepsilon),\ \forall x \in I: |f_n(x) - f(x)| < \varepsilon$

* $f_n(x) = x^n$
$x \in [0,1]$

$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$



$f(x)$ není spojitá, a f_n jsou spojité, tedy nejde o rovnoměrnou konvergenci pouze o bodovou

- $f_n(x) \rightrightarrows f(x) \iff \lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0$ ~~(Weierstrassovo kritérium)~~

* $f_n(x) = \frac{1}{1+nx}$, $x \in [1,\infty)$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = 0 \quad \text{pro } \forall x \in [1,\infty)$$

$$f(x) \equiv 0 \text{ na } [1,\infty)$$

$$\sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup_{x\in[1,\infty)} \left|\frac{1}{1+nx}\right| = \sup_{x\in[1,\infty)} \frac{1}{1+nx} = \frac{1}{1+n}$$

$$\lim_{n\to\infty} \frac{1}{1+n} = 0$$

tedy $f_n(x) \rightrightarrows f(x)$

- $B[a,b]$, $\rho_c(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$ metrika stejnoměrné konvergence

pro spoj. fce. lze brát max místo sup

* $f_n(x) = x^n - x^{2n}$, $x \in [0,1]$

$$\lim_{n\to\infty} (x^n - x^{2n}) = \begin{cases} 0 & \text{pro } x<1 \\ 0 & \text{pro } x=1 \end{cases}$$

$$f(x) = 0$$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} |x^n - x^{2n}| = \max_{x\in[0,1]} (x^n - x^{2n}) = \frac{1}{4}$$

$$f_n'(x) = nx^{n-1} - 2nx^{2n-1} = 0$$

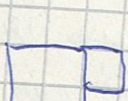
$$nx^{n-1}(1-2x^n) = 0$$

$$x^n = \frac{1}{2} \qquad x_0 = \frac{1}{\sqrt[n]{2}} \qquad f_n(x_0) = \frac{1}{2} - \frac{1}{2^2} = \frac{1}{4}$$

$$\lim_{n\to\infty} \frac{1}{4} \neq 0$$

nejedná se o stejnoměrnou konv.

$\boxed{\text{PR!}}$

- bacha, např.  je elementární množina 

- lemma 1,2 se slidu 20 v integrálech není třeba umět důkazy

$\boxed{\text{PŘ.}}$

- $\forall m,n \in \mathbb{N}: S_m(f,A) \geq s_n(f,A)$

- $$\overline{\iint_A} f(x,y)\,dx\,dy \geq \underline{\iint_A} f(x,y)\,dx\,dy$$

- slide 42: $D_j^n = (D^n \cap B) \cup (D_j^n \cap (A \setminus B))$ disj. $\cup$ $\quad m(D_j^n) = m(D_j^n \cap B) + m(D_j^n \cap (A \setminus B))$

$$s_n(f,B) = \sum_{j=1}^{m} \inf_{D_j^n \cap B} f \cdot m(D_j^n \cap B)$$

$$s_n(f, A \setminus B) = \sum_{j=1}^{m} \inf_{D_j^n \cap (A \setminus B)} f \cdot m(D_j^n \cap (A \setminus B))$$

$$s_n(f,B) + s_n(f, A \setminus B) \geq \sum_{j=1}^{m} \inf_{D_j^n} f \cdot m(D_j^n \cap B) + \sum_{j=1}^{m} \inf_{D_j^n} f \cdot m(D_j^n \cap (A \setminus B)) =$$

$$= \sum_{j=1}^{m} \inf_{D_j^n} f \cdot m(D_j^n) = s_n(f,A)$$

- slide 49: $\quad -M \leq f(x,y) \leq M$

$$s_n(f,A) = \sum_{j=1}^{m} \inf f \cdot m(D_j^{m}) \geq \sum_{j=1}^{m} (-M) \cdot m(D_j^{m}) = -M \cdot \sum_{j=1}^{m} m(D_j^{m}) = M \cdot m(A)$$

$$\cancel{S_m(f,A)} = S_n(f,A) \leq M \sum_{j=1}^{m} m(D_j^n) = M \cdot m(A)$$

- elementární množina

$\boxed{\text{6. CV.}}$

* $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in \mathbb{R}$ $\quad$ zjistěte obor konvergence, limitní funkci a zjistěte stejnoměrnou posloupnosti

sin nx ohraničená, $\frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{\sin nx}{\sqrt{n}} = 0$$

$f(x) \equiv 0$, $x \in \mathbb{R}$ ... obor konv.

protože $\left|\frac{\sin nx}{\sqrt{n}}\right| \le \frac{1}{\sqrt{n}}$

$$0 \le \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\frac{\sin nx}{\sqrt{n}}\right| \le \cancel{\sup_{x\in\mathbb{R}} \frac{1}{\sqrt{n}}} \frac{1}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

věta o třech limitách $\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\frac{\sin nx}{\sqrt{n}}\right| = 0$

$$f_n(x) \rightrightarrows f(x)$$

* -"- $f_n(x) = \sin\frac{x}{n}$, $x \in \mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sin\frac{x}{n} = \sin 0 = 0$$

$$f(x) \equiv 0, \quad x \in \mathbb{R}$$

$x = \frac{n\pi}{2} \Rightarrow \sin\frac{x}{n} = 1$

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\sin\frac{x}{n}\right| = 1$$

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\sin\frac{x}{n}\right| = 1$$

nejedná se o stejnoměrnou konv. (pouze bodovou)

$$f_n(x) \not\rightrightarrows f(x)$$

* -"- $\cancel{f_n(x) = }$

$$f_n(x) = 2n^2 x e^{-n^2x^2}, \quad x \in [0,1]$$

pro $x \neq 0$ — e^n roste rychle

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2n^2x}{e^{n^2x^2}} = \left|\frac{\infty}{\infty}\right| = 0$$

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{pro } x = 0$$

$$f(x) \equiv 0, \quad x \in [0,1]$$

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = \sup_{x\in[0,1]} \left|\frac{2n^2x}{e^{n^2x^2}}\right| = \max_{x\in[0,1]} \frac{2n^2x}{e^{n^2x^2}}$$

$$f_n'(x) = 2n^2 e^{-n^2x^2} + 2n^2 \cdot x \cdot e^{-n^2x^2} \cdot (-2xn^2) = 0$$

$$1 + x \cdot (-2xn^2) = 0$$

$$x^2 = \frac{1}{2n^2}$$

$$x = \frac{\sqrt{2}}{2n} \in [0,1] \quad \cancel{x \in [0,1] \Rightarrow x = \frac{\sqrt{2}}{2}}$$

$f_n(0)=0$, $f_n(1)=2n^2e^{-n^2}$

$f_n\left(\frac{\sqrt{2}}{2n}\right)=\frac{\sqrt{2}n}{\sqrt{e}}$ $\qquad \frac{\sqrt{2}n}{\sqrt{e}} \geq \frac{2n^2}{e^{n^2}}$ pro dost velké n

$$\lim_{n\to\infty} \max_{x\in[0,1]} \frac{2n^2x}{e^{n^2x^2}} = \lim_{n\to\infty} \frac{\sqrt{2}n}{\sqrt{e}} = \infty$$

$f_n(x) \not\rightrightarrows f(x)$ konverguje bodově, ale ne stejnoměrně

*DÚ $f_n(x)=\frac{2nx}{1+n^2x^2}$, $x\in[0,1]$
-"-

$f_n(x)=nxe^{-nx}$, $x[0,\infty)$

$$f_n(x)=\begin{cases} \frac{1}{x} & x\in[-1,-\frac{1}{n}]\cup[\frac{1}{n},1] \\ n^2x & x\in(-\frac{1}{n},\frac{1}{n}) \end{cases}$$

$x\in(-1$ $x\in[-1,1]$

.. stejnoměrná je hodně těžká

- $\sum_{n=1}^{\infty} f_n(x)=s(x)$ konverguje na oboru konvergence řady

$s(x)=\lim_{n\to\infty} s_n(x)$, $s_n(x)=\sum_{k=1}^{n} f_k(x)$ $\qquad$ lze zkoumat i stejnoměrnost

- Weierstrassovo kritérium $\forall n\in\mathbb{N}, \forall x\in\mathbb{R}: |f_n(x)|\leq a_n$ a $\sum_{n=1}^{\infty} a_n$ konv. $\Rightarrow \sum_{n=1}^{\infty} f_n(x)$ stejnoměrně konverguje

majorantní řada

* $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$, $x\in\mathbb{R}$, zjistěte zda konv. stejnoměrně

$\forall x\in\mathbb{R}, \forall n\in\mathbb{N}$: $\left|\frac{\sin nx}{n^2}\right|\leq\frac{1}{n^2}$ $\qquad$ majoranta $\sum_{n=1}^{\infty}\frac{1}{n^2}$ konverguje

tedy $\sum_{n=1}^{\infty}\frac{\sin nx}{n^2}$ konv. stejnoměrně (a tedy i bodově)

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n^2} \rightrightarrows \text{ na } \mathbb{R}$$

* $\sum_{n=1}^{\infty} \frac{1}{n^2+x^2}$, $x \in \mathbb{R}$ vyšetřete konvergenci

$\left|\frac{1}{n^2+x^2}\right| \leq \frac{1}{n^2}$ $\quad\sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje

tedy $\sum \frac{1}{n^2+x^2}$ $\quad\sum_{n=1}^{\infty} \frac{1}{n^2+x^2} \rightrightarrows$ na $\mathbb{R}$

* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1}$, $x \in \mathbb{R}$ vyšetřete konvergenci

$\left|\frac{\sin nx}{n^2+1}\right| \leq \frac{1}{n^2}$ $\quad\sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje

$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2+1} \rightrightarrows$ na $\mathbb{R}$

* $\sum_{n=1}^{\infty} \frac{\sin nx}{n^n}$; $x \in \mathbb{R}$

$\left|\frac{\sin nx}{n^n}\right| \leq \frac{1}{n^2}$ ~~pro dost~~ ~~dost velké n~~ $\quad\sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje

$\sum_{n=1}^{\infty} \frac{\sin nx}{n^n} \rightrightarrows$ na $\mathbb{R}$

* $\sum_{n=1}^{\infty} \frac{\sin^2 nx}{\sqrt[3]{n^4+x^4}}$, $x \in \mathbb{R}$

$\left|\frac{\sin^2 nx}{\sqrt[3]{n^4+x^4}}\right| \leq \frac{1}{n^{4/3}}$ $\quad\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ konverguje

$\sum_{n=1}^{\infty} \frac{\sin^2 nx}{\sqrt[3]{n^4+x^4}} \rightrightarrows$ na $\mathbb{R}$

* $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, $x \in \mathbb{R}$ zjistěte obor konvergence a stejnoměrnost

víme, že obor konvergence je $(1,\infty)$

zvolme a na $(1,\infty)$, $x \in [a,\infty)$

$$\frac{1}{n^x} \leq \frac{1}{n^a} \qquad \sum_{n=1}^{\infty} \frac{1}{n^a} \text{ ... konverguje}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^x} \rightrightarrows \text{ na } [a,\infty)$$

ale lze ukázat, že na $(1,\infty)$ nekonverguje stejnoměrně

* $\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2}$, $x \in [0,\infty)$

$$\left|\frac{x}{1+n^4x^2}\right| = \frac{x}{1+n^4x^2} \leq \frac{1}{2n^2} \iff (xn^2-1)^2 \geq 0$$

$$\sum_{n=1}^{\infty} \frac{1}{2n^2} \text{ ... konv.}$$

$$\sum_{n=1}^{\infty} \frac{x}{1+n^4x^2} \rightrightarrows \text{ na } x \in [0,\infty)$$

$\boxed{7. \text{CV}}$

* $1 - 1 + \frac{1}{3} - \frac{1}{3^2} + \frac{1}{5} - \frac{1}{5^2} + \frac{1}{7} - \frac{1}{7^2} + \dots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$

ukázka ~~nutnosti~~ nenahraditelnosti

$a_n \searrow$ v Leibnizovi

$$a_n = \begin{cases} \frac{1}{n} \\ \frac{1}{(n-1)^2} \end{cases} \qquad \lim_{n\to\infty} a_n = 0$$

ale řada diverguje

$$(1-1) + \left(\frac{1}{3} - \frac{1}{3^2}\right) + \left(\frac{1}{5} - \frac{1}{5^2}\right) + \dots = \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{(2k-1)^2}\right) = \sum_{k=1}^{\infty} \frac{2k-2}{(2k-1)^2} \qquad \frac{2k-2}{(2k-1)^2} \gtrsim \frac{1}{2k} \text{ , div.}$$

$$\lim_{k\to\infty} \frac{\frac{2k-2}{(2k-1)^2}}{\frac{1}{2k}} = 1$$

tedy původní ~~rozlo~~ řada diverguje

ta posloupnost není nerostoucí

* $\sum_{n=0}^{\infty} (3-x^2)^n \ , \ x \in \mathbb{R}$

určete obor konvergence

je to geom. řada

konv. právě když

$-1 < 3-x^2 < 1$

$-1 < x^2-3 < 1$

$2 < x^2 < 4$

$x \in (\sqrt{2}, 2) \cup (-2, -\sqrt{2})$ ... obor konvergence

* $\sum_{n=1}^{\infty} e^{-n^2 x}$, určete obor konvergence

$e^{-n^2x} = \frac{1}{e^{n^2x}}$

$$\lim_{n\to\infty} \frac{1}{e^{n^2x}} = \begin{cases} L=1 & x=0 \\ L=\infty & x<0 \\ L=0 & x>0 \end{cases}$$

musí tedy $x>0$

$x>0$: $\lim_{n\to\infty} \sqrt[n]{\frac{1}{e^{n^2x}}} = \lim_{n\to\infty} \frac{1}{e^{nx}} = 0 < 1$

obor konvergence (bodové) $(0,\infty)$

konverguje stejnoměrně na $(0,\infty)$? Ne! Zkusit dokázat, ale konv. na $[a,\infty)$, $a>0$.

dle Weierstrassova kritéria

$|e^{-n^2x}| = e^{-n^2x} \leq e^{-n^2a} \quad \forall x \in [a,\infty)$

ale $\sum e^{-n^2a}$ konvergují dle předchozího

– $\sum_{n=0}^{\infty} a_n \cdot (x-x_0)^n$

$x_0-r \quad x_0 \quad x_0+r$

$(\,,) = J$ ... interval konvergence

(to není vždy obor konvergence)

r ... poloměr konvergence

$$r = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}$$

pokud $\exists \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \implies r = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$

* $\sum_{n=1}^{\infty} \frac{n+1}{n} \cdot x^n$ zjistěte poloměr a obor konvergence

$$r = \frac{1}{\limsup \sqrt[n]{\frac{n+1}{n}}} = \frac{1}{1} = 1$$

$J = (-1, 1)$

$x = 1$: $\sum_{n=1}^{\infty} \frac{n+1}{n}$ ... diverguje $\lim_{n\to\infty} \frac{n+1}{n} = 1$

$x = -1$: $\sum_{n=1}^{\infty} \frac{n+1}{n}(-1)^n$ ... ~~div~~ osciluje

$\limsup_{n\to\infty} \frac{n+1}{n} \cdot (-1)^n = 1$

$\liminf_{n\to\infty} \frac{n+1}{n} \cdot (-1)^n = -1$

lepší je Leibniz. $\lim \frac{n+1}{n} = 1$

$\frac{n+1}{n}$ je nerostoucí

obor konvergence: $(-1, 1)$

* $\sum_{n=1}^{\infty} \frac{x^{4n-3}}{4n-3}$

$$a_k = \begin{cases} 0 & k \not\equiv 1 \pmod 4 \\ \frac{1}{k} & k \equiv 1 \pmod 4 \end{cases} \qquad \sqrt[k]{|a_k|} = \begin{cases} 0 & k \not\equiv 1 \pmod 4 \\ \frac{1}{\sqrt[k]{k}} & k \equiv 1 \pmod 4 \end{cases}$$

$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = 1$, protože $\lim_{k\to\infty} \frac{1}{\sqrt[k]{k}} = 1$.

$r = \frac{1}{1} = 1$ $J = (-1, 1)$

$x = 1$: $\sum_{n=1}^{\infty} \frac{1}{4n-3}$ ... diverguje

protože $\sum_{n=1}^{\infty} \frac{1}{n}$ diverguje

$x = -1$: $\sum_{n=1}^{\infty} \frac{-1}{4n-3}$ ... diverguje

obor konvergence $= (-1, 1)$

* $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right) \cdot x^n$

$a_n = \sin\left(\frac{1}{n}\right)$

$$\sin\frac{1}{n} \approx \frac{1}{n}$$

$$\limsup_{n\to\infty} \underbrace{\sqrt[n]{\left|\sin\frac{1}{n}\right|}}_{\approx} = \lim_{n\to\infty} \sqrt[n]{\left|\sin\frac{1}{n}\right|} = 1$$

$$\sqrt[n]{\frac{1}{n}} \xrightarrow{n\to\infty} 1$$

$$r = \frac{1}{1} = 1 \qquad J = (-1,1)$$

$x = -1$: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)\cdot(-1)^n$ ... konverguje $\left(\sin\left(\frac{1}{n}\right)\right)_{n=1}^{\infty}$ je nerostoucí pro velké n

$x = 1$: $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ ... ~~konverguje~~ diverguje $\quad \lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$

$$\sin\frac{1}{n} \approx \frac{1}{n} \qquad \cancel{\sin\frac{1}{n} < \frac{1}{n}} \quad \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$$

obor konvergence $\cancel{[-1,1]}$

$[-1,1)$

$$* \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$$

$$a_n = \frac{(n!)^2}{(2n)!}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}$$

$$r = \frac{1}{1/4} = 4 \qquad J = (-4,4)$$

$x = 4$: $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\cdot 4^n$ ... ~~konverguje~~ diverguje dle Raaba $\quad \frac{(n!)^2 \cdots}{(n+1)\cdots(2n)} \cdot 4^n$

$$n\left(1 - \frac{a_{n+1}}{a_n}\right) = n\left(1 - \frac{4(n+1)^2}{(2n+1)(2n+2)}\right) = n\left(1 - \frac{4(n+1)}{2(2n+1)}\right) =$$

$$= n\left(1 - \frac{4(n+1)}{4n+2}\right) = n\left(\frac{-2}{4n+2}\right) \xrightarrow{n\to\infty} -\frac{1}{2}$$

$x = -4$: $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}\cdot(-4)^n$ ... ~~konverguje abs. (dle $x=4$)~~ ... diverguje

$$= \sum (-1)^n \cdot b_n \qquad \frac{((n+1)!)^2}{(2n+2)!} 4^{n+1} \qquad \left|\frac{b_{n+1}}{b_n}\right| = \frac{4n+4}{4n+2} > 1$$

~~obor konvergence ... $[-4,4]$~~

$|b_{n+1}| > |b_n|$

tedy $\lim\limits_{n\to\infty} b_n \cdot (-1)^n \neq 0$.

obor konvergence $(-4,4)$

DÚ: $\sum\limits_{n=1}^{\infty} 3^{n^2} \cdot x^{n^2}$

$\sum\limits_{n=0}^{\infty} \frac{(x+1)^n}{2^n(n+1)(n+2)}$

$\boxed{8\ \text{CV.}}$

* Najděte součet $s(x) = \sum\limits_{n=1}^{\infty} n x^{n-1} = \sum\limits_{m=0}^{\infty} (m+1)x^m$

$a_m = m+1$

$$r = \frac{1}{\limsup\limits_{m\to\infty} \sqrt[m]{|a_m|}} = \frac{1}{\lim\limits_{m\to\infty} \sqrt[m]{m+1}} = \frac{1}{1} = 1$$

interval konvergence $I = (-1,1)$ (využijeme stejnoměrnost)

$$s(x) = \sum_{n=1}^{\infty} (x^n)', \quad \forall x \in I$$

$$s(x) = \left(\sum_{n=1}^{\infty} x^n\right)' = \left(\frac{x}{1-x}\right)' = \left(-1 + \frac{1}{1-x}\right)' = \frac{-1}{(1-x)^2}\cdot(-1) = \frac{1}{(1-x)^2}, \quad \forall x \in I$$

např. $x = \frac{1}{2}$: $\sum\limits_{n=1}^{\infty} \frac{n}{2^{n-1}} = \frac{1}{(1-\frac{1}{2})^2} = 4$

* $\sum\limits_{n=1}^{\infty} n^2 x^n$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{n^2}} = \frac{1}{1} = 1$$

interval konvergence $I = (-1,1)$

$\sum\limits_{n=1}^{\infty} x^n = \frac{x}{1-x}$ $\quad$ $\sum\limits_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ $\quad$ $\sum\limits_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2}$

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \qquad \forall x \in (-1,1)$$

* $$\sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{2^n}$$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\frac{n}{2^n}}} = \frac{1}{\frac{1}{2}} = 2$$

$I = (-2,2)$ interval konvergence

$$\cancel{\sum_{n=1}^{\infty} x^{n} = \frac{x}{1-x}} \qquad (x^n)' = n x^{n-1}$$

$$(x^n)' = n \cdot x^{n-1}$$

$$\cancel{\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{\ }} \qquad \left(\frac{x^n}{2^n}\right)' = \frac{n x^{n-1}}{2}$$

$$\left(\left(\frac{x}{2}\right)^n\right)' = \frac{n x^{n-1}}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2^n} = \sum_{n=1}^{\infty} \left(\left(\frac{x}{2}\right)^n\right)' = \left(\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n\right)' = \left(\frac{x/2}{1-\frac{x}{2}}\right)' = \left(\frac{x}{2-x}\right)' =$$

$$= \left(-1 + \frac{2}{2-x}\right)' = \frac{2\cdot(-1)}{(2-x)^2} \cdot (-1) = \frac{2}{(2-x)^2} \qquad \forall x \in (-2,2)$$

* $$\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2n} = \frac{(x-3)^2}{2} + \frac{(x-3)^4}{4} + \dots$$

$$r = \frac{\cancel{(x-3)^{2n}}\ 1}{\limsup\limits_{n\to\infty} \sqrt[n]{a_n}} = \frac{1}{\frac{1}{1}} = 1 \qquad a_n = \begin{cases} 0 & 2 \nmid n \\ \frac{1}{n} & 2 \mid n \end{cases} \quad n \geq 1$$

$I = (2,4)$ ... interval konvergence

$$\int_3^x \frac{(t-3)^{2n-1}}{\cancel{2n}}\, dt = \frac{(x-3)^{2n}}{2n}$$

$$\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2n} = \sum_{n=1}^{\infty} \int_3^x (t-3)^{2n-1} = \int_3^x \sum_{n=1}^{\infty} (t-3)^{2n-1}\, dt = \int_3^x \frac{t-3}{1-(t-3)^2}\, dt =$$

$$= \frac{-1}{2}\int_3^x \frac{-2(t-3)}{1-(t-3)^2}\, dt = -\frac{1}{2}\Big[\ln\big(|1-(t-3)^2|\big)\Big]_3^x = -\frac{1}{2}\ln\big(|1-(x-3)^2|\big)$$

$\forall x \in (-2,2)$

* $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = x - x^3 + x^5 - x^7 + \dots$

geom. rada s ~~koef~~ $-x^2$

$$a_n = \begin{cases} 0 & 2 \mid n \\ 1 & n \equiv 1 \pmod 4 \\ -1 & n \equiv 3 \pmod 4 \end{cases}$$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} = 1$$

$I = (-1,1)$ ... interval konvergence

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = \frac{x}{1+x^2} \qquad \forall x \in (-1,1)$$

* DÚ $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

\- $f(x)$, $x_0 \in D(f)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n = f(x)$$

neplatí vždy

postačující podmínka, např. $|f^{(n)}(x)| \le K$, $\forall x \in O(x_0)$, $\forall n \in \mathbb{N}_0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \forall x \in \mathbb{R}$$

$a_n = \frac{1}{n!} \qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \xrightarrow{n\to\infty} 0 \qquad r = \infty \quad I = \mathbb{R}$

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1-x)^2+2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \qquad \forall x \in (-1,1)$$

* $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{2^n}$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \cancel{\sqrt[n]{\frac{n}{2^n}}}\sqrt[n]{\frac{n}{2^n}}} = \frac{1}{\frac{1}{2}} = 2$$

$I = (-2,2)$ interval konvergence

$$\cancel{\sum_{n=1}^{\infty} x^{n} = \frac{x}{1-x}} \qquad (x^n)' = nx^{n-1}$$

$$\cancel{\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{}} \qquad (x^n)' = n \cdot x^{n-1}$$

$$\left(\frac{x^n}{2^n}\right)' = \frac{nx^{n-1}}{2}$$

$$\left(\left(\frac{x}{2}\right)^n\right)' = \frac{nx^{n-1}}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{nx^{n-1}}{2^n} = \sum_{n=1}^{\infty}\left(\left(\frac{x}{2}\right)^n\right)' = \left(\sum_{n=1}^{\infty}\left(\frac{x}{2}\right)^n\right)' = \left(\frac{x/2}{1-\frac{x}{2}}\right)' = \left(\frac{x}{2-x}\right)' =$$

$$= \left(-1+\frac{2}{2-x}\right)' = \frac{2\cdot(-1)}{(2-x)^2}\cdot(-1) = \frac{2}{(2-x)^2} \qquad \forall x \in (-2,2)$$

* $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2n} = \frac{(x-3)^2}{2} + \frac{(x-3)^4}{4} + \dots$

$$a_n = \begin{cases} 0 & 2\nmid n \\ \frac{1}{n} & 2\mid n \end{cases} \qquad n \geq 1$$

$$r = \frac{\cancel{(x-3)^{2n}}\,1}{\limsup\limits_{n\to\infty}\sqrt[n]{a_n}} = \frac{1}{\frac{1}{1}} = 1$$

$I = (2,4)$ ... interval konvergence

$$\int_3^x \cancel{\frac{(t-3)^{2n-1}}{2n}}\, dt = \frac{(x-3)^{2n}}{2n}$$

$$\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2n} = \sum_{n=1}^{\infty} \int_3^x (t-3)^{2n-1}\,dt = \int_3^x \sum_{n=1}^{\infty} (t-3)^{2n-1}\,dt = \int_3^x \frac{t-3}{1-(t-3)^2}\,dt =$$

$$= \frac{-1}{2}\int_3^x \frac{-2(t-3)}{1-(t-3)^2}\,dt = -\frac{1}{2}\Big[\ln\big(|1-(t-3)^2|\big)\Big]_3^x = -\frac{1}{2}\ln\big(|1-(x-3)^2|\big)$$

$\forall x \in (-2,2)$

* $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = x - x^3 + x^5 - x^7 + \dots$

geom. řada s kv. $-x^2$

$$a_n = \begin{cases} 0 & 2 \mid n \\ 1 & n \equiv 1 \pmod 4 \\ -1 & n \equiv 3 \pmod 4 \end{cases}$$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} = 1$$

$I = (-1,1)$ ... interval konvergence

$$\cancel{(-1)^{n-1} x^{2n-1} = x \cdot (-1)^{n-1} \cdot (x}$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = \frac{x}{1+x^2} \qquad \forall x \in (-1,1)$$

* DÚ $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

*
– $f(x)$, $x_0 \in D(f)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}\cdot(x-x_0)^n = f(x)$$

└ neplatí vždy

postačující podmínka, např. $|f^{(n)}(x)| \le K$, $\forall x \in O(x_0)$, $\forall n \in \mathbb{N}_0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad \forall x \in \mathbb{R}$$

$a_n = \frac{1}{n!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \xrightarrow{n\to\infty} 0 \quad r = \infty \quad I = \mathbb{R}$

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1-x)^2+2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \qquad \forall x \in (-1,1)$$

* $$\sum_{n=1}^{\infty} \frac{n\cdot x^{n-1}}{2^n}$$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\frac{n}{2^n}}} = \frac{1}{\frac{1}{2}} = 2$$

$I = (-2,2)$ interval konvergence

$$(x^n)' = n x^{n-1}$$

$$(x^n)' = n\cdot x^{n-1}$$

$$\left(\frac{x^n}{2^n}\right)' = \frac{n x^{n-1}}{2}$$

$$\left(\left(\frac{x}{2}\right)^n\right)' = \frac{n x^{n-1}}{2^n}$$

$$\sum_{n=1}^{\infty} \frac{n x^{n-1}}{2^n} = \sum_{n=1}^{\infty} \left(\left(\frac{x}{2}\right)^n\right)' = \left(\sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^n\right)' = \left(\frac{x/2}{1-\frac{x}{2}}\right)' = \left(\frac{x}{2-x}\right)' =$$

$$= \left(-1 + \frac{2}{2-x}\right)' = \frac{2\cdot(-1)}{(2-x)^2}\cdot(-1) = \frac{2}{(2-x)^2} \qquad \forall x \in (-2,2)$$

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$$a_n = \begin{cases} 0 & 2 \nmid n \\ \frac{1}{n} & 2 \mid n \end{cases} \qquad n \geq 1$$

$$r = \frac{\cancel{(x-3)^{2n}}\ 1}{\limsup\limits_{n\to\infty} \sqrt[n]{a_n}} = \frac{1}{1} = 1$$

$I = (2,4)$ ... interval konvergence

$$\int_3^x \frac{(t-3)^{2n-1}}{~~2n~~}\, dt = \frac{(x-3)^{2n}}{2n}$$

$$\sum_{n=1}^{\infty} \frac{(x-3)^{2n}}{2n} = \sum_{n=1}^{\infty} \int_3^x (t-3)^{2n-1}\,dt = \int_3^x \sum_{n=1}^{\infty} (t-3)^{2n-1}\,dt = \int_3^x \frac{t-3}{1-(t-3)^2}\,dt =$$

$$= -\frac{1}{2} \int_3^x \frac{-2(t-3)}{1-(t-3)^2}\,dt = -\frac{1}{2}\Big[\ln\big(|1-(t-3)^2|\big)\Big]_3^x = -\frac{1}{2}\ln\big(|1-(x-3)^2|\big)$$

$\forall x \in (-2,2)$

* $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = x - x^3 + x^5 - x^7 + \dots$

geom. rada s $-x^2$

$$a_n = \begin{cases} 0 & 2 \mid n \\ 1 & n \equiv 1 \pmod 4 \\ -1 & n \equiv 3 \pmod 4 \end{cases}$$

$$r = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{1} = 1$$

$I = (-1,1)$ ... interval konvergence

$$~~(-1)^{n-1} x^{2n-1} = x \cdot (-1)^{n-1} \cdot (x~~$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot x^{2n-1} = \frac{x}{1+x^2} \qquad \forall x \in (-1,1)$$

* DÚ $\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$

*

$-$ $f(x)$, $x_0 \in D(f)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x-x_0)^n = f(x)$$

neplatí vždy

postačující podmínka, např. $|f^{(n)}(x)| \leq K$, $\forall x \in O(x_0)$, $\forall n \in \mathbb{N}_0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad \forall x \in \mathbb{R}$$

$a_n = \frac{1}{n!}$ $\qquad \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{n+1} \xrightarrow{n\to\infty} 0 \qquad r = \infty \qquad I = \mathbb{R}$

jde spočítat i $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(ale bacha na cirkulární argumentaci)

takto $\frac{e^x-1}{x}=1+\frac{x}{2!}+\frac{x^2}{3!}+\dots$

Taylorovy řady jsou jednoznačné [illegible]

$$\sin x=\sum_{n=0}^{\infty}\frac{x^{2n-1}}{(2n-1)!}\cdot(-1)^{n-1}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots$$

$\forall x\in\mathbb{R}$

$$\cos x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\cdot(-1)^{n-1}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\qquad\forall x\in\mathbb{R}$$

$$\ln(1+x)=\sum_{n=1}^{\infty}\frac{x^n}{n}(-1)^{n-1}=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\qquad\forall x\in(-1,1]$$

$$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}\cdot x^n\qquad |x|<1$$

$r\in\mathbb{R}$

$$\binom{r}{n}:=\frac{r(r-1)\cdots(r-(n-1))}{n!}$$

* najděte Maclaurina $f(x)=e^{-x^2}$

$$e^t=\sum_{n=0}^{\infty}\frac{t^n}{n!},\quad\forall t\in\mathbb{R}$$

$t=-x^2$

$$e^{-x^2}=\sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot x^{2n}$$

* $f(x)=\sin^2 x,\ x_0=0$

(jde to i vynásobením řad pro $\sin x$ či z definice)

$$\sin^2 x=\frac{1-\cos 2x}{2}=\frac{1}{2}-\frac{1}{2}\cos 2x=\frac{1}{2}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{(2x)^{2n}}{(2n)!}(-1)^{n-1}=$$

$$=\frac{1}{2}-\frac{1}{2}\cdot 1-\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{2n}x^{2n}}{(2n)!}\cdot(-1)^{n-1}=\sum_{n=1}^{\infty}(-1)^n\cdot\frac{2^{2n-1}}{(2n)!}\cdot x^{2n}$$

$\forall x\in\mathbb{R}$

lze využít i $(\sin^2 x)'=2\sin x\cos x=\sin 2x$
a pak integrovat

* $f(x) = (1+e^x)^2$, $x_0 = 0$

~~$f'(x) = 2(1+e^x)\cdot e^x$~~

$$f(x) = 1 + 2e^x + e^{2x} = 1 + 2\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = 4 + \sum_{n=1}^{\infty} \frac{(2+2^n)}{n!}\cdot x^n$$

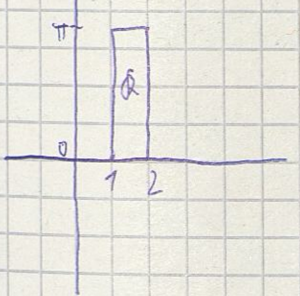
$\boxed{\text{PŘ.}}$

příklady

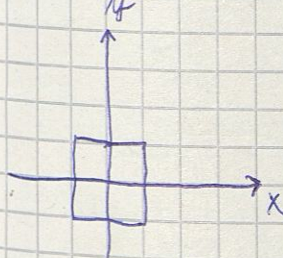
* $$\iint_D y\sin(xy)\,dx\,dy = \int_0^{\pi}\left(\int_1^2 y\sin(xy)\,dx\right)dy = \int_0^{\pi} y\cdot\left[-\frac{\cos(xy)}{y}\right]_1^2 dy = \int_0^{\pi}\left[-\cos(xy)\right]_1^2 dy =$$

$D = [1,2]\times[0,\pi]$

$$= \int_0^{\pi}(-\cos(2y)+\cos y)\,dy = \left[-\frac{\sin(2y)}{2} + \sin y\right]_0^{\pi} = 0$$



* Vypočtěte objem $V = ?$ tělesa pod plochou $z = 4 - x^2 - y^2$
nad čtvercem $[-1,-1], [-1,1], [1,1], [1,-1]$

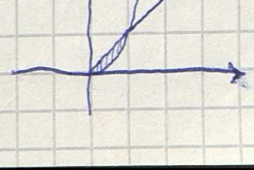


$$\int_{-1}^{1}\int_{-1}^{1}\int_0^{4-x^2-y^2} 1\,dz\,dy\,dx = \int_{-1}^{1}\int_{-1}^{1}(4-x^2-y^2)\,dy\,dx = \int_{-1}^{1}\left[4y - x^2y - \frac{y^3}{3}\right]_{-1}^{1} dx =$$

$$= \int_{-1}^{1}\left(\frac{22}{3} - 2x^2\right)dx = \left[\frac{22}{3}x - 2\frac{x^3}{3}\right]_{-1}^{1} = \frac{40}{3}$$

* $$\iint_M (x+y)\,dx\,dy = \int_0^1\left(\int_{x^2}^{x} x+y\,dy\right)dx = \int_0^1\left[x\cdot y + \frac{y^2}{2}\right]_{x^2}^{x} dx =$$

$M:\ y = x^2,\ y = x$

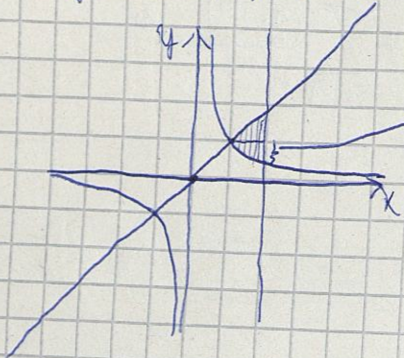


$$= \int_0^1 \frac{3}{2}x^2 - x^3 - \frac{x^4}{2}\,dx = \left[\frac{x^3}{2} - \frac{x^4}{4} - \frac{1}{2}\cdot\frac{x^5}{5}\right]_0^1 = \frac{3}{20}$$

* $\iint_M \frac{x^2}{y^2}\,dx\,dy = \int_1^2\left(\int_{1/x}^{x} \frac{x^2}{y^2}\,dy\right)dx = \int_1^2 x^2\cdot\left[-\frac{1}{y}\right]_{\frac{1}{x}}^{x} dx = \int_1^2 x^2\cdot\left(-\frac{1}{x}+x\right)dx = \int_1^2 (x^3-x)\,dx =$

$= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_1^2 = -\frac{4}{2}+\frac{16}{4}-\left(-\frac{1}{2}+\frac{1}{4}\right) = \frac{9}{4}$

M: $y=x,\ x=2,\ xy=1$

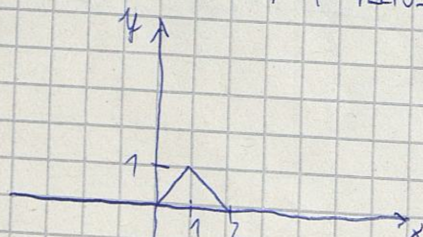


prohození integrace: $\int_{\frac{1}{2}}^{1}\left(\int_{1/y}^{2}\frac{x^2}{y^2}\,dx\right)dy + \int\int_{\Delta}$ $\int_1^2\int_y^2 \frac{x^2}{y^2}\,dx\,dy$ $= \frac{1}{2}$

* $\iint_M xy\,dx\,dy = \int_0^1\left(\int_0^x xy\,dy\right)dx + \int_1^2\int_0^{2-x} xy\,dy\,dx = \dots = \frac{1}{3}$

M: Δ $[0,0]$, $[1,1]$, $[2,0]$

alt. pořadí: $\int_0^1\left(\int_y^{2-y} xy\,dx\right)dy$



* $\iint_M \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^1\int_0^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\,dy\,dx$

M: čtvrtkruh jednotkový v prvním kvadrantě

$x^2+y^2\le 1$

jde to i polárními souřadnicemi: $x=\rho\cdot\cos\varphi$, $y=\rho\sin\varphi$ $\quad |J|=\rho$

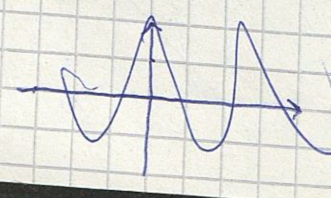
$x^2+y^2=\rho^2(\cos^2\varphi+\sin^2\varphi)=\rho^2$

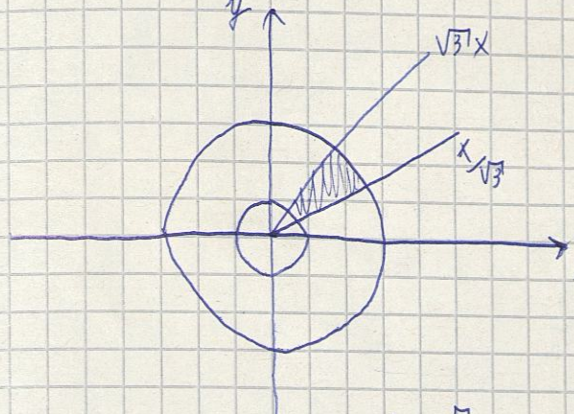
$\rho\in(0,1)$, $\varphi\in(0,\pi/2)$

$\iint_M \sqrt{1-x^2-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^1 \sqrt{1-\rho^2}\,\rho\,d\rho\,d\varphi = \dots = \frac{\pi}{6}$

* $\iint_M (x^2+y^2)\,dx\,dy$

M: $1\le x^2+y^2\le 3$, $\frac{x}{\sqrt{3}}\le y\le \sqrt{3}\,x$





$\rho \in [1, \sqrt{3}]$

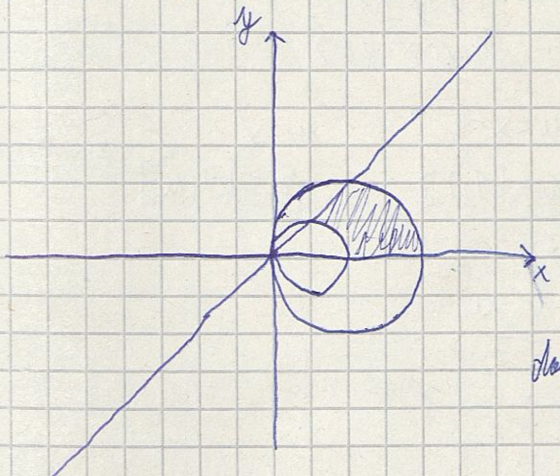
$\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

$tg\,\varphi = \frac{1}{\sqrt{3}} \Rightarrow \varphi = \frac{\pi}{6}$

$$\int_{1}^{\sqrt{3}}\int_{\pi/6}^{\pi/3} \rho^2 \cdot \rho \, d\varphi \, d\rho = \left(\frac{\pi}{3} - \frac{\pi}{6}\right) \cdot \left[\frac{\rho^4}{4}\right]_1^{\sqrt{3}} = \dots = \frac{\pi}{3}$$

$* \; S = ?$ $\quad \cancel{x^2 + y}$ $x^2 + y^2 = 2x, \; x^2 + y^2 = 4x, \; y = x, \; y = 0$

$(x-1)^2 + y^2 = 1, \quad (x-2)^2 + y^2 = 4, \; y = x, \; y = 0$



$\varphi \in [0, \pi/3]$

$\rho \in [2\cos\varphi, 4\cos\varphi]$

dolní mez ρ:

$x^2 + y^2 = 2x$

$\rho^2 = 2\rho\cos\varphi$

$\rho = 2\cos\varphi$

horní mez ρ:

$x^2 + y^2 = 4x$

$\rho^2 = 2\rho\cos\varphi$

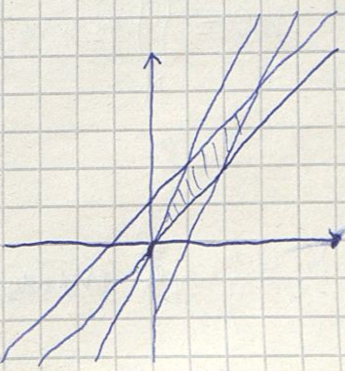
$\rho = 2\cos\varphi$

$$\int_0^{\pi/4} \left(\int_{2\cos\varphi}^{4\cos\varphi} 1 \cdot \rho \, d\rho \right) d\varphi = \int_0^{\pi/4} \left[\frac{\rho^2}{2}\right]_{2\cos\varphi}^{4\cos\varphi} d\varphi = \int_0^{\pi/4} \left(\frac{16\cos^2\varphi}{2} - \frac{4\cos^2\varphi}{2} \right) d\varphi = 6 \int_0^{\pi/4} \cos^2\varphi \, d\varphi =$$

$$= 6 \cdot \int \frac{1 + \cos 2\varphi}{2} d\varphi = \dots = \frac{3}{4} \cdot \pi + \frac{3}{2}$$

$* \; S = ?, \quad x \leq y \leq x + 1$

$2x - 2 \leq y \leq 2x$



$u = y - x$

$v = y - 2x$

$x = u - v$

$y = 2u - v$

$0 \leq y - x \leq 1 \qquad u \in [0, 1]$

$-2 \leq y - 2x \leq 0 \qquad \cancel{y - 2x \leq}$

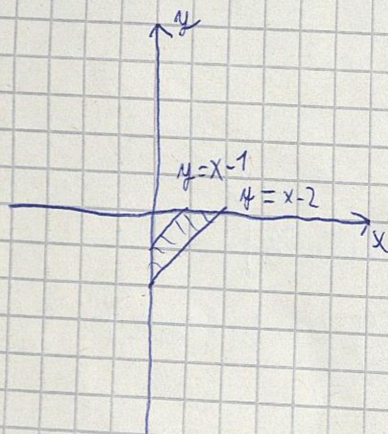
$v \in [-2, 0]$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ & \end{vmatrix} = 1$$

$$S = \int_0^1 \left(\int_{-1}^{0} 1 \cdot 1 \, dv \right) du = (1-0)\cdot(0+2) = 2$$

* $\iint_M e^{\frac{x+y}{x-y}} \, dx\, dy$

M: lichoběžník s vrcholy: $[1,0], [2,0], [0,-2], [0,-1]$



$x-1 \geq y \geq x-2$

$-1 \geq y-x \geq -2$

$1 \leq x-y \leq 2$

$u = x-y$ $v = x$ $y = 0$

$u \in [1,2]$ $v \in [0,u]$ $y = \cancel{u}\ v-u$

$x = v$

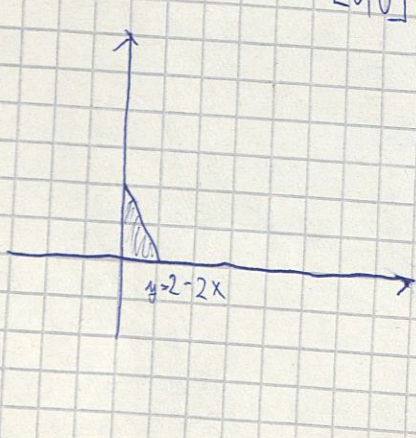
$y = v-u$

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = 1$$

$$\int_1^2 \int_0^u e^{\frac{2v-u}{u}} \cdot 1 \, dv\, du = \frac{1}{e} \int_1^2 \int_0^u e^{\frac{2v}{u}} \, dv\, du = \dots = \frac{3\cdot(e^2-1)}{4e}$$

* $m = ?,\ T = ?$

$\Delta: [0,0], [1,0], [0,2]$ $\rho(x,y) = 1+3x+y$



$$m = \iint_M 1+3x+y \, dx\, dy = \int_0^1 \int_0^{2-2x} 1+3x+3y \, dy\, dx = \dots$$

$$= \frac{8}{3}$$

$$M_x = \iint_M y\, \rho(x,y) \, dx\, dy = \dots = \frac{11}{6}$$

$$M_y = \iint_M x \cdot \rho(x,y)\, dx\, dy = \dots = 1$$

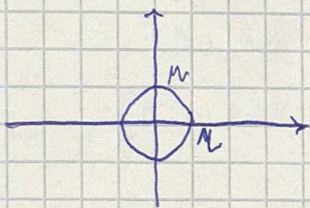
$$T = \left[\frac{M_y}{m}, \frac{M_x}{m} \right] = \left[\frac{9}{8}, \frac{11}{16} \right]$$

$\boxed{\text{IŘ}}$

příklady

* $\rho(x,y) = K$

homogenní



$x = \rho \cos\varphi$ $\qquad \varphi \in [0, 2\pi]$

$y = \rho \cdot \sin\varphi$ $\qquad \rho \in [0, r]$

$|J| = \rho$

$$m = \iint_M \rho(x,y)\, dx\, dy = \int_0^r \int_0^{2\pi} K\rho\, d\varphi\, d\rho = 2\pi K \left[\frac{\rho^2}{2} \right]_0^r = \pi \cdot K \cdot r^2$$

$$I_x = \iint_M y^2 \cdot \rho(x,y)\, dx\, dy = \int_0^r \int_0^{2\pi} \rho^2 \sin^2\varphi \cdot K \cdot \rho\, d\varphi\, d\rho = \dots =$$

kvadr. moment

$$I_y = \iint_M x^2 \rho(x,y)\, dx\, dy = \int_0^r \int_0^{2\pi} \rho^2 \cos^2\varphi \cdot K \cdot \rho\, d\varphi\, d\rho = \dots =$$

$$I_0 = \iint_M (x^2+y^2)\, \rho(x,y)\, dx\, dy = \int_0^r \int_0^{2\pi} \rho^2 \cdot K \cdot \rho\, d\varphi\, d\rho = \dots =$$

polární moment, netřeba znát, třeba znát věci z prezentace

* $L = 0{,}000013\ m v^2$

délka brzdných stop

$x = m \in [3000, 4000]$

$y = v \in [50, 60]$

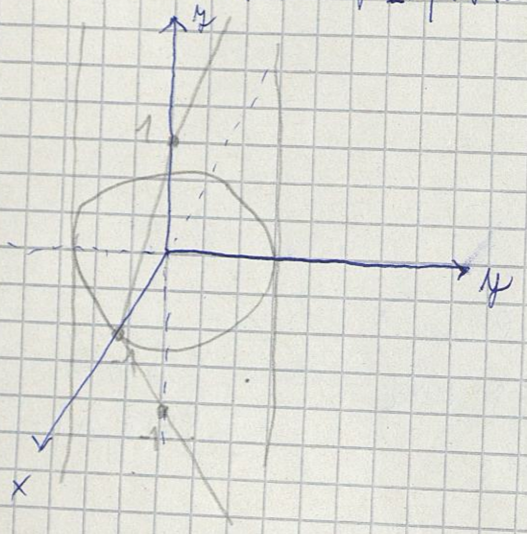
$3000 \le m \le 4000$

$50 \le v \le 60$

$$avL = \frac{1}{m(M)} \iint_M b(x,y)\, dx\, dy = 10^{-4} \cdot \int_{3000}^{4000} \int_{50}^{60} 0{,}000013\ m \cdot v^2\, dv\, dm = \dots = 138{,}01667$$

$*$ $$\iiint_M (x^2+y^2)\,dx\,dy\,dz = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x-1}^{1-x}(x^2+y^2)\,dz\,dy\,dx = \dots = \pi$$

$$M=\{[x,y,z]\in\mathbb{R}^3 \mid x^2+y^2\le 1,\ |z|\le 1-x\}$$



válcové souřadnice: $x=\rho\cdot\cos\varphi$

$y=\rho\cdot\sin\varphi$

$z=z$

$|J|=\rho$

$\rho\in[0,1]$

$\varphi\in[0,2\pi]$

$z=1-x=1-\rho\cos\varphi$

$z=x-1=\rho\cos\varphi-1$

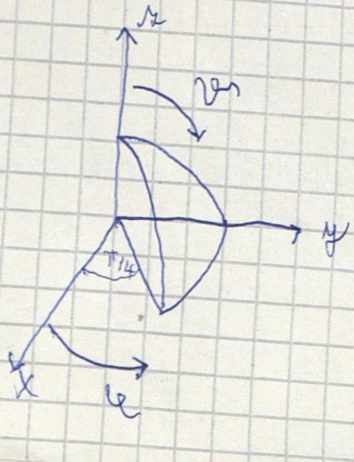
$$\int_0^1\int_0^{2\pi}\int_{\rho\cos\varphi-1}^{1-\rho\cos\varphi}\rho^2\cdot\rho\,dz\,d\varphi\,d\rho=\int_0^1\int_0^{2\pi}\rho^3\cdot(1-\rho\cdot\cos\varphi-\rho\cdot\cos\varphi+1)\,d\varphi\,d\rho=$$

$$=2\int_0^1\int_0^{2\pi}(\rho^3-\rho^4\cdot\cos\varphi)\,d\varphi\,d\rho=2\int_0^1\left[\rho^3\varphi-\rho^4\sin\varphi\right]_0^{2\pi}d\varphi=$$

$$=\dots=\pi$$

$*$ $$\iiint_A \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$

A: $0\le x\le y,\ z\ge 0,\ x^2+y^2+z^2\le 1$



$x=r\cdot\cos\varphi\cdot\sin\vartheta$

$y=r\cdot\sin\varphi\cdot\sin\vartheta$

$z=r\cdot\cos\vartheta$

$|J|=r^2\cdot\sin\vartheta$

$r\in[0,1]$

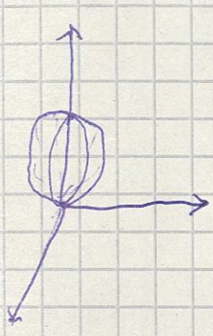
$\vartheta\in[0,\frac{\pi}{2}],\ \varphi\in[\frac{\pi}{4},\frac{\pi}{2}]$

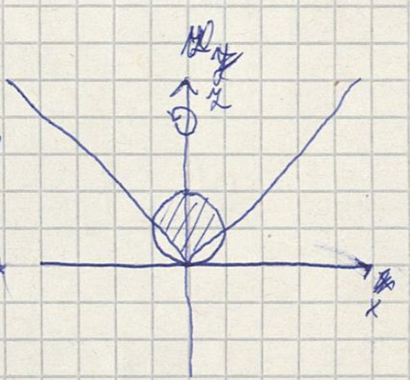
$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^1 r\cdot r^2\cdot \sin\vartheta \, dr\, d\varphi\, d\vartheta = \dots =$$

* A: $x^2+y^2+z^2 \le 2z$ − koule

$z \ge \sqrt{x^2+y^2}$ − kužel

$x^2+y^2+(z-1)^2 \le 1$





a) válc.

$r \in [0,1]$

$\varphi \in [0,2\pi]$

$z \in [\sqrt{x^2+y^2}, 1+\sqrt{1-x^2-y^2}] = [r, 1+\sqrt{1-r^2}]$

$$\int_0^1\int_0^{2\pi}\int_r^{1+\sqrt{1-r^2}} 1\cdot r\, dz\, d\varphi\, dr = \int_0^1\int_0^{2\pi} r - r^2 + r\sqrt{1-r^2}\, d\varphi\, dr = 2\pi\int_0^1 r - r^2 + r\sqrt{1-r^2}\, dr =$$

$$= 2\pi\left[\frac{r^2}{2}-\frac{r^3}{3}\right]_0^1 + \frac{2}{3}\cdot\frac{1}{2}\cdot 2\pi = \cancel{2\pi} \dots = \pi$$

$t = 1-r^2$

$dt = -2r\,dr$

$r=1 \to t=0$

$r=0 \to t=1$

$$\int_1^0 -\frac{1}{2} t^{\frac{1}{2}}\, dt = -\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]_1^0$$

b) sférické

$\varphi \in [0,2\pi]$

$\vartheta \in [0,\frac{\pi}{4}]$

$r \in [0, 2\cos\vartheta]$

$x^2+y^2+z^2 = 2z$

$r^2 = 2r\cdot\cos\vartheta$

$r = 2\cos\vartheta$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\vartheta} 1\cdot r^2\cdot\sin\vartheta\, dr\, d\vartheta\, d\varphi = \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{r^3}{3}\sin\vartheta\right]_0^{2\cos\vartheta} d\vartheta\, d\varphi =$$

$$= \frac{8}{3}\int_0^{2\pi}\int_0^{\pi/4}\cos^3\vartheta\,\sin\vartheta\, d\vartheta\, d\varphi = \left| \begin{matrix} t=\cos\vartheta \\ dt = -\sin\vartheta\, d\vartheta \end{matrix}\right| =$$

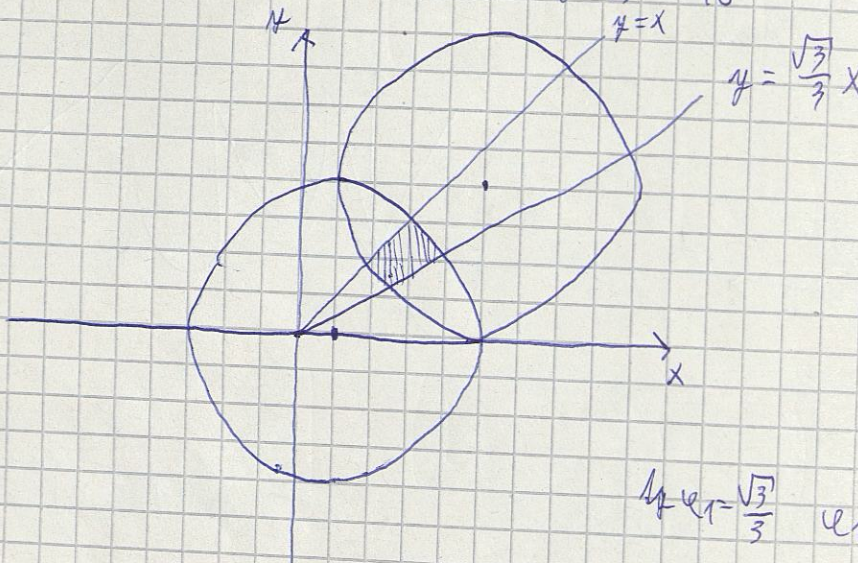
$$= \frac{8}{3}\int_0^{2\pi}\int_1^{\frac{\sqrt{2}}{2}} (-t^3)\, dt = -\frac{8}{3}\cdot\int_0^{2\pi}\left[\frac{t^4}{4}\right]_1^{\frac{\sqrt{2}}{2}} d\varphi = -\frac{8}{3}\cdot 2\pi\cdot\frac{1}{4}\left(\frac{1}{4}-1\right) = \pi$$

* $\iint_A f(x,y)\,dx\,dy \overset{\text{píše se někdy takto}}{=} \iint_A f(x,y)\,dA$

A: $x^2+y^2 \leq 15+2x$
$x^2+y^2 \leq 10x+8y-25$
$y \geq \frac{\sqrt{3}}{3}x$
$y \leq x$

udělejte převod do pol. souřadnic

$(x-1)^2+y^2 \leq 16$
$(x-5)^2+(y-4)^2 \leq 16$



$\operatorname{tg}\varphi_1 = \frac{\sqrt{3}}{3}$ $\varphi_1 = \frac{\pi}{6}$

$\operatorname{tg}\varphi_2 = 1$ $\varphi_2 = \frac{\pi}{4}$

$\varphi \in \left[\frac{\pi}{6}, \frac{\pi}{4}\right]$

$r \leq ?$.

$x^2+y^2 = 15+2x$

$r^2 = 15+2\cdot r\cos\varphi$

$r^2 - 2r\cos\varphi - 15 = 0$

$D = 4\cos^2\varphi + 60 > 0$

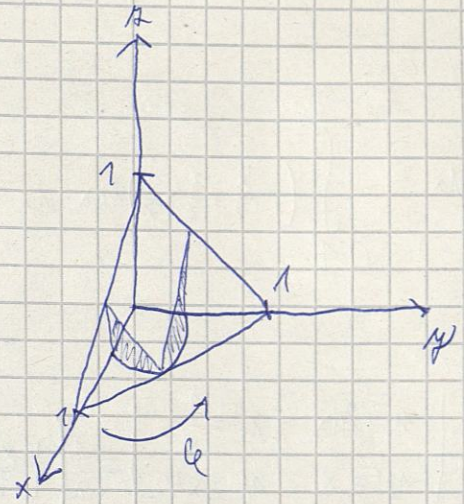
$r_{1,2} = \frac{2\cos\varphi \pm \sqrt{4\cos^2\varphi+60}}{2} =$

$= \cos\varphi \pm \sqrt{\cos^2\varphi + 15}$

$r \in [\ (\ , \cos\varphi + \sqrt{\cos^2\varphi+15}\ ]$
dolní mez podobně

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \int_{5\cdot\cos\varphi+4\cdot\sin\varphi-\sqrt{(5\cos\varphi+4\sin\varphi)^2-25}}^{\cos\varphi+\sqrt{\cos^2\varphi+15}} f(r\cdot\cos\varphi, r\sin\varphi)\cdot r\,dr\,d\varphi$$

* $V = ?$, $x \geq 0$, $x+y+z \leq 1$
$y \geq 0$, $x^2+y^2 \leq \frac{1}{2}$
$z \geq 0$



$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{\sqrt{\frac{1}{2}-x^2}} \int_0^{1-x-y} 1 \, dz \, dy \, dx = \dots =$$

b) cylindrické souřadnice

$r \in [0, \frac{\sqrt{2}}{2}]$

$\varphi \in [0, \frac{\pi}{2}]$

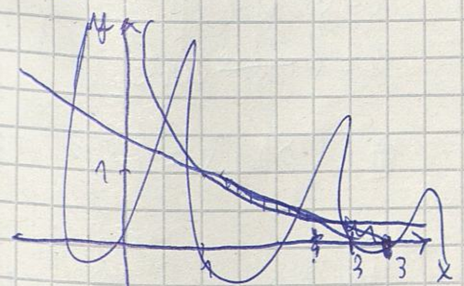
$z \in [0, 1-x-y] = [0, 1 - r\cos\varphi - r\sin\varphi]$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{\frac{\pi}{2}} \int_0^{1-r(\cos\varphi+\sin\varphi)} 1 \cdot r \, dz \, d\varphi \, dr = \int_0^{\sqrt{2}/2} \int_0^{\pi/2} r \cdot (1 - r(\cos\varphi + \sin\varphi)) \, d\varphi \, dr = \int_0^{\sqrt{2}/2} \left[r\varphi - r^2(\sin\varphi + \cos\varphi) \right]_0^{\frac{\pi}{2}} dr$$

$$= \dots = \frac{\pi}{8} - \frac{\sqrt{2}}{6}$$

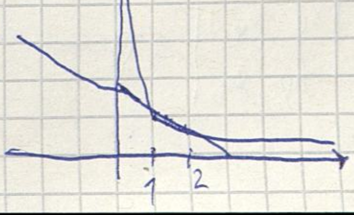
$\boxed{10. \text{CV.}}$

* $\iint_\Omega \ln x \, dx \, dy$, Ω: $y = \frac{1}{x}$, $x + 2y = 3$



$y = -\frac{x}{2}$ $x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$



$x \in [1, 2]$

$\frac{1}{x} \leq y \leq -\frac{x}{2} + \frac{3}{2}$

$$\int_1^2 \int_{\frac{1}{x}}^{-\frac{x}{2}+\frac{3}{2}} \ln x \, dy\, dx = \int_1^2 \left(-\frac{x}{2}+\frac{3}{2}-\frac{1}{x}\right)\ln x\, dx = -\int_1^2 \frac{\ln x}{x}\,dx + \int_1^2 \frac{3}{2}\ln x\,dx - \frac{1}{2}\int_1^2 x\ln x$$

$$-\frac12\int_1^2 x\ln x\,dx = \begin{vmatrix} u=\ln x & u'=\frac1x \\ v'=x & v=x^2/2\end{vmatrix} = \frac12\left(\left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^2 \frac{x}{2}\,dx\right) = \frac12\left(2\ln 2 - 1 + \frac14\right) = -\ln 2$$

$$\frac32\int_1^2 \ln x\,dx = \begin{vmatrix} u=\ln x & u'=\frac1x \\ v'=1 & v=x\end{vmatrix} = \frac32\left(\left[x\ln x\right]_1^2 - \int_1^2 x\frac1x\,dx\right) = \frac32(2\ln 2 - 1) =$$

$$= 3\ln 2 - \frac32$$

jde to i substitucí $s=\ln x$

$$-\int_1^2 \frac{\ln x}{x}\,dx = \begin{vmatrix} u=\ln x & u'=\frac1x \\ v'=\frac1x & v=\ln x\end{vmatrix} = -\left(\left[\ln^2 x\right]_1^2 - \int_1^2 \frac{\ln x}{x}\,dx\right)$$

po druhé per partes je to cykl. int.

$$-2\int_1^2 \frac{\ln x}{x}\,dx = \left[-\ln^2 x\right]_1^2$$

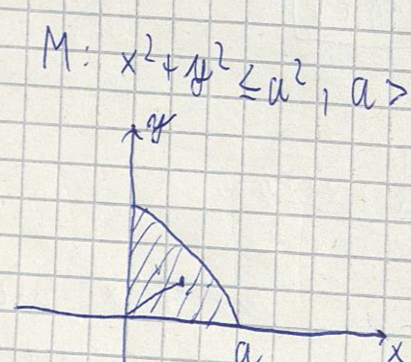
$$\parallel$$

$$-\frac12(\ln 2)^2$$

$$\int_1^2 \int_{\frac{1}{x}}^{-\frac{x}{2}+\frac{3}{2}} \ln x\, dy\, dx = 2\ln 2 - \frac12(\ln 2)^2 - \frac98$$

- polární transformace jsou dobré na zjednodušení integračního oboru

* $\iint_M (x^2+y^2)\,dx\,dy$ M: $x^2+y^2 \le a^2$, $a>0$, $x\ge 0$, $y \ge 0$



$x = \rho\cos\varphi$

$y = \rho\, \cancel{\cos\varphi}\, \sin\varphi$

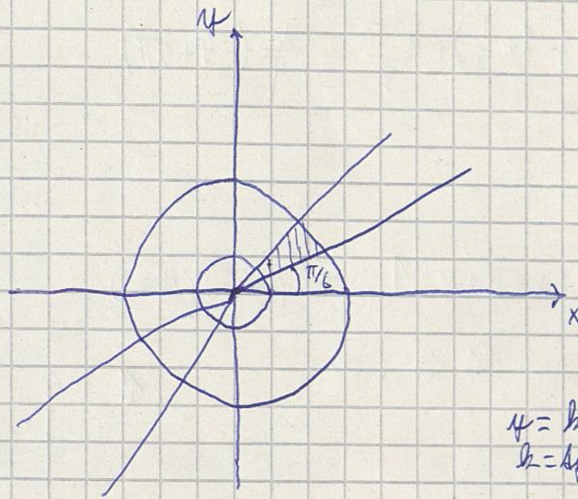
~~$\rho \ge a$, $\varphi \in$~~ $\rho \ge 0$, $\varphi \in [-\pi, \pi$

$$J = \begin{vmatrix} x_\rho & x_\varphi \\ y_\rho & y_\varphi \end{vmatrix} = \rho$$

M: $0 \le \rho \le a$, $0 \le \varphi \le \frac{\pi}{2}$

$$\int_0^{\pi/2}\int_0^a \rho^2 \rho \, d\rho \, d\varphi = \int_0^{\pi/2} \frac{a^4}{4}\, d\varphi = \frac{a^4}{4}\cdot\frac{\pi}{2} = \frac{\pi a^4}{8}$$

* $\iint_M \operatorname{arctg}\frac{y}{x}\, dx\, dy$, M: $1 \le x^2+y^2 \le 9$, $y \le x\sqrt{3}$, $y \ge x/\sqrt{3}$



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$1 \le \rho^2 \le 9$

$1 \le \rho \le 3$

$y = kx$

$k = \operatorname{tg}\varphi$

M: $1 \le \rho \le 3$, $\frac{\pi}{6} \le \varphi \le \frac{\pi}{3}$

$\operatorname{arctg}\frac{y}{x} = \cancel{\cos}\ \operatorname{arctg}\operatorname{tg}\varphi = \varphi$

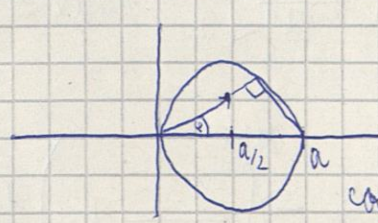
$$\int_{\pi/6}^{\pi/3}\int_1^3 \varphi\cdot\rho\, d\rho\, d\varphi = \int_{\pi/6}^{\pi/3}\left[\varphi\frac{\rho^2}{2}\right]_1^3 d\varphi = \int_{\pi/6}^{\pi/3} 4\varphi\, d\varphi = \left[2\varphi^2\right]_{\pi/6}^{\pi/3} = \frac{2\pi^2}{9} - \frac{2\pi^2}{36} =$$

$$= 2\pi^2\cdot\frac{3}{36} = \frac{\pi^2}{6}$$

$\boxed{\text{11.CV}}$

* $\iint_M \sqrt{a^2-x^2-y^2}\, dx\, dy$, M: $x^2+y^2 \le a\cdot x$, $a>0$

$$\left(x-\frac{a}{2}\right)^2 + y^2 \le \frac{a^2}{4}$$



$\cos\varphi = \frac{\rho}{a}$

$\rho = a\cos\varphi$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\sqrt{a^2-x^2-y^2} = \sqrt{a^2-\rho^2}$

M: $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

$0 \le \rho \le a\cos\varphi$ ... lze zjistit i dosazením do $x^2+y^2 \le ax$

na pořadí záleží

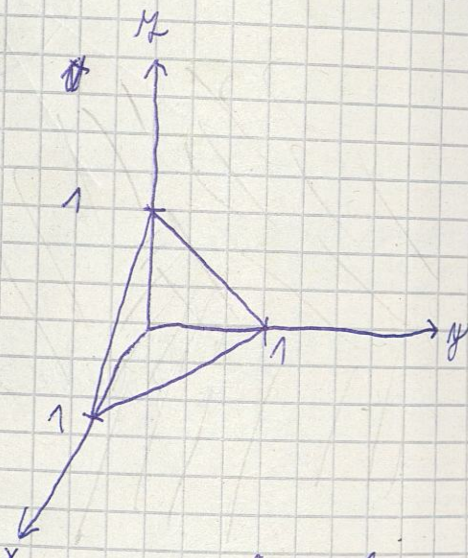
$$\int_{-\pi/2}^{\pi/2}\int_0^{a\cos\varphi} \sqrt{a^2-\rho^2}\,\rho\, d\rho\, d\varphi = \left| \begin{array}{l} a^2-\rho^2 = t \\ -2\rho\, d\rho = dt \end{array} \right| = -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_{a^2}^{a^2\sin^2\varphi} \sqrt{t}\, dt\, d\varphi = -\frac{1}{2}\int_{-\pi/2}^{\pi/2}\left[\frac{2t^{3/2}}{3}\right]_{a^2}^{a^2\sin^2\varphi} d\varphi$$

$$= \frac{-a^3}{3}\int_{-\frac{\pi}{2}}^{\pi/2} |\sin^3\varphi| - 1\, d\varphi = \frac{-2a^3}{3}\int_0^{\pi/2} \sin^3\varphi - 1\, d\varphi = \left|\begin{matrix}\cos\varphi = s\\ -\sin\varphi\, d\varphi = ds\end{matrix}\right| = \frac{2a^3}{3}\int_1^0 1-s^2\, ds + \frac{2a^3}{3}\int_0^{\pi/2} 1\, d\varphi$$

$$= \frac{2a^3}{3}\left[s - \frac{s^3}{3}\right]_1^0 + \frac{a^3\pi}{3} = \frac{2a^3}{3}\left(-1+\frac{1}{3}\right) + \frac{a^3\pi}{3} = \frac{a^3}{9}(-4+3\pi)$$

$*\ \iiint_M \frac{1}{x+y+1}\,dx\,dy\,dz$, $\quad M: x+y+z=1,\ x=0,\ y=0,\ z=0$

je to simplex



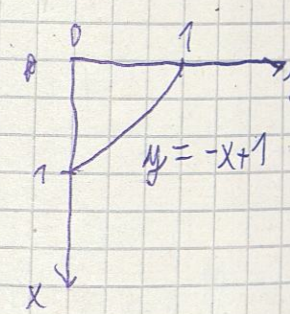
$M: 0 \le z \le 1-x-y$

$0 \le y \le 1-x$

$0 \le x \le 1$

na pořadí záleží

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} \cancel{\frac{1}{x+y+1}\,dx\,dy\,dz} \qquad \int_0^1\int_0^{1-x}\int_0^{1-x-y}\frac{1}{x+y+1}\,dz\,dy\,dx =$$



$$= \int_0^1\int_0^{1-x}\left[\frac{z}{x+y+1}\right]_0^{1-x-y} dy\,dx = \int_0^1\int_0^{1-x}\frac{1-x-y}{x+y+1}\,dy\,dx =$$

$$= \int_0^1 -\left(\int_0^{1-x} 1\,dy - 2\int_0^{1-x}\frac{1}{x+y+1}\,dy\right)dx = \int_0^1 -\left(1-x-2\left[\ln|x+y+1|\right]_0^{1-x}\right)dx =$$

$$= \int_0^1 -\left(1-x-2\left(\ln 2 - \ln|1+x|\right)\right)dx = \int_0^1 x-1+2\ln 2 - 2\ln|x+1|\,dx =$$

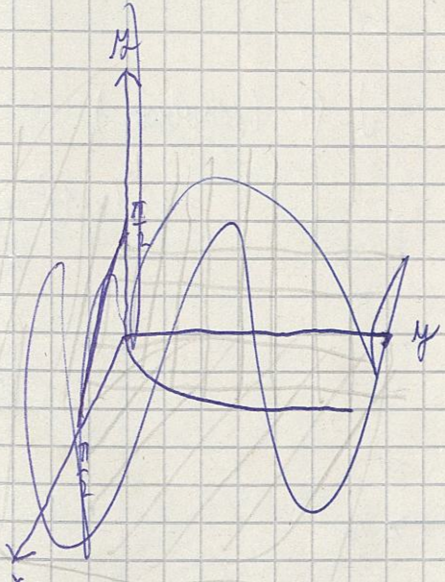
$$= 2\ln 2 - 1 + \left[\frac{x^2}{2}\right]_0^1 - 2\int_0^1 \ln|x+1|\,dx = 2\ln 2 - \frac{1}{2} - 2\int_0^1 \ln|x+1|\,dx$$

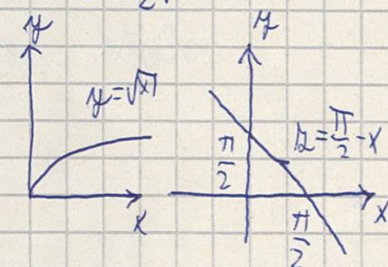
$$\int_0^1 \ln|x+1|\,dx = \left| \begin{matrix} u=\ln|x+1| & u'=\frac{1}{x+1} \\ v'=1 & v=x+1 \end{matrix} \right| = \Big[\ln|x+1|\cdot(x+1)\Big]_0^1 - \int_0^1 1\,dx =$$

$$= 2\ln 2 - 1$$

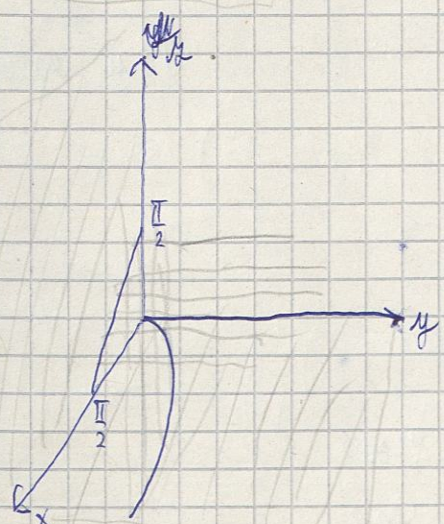
$$2\ln 2 - \frac{1}{2} - 2\int_0^1 \ln|x+1|\,dx = 2\ln 2 - \frac{1}{2} - 4\ln 2 + 2 = -2\ln 2 + \frac{3}{2}$$

* $\iiint_M y\cos(z+x)\,dx\,dy\,dz$ M: $y=\sqrt{x}$, $y=0$, $z=0$, $x+z=\frac{\pi}{2}$ $x+z=\frac{\pi}{2}$





$x \in [0, \frac{\pi}{2}]$
$z \in [0, \frac{\pi}{2} - x]$
$y \in [0, \sqrt{x}]$



$$\int_0^{\pi/2}\int_0^{\pi/2-x}\int_0^{\sqrt{x}} y\cos(z+x)\,dy\,dz\,dx =$$

$$= \int_0^{\pi/2}\int_0^{\frac{\pi}{2}-x}\int_0^{\sqrt{x}} \left[\frac{\cos(z+x)\,y^2}{2}\right]_0^{\sqrt{x}} dz\,dx =$$

$$= \int_0^{\pi/2}\int_0^{\frac{\pi}{2}-x} \frac{x\cos(z+x)}{2}\,dz\,dx = \int_0^{\pi/2} \frac{x}{2}\Big[\sin(z+x)\Big]_0^{\frac{\pi}{2}-x} dx =$$

$$= \int_0^{\pi/2} \frac{x}{2}\left(\sin\frac{\pi}{2} - \sin x\right)dx = \int_0^{\frac{\pi}{2}} \frac{x}{2} - \frac{x}{2}\sin x\,dx = \left[\frac{1}{4}x^2\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\pi/2} \underset{u}{x}\,\underset{v'}{\sin x}\,dx =$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left(\left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\pi/2}\cos x\,dx\right) = \frac{\pi^2}{16} - \frac{1}{2}\left[\sin x\right]_0^{\pi/2} = \frac{\pi^2}{16} - \frac{1}{2}$$

* DÚ $\iiint_M xyz\,dx\,dy\,dz$ $M: x^2+y^2+z^2 \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0$

* $\iiint_M z\sqrt{x^2+y^2}\,dx\,dy\,dz$ $M: x^2+y^2 = 2x$

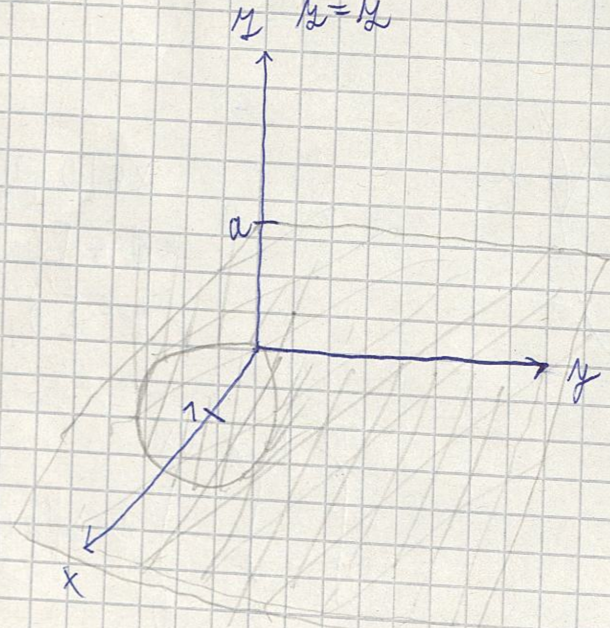
$z = 0,\ z = a,\ a > 0$

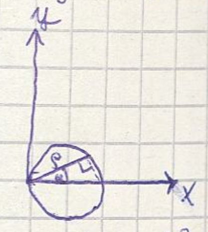
válcové souř.: $x = \rho\cos\varphi$

$y = \rho\sin\varphi$ $J = \rho$ (jakobián)

$z = z$

$(x-1)^2 + y^2 = 1$





$\cos\varphi = \frac{\rho}{2}$

M je válec od $z=0$ po $z=a$

$z \in [0, a]$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\rho \in [0, 2\cos\varphi]$ $\rho^2 \leq 2\rho\cos\varphi$

$\rho \leq 2\cos\varphi$

$$\int_0^a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\varphi} z\,\rho\cdot\overset{\text{Jakobián}}{\rho}\, d\rho\, d\varphi\, dz = \int_0^a z \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\frac{\rho^3}{3}\right]_0^{2\cos\varphi} d\varphi\, dz =$$

$$= \frac{8}{3}\int_0^a z \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi\, dz = \frac{8}{3}\int_0^a z \left| \begin{matrix} \sin\varphi = t,\ \sin\varphi = 1 \\ \cos\varphi\, d\varphi = dt \end{matrix} \right| = \frac{8}{3}\int_0^a z \int_{-1}^{1} 1 - t^2\, dt\, dz =$$

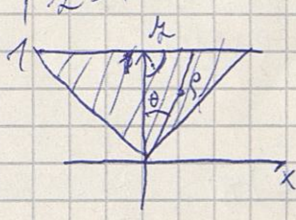
$$= \frac{8}{3}\int_0^a z \left[t - \frac{t^3}{3}\right]_{-1}^{1} dz = \frac{8}{3}\int_0^a z \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) dz = \frac{32}{9}\int_0^a z\, dz = \frac{32}{9}\cdot\frac{a^2}{2} =$$

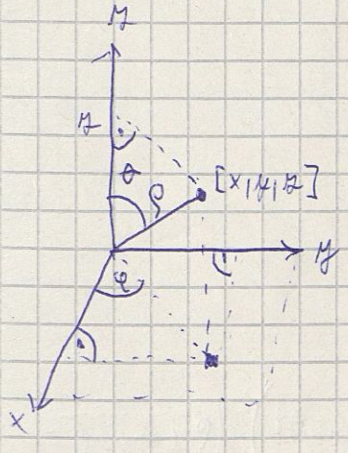
$$= \frac{16a^2}{9}$$

– 3. písemka: dvojné int., transformace, záměna pořadí integrace, lehčí trojné int.

$\boxed{\text{12. CV.}}$

* $I = \iiint_M z\, dx\, dy\, dz$, M: $z = \sqrt{x^2+y^2}$, $z = 1$





$x = (\rho\sin\theta)\cos\varphi$

$y = (\rho\sin\theta)\sin\varphi$

$z = \rho\cos\theta$

$J = -\rho^2\sin\theta$

$0 \le \varphi \le 2\pi$

$0 \le \theta \le \frac{\pi}{4}$

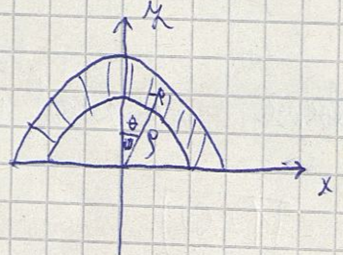
$0 \le \rho \le \frac{1}{\cos\theta}$

$$I = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\theta}} \rho\cos\theta\, |\rho^2\sin\theta|\, d\rho\, d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\cos\theta}} \rho^3\cos\theta\sin\theta\, d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{\rho^4}{4}\cos\theta\sin\theta\right]^{\frac{1}{\cos\theta}} d\theta\, d\varphi = \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta}\, d\theta\, d\varphi = \left| \begin{array}{l} t=\cos\theta \\ dt = -\sin\theta\, d\theta \\ 0 \rightsquigarrow 1 \\ \frac{\pi}{4} \rightsquigarrow \frac{1}{\sqrt2}\end{array}\right| =$$

$$= \frac14 \int_0^{2\pi}\int_1^{\frac{1}{\sqrt2}} \frac{-dt}{t^3}\, d\varphi = \frac14\int_0^{2\pi}\left[-\frac{t^{-2}}{-2}\right]_1^{\frac{1}{\sqrt2}} d\varphi = \frac14\int_0^{2\pi}\left(\frac12\cdot\frac{1}{\frac12} - \frac12\cdot 1/1\right)d\varphi = \frac18\cdot 2\pi = \frac{\pi}{4}$$

* $\iiint\limits_M (x^2+y^2)\,dx\,dy\,dz$, $\quad M: z\ge 0,\ 4\le x^2+y^2+z^2\le 9$



$$x = \rho\sin\theta\cos\varphi$$
$$y = \rho\sin\theta\sin\varphi$$
$$z = \rho\cos\theta$$

$$M:\ 0\le\varphi\le 2\pi$$
$$0\le\theta\le \pi/2$$
$$2\le\rho\le 3$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_2^3 \left(\left((\rho\sin\theta\cos\varphi)^2 + (\rho\sin\theta\sin\varphi)^2\right)\cdot\rho^2|\sin\theta|\right) d\rho\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_2^3 \rho^4\cdot\sin^3\theta\, d\rho\, d\theta\, d\varphi = \left(\int_0^{2\pi} d\varphi\right)\left(\int_0^{\pi/2}\sin^3\theta\, d\theta\right)\left(\int_2^3\rho^4\, d\rho\right) =$$

$$= 2\pi\left[\frac{\rho^5}{5}\right]_2^3\int_0^{\pi/2}\sin^3\theta\, d\theta = 2\pi\frac{243-32}{5}\int_0^{\pi/2}\sin^3\theta\, d\theta = \frac{422\pi}{5}\int_0^{\pi/2}\sin^3\theta\, d\theta =$$

$$= \left| \begin{array}{l} s = \cos\theta \\ ds = -\sin\theta \, d\theta \\ 0 \rightsquigarrow 1 \\ \frac{\pi}{2} \rightsquigarrow 0 \end{array} \right| = \frac{422\pi}{5} \int_1^0 (1-s^2)(-ds) = \frac{422\pi}{5} \int_0^1 (1-s^2)\,ds = \frac{422\pi}{5} \left[s - \frac{s^3}{3} \right]_0^1 =$$

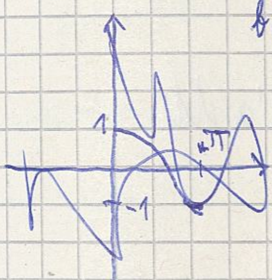
$$= \frac{422\pi}{5} \cdot \frac{2}{3} = \frac{844\pi}{15}$$

$\boxed{13. CV.}$

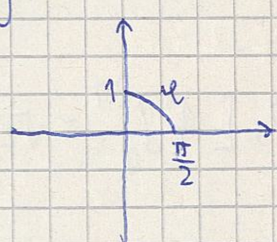
KI prvního druhu

* $I = \int_{\varphi} \sin 2x \, dl \,, \quad \varphi : y = \cos x \,, \; x \in [0, \frac{\pi}{2}]$

$f(x,y) = \sin 2x$



$\varphi : x = t$
$\quad y = \cos t$



$dl = \sqrt{(x')^2 + (y')^2}\, dt$

$x' = 1$
$y' = -\sin t$

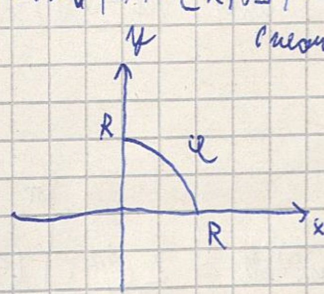
$dl = \sqrt{1 + \sin^2 t}\, dt$

$$I = \int_0^{\pi/2} \sin 2t \cdot \sqrt{1+\sin^2 t}\, dt = \left| \begin{array}{l} u = 1 + \sin^2 t \\ du = 2 \sin t \cos t \, dt = \sin 2t \, dt \\ 0 \rightsquigarrow 1 \,, \; \frac{\pi}{2} \rightsquigarrow 2 \end{array} \right| = \int_1^2 \sqrt{u}\, du = \left[\frac{2}{3} u^{3/2} \right]_1^2 =$$

$$= \frac{2}{3} (2\sqrt{2} - 1)$$

* $\int_{\varphi} x^2 y \, dl \,, \quad \varphi : x^2 + y^2 = R^2 \,, \; x \geq 0 \,, \; A = [R, 0] \,, \; B = [0, R]$ krajní body, $R > 0$ konst.
(neorientované)

$f(x,y) = x^2 y$



$\varphi : \begin{array}{l} x = R \cos t \\ y = R \sin t \end{array} \quad t \in [0, \frac{\pi}{2}]$

$x' = -R \sin t$
$y' = R \cos t$

$$dl = \sqrt{(x')^2 + (y')^2}\, dt = \sqrt{(-R \cos t)^2 + (R \sin t)^2}\, dt = R\, dt$$

$$I = \int_0^{\pi/2} (R\cos t)^2 \cdot R\sin t \cdot R\,dt = R^4 \int_0^{\pi/2} \cos^2 t \sin t\,dt = \left| \begin{array}{l} \cos t = u \\ du = -\sin t\,dt \\ 0 \rightsquigarrow 1 \\ \frac{\pi}{2} \rightsquigarrow 0 \end{array} \right| = R^4 \int_1^0 (-u^2)\,du =$$

$$= R^4 \left[\frac{u^3}{3} \right]_0^1 = \frac{R^4}{3}$$

* $I = \int_\varphi \frac{z^2}{x^2+y^2}\,dl$, $\quad \varphi:\ x = a\cos t$
$\quad y = a\sin t \quad t \in [0, 2\pi]$ šroubovice
$\quad z = at \quad a > 0$ konst.

$$f(x,y,z) = \frac{z^2}{x^2+y^2}$$

$$dl = \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt = \sqrt{a^2\cos^2 t + a^2\sin^2 t + a^2}\,dt = \sqrt{2}\,a\,dt$$

$$I = \int_0^{2\pi} \frac{a^2 t^2}{a^2} \cdot \sqrt{2}\,a\,dt = \sqrt{2}\,a \int_0^{2\pi} t^2\,dt = \sqrt{2}\,a \left[\frac{t^3}{3} \right]_0^{2\pi} = \sqrt{2}\,a\,\frac{8\pi^3}{3} = \frac{\sqrt{2}\cdot 8\cdot a\cdot \pi^3}{3}$$

délka šroubovice je $L = \int_\varphi dl = \int_0^{2\pi} \sqrt{2}\,a\,dt = 2\sqrt{2}\,a\pi$

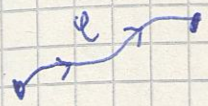
* $I = \int_\varphi \sqrt{2y}\,dl \qquad \varphi:\ x = a(t - \sin t)$
$\quad y = a(1 - \cos t) \quad t \in [0, 2\pi]$ cykloida

$$dl = \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\,dt = a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt = a\sqrt{2 - 2\cos t}\,dt$$

$$= \sqrt{2}\,a\sqrt{1-\cos t}\,dt$$

$$I = \int_0^{2\pi} \sqrt{2a(1-\cos t)} \cdot \sqrt{2}\,a \cdot \sqrt{1-\cos t}\,dt = 2a^{3/2} \cdot \int_0^{2\pi} (1 - \cos t)\,dt = 2a^{3/2} \cdot [t - \sin t]_0^{2\pi} =$$

$$= 4a^{3/2} \cdot \pi$$



$C = C(t),\ t \in [a,b]$

KI druhého druhu

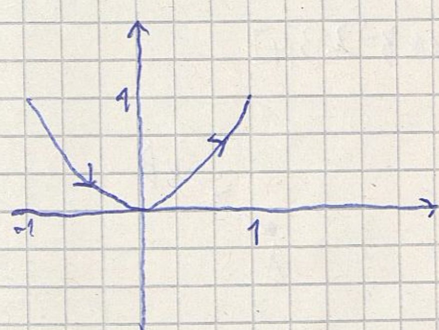
* ~~$I = \int_C f(x,y) \cdot d\vec{r}$~~ skal.

$I = \int_C f(x,y) \cdot d\vec{r}$

skal. součin

$f(x,y) = (x^2-2xy,\ y^2-2xy)$

$C:\ y = x^2,\ x \in [-1,1]$, parabola je orientovaná ve směru růstu x.



$C:\ x = t,\ y = t^2 \quad t \in [-1,1]$

param. zvolená souhlasná s orientací

$$d\vec{r} = (x', y')\,dt = (1, 2t)\,dt$$

$$I = +\int_{-1}^{1} (t^2 - 2t\cdot t^2,\ t^4 - 2t\cdot t^2) \cdot (1, 2t)\,dt = \int_{-1}^{1} (t^2 - 2t^3 + 2t^5 - 4t^4)\,dt =$$

$$= \int_{-1}^{1} \left[\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4t^5}{5}\right]_{-1}^{1} = \frac{1}{3} + \frac{1}{3} - \frac{4}{5} - \frac{4}{5} = \frac{2}{3} - \frac{8}{5} = \frac{10-24}{15} = \frac{-14}{15}$$

* ~~$I = \int_C [y\,dx + z\,dz + \dots$~~

$$I = \int_C [y\,dx + z\,dy + x\,dz] = \underbrace{(y, z, x)}_{f(x,y,z)} \cdot \underbrace{(dx, dy, dz)}_{d\vec{r}}$$

$C:\ x = a\cos t,\ y = a\sin t,\ z = b\cdot t$

$a, b > 0$ konst. $t \in [0, 2\pi]$

param. zadaná souhlasně s orientací

$dx = -a \sin t \, dt$

$dy = a \cos t \, dt$

$dz = b \, dt$

$$I = + \int_0^{2\pi} \left[a \sin t(-a \sin t \, dt) + bt a \cos t \, dt + a \cos t \cdot b \, dt \right] =$$

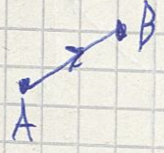
$$= \int_0^{2\pi} \left[-a^2 \sin^2 t + abt \cos t + ab \cos t \right] dt = \int_0^{2\pi} \left[-a^2 \sin^2 t + ab(t+1) \cos t \right] dt =$$

$$= \dots = -a^2 \pi$$

* $I = \int_c \left[x\,dx + y\,dy + (x+y-1)\,dz \right]$

c: úsečka AB, orientace od $A=[1,1,1]$ ~~do~~ po $B=[2,3,4]$

c: $x = 1+t$, $y = 1+2t$, $z = 1+3t$, $t \in [0,1]$



parametrizace zřejmě souhlasná s orientací

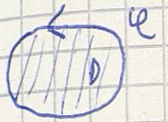
$dx = 1\,dt$, $dy = 2\,dt$, $dz = 3\,dt$

$$\cancel{I = \int_0^1 (x+y) 1 + 2 + 3t + 3 - 1}$$

$$I = + \int_0^1 \left[(1+t)\,dt + (1+2t)\,2\,dt + (1+t+1+2t-1)\,3\,dt \right] = \int_0^1 (1+t+2+4t+3+9t)\,dt =$$

$$= \int_0^1 (14t+6)\,dt = \left[7t^2 + 6t \right]_0^1 = 7+6 = 13$$

- Greenova věta



Jordanova křivka

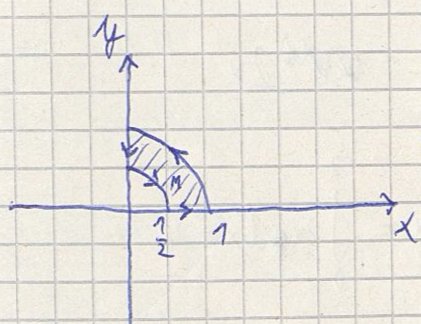
$$f(x,y) = (P(x,y), Q(x,y))$$

$$I = \oint_{\mathcal{C}} P(x,y)\,dx + Q(x,y)\,dy$$

P, Q, P_y, Q_x jsou spojité na $\overline{\text{Int}\,\mathcal{C}}$ $\Rightarrow I = \iint_D (Q_x - P_y)\,dx\,dy$

$$* \quad I = \oint_{\mathcal{C}} \left(\sqrt{x^2+y^2}\,dx + y\left(xy + \ln\left(\sqrt{x^2+y^2}+x\right)\right) dy\right)$$

$\mathcal{C}$: kladně orient. (↺) hranice množiny M: $\frac{1}{4} \le x^2+y^2 \le 1,\ x \ge 0, y \ge 0$



je Jordanova

$$\left.\begin{aligned} P &= \sqrt{x^2+y^2} \\ Q &= y\left(xy + \ln\left(\sqrt{x^2+y^2}+x\right)\right) \end{aligned}\right\} \text{ spojité na } M$$

$$P_y = \frac{y}{\sqrt{x^2+y^2}} \qquad Q_x = y^2 + \frac{y}{\sqrt{x^2+y^2}} \quad \text{spojité na } M$$

$$Q_x - P_y = y^2$$

Green: $I = \iint_M y^2\,dx\,dy$

pol. souř. M: $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ $\qquad \frac{1}{2} \le \rho \le 1$, $0 \le \varphi \le \frac{\pi}{2}$

$$I = \int_0^{\frac{\pi}{2}} \int_{\frac{1}{2}}^{1} \rho^2 \sin^2\varphi \cdot \rho\,d\rho\,d\varphi = \dots = \frac{15\pi}{256}$$

$\boxed{14.\ \text{CV.}}$

* $\int\limits_{\mathcal{C}} \left((1+xy)\, e^{xy}\, dx + (1+e^{xy})\, dy \right)$

$\mathcal{C}$: kladně orient. obdélník $A=[0,0]$, $B=[2,0]$, $C=[2,1]$, $D=[0,1]$ [illegible]

$P = (1+xy)\, e^{xy}$

$Q = (1+e^{xy})\, x^2$ spojité a spoj. dif. [illegible]

$P_y = x e^{xy} + (1+xy) e^{xy} x$

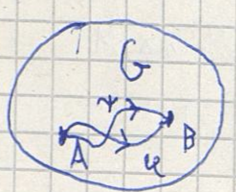
$Q_x = y e^{xy} x^2 + (1+e^{xy}) 2x$

$M := \overline{\operatorname{Int} \mathcal{C}}$

$Q_x - P_y = \cancel{y e^{xy}}\, 2x$

$$I = \iint\limits_M 2x \, dx\, dy = \int_0^2 \int_0^1 2x \, dy\, dx = \int_0^2 2x \, dx = 4$$

– nezávislost na int. cestě



$I = \int\limits_{\mathcal{C}} (P\, dx + Q\, dy)$

Když I nezávisí na int. cestě v $G \Rightarrow P_y = Q_x$ v G

Pokud G je jednoduše souvislá, pak platí i "$\Leftarrow$"

$f = (P, Q) \ \dots \ \exists F(x,y)$: $P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$ … ekviv. nezávislosti pro lib. oblast.

~~potenciál~~ (kmenová funkce)

potenciál je $-F$? (zjistit ve slidech)

* $\int_{c} \left((2x+3y)dx + (3x-4y)dy \right)$ $\qquad G=\mathbb{R}^2$

$P = 2x+3y$ $\qquad Q = 3x-4y$

$P_y = 3$ $\qquad Q_x = 3$ $\qquad \Rightarrow P_y = Q_x$ v $\mathbb{R}^2$ $\Rightarrow$ nezávisí na integ. cestě

$f = (P,Q) = (2x+3y, 3x-4y)$

$F_x = P = 2x+3y$ $\qquad F = \int (2x+3y)dx = x^2+3xy + C(y)$

$F_y = Q = 3x-4y$ $\qquad F_y = 3x + C'(y)$

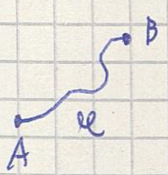
$\qquad \overset{\parallel}{3x-4y}$

$C'(y) = -4y$

$C(y) = -2y^2 + K \qquad K \in \mathbb{R}$

$F(x,y) = x^2+3xy-2y^2+K, \; K \in \mathbb{R}$

to nám umožní takto počítat

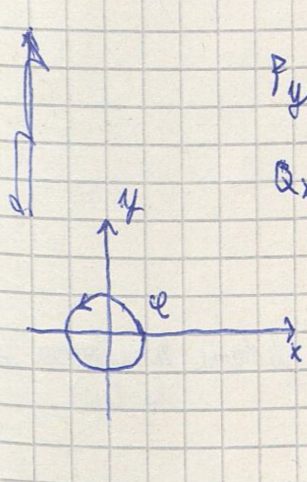


$\int_{c} \left((2x+3y)dx + (3x-4y)dy \right) = F(B) - F(A)$

* $I = \int_{c} \left(\frac{x}{x^2+y^2} dy \; - \frac{y}{x^2+y^2} dx \right)$

$G = \mathbb{R}^2 \setminus \{[0,0]\}$ ... není jednoduše souvislá

$f = (P,Q) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$



$P_y = -\frac{1}{x^2+y^2} - \frac{-y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$Q_x = \frac{y^2-x^2}{(x^2+y^2)^2}$

$P_y = Q_x$ v G, ale G není jednoduše souvislá, tedy int. může záviset na int. cestě, dle a vskutku závisí

$c: x = \cos t$
$\quad y = \sin t$
$\quad t \in [0, 2\pi]$

c ... uzavřená

$$I = +\int_0^{2\pi}\left(\frac{\cos t}{1}\cos t\,dt - \frac{\sin t}{1}(-\sin t)\,dt\right) = \int_0^{2\pi} 1\,dt = 2\pi \neq 0$$

tedy I závisí na int. cestě

– $\int_{\mathcal{C}} (P\,dx + Q\,dy + R\,dz)$

$f = (P, Q, R)$, $P = P(x,y,z)$
$Q = Q(x,y,z)$
$R = R(x,y,z)$

"$\Leftarrow$" platí pro jednodušše souvisl. oblast

pokud ~~tedy~~ I nezávisí na int. cestě $\Rightarrow$ $\underbrace{P_y = Q_x,\ R_x = P_z,\ R_y = Q_z}$ lze si to "odůvodnit" pomocí $R = \frac{\partial F}{\partial z}$

$\Updownarrow$

$P = \frac{\partial F}{\partial x}$, $Q = \frac{\partial F}{\partial y}$, $R = \frac{\partial F}{\partial z}$

* $I = \int_{\mathcal{C}} (yz\,dx + zx\,dy + xy\,dz)$, $G = \mathbb{R}^3$ ... jednodušše souvisl.

~~$R(y,z)$~~ $P = yz$, $Q = zx$, $R = xy$

$P_y = z$ $\quad Q_x = z$ $\quad R_y = x$
$P_z = y$ $\quad Q_z = x$ $\quad R_x = y$

$P_y = Q_x$, $R_x = P_z$, $R_y = Q_z$

$\Rightarrow$ int. nezávisí na int. cestě

$F(x,y,z)$: $F_x = P = yz$
$F_y = Q = zx$
$F_z = R = xy$

$F = \int yz\,dx = xyz + H(y,z)$
$F_y = zx + H_y = zx \Rightarrow H_y = 0$
$F_z = yx + H_z = xy \Rightarrow H_z = 0$ } hledáme kmenovou funkci k 0,0

$$H = \int 0\,dy = 0 + C(z) = C(z)$$

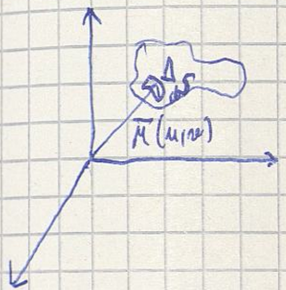
$$H_z = C'(z) = 0 \Rightarrow C(z) = K \in \mathbb{R}$$

$$H(x,y,z) = xyz + K, \quad K \in \mathbb{R}$$

* $I = \int\limits_{\sigma} \left(z + 2x + \frac{4}{3}y\right) dS,$ $\quad \sigma$ (plocha) $: \bar{r}(u,v) = \left(u, v, 4 - 2u - \frac{4}{3}v\right)$ $\quad \Leftrightarrow x = u$

$0 \le u \le 2, \quad 0 \le v \le 3 - \frac{3u}{2}$ $\quad y = v$

$z = 4 - 2u - \frac{4}{3}v$



plošný integrál 1. druhu

$$dS = \|\bar{n}(u,v)\|\,du\,dv$$

norm. vektor plochy

$$\bar{n}(u,v) = \bar{r}_u(u,v) \times \bar{r}_v(u,v)$$

vekt. součin

pozn. $\bar{a} = (a_1, a_2, a_3) = a_1\bar{i} + a_2\cdot\bar{j} + a_3\cdot\bar{k}$

$\bar{b} = (b_1, b_2, b_3) = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$

$\bar{i} = (1,0,0), \ \bar{j} = (0,1,0), \ \bar{k} = (0,0,1)$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \bar{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \bar{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

formální determinant

$\bar{r}_u = (1, 0, -2)$

$\bar{r}_v = \left(0, 1, -\frac{4}{3}\right)$

$$\bar{r}(u,v) = (1,0,-2) \times \left(0,1,-\frac{4}{3}\right) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & -2 \\ 0 & 1 & -\frac{4}{3} \end{vmatrix} = \bar{i}\cdot 2 - \bar{j}\left(-\frac{4}{3}\right) + \bar{k}\cdot 1 =$$

$$= \left(2, \frac{4}{3}, 1\right)$$

$$\|\bar{n}(u,v)\| = \sqrt{4 + \frac{16}{9} + 1} = \sqrt{\frac{61}{9}} = \cancel{\frac{\sqrt{61}}{9}}\ \frac{\sqrt{61}}{3}$$

$$I = \iint_M \left(4 - 2u - \frac{4}{3}v + 2u + \frac{4}{3}v\right) \cdot \frac{\sqrt{61}}{3} \, du\,dv =$$

$M: 0 \leq u \leq 2$

$0 \leq v \leq 3 - \frac{3u}{2}$

$$I = \int_0^2 \int_0^{3-\frac{3u}{2}} 4 \cdot \frac{\sqrt{61}}{3} \, dv\,du = 4 \cdot \frac{\sqrt{61}}{3} \cdot \int_0^2 \left(3 - \frac{3u}{2}\right) du = 4 \cdot \frac{\sqrt{61}}{3} \left[3u - \frac{3u^2}{4}\right]_0^2 = 4\sqrt{61}(2-1) =$$

$$= 4\sqrt{61}$$

– ještě si procvičit plošný int. 2. druhu