

SURPRISES ABOUT SOME ELEMENTARY FUNCTIONS: UNIFORM LINEAR APPROXIMATIONS

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ABSTRACT. Some elementary functions, like logarithms and exponentials, can be uniformly approximated by a piecewise linear functions, which have surprising invariants. For the logarithmic function $y = \log_a x$, the maximal error is bounded by an explicit constant, which depends only on the basis a and not on the approximation interval. For the exponential function $y = a^x$, the maximal error depends only on the index of the approximation interval and not on the basis a . Two open problems generalizing introduced ideas are presented.

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1. MOTIVATION

Given any positive real number x , we may compute $\log_2 x$ by the formula

$$\log_2 x \approx \frac{x}{2^k} + k - 0.957 \quad (1)$$

with an error at most ± 0.043 , where k is the integer satisfying $x \in [2^k, 2^{k+1}]$. E.g., $\log_2 3 \approx 1.543$ ($k = 1$), $\log_2 10 \approx 3.293$ ($k = 3$). For any real number x with $x \in [\log_2 k, \log_2(k + 1)]$ for some $k \in \mathbb{N}$, we may compute 2^x by a formula similar to (1).

In this contribution we would like to present some interesting approximation results concerning logarithmic and exponential functions. In Section 2 we derive a uniform piecewise linear approximation of the logarithmic function $y = \log_a x$. We show that the maximal error can be estimated by a certain explicit constant, which depends only on the basis a and not on the approximation interval. We examine the behavior of this constant as a function of a , and compare it to classical linear approximation by tangent lines (differentials): our result yields smaller approximation error. By similar computations, we show in Section 3 that the maximal error for $y = a^x$ depends only on the index of the approximation interval and not on the basis a . Section 4 contains a general approach and open problems, whether there are other functions having these approximation invariants.

2. LOGARITHMIC FUNCTION

Let $a \in \mathbb{R}$, $a > 1$, be the basis of the logarithmic function $f(x) = \log_a x$. Let $I_k := [a^k, a^{k+1}]$ for $k \in \mathbb{Z}$, so that the union of all I_k 's is $(0, \infty)$. On each of the intervals I_k we approximate $f(x)$ by a linear function $\ell(x)$, depending on a and k , through the points (a^k, k) and $(a^{k+1}, k + 1)$. I.e.,

$$\ell(x) = \frac{a^{-k}}{a-1} x + k - \frac{1}{a-1}, \quad x \in I_k.$$

Since $f(x)$ is concave, the approximation error between $f(x)$ and $\ell(x)$ is $e(x) := f(x) - \ell(x) \geq 0$ for all $x \in I_k$, $k \in \mathbb{Z}$.

Theorem 1. *For any basis $a > 1$, the maximal approximation error of $f(x) = \log_a x$ on $(0, \infty)$ is*

$$\gamma^+(a) = \frac{\ln(a-1)}{\ln a} - \frac{\ln \ln a}{\ln a} - \frac{1}{\ln a} + \frac{1}{a-1} \quad (2)$$

(the superscript $+$ reminds on the values $a > 1$). Hence, the uniform piecewise linear approximation of $f(x)$ with an error at most $\gamma^+(a)/2$ is $\ell(x) + \gamma^+(a)/2$, i.e. for $x \in [a^k, a^{k+1}]$, $k \in \mathbb{Z}$,

$$\log_a x \approx \frac{a^{-k}}{a-1} x + k - \frac{1}{a-1} + \frac{\gamma^+(a)}{2}. \quad (3)$$

Proof. Clearly, $e(a^k) = 0 = e(a^{k+1})$, and thus $e(x)$ attains its local maximum in I_k at some point \bar{x} . Solving $e'(x) = 0$ we have

$$\bar{x} = \frac{a^k(a-1)}{\ln a}. \quad (4)$$

It is easily seen that \bar{x} is the point from Lagrange mean value theorem. Denote by $\gamma^+(a) := e(\bar{x})$ the value of maximal error. Then $\gamma^+(a)$ reads as (2) and is *independent* on k . ■

For bases $a \in (0, 1)$ we may proceed in the same way as above, except that the error function would be $\ell(x) - f(x) \geq 0$. In this case the maximal error is

$$\gamma^-(a) = \frac{\ln(-\ln a)}{\ln a} - \frac{\ln(1-a)}{\ln a} + \frac{1}{\ln a} + \frac{1}{1-a} \quad (5)$$

(the superscript $-$ reminds on the values $a < 1$).

Define the function $\gamma : a \mapsto \gamma(a)$ by $\gamma(a) := \gamma^+(a)$ if $a > 1$, and $\gamma(a) := \gamma^-(a)$ if $0 < a < 1$. For illustration we list some values of $\gamma(a)$ for some particular bases (rounded for 10^{-8}):

$$\begin{aligned} \gamma(2) &= 0.08607133, & \gamma(10) &= 0.26884346, \\ \gamma(e) &= 0.12330156, & \gamma(10^{10}) &= 0.82034899. \end{aligned}$$

Remark 1. This method is related to the convergence of the series $\sum_0^\infty A_{-k}$, where A_k is the area between $f(x)$ and $\ell(x)$ for $x \in I_k$, i.e.

$$A_k = \int_{a^k}^{a^{k+1}} e(x) dx = \frac{a^k(a+1)}{2} - \frac{a^k(a-1)}{\ln a}, \quad (6)$$

see [1] for details. More precisely, [1] shows that $\sum_0^\infty A_{-k} < \frac{a-1}{2}$. After a simple computation we get in fact easily

$$\sum_{k=0}^{\infty} A_{-k} = \frac{a(a+1)}{2(a-1)} - \frac{a}{\ln a}.$$

Next we shortly examine the behavior of the maximal error $\gamma(a)$ as a function of a .

First, observe that $\gamma(a) = \gamma(\frac{1}{a})$, i.e. it is reciprocal symmetric function. Second, observe that

$$\lim_{a \rightarrow 1} \gamma(a) = 0,$$

which is not a surprising fact, since for the basis $a = 1$ the logarithm is deformed to the linear function $x = 1$, which has zero distance from its linear approximation. Furthermore,

$$\lim_{a \rightarrow 0^+} \gamma(a) = 1 = \lim_{a \rightarrow \infty} \gamma(a).$$

Define $\gamma(0) := 1$ and $\gamma(1) := 0$. Then $\gamma : [0, \infty) \rightarrow [0, 1]$ is continuous (even differentiable of all orders except at $a = 1$), and bounded by 1. It decreases to 0 as $a \rightarrow 1$ with slopes $-\frac{1}{8}$ from left, and $\frac{1}{8}$ from right, respectively. Function $\gamma(a)$ approaches 1 with slope $-\infty$ as $a \rightarrow 0^+$, see Figure 1.

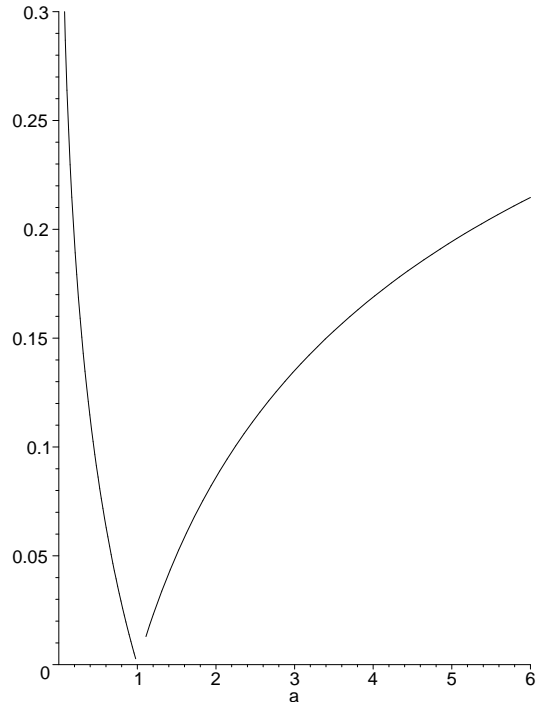


FIGURE 1. The function $\gamma(a)$.

We finish this section by comparing our result with standard linear approximations by tangent lines (or differentials). The formula

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = \log_a x + \frac{x - x_0}{x_0}, \quad x \in I_k, \quad (7)$$

gives a linear approximation of $f(x) = \log_a x$. We denote by $T_k(x)$ and $T_{k+1}(x)$ the approximation (7) with $x_0 = a^k$ and $x_0 = a^{k+1}$, respectively. For $x \in I_k$, the maximal error $\delta(a)$ of T_k is attained at $\bar{x} = a^{k+1}$, and the maximal error $\sigma(a)$ of T_{k+1} is attained at $\bar{x} = a^k$. Simple computations show that

$$\delta(a) = \begin{cases} \frac{a-1}{\ln a} - 1 & \text{if } a > 1, \\ 1 - \frac{a-1}{\ln a} & \text{if } 0 < a < 1, \end{cases} \quad \sigma(a) = \begin{cases} 1 - \frac{a-1}{a \ln a} & \text{if } a > 1, \\ \frac{a-1}{a \ln a} - 1 & \text{if } 0 < a < 1. \end{cases} \quad (8)$$

Observe that both $\delta(a)$ and $\sigma(a)$ are *independent* on k . Moreover, $\lim_{a \rightarrow 1} \delta(a) = 0 = \lim_{a \rightarrow 1} \sigma(a)$, $\lim_{a \rightarrow \infty} \delta(a) = \infty = \lim_{a \rightarrow 0^+} \sigma(a)$, and $\lim_{a \rightarrow 0^+} \delta(a) = 1 = \lim_{a \rightarrow \infty} \sigma(a)$. We set then $\delta(1) := 0$, $\sigma(1) := 0$, and $\delta(0) := 1$. Comparing the values of $\delta(a)$, $\sigma(a)$ with our $\gamma(a)$ we obtain the following theorem.

Theorem 2. *Let $\gamma(a)$ and $\delta(a)$, $\sigma(a)$ be defined by (2), (5) and (8), respectively. Then the maximal error of the linear approximation (3) of $y = \log_a x$ is always smaller than the maximal error of (7). More precisely,*

$$\gamma(a) < \sigma(a) < \delta(a) \quad \text{if } a > 1, \quad \text{and} \quad \gamma(a) < \delta(a) < \sigma(a) \quad \text{if } 0 < a < 1,$$

with the equalities only at $a \in \{0, 1\}$, i.e.

$$\gamma(1) = \delta(1) = \sigma(1) = 0 \quad \text{and} \quad \gamma(0) = \delta(0) = 1.$$

Proof. Straightforward computations. ■

Remark 2. The point where the linear approximations T_k and T_{k+1} intersect is

$$\tilde{x} = \frac{a^{k+1} \ln a}{a - 1}.$$

This means that both tangent approximations T_k and T_{k+1} have the same error at \tilde{x} . It is interesting to note that the value of this error is exactly $\gamma(a)$ given by (2), (5).

3. EXPONENTIAL FUNCTION

Let $a \in \mathbb{R}$, $a > 1$ be the basis of exponential function $g(x) = a^x$. We approximate $g(x)$ on intervals $J_k := [\log_a k, \log_a(k+1)]$, $k \in \mathbb{N}$, by the same method as in Section 2. We do not repeat all the details again. The local maximum of the error function is attained in J_k at the point

$$\bar{x} = -\frac{\ln \ln \frac{k+1}{k}}{\ln a},$$

and the maximal error $\omega(k) = e(\bar{x})$ is then

$$\omega(k) = k - \frac{\ln \ln \frac{k+1}{k}}{\ln \frac{k+1}{k}} - \frac{\ln k}{\ln \frac{k+1}{k}} - \frac{1}{\ln \frac{k+1}{k}},$$

which is a function *independent* on the basis a of $g(x)$.

Examining the behavior of the function $\omega : [0, \infty) \rightarrow (0, 1]$, where $\omega(0) := \lim_{k \rightarrow 0^+} \omega(k) = 1$, we realize the following interesting fact:

$$\omega(1) = 1 - \frac{\ln \ln 2}{\ln 2} - \frac{1}{\ln 2} = 0.086071332 = \gamma(2),$$

i.e. the maximal error of $\log_a x$ for basis $a = 2$ (independently on k) equals to the maximal error of a^x on the first approximation interval $I_1 = [0, \log_a 2]$ (independently on a). The value $\omega(0) = 1$ is not again surprising, since it is the maximal error of the linear approximation $y = 1$ of $y = a^x$ over the interval $(-\infty, 0]$.

4. GENERALIZATION AND OPEN PROBLEMS

When trying to find a general description of functions $f(x)$ having similar approximation invariants as the logarithmic, exponential, and linear functions (this is the limiting case in our computations), we proceed as follows. If $f(x)$ is monotone with continuous $f''(x) \neq 0$, we approximate $f(x)$ by a piecewise linear function on each of the intervals $\mathcal{I}_k := [g(k), g(k+1)]$, through the points $(g(k), k)$ and $(g(k+1), k+1)$, where $g = f^{-1}$ is the inverse of f . It is easy to see that the linear approximation is

$$\ell_f(x) = \frac{1}{g(k+1) - g(k)} x + k - \frac{g(k)}{g(k+1) - g(k)}, \quad x \in \mathcal{I}_k.$$

The error function is then $e_f(x) = |f(x) - \ell_f(x)|$, $x \in \mathcal{I}_k$. We have $e_f(g(k)) = 0 = e_f(g(k+1))$, and solving $e'_f(x) = 0$ we get the point of local maximum

$$\bar{x}_f = f'^{-1} \left(\frac{1}{g(k+1) - g(k)} \right). \quad (9)$$

Observe that f' is injective and that \bar{x}_f depends on k . Then the maximal error is $\gamma_f := e_f(\bar{x}_f)$, i.e.

$$\gamma_f = f(\bar{x}_f) - \frac{\bar{x}_f}{g(k+1) - g(k)} - k + \frac{g(k)}{g(k+1) - g(k)}. \quad (10)$$

The question is for which functions f is γ_f actually *independent* on k . In this case, we say that f possesses a *linear approximation independence property*. However, in spite of that the following theorem contains a complete characterization of such functions, we did not find any other example than $y = \log_a x$ and $y = ax + b$ (the function $y = a^x$ belongs among these somehow, as explained in Section 3).

Theorem 3. *Let f be monotone and twice continuously differentiable with $f''(x) \neq 0$. Let $g = f^{-1}$ be the inverse of f . Let γ_f be the maximal error of the piecewise linear approximation ℓ_f of f given by (10). Then f has the linear approximation independence property (i.e. γ_f is independent on k) if and only if*

$$\bar{x}_f = \frac{[g(k+1) - g(k)]^2 - g'(k)g(k+1) + g(k)g'(k+1)}{g'(k+1) - g'(k)}. \quad (11)$$

Proof. Observe that we may actually differentiate γ_f with respect to k (a discrete variable), since k is not, in fact, discrete at all – an infinitesimal change of k forces an infinitesimal change both of \bar{x}_f and $g(k)$. By using (9), a short computation shows that

$$\frac{d}{dk} \gamma_f = \frac{\bar{x}_f [g'(k+1) - g'(k)] + g'(k)g(k+1) - g(k)g'(k+1)}{[g(k+1) - g(k)]^2} - 1. \quad (12)$$

Now, if γ_f is independent on k , i.e. $\frac{d}{dk} \gamma_f = 0$, then (11) follows immediately from (12). Conversely, if \bar{x}_f satisfies (11) then $\frac{d}{dk} \gamma_f = 0$ just by substituting into the above computations, and so γ_f is independent on k . ■

For $f(x) = \log_a x$ the expression (11) holds and reduces to (9), (4). For $f(x) = ax + b$ we interpret equality (11) as $0 \cdot \bar{x}_f = 0$ (since $g'(x) = \frac{1}{a}$ is then constant), which holds for any $\bar{x}_f \in \mathbb{R}$. For $f(x) = x^p$, $p \in \mathbb{R}$, equality (11) does not hold, unless $p \in \{0, 1\}$.

Open Problem 1. *Find another function(s) having the linear approximation independence property, or prove that $y = \log_a x$ and $y = ax + b$ are the only ones.*

Denote the tangent approximations (7) of f at $g(k)$ and $g(k+1)$ by T_k^f and T_{k+1}^f , respectively. Let \tilde{x}_f be the point where they intersect, i.e.,

$$\tilde{x}_f = \frac{f'(g(k+1))g(k+1) - f'(g(k))g(k) - 1}{f'(g(k+1)) - f'(g(k))}.$$

Then the errors of T_k^f and T_{k+1}^f at \tilde{x}_f are the same and the considerations from Remark 2 rise the following question.

Open Problem 2. *For which function(s) is the error γ_f of the secant line through the points $[g(k), k]$ and $[g(k+1), k+1]$ exactly the same as the error of T_k^f (or T_{k+1}^f) at \tilde{x}_f ?*

Open Problem 3. *Is there any relation between the functions from Open Problem 2 and the linear approximation independence property?*

Remark 3. Similarly as in Section 2, paper [1] describes the convergence of $\sum A_k$ for this general f , where A_k is the area between $f(x)$ and $\ell_f(x)$ for $x \in \mathcal{I}_k$.

REFERENCES

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