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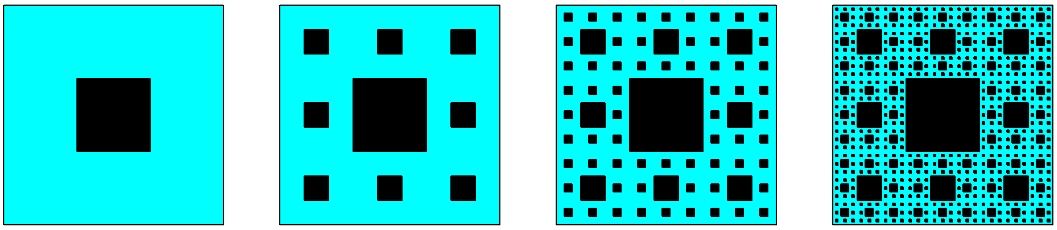

Collection of examples about infinite series

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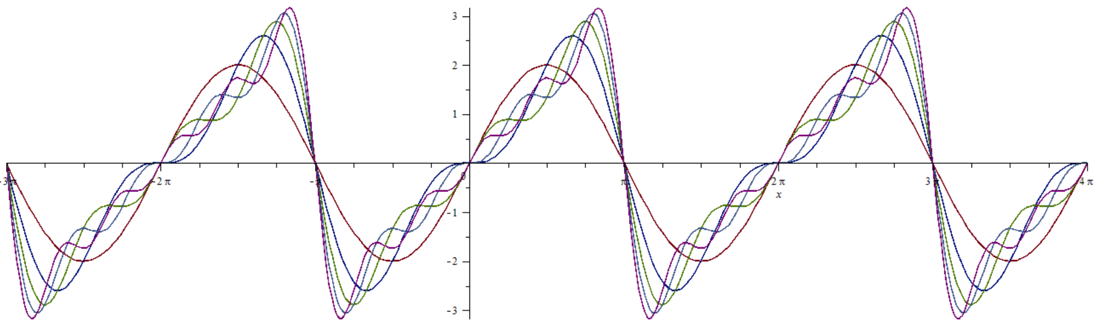

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The purpose of this collection of examples is to help students of basic courses of mathematical analysis to develop calculus routines and to improve their knowledge and understanding of infinite series. The document contains more than 300 solved examples and several dozens of unsolved problems with the correct results.

The examples in this document were collected by the authors during their lectures of courses containing the theory of infinite series. Some of the examples are taken from exams and consultations and last but not least written for this document. Regarding the used literature sources (especially, with the theory background), we mention the references (Czech and English) at the end of the document (page 189).

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Chapter

1 Numerical series

§ 1.1 Sums of series

Definition 1.

The infinite sum

$$a_N + a_{N+1} + \cdots + a_n + \cdots = \sum_{n=N}^{\infty} a_n$$

is called an infinite series. The value a_n is called the n-th term and

$$s_n := a_N + a_{N+1} + \cdots + a_n$$

is called the n-th partial sum of the given series.

Definition 2.

A geometric series is of the form

$$a + aq + aq^2 + aq^3 + \cdots + aq^n + \cdots = \sum_{n=1}^{\infty} aq^{n-1} = \sum_{n=0}^{\infty} aq^n,$$

where $a, q \in \mathbb{R}$, i.e., it is the infinite series for

$$a_n := aq^n, \qquad n = 0, 1, 2, \ldots$$

The number q is called the common ratio (quotient) of the given geometric series.

Example 1.

Write the partial sums

$$\sigma_n := a + aq + aq^2 + aq^3 + \cdots + aq^{n-1}, \qquad n \in \mathbb{N},$$

for the series

$$\sum_{n=0}^{\infty} \frac{1}{5^n} = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots + \frac{1}{5^n} + \cdots$$

Solution. It is a geometric series with $q \neq 1$. To get the sequence of the partial sums, we write

$$\begin{aligned} \sigma_n &= a + aq + aq^2 + \cdots + aq^{n-2} + aq^{n-1}, \\ q\sigma_n &= aq + aq^2 + aq^3 + \cdots + aq^{n-1} + aq^n. \end{aligned}$$

Subtracting the second equation from the first, we obtain

$$\sigma_n - q\sigma_n = a - aq^n,$$

i.e.,

$$\begin{aligned} \sigma_n (1-q) &= a\,(1-q^n), \\ \sigma_n &= a\,\frac{1-q^n}{1-q}. \end{aligned}$$

It is the geometric series with $a = 1$ and $q = 1/5$. Hence,

$$\sigma_n = \frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}} = \frac{1-\left(\frac{1}{5}\right)^n}{\frac{4}{5}} = \frac{5}{4} - \frac{1}{4 \cdot 5^{n-1}}, \qquad n \in \mathbb{N}.$$

■

Example 2.

Write the partial sums

$$\sigma_n := a + aq + aq^2 + aq^3 + \cdots + aq^{n-1}, \qquad n \in \mathbb{N},$$

for the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots + \frac{(-1)^n}{4^n} + \cdots$$

Solution. We use the solution of Example 1. We have the geometric series with $a = 1$ and $q = -1/4$. Thus,

$$\begin{aligned}\sigma_n &= \frac{1-\left(-\frac{1}{4}\right)^n}{1+\frac{1}{4}} = \frac{4^n-(-1)^n}{4^n}\cdot\frac{4}{5}\\ &= \frac{1}{5}\cdot\frac{4^n+(-1)^{n+1}}{4^{n-1}} = \frac{4}{5}+\frac{(-1)^{n+1}}{5\cdot 4^{n-1}}, \qquad n\in\mathbb{N}.\end{aligned}$$

■

Definition 3.

If the limit $\lim_{n\to\infty} s_n$ exists and is finite with a value s, we say that the infinite series $\sum_{n=0}^{\infty} a_n$ converges (is convergent) with the sum (or value) s. We write

$$\sum_{n=0}^{\infty} a_n = s.$$

If the limit $\lim_{n\to\infty} s_n$ is not finite, i.e., it equals to $\pm\infty$, we say that the infinite series $\sum_{n=0}^{\infty} a_n$ diverges (is divergent) to $\pm\infty$. We write

$$\sum_{n=0}^{\infty} a_n = \pm\infty.$$

If the limit $\lim_{n\to\infty} s_n$ does not exist, we say that the infinite series $\sum_{n=0}^{\infty} a_n$ oscillates (is oscillatory).

Example 3.

If it is possible, find the sum of the so-called Grandi series

$$\sum_{n=1}^{\infty}(-1)^n.$$

Solution. The partial sums for $n \in \mathbb{N}$ are

$$s_n = a_1+\cdots+a_n = \begin{cases} -1 & \text{for } n \text{ odd;}\\ 0 & \text{for } n \text{ even.}\end{cases}$$

The limit $\lim_{n\to\infty} s_n$ does not exist. Therefore, the Grandi series oscillates. ■

Example 4.

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right).$$

Solution. For $n \in \mathbb{N}$, the partial sum of the series is

$$\begin{aligned} s_n = \left(\sqrt{3}-2\sqrt{2}+\sqrt{1}\right)+\left(\sqrt{4}-2\sqrt{3}+\sqrt{2}\right)+\left(\sqrt{5}-2\sqrt{4}+\sqrt{3}\right) \\ +\left(\sqrt{6}-2\sqrt{5}+\sqrt{4}\right)+\cdots+\left(\sqrt{n}-2\sqrt{n-1}+\sqrt{n-2}\right) \\ +\left(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\right)+\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right). \end{aligned}$$

Immediately, we get

$$s_n = \sqrt{1}-\sqrt{2}+\sqrt{n+2}-\sqrt{n+1}, \qquad n \in \mathbb{N}.$$

The final sum is

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sqrt{1}-\sqrt{2}+\sqrt{n+2}-\sqrt{n+1}\right) = 1-\sqrt{2},$$

where

$$\begin{aligned} &\lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n+1}\right) \\ &= \lim_{n\to\infty}\left(\sqrt{n+2}-\sqrt{n+1}\right)\frac{\sqrt{n+2}+\sqrt{n+1}}{\sqrt{n+2}+\sqrt{n+1}} \\ &= \lim_{n\to\infty}\frac{1}{\sqrt{n+2}+\sqrt{n+1}} = 0. \end{aligned}$$

■

Example 5.

Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{2}{\arctan n}.$$

Solution. Since

$$\sum_{n=1}^{\infty}\frac{2}{\arctan n} \geq \sum_{n=1}^{\infty}\frac{2}{\frac{\pi}{2}} = \sum_{n=1}^{\infty}\frac{4}{\pi} = +\infty,$$

it is clear that the given series diverges to $+\infty$. ■

Example 6.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^3 + 2n + 5}{n^2 - 2n^3}.$$

Solution. Using the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3 + 2n + 5}{n^2 - 2n^3} = -\frac{1}{2},$$

we have the existence of $N \in \mathbb{N}$ with the property that

$$\sum_{n=1}^{\infty} \frac{n^3 + 2n + 5}{n^2 - 2n^3} \leq \sum_{n=N}^{\infty} -\frac{1}{4} = -\infty.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{n^3 + 2n + 5}{n^2 - 2n^3} = -\infty.$$

We remark that the given series has only negative terms. ■

Example 7.

Find out whether the series

$$\sum_{n=1}^{\infty} ((-1)^n - 2)$$

converges, diverges, or oscillates.

Solution. We study the limit behaviour of the terms of the given infinite series (although $\lim_{n\to\infty} a_n$ does not exist). Obviously,

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} ((-1)^n - 2) = -1 < 0.$$

Thus, the series is divergent with the sum $-\infty$. ■

Example 8.

Find out whether the series

$$\sum_{n=1}^{\infty} (2 + (-1)^n)^n$$

converges, diverges, or oscillates.

Solution. We study the limit behaviour of the terms of the given infinite series (although $\lim_{n\to\infty} a_n$ does not exist). We have

$$\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} (2 + (-1)^n)^n \geq 1 > 0,$$

which implies that the series diverges to $+\infty$. ■

Example 9.

Find out whether the series

$$\sum_{n=1}^{\infty} (-3)^n$$

converges, diverges, or oscillates.

Solution. We study the limit behaviour of the terms of the given infinite series (although $\lim_{n\to\infty} a_n$ does not exist). With regard to the parity of n, we have

$$\liminf_{n\to\infty} a_n = \liminf_{n\to\infty} (-3)^n = -\infty,$$

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} (-3)^n = +\infty.$$

Evidently, the given series is oscillatory. ■

Example 10.

Find the sum of the series

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right).$$

Solution. The partial sums are

$$\begin{aligned} s_n = \sum_{j=1}^{n} \ln\left(1 + \frac{1}{j}\right) &= \ln\left(\prod_{j=1}^{n}\left[1 + \frac{1}{j}\right]\right) = \ln\left(\prod_{j=1}^{n} \frac{j+1}{j}\right) \\ &= \ln\left(\frac{2}{1} \cdot \frac{3}{2} \cdots \frac{n+1}{n}\right) = \ln(n+1), \qquad n \in \mathbb{N}. \end{aligned}$$

Therefore,

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right) = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = +\infty.$$

■

Example 11.

Decide whether the infinite series

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\cdots$$

converges.

Solution. We have

$$a_{n-1}=\frac{2^n-1}{2^n}=1-\frac{1}{2^n}, \qquad n\in\mathbb{N}.$$

Since

$$\lim_{n\to\infty}\frac{1}{2^n}=0,$$

we get

$$\lim_{n\to\infty}a_n=1.$$

The sum is $+\infty$. ■

Remark. An arithmetic sequence is a sequence whose difference of subsequent terms is constant. The partial sum of an arithmetic series with a difference d is

$$\begin{aligned} s_n=a_1+a_2+\cdots+a_n &= a_1+(a_1+d)+(a_1+2d)+(a_1+3d)+\cdots+(a_1+(n-1)d)\\ &= na_1+\frac{1}{2}n(n-1)d=\frac{n(a_1+a_n)}{2}. \end{aligned}$$

Example 12.

Find the limit

$$\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right).$$

Solution. Directly (using partial sums of an arithmetic series), we obtain

$$\begin{aligned} \lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\cdots+\frac{n-1}{n^2}\right) &= \lim_{n\to\infty}\left[\frac{1}{n^2}(1+2+\cdots+n-1)\right]\\ &= \lim_{n\to\infty}\left[\frac{1}{n^2}\cdot\frac{n(n-1)}{2}\right]=\lim_{n\to\infty}\frac{n^2-n}{2n^2}=\frac{1}{2}. \end{aligned}$$

■

Example 13.

Decide, for which values of the parameter $a_1 \in \mathbb{R}$ and the difference $d \in \mathbb{R}$, the series

$$\sum_{n=0}^{\infty}(a_1 + nd)$$

is convergent.

Solution. The series converges only for $a_1 = d = 0$, where the sum is 0.

(i) If $a_1 \neq 0$ and $d \neq 0$, then

$$\sum_{n=0}^{\infty}(a_1 + nd) = (\operatorname{sgn} d) \cdot (+\infty).$$

(ii) If $a_1 \neq 0$ and $d = 0$, then

$$\sum_{n=0}^{\infty} a_1 = (\operatorname{sgn} a_1) \cdot (+\infty).$$

(iii) If $a_1 = 0$ and $d \neq 0$, then

$$\sum_{n=0}^{\infty} nd = (\operatorname{sgn} d) \cdot (+\infty).$$

■

Theorem 1.

Let $a \neq 0$. The geometric series $\sum_{n=0}^{\infty} aq^n$ converges if and only if $|q| < 1$. Moreover,

$$\sum_{n=0}^{\infty} aq^n = \frac{a}{1-q}, \qquad |q| < 1.$$

Example 14.

Study the convergence of the series

$$9 - 12 + 16 - \frac{64}{3} + \frac{256}{9} - \cdots$$

Solution. It is a geometric series with $q = -4/3$. Since $|q| = 4/3 \geq 1$, the series does not converge. ■

Example 15.

Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{1}{2^n}.$$

Solution. It is a geometric series. Hence,
$$a = \frac{1}{2}, \qquad q = \frac{1}{2}, \qquad \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$
■

Example 16.

Find the sum of the series
$$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.$$

Solution. It is a geometric series, where
$$a = \frac{2}{3}, \qquad q = \frac{2}{3}, \qquad \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}}{1 - \frac{2}{3}} = 2.$$
■

Example 17.

Find the sum of the series
$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n.$$

Solution. It is a geometric series, for which
$$q = \frac{3}{2} > 1, \qquad \sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = +\infty.$$
■

Example 18.

Find the sum of the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{4}{5}\right)^n.$$

Solution. It is a geometric series. We have

$$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{4}{5}\right)^n = -\sum_{n=1}^{\infty}\left(-\frac{4}{5}\right)^n = -\frac{-\frac{4}{5}}{1+\frac{4}{5}} = \frac{4}{9},$$

where

$$a = -\frac{4}{5}, \qquad q = -\frac{4}{5}.$$

■

Example 19.

If it is possible, find the sum of the series

$$\sum_{n=3}^{\infty}\left(\frac{3}{7}\right)^n.$$

Solution. We have

$$\sum_{n=3}^{\infty}\left(\frac{3}{7}\right)^n = \sum_{n=1}^{\infty}\left(\frac{3}{7}\right)^n - \sum_{n=1}^{2}\left(\frac{3}{7}\right)^n = \frac{\frac{3}{7}}{1-\frac{3}{7}} - \frac{3}{7} - \left(\frac{3}{7}\right)^2 = \frac{3}{4} - \frac{3}{7} - \frac{9}{49} = \frac{27}{196},$$

where

$$a = \frac{3}{7}, \qquad q = \frac{3}{7}.$$

Similarly, we have

$$a = \left(\frac{3}{7}\right)^3, \qquad q = \frac{3}{7}, \qquad \sum_{n=3}^{\infty}\left(\frac{3}{7}\right)^n = \frac{\left(\frac{3}{7}\right)^3}{1-\frac{3}{7}} = \frac{3^3}{7^2\cdot 4} = \frac{27}{196}.$$

■

Example 20.

If it is possible, find the sum of the series

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n.$$

Solution. We have

$$a = 1, \qquad q = \frac{1}{3}, \qquad \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}.$$

■

Example 21.

If it is possible, find the sum of the series

$$\sum_{n=-2}^{\infty} \left(\frac{-1}{5}\right)^n.$$

Solution. It holds

$$\sum_{n=-2}^{\infty} \left(\frac{-1}{5}\right)^n = \sum_{n=-2}^{0} \left(\frac{-1}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{-1}{5}\right)^n =$$

$$= \left(\frac{-1}{5}\right)^{-2} + \left(\frac{-1}{5}\right)^{-1} + \left(\frac{-1}{5}\right)^{0} + \frac{\frac{-1}{5}}{1+\frac{1}{5}} = 25 - 5 + 1 - \frac{1}{6} = \frac{125}{6}.$$

We can also proceed as follows

$$a = \left(\frac{-1}{5}\right)^{-2}, \qquad q = -\frac{1}{5}, \qquad \sum_{n=-2}^{\infty} \left(\frac{-1}{5}\right)^n = \frac{\left(\frac{-1}{5}\right)^{-2}}{1+\frac{1}{5}} = \frac{25}{1+\frac{1}{5}} = \frac{125}{6}.$$

■

Example 22.

If it is possible, find the sum of the series

$$\sum_{n=4}^{\infty} \left(\frac{6}{5}\right)^n.$$

Solution. It is a divergent series with the sum $+\infty$, because

$$\sum_{n=4}^{\infty} \left(\frac{6}{5}\right)^n \geq \sum_{n=4}^{\infty} 1 = +\infty.$$

■

Example 23.

Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{e^n}}.$$

Solution. It is a geometric series with $q = 1/\sqrt{e}$, i.e., $|q| < 1$. Therefore, we calculate

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{e^n}} = \sum_{n=2}^{\infty} \left(\frac{1}{\sqrt{e}}\right)^n = \frac{1}{1-\frac{1}{\sqrt{e}}} - \left(1 + \frac{1}{\sqrt{e}}\right)$$
$$= \frac{\sqrt{e}}{\sqrt{e}-1} - \frac{\sqrt{e}+1}{\sqrt{e}} = \frac{1}{\sqrt{e}\,(\sqrt{e}-1)}.$$

■

Example 24.

Find the sum of the series

$$\sum_{n=1}^{\infty} e^{3n}.$$

Solution. It is a geometric series with $q = e^3$, i.e., $q > 1$. Therefore, it is a divergent series with the sum $+\infty$. We can also estimate

$$\sum_{n=1}^{\infty} e^{3n} \geq \sum_{n=1}^{\infty} 1 = +\infty.$$

Hence, $\sum_{n=1}^{\infty} e^{3n}$ diverges to $+\infty$. ■

Example 25.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \ln^n x.$$

If it converges, find its sum s.

Solution. It is a geometric series with $q = \ln x$. Hence, it converges whenever

$$|\ln x| < 1, \qquad \text{i.e.,} \qquad x \in \left(\frac{1}{e}, e\right).$$

The sum is

$$s = \frac{\ln x}{1 - \ln x}, \qquad x \in \left(\frac{1}{\mathrm{e}}, \mathrm{e}\right).$$

■

Example 26.

Consider the series

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{1}{5^4} + \frac{2}{5^5} + \frac{3}{5^6} + \cdots$$

Find its sum.

Solution. Since all terms are non-negative, the reordering of them does not change the result. Hence, we rearrange them as follows

$$\frac{1}{5} + \frac{1}{5^4} + \frac{1}{5^7} + \cdots + \frac{2}{5^2} + \frac{2}{5^5} + \frac{2}{5^8} + \cdots + \frac{3}{5^3} + \frac{3}{5^6} + \frac{3}{5^9} + \cdots$$

We obtain three geometric series, each of them with the common ratio $q = 1/5^3$, but with different first terms. Hence,

$$s_1 = \frac{\frac{1}{5}}{1 - \frac{1}{5^3}} = \frac{1}{5} \cdot \frac{5^3}{124} = \frac{25}{124},$$

$$s_2 = \frac{\frac{2}{5^2}}{1 - \frac{1}{5^3}} = \frac{2}{5^2} \cdot \frac{5^3}{124} = \frac{10}{124},$$

$$s_3 = \frac{\frac{3}{5^3}}{1 - \frac{1}{5^3}} = \frac{3}{5^3} \cdot \frac{5^3}{124} = \frac{3}{124}.$$

The final sum is

$$s = s_1 + s_2 + s_3 = \frac{25}{124} + \frac{10}{124} + \frac{3}{124} = \frac{38}{124} = \frac{19}{62}.$$

■

Example 27.

Find out for which $x \in \mathbb{R}$ the identity

$$\ln x + \ln \sqrt{x} + \ln \sqrt[4]{x} + \cdots = 2$$

holds.

Solution. Directly, we get

$$\begin{aligned} \ln x + \frac{1}{2}\ln x + \frac{1}{4}\ln x + \cdots &= 2, \\ \ln x\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) &= 2, \\ \ln x \sum_{n=0}^{\infty} \frac{1}{2^n} &= 2, \\ 2\ln x &= 2, \\ \ln x &= 1, \\ x &= \mathrm{e}\,. \end{aligned}$$

We have used a geometric series with the common ratio 1/2 and the first term 1. ∎

Theorem 2.

Let $c \in \mathbb{R}$ and

$$\sum_{n=0}^{\infty} a_n = A, \qquad \sum_{n=0}^{\infty} b_n = B.$$

It holds:

(i)

$$\sum_{n=0}^{\infty} c \cdot a_n = c\sum_{n=0}^{\infty} a_n = c \cdot A;$$

(ii)

$$\sum_{n=0}^{\infty} (a_n \pm b_n) = \sum_{n=0}^{\infty} a_n \pm \sum_{n=0}^{\infty} b_n = A \pm B,$$

whenever the terms on the right-hand side are reasonable, i.e., unless $0 \cdot \infty$, $\infty - \infty$.

Example 28.

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{3 + (-1)^n}{(-3)^n}.$$

Find its sum.

Solution. We split the given series into two geometric series

$$\sum_{n=1}^{\infty} \frac{3 + (-1)^n}{(-3)^n} = \sum_{n=1}^{\infty} \left(\frac{3}{(-3)^n} + \frac{(-1)^n}{(-3)^n}\right) = 3\sum_{n=1}^{\infty} \left(-\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.$$

For both of them, we have $|q| < 1$. Thus,

$$3 \cdot \frac{-\frac{1}{3}}{1+\frac{1}{3}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} = -\frac{3}{4} + \frac{1}{2} = -\frac{1}{4}.$$

■

Example 29.

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{5^n}.$$

Solution. It is a geometric series with $q = 1/5$. Since $|q| < 1$, we obtain

$$s = \frac{a}{1-q} = \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4}.$$

■

Example 30.

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{9 \cdot 3^n - 2^{n+1}}{6^n}.$$

Solution. We split the given series into two geometric series and we get

$$\sum_{n=0}^{\infty} \frac{9 \cdot 3^n - 2^{n+1}}{6^n} = \sum_{n=0}^{\infty} \frac{9 \cdot 3^n}{6^n} - \sum_{n=0}^{\infty} \frac{2^{n+1}}{6^n} = 9 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 2 \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n.$$

These series are geometric series with $a = 1$ and $q = 1/2$, $q = 1/3$, respectively. In both cases, we have $|q| < 1$. Hence, the result is

$$9 \frac{1}{1-\frac{1}{2}} - 2 \frac{1}{1-\frac{1}{3}} = 15.$$

■

Example 31.

Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{2n+5}}.$$

Solution. We have

$$\sum_{n=2}^{\infty} \frac{3^{n+1}}{2^{2n+5}} = \sum_{n=2}^{\infty} \frac{3 \cdot 3^n}{2^5 \cdot 4^n} = \frac{3}{2^5} \cdot \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n.$$

We have obtained a geometric series with $q = 3/4 \in (-1, 1)$. Hence,

$$\frac{3}{2^5} \cdot \sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{2^5} \cdot \frac{9}{16} \cdot \frac{1}{1 - \frac{3}{4}} = \frac{3}{2^5} \cdot \frac{9}{16} \cdot 4 = \frac{27}{128}.$$

■

Example 32.

Find the sum of the series

$$\sum_{n=1}^{\infty} \left(\frac{5}{3^{2n+1}} + \frac{2^n}{6^{2n}} \right).$$

Solution. We have

$$\sum_{n=1}^{\infty} \left(\frac{5}{3^{2n+1}} + \frac{2^n}{6^{2n}} \right) = \frac{5}{3} \sum_{n=1}^{\infty} \frac{1}{9^n} + \sum_{n=1}^{\infty} \frac{2^n}{(6^2)^n} = \frac{5}{3} \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)^n + \sum_{n=1}^{\infty} \left(\frac{1}{2 \cdot 3^2}\right)^n$$

$$= \frac{5}{3} \cdot \frac{\frac{1}{9}}{1 - \frac{1}{9}} + \frac{\frac{1}{18}}{1 - \frac{1}{18}} = \frac{5}{24} + \frac{1}{17} = \frac{109}{408}.$$

■

Example 33.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{4^{2n+1} + 3^{n-2}}{12^{2n}}.$$

Solution. We have

$$\sum_{n=1}^{\infty} \frac{4^{2n+1} + 3^{n-2}}{12^{2n}} = 4 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{2n} + \frac{1}{3^2} \sum_{n=1}^{\infty} \left(\frac{1}{3 \cdot 4^2}\right)^n$$

$$= \frac{4}{9} \cdot \frac{9}{8} + \frac{1}{9 \cdot 48} \cdot \frac{48}{47} = \frac{4}{8} + \frac{1}{423} = \frac{425}{846},$$

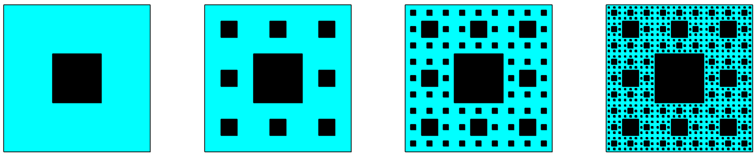
where one can see two geometric series with

$$q = \frac{1}{9},\ a = \frac{1}{9} \qquad \text{and} \qquad q = \frac{1}{48},\ a = \frac{1}{48}.$$

■

Example 34.

Consider the so-called Sierpiński carpet, i.e., consider the square with the sides of the length 1 unit which is divided into 9 smaller squares from whose the central one is removed. The process continues "infinitely many times" on every remaining square (see the picture, black parts are the removed ones). What is the area S of the Sierpiński carpet?



Solution. The initial area of the first square is 1. In the first step, we remove the area 1/9. In the second step, we remove the area $1/9^2$ from each of 8 remaining squares. In the third step, we remove the area $1/9^3$ from 8^2 remaining squares, etc. The final area is

$$S = 1 - \sum_{n=1}^{\infty} \frac{8^{n-1}}{9^n} = 1 - \frac{1}{8} \sum_{n=1}^{\infty} \left(\frac{8}{9}\right)^n = 1 - \frac{1}{8} \cdot \frac{\frac{8}{9}}{1 - \frac{8}{9}} = 1 - 1 = 0.$$

■

Example 35.

From the height of $A > 0$ metres above a flat surface, we release a ball. Each time the ball bounces from the height h, it goes to the height bh, where $b \in (0, 1)$. Find the total length of the trajectory s of the ball if it bounces infinitely many times.

Solution. The ball starts at the height $A > 0$. Hence, it goes A metres before the first bounce. Then, it goes to bA metres and back, i.e., $A + 2bA$ metres. The process is repeated. The total length of its trajectory is

$$s = A + 2bA + 2b^2A + 2b^3A + \cdots = A + \sum_{n=1}^{\infty} 2Ab^n.$$

It is a geometric series with $a = 2bA$ and $q = b$, i.e.,

$$s = A + \frac{2bA}{1-b} \text{ m} = A\,\frac{1 - b + 2b}{1-b} \text{ m} = A\,\frac{1+b}{1-b} \text{ m}.$$

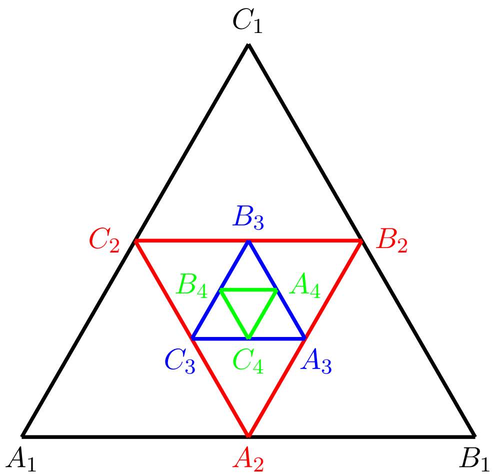
■

Example 36.

Consider the equilateral triangle $A_1\,B_1\,C_1$ with the sides of the length 4 (units). We construct the second triangle $A_2\,B_2\,C_2$ by joining the centres of the sides of the first one. We will proceed further in this manner and we obtain infinitely many triangles (see Pic. 1.1). Find:

(a) the sum of their circumferences;

(b) the sum of their areas.



Pic. 1.1: Sequence of triangles

Solution. The length of all subsequent triangle's side equals the half of the length of the side of the previous triangle, i.e., if the length of each side in $A_1\,B_1\,C_1$ is 4, then the length of the sides in $A_2\,B_2\,C_2$ is 2, etc.

(a) The circumference of $A_1\,B_1\,C_1$ is $4+4+4=12$, the circumference of $A_2\,B_2\,C_2$ is $2+2+2=6$, the circumference of $A_3\,B_3\,C_3$ is $1+1+1=3$, etc. This observation leads to the series
$$12+6+3+\frac{3}{2}+\cdots,$$
which is a geometric series with $a=12$ and $q=1/2<1$. Hence, the sum of all circumferences is
$$s=\frac{a}{1-q}=\frac{12}{1-\frac{1}{2}}=\frac{12}{\frac{1}{2}}=24.$$

(b) The area S of an equilateral triangle is given by
$$S=\frac{a\cdot v_a}{2}=\frac{a\cdot a\,\frac{\sqrt{3}}{2}}{2}=a^2\,\frac{\sqrt{3}}{4},$$

where a is the length of its side and v_a is its height. Therefore, the areas of the given triangles are

$$S_{A_1B_1C_1} = 4^2\,\frac{\sqrt{3}}{4} = 4\sqrt{3},$$
$$S_{A_2B_2C_2} = 2^2\,\frac{\sqrt{3}}{4} = \sqrt{3},$$
$$S_{A_3B_3C_3} = 1^2\,\frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{4},$$
$$\vdots$$

These numbers form the infinite geometric series

$$4\sqrt{3} + \sqrt{3} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{16} + \cdots,$$

where $a = 4\sqrt{3}$ and $q = 1/4 < 1$. Hence, the resulting area is

$$s = \frac{a}{1-q} = \frac{4\sqrt{3}}{1-\frac{1}{4}} = \frac{4\sqrt{3}}{\frac{3}{4}} = 16\,\frac{\sqrt{3}}{3}.$$

■

Example 37.

Calculate the length d of the "infinite" polygonal chain given by the line segments A_1A_2, A_2A_3, A_3A_4, A_4A_5, …, where $A_1 = [1; 0]$, $A_2 = [1; 1]$, $A_3 = [0; 1]$, $A_4 = [0; 1/2]$, $A_5 = [1/2; 1/2]$, $A_6 = [1/2; 3/4]$, $A_7 = [1/4; 3/4]$, … See Pic. 1.2.

Solution. The lengths of the line segments A_iA_{i+1} (for $i \in \mathbb{N}$) are

$$|A_1A_2| = 1; \qquad |A_2A_3| = 1; \qquad |A_3A_4| = \frac{1}{2}; \qquad |A_4A_5| = \frac{1}{2};$$
$$|A_5A_6| = \frac{1}{4}; \qquad |A_6A_7| = \frac{1}{4}; \qquad |A_7A_8| = \frac{1}{8}; \qquad |A_8A_9| = \frac{1}{8};$$
$$\vdots$$

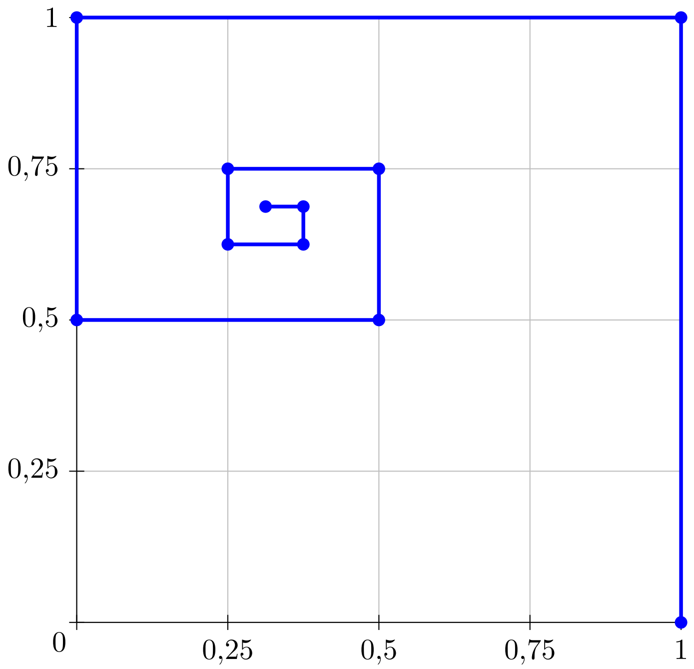
Therefore, d is the sum

$$d = 2\cdot 1 + 2\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 2\cdot\frac{1}{8} + \cdots,$$

which is a geometric series with $q = 1/2 < 1$ and $a = 2$. The sum of this series is

$$s = \sum_{n=0}^{\infty} 2\cdot\left(\frac{1}{2}\right)^n = \frac{2}{1-\frac{1}{2}} = 4.$$

■



Pic. 1.2: Polygonal chain

Example 38.

An “infinite” spiral consists of semicircles. The radius of the first semicircle is 6 cm, the radius of the next subsequent semicircle is a third less than the previous one. Calculate the total length of the spiral (see Pic. 1.3).

Solution. The length of a semicircle is given by $d = \pi \cdot r$, where r is its radius. The length of the first semicircle is $d_1 = \pi \cdot 6$. For the second one, we have

$$d_2 = \pi \cdot \frac{2}{3} \cdot 6 = \frac{12}{3}\pi.$$

For the third one, we can see

$$d_3 = \pi \cdot \frac{2}{3} \cdot \frac{12}{3} = \frac{24}{9}\pi,$$

for the fourth one,

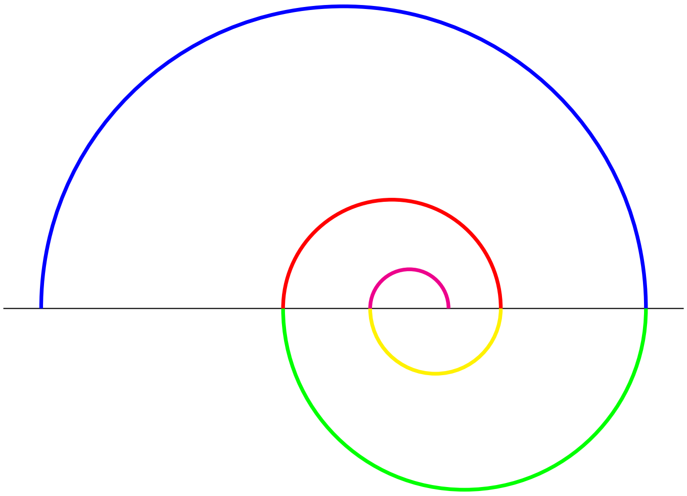
$$d_4 = \pi \cdot \frac{2}{3} \cdot \frac{24}{9} = \frac{48}{27}\pi,$$

for the fifth one,

$$d_5 = \pi \cdot \frac{2}{3} \cdot \frac{48}{27} = \frac{96}{81}\pi,$$

etc. The total length of the spiral is

$$d = d_1 + d_2 + d_3 + \cdots = 6\pi + \frac{12}{3}\pi + \frac{24}{9}\pi + \frac{48}{27}\pi + \frac{96}{81}\pi + \cdots$$



Pic. 1.3: Infinite spiral

It is a geometric series with $a = 6\pi$ and $q = 2/3 < 1$. Hence, the result is

$$d = 6\pi \cdot \frac{1}{1 - \frac{2}{3}} = \frac{6\pi}{\frac{1}{3}} = 18\pi.$$

■

Example 39.

Show that the infinite decimal $0.999\,9\ldots$ is equal to 1.

Solution. It holds

$$0.999\,9\cdots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = 1.$$

■

Example 40.

Rewrite $0.\overline{15}$ as a simple fraction.

Solution. We have

$$0.\overline{15} = 0.15 + 0.001\,5 + 0.000\,015 + \cdots = s,$$

which is a geometric series with $a = 0.15$ and $q = 0.01 < 1$. Calculating the sum, we obtain

$$s = \frac{a}{1-q} = \frac{0.15}{1-0.01} = \frac{0.15}{0.99} = \frac{15}{99} = \frac{5}{33}.$$

■

Example 41.

Rewrite $4.4\overline{62}$ as a simple fraction.

Solution. At first, we consider the periodic part of the given number. We have

$$0.0\overline{62} = 0.062 + 0.00062 + 0.0000062 + \cdots$$

It is a geometric series with $a = 0.062$ and $q = 0.01 < 1$. Its sum is

$$\frac{a}{1-q} = \frac{0.062}{1-0.01} = \frac{0.062}{0.99} = \frac{62}{990}.$$

We add the non-periodic part with the result

$$4.4\overline{62} = \frac{44}{10} + \frac{62}{990} = \frac{2209}{495}.$$

■

Example 42.

Evaluate the infinite repeating decimal $a = 0.215626262\ldots$ (rewrite it as a simple fraction).

Solution. Obviously,

$$a = 0.215 + \frac{62}{10^5} + \frac{62}{10^7} + \frac{62}{10^9} + \cdots$$

Since

$$\frac{62}{10^5} + \frac{62}{10^7} + \frac{62}{10^9} + \cdots = \frac{\frac{62}{10^5}}{1 - \frac{1}{10^2}} = \frac{62}{99000},$$

we obtain

$$a = \frac{215}{1000} + \frac{62}{99000} = \frac{21347}{99000}.$$

■

Example 43.

Find $x \in \mathbb{R}$ for which the series

$$\sum_{n=0}^{\infty} 2\,(3x)^n$$

converges. If it is possible, find its sum.

Solution. It is a geometric series with $a = 2$ and $q = 3x$. We need $|q| < 1$, i.e.,

$$|3x| < 1, \qquad \text{i.e.,} \qquad |x| < \frac{1}{3}.$$

Hence, for

$$x \in \left(-\frac{1}{3}, \frac{1}{3}\right),$$

the series $\sum_{n=0}^{\infty} 2\,(3x)^n$ converges with the sum

$$s = \frac{a}{1-q} = \frac{2}{1-3x}.$$

■

Example 44.

Find $x \in \mathbb{R}$ for which the series

$$\sum_{n=0}^{\infty} 3^n x$$

converges. If it is possible, find its sum.

Solution. We have $a = x$ and $q = 3$. We need $|q| < 1$, which is impossible here. Therefore, the series converges only for $x = 0$ with the sum 0. ■

Example 45.

Find the value of $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty} (4-3x)^n$$

has the sum

$$s = -\frac{1}{2x}.$$

Solution. The sum of a geometric series is

$$s = \frac{a}{1-q}.$$

It leads to

$$\begin{aligned} -\frac{1}{2x} &= \frac{4-3x}{1-(4-3x)}, \\ -\frac{1}{2x} &= \frac{4-3x}{3x-3}, \\ -(3x-3) &= 2x(4-3x), \\ 3-3x &= 8x-6x^2, \\ 6x^2-11x+3 &= 0. \end{aligned}$$

Therefore,

$$\left(x-\frac{3}{2}\right)\left(x-\frac{1}{3}\right) = 0, \qquad \text{i.e.,} \qquad x_1 = \frac{3}{2},\, x_2 = \frac{1}{3}.$$

The common ratio q has to satisfy $|q| < 1$. We check both of the obtained values

$$\begin{aligned} x &= \frac{3}{2}: \qquad q = 4 - 3\cdot\frac{3}{2} = -\frac{1}{2}, \\ x &= \frac{1}{3}: \qquad q = 4 - 3\cdot\frac{1}{3} = 3. \end{aligned}$$

The only possible result is $x = 3/2$. ■

Example 46.

Solve the equation

$$\sin x + \sin^2 x + \sin^3 x + \cdots = -\frac{1}{3}.$$

Solution. The left-hand side is a geometric series with $q = \sin x$. The common ratio has to be $|q| < 1$. We easily obtain

$$\begin{aligned} \frac{\sin x}{1-\sin x} &= -\frac{1}{3}, \\ 3\sin x &= \sin x - 1, \\ 2\sin x &= -1, \\ \sin x &= -\frac{1}{2}, \end{aligned}$$

which leads to the result

$$x_1 = \frac{7\pi}{6} + 2k\pi; \qquad x_2 = \frac{11\pi}{6} + 2k\pi, \qquad k \in \mathbb{Z}.$$

■

Example 47.

Solve the equation

$$4x + x + \frac{x}{4} + \cdots = 32.$$

Solution. The left-hand side is a geometric series with $a = 4x$ and $q = 1/4 < 1$. Its sum is

$$s = \frac{a}{1-q} = \frac{4x}{1-\frac{1}{4}} = \frac{4x}{\frac{3}{4}} = \frac{16}{3}\,x.$$

Replacing the left-hand side by this sum, we obtain

$$\frac{16}{3}\,x = 32.$$

The result is $x = 6$. ∎

Example 48.

Solve the equation

$$2 - x + \frac{x^2}{2} - \frac{x^3}{4} + \cdots = \frac{4}{3}\,x.$$

Solution. The left-hand side is a geometric series with $a = 2$ and $q = -x/2$. It converges for

$$\left|-\frac{x}{2}\right| < 1, \qquad \text{i.e.,} \qquad x \in (-2, 2).$$

For such x, the sum is

$$s = \frac{a}{1-q} = \frac{2}{1-\left(-\frac{x}{2}\right)} = \frac{2}{1+\frac{x}{2}} = \frac{2}{\frac{2+x}{2}} = \frac{4}{2+x}.$$

Replacing the left-hand side by this sum, we obtain

$$\frac{4}{2+x} = \frac{4}{3}\,x.$$

Immediately, we get the equation

$$x^2 + 2x - 3 = 0, \qquad \text{i.e.,} \qquad (x+3)\,(x-1) = 0,$$

with the solutions $x_1 = -3$ and $x_2 = 1$. Since $x_1 \notin (-2, 2)$, the only result is $x = 1$. ∎

Example 49.

Solve the equation

$$(x-2) + (x-2)^2 + (x-2)^3 + \cdots = 1.$$

Solution. The left-hand side is a geometric series with $a = x - 2$ and the common ratio is also $q = x - 2$. For $|q| < 1$, the series converges. Hence, for $x \in (1, 3)$, we have the sum

$$s = \frac{a}{1-q} = \frac{x-2}{1-(x-2)} = \frac{x-2}{3-x}.$$

Replacing the left-hand side by this sum, for $x \in (1, 3)$, we obtain

$$\frac{x-2}{3-x} = 1, \qquad \text{i.e.,} \qquad x - 2 = 3 - x.$$

Therefore, the only solution is $x = 5/2$. ■

Example 50.

Solve the equation

$$\ln x + \ln \sqrt[3]{x} + \ln \sqrt[9]{x} + \ln \sqrt[27]{x} + \cdots = \frac{3}{2}.$$

Solution. We have the equation

$$\sum_{n=0}^{\infty} \frac{1}{3^n} \ln x = \frac{3}{2}.$$

Considering the sum of the geometric series with $q = 1/3$, we have

$$\frac{1}{1-\frac{1}{3}} \ln x = \frac{3}{2}.$$

Hence,

$$\ln x = 1, \qquad \text{i.e.,} \qquad x = \mathrm{e}.$$

■

Example 51.

Simplify

$$2 \cdot \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdots$$

Solution. We have

$$2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}.$$

Using the geometric series $1 + 1/2 + 1/4 + 1/8 + \cdots$ with $a = 1$, $q = 1/2 < 1$, we get

$$s = \frac{a}{1-q} = \frac{1}{1-\frac{1}{2}} = 2.$$

The solution is

$$2 \cdot \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdots = 2 \cdot 2^{\frac{1}{2}} \cdot 2^{\frac{1}{4}} \cdot 2^{\frac{1}{8}} \cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = 2^2 = 4.$$

■

Example 52.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty}(-1)^n \left(\frac{5}{7}\right)^n.$$

Solution. We have $\sum_{n=1}^{\infty}\left(-\frac{5}{7}\right)^n$, which is a geometric series with $a = -5/7$ and $q = -5/7$. Since $|q| < 1$, we obtain

$$\sum_{n=1}^{\infty}(-1)^n \left(\frac{5}{7}\right)^n = \frac{-\frac{5}{7}}{1+\frac{5}{7}} = -\frac{5}{12}.$$

■

Example 53.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty}\frac{2n-1}{2^n}.$$

Solution. This is not a geometric series. Thus, we use directly the definition, i.e., the limit of the partial sums s_n as $n \to \infty$. We have

$$s_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \frac{9}{2^5} + \cdots + \frac{2(n-1)-1}{2^{n-1}} + \frac{2n-1}{2^n}, \qquad n \in \mathbb{N}.$$

We divide this identity by 2 and subtract it from s_n. For $n \in \mathbb{N}$, we obtain

$$\begin{aligned} s_n &= \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{7}{2^4} + \frac{9}{2^5} + \cdots + \frac{2n-3}{2^{n-1}} + \frac{2n-1}{2^n}, \\ \frac{s_n}{2} &= \frac{1}{2^2} + \frac{3}{2^3} + \frac{5}{2^4} + \frac{7}{2^5} + \cdots + \frac{2n-5}{2^{n-1}} + \frac{2n-3}{2^n} + \frac{2n-1}{2^{n+1}}, \\ \frac{s_n}{2} &= \frac{1}{2} + \frac{2}{2^2} + \frac{2}{2^3} + \frac{2}{2^4} + \frac{2}{2^5} + \cdots + \frac{2}{2^{n-1}} + \frac{2}{2^n} - \frac{2n-1}{2^{n+1}}. \end{aligned}$$

Further, we multiple both sides by 2 which yields

$$s_n = 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-3}} + \frac{1}{2^{n-2}} - \frac{4n-2}{2^{n+1}}, \qquad n \in \mathbb{N},$$

and we find the limit of the partial sums

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n-2}} - \frac{2n-1}{2^n}\right) = 1 + \sum_{n=0}^{\infty}\frac{1}{2^n}.$$

Now, the right-hand side contains a geometric series with $|q| < 1$. Therefore,

$$\lim_{n\to\infty} s_n = 1 + \frac{1}{1-\frac{1}{2}} = 3,$$

which is the desired sum.

We remark that, for $a > 1$, any series of the form $\sum_{n=N}^{\infty} \frac{n}{a^n}$ is called the arithmetic–geometric series. To find its sum, we can apply the procedure above (we divide the partial sums with the considered a). ■

Example 54.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2\,(n+1)^2}.$$

Solution. By the partial fraction decomposition (expansion), we get

$$\frac{2n+1}{n^2\,(n+1)^2} = \frac{A}{n^2} + \frac{B}{n} + \frac{C}{(n+1)^2} + \frac{D}{n+1}$$

with real coefficients A, B, C, D. It holds

$$\begin{aligned}
2n+1 &= A(n+1)^2 + Bn(n+1)^2 + Cn^2 + Dn^2(n+1),\\
2n+1 &= An^2 + 2An + A + Bn^3 + 2Bn^2 + Bn + Cn^2 + Dn^3 + Dn^2,\\
2n+1 &= n^3(B+D) + n^2(A+2B+C+D) + n(2A+B) + A.
\end{aligned}$$

Hence,

$$\begin{aligned}
n^3 &: \quad 0 = B + D,\\
n^2 &: \quad 0 = A + 2B + C + D,\\
n^1 &: \quad 2 = 2A + B,\\
n^0 &: \quad 1 = A.
\end{aligned}$$

Finally,

$$A = 1, \qquad B = 0, \qquad C = -1, \qquad D = 0.$$

Therefore, we obtain

$$\sum_{n=1}^{\infty} \frac{2n+1}{n^2\,(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right).$$

The limit of the partial sums is

$$\begin{aligned}
\lim_{n\to\infty} s_n = \lim_{n\to\infty} \Bigg[&\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \left(\frac{1}{4^2} - \frac{1}{5^2}\right)\\
&+ \cdots + \left(\frac{1}{(n-1)^2} - \frac{1}{n^2}\right) + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)\Bigg].
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{(n+1)^2}\right) = 1.$$

The desired sum is $\sum_{n=1}^{\infty} \frac{2n+1}{n^2\,(n+1)^2} = 1$. ■

Example 55.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n-2)}.$$

Solution. By the partial fraction decomposition, we get

$$\frac{1}{(3n+1)(3n-2)} = \frac{A}{3n+1} + \frac{B}{3n-2}$$

with real coefficients A, B. We obtain

$$1 = 3An - 2A + 3Bn + B$$

and, consequently,

$$\begin{aligned} 1 &= B - 2A, \\ 0 &= 3A + 3B, \end{aligned}$$

which implies $A = -1/3, B = 1/3$. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n-2)} = \sum_{n=1}^{\infty}\left(\frac{-1/3}{3n+1} + \frac{1/3}{3n-2}\right) = -\frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{3n+1} - \frac{1}{3n-2}\right).$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned} s_n = -\frac{1}{3}\Bigg[&\left(\frac{1}{4} - \frac{1}{1}\right) + \left(\frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{10} - \frac{1}{7}\right) + \left(\frac{1}{13} - \frac{1}{10}\right) \\ &+ \left(\frac{1}{16} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{3(n-2)+1} - \frac{1}{3(n-2)-2}\right) \\ &+ \left(\frac{1}{3(n-1)+1} - \frac{1}{3(n-1)-2}\right) + \left(\frac{1}{3n+1} - \frac{1}{3n-2}\right)\Bigg] \end{aligned}$$

and

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} -\frac{1}{3}\left[-1 + \frac{1}{3n+1}\right] = \frac{1}{3}.$$

The sum of $\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n-2)}$ is $1/3$. ■

Example 56.

Find the sum of the infinite series
$$\sum_{n=1}^{\infty} \frac{n}{5^n}.$$

Solution. It is an arithmetic–geometric series. Thus, we can proceed as in Example 53. The partial sum
$$s_n = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \cdots + \frac{n-1}{5^{n-1}} + \frac{n}{5^n}, \qquad n \in \mathbb{N},$$
is divided by 5 which gives
$$\frac{s_n}{5} = \frac{1}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots + \frac{n-2}{5^{n-1}} + \frac{n-1}{5^n} + \frac{n}{5^{n+1}}, \qquad n \in \mathbb{N}.$$
These two identities are subtracted. We obtain
$$\frac{4}{5}\, s_n = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots + \frac{1}{5^{n-1}} + \frac{1}{5^n} - \frac{n}{5^{n+1}}, \qquad n \in \mathbb{N}.$$
The limit of the partial sums is
$$\begin{aligned}\lim_{n\to\infty} s_n &= \lim_{n\to\infty} \frac{5}{4}\left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots + \frac{1}{5^{n-1}} + \frac{1}{5^n} - \frac{n}{5^{n+1}}\right) \\ &= \frac{5}{4}\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{5}{4} \cdot \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{5}{4} \cdot \frac{1}{4} = \frac{5}{16}.\end{aligned}$$

■

Example 57.

If it is possible, find the sum of the infinite series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}.$$

Solution. We find the limit of the partial sums. By the partial fraction expansion, we have
$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \sum_{n=1}^{\infty} \frac{1/2}{n} - \frac{1/2}{n+2}.$$
The partial sums are
$$\begin{aligned}s_n &= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \cdots \right. \\ &\qquad \left. + \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right), \qquad n \in \mathbb{N}.\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n} = \lim_{n\to\infty} s_n = \frac{3}{4}.$$

■

Example 58.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots$$

Solution. By the partial fraction decomposition, we get

$$\frac{1}{n\,(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, we have the partial sum

$$\begin{aligned} s_{n+1} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)\,(n+2)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= 1 - \left(\frac{1}{2} - \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{4}\right) - \cdots - \left(\frac{1}{n+1} - \frac{1}{n+1}\right) - \frac{1}{n+2}, \end{aligned}$$

i.e.,

$$s_{n+1} = 1 - \frac{1}{n+2}, \qquad n \in \mathbb{N}.$$

The given series converges with the sum

$$\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right) = 1.$$

■

Example 59.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)}.$$

Solution. We can see

$$\frac{2n+1}{n(n+1)} = \frac{1}{n} + \frac{1}{n+1}, \qquad n \in \mathbb{N}.$$

For $n \in \mathbb{N}$, the partial sums are

$$s_n = \left(\frac{1}{1} + \frac{1}{2}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{4} + \frac{1}{5}\right) + \cdots + (-1)^{n+1}\left(\frac{1}{n} + \frac{1}{n+1}\right),$$

i.e.,

$$s_n = 1 - \frac{1}{n+1} \qquad \text{or} \qquad s_n = 1 + \frac{1}{n+1}.$$

The both cases imply that the sequence of the partial sums converges to 1, i.e., we have

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2n+1}{n(n+1)} = 1.$$

■

Theorem 3.

If the series $\sum_{n=0}^{\infty} a_n$ converges, then

$$\lim_{n \to \infty} a_n = 0. \tag{1.1}$$

The condition (1.1) is necessary, not sufficient. There exist series that do not converge, but the condition (1.1) is fulfilled for them. A basic example of such a series follows.

Example 60.

Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

It is the so-called harmonic series.

Solution. We change the values $a_n = 1/n$ as follows. The terms in the form $1/2^i, i \in \mathbb{N} \cup \{0\}$, remain unchanged. All other terms are changed—the denominator is replaced by the nearest higher power of 2. We obtain

$$\begin{aligned}
&1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots + \frac{1}{16} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + 8 \cdot \frac{1}{16} + \cdots
\end{aligned}$$

From the third term, the partial sums of the new series are lower than the partial sums of the harmonic series. Hence,

$$s_{2^n} \geq 1 + n \cdot \frac{1}{2} \to +\infty \qquad \text{as} \qquad n \to \infty,$$

i.e., the harmonic series is divergent with the sum $+\infty$. ∎

Example 61.

Justify that the series

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+1}}$$

does not converge.

Solution. The series does not meet the necessary condition for the convergence (see (1.1)), because

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1.$$

∎

Example 62.

Let $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Find the sum of the series

$$\sum_{n=1}^{\infty} (a_{n+1} - a_n).$$

Solution. For $n \in \mathbb{N}$, the partial sums are

$$s_n = \sum_{j=1}^{n} (a_{j+1} - a_j) = (a_2 - a_1) + (a_3 - a_2) + (a_4 - a_3) + \cdots + (a_{n+1} - a_n).$$

Therefore,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} (a_{n+1} - a_1) = a - a_1,$$

which is the desired sum. ∎

§ 1.2 Series with non-negative terms

We consider $\sum_{n=N}^{\infty} a_n$ for $a_n \geq 0$, $n \in \{N, N+1, N+2, \dots\}$. We observe that the sequence of the partial sums is non-decreasing, because

$$s_n = a_N + \cdots + a_n, \qquad s_{n+1} = a_N + \cdots + a_n + a_{n+1} = s_n + a_{n+1} \geq s_n, \qquad n \geq N,\, n \in \mathbb{N}.$$

Hence, any series with non-negative terms is convergent or it diverges to $+\infty$ (it cannot diverge to $-\infty$ or oscillate).

Example 63.

Does the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}$$

converge?

Solution. We consider

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}.$$

It holds

$$\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1 - \frac{1}{2}} = 1 + 2 = 3.$$

Hence, the given series is convergent (with a sum less than 3). ■

Example 64.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution. For $n \geq 2$, we have

$$\frac{1}{n^2} < \frac{1}{n(n-1)}$$

and, for large $m \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=2}^{m} \frac{1}{n(n-1)} &= \sum_{n=2}^{m} \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-2} - \frac{1}{m-1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\ &= 1 - \frac{1}{m}.\end{aligned}$$

Therefore,

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{m\to\infty} \sum_{n=2}^{m} \frac{1}{n(n-1)} = \lim_{m\to\infty} \left(1 - \frac{1}{m}\right) = 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < 1 + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 2.$$

The given series is convergent (with a sum from the interval $(0, 2)$). ∎

Example 65.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{7n+3}.$$

Solution. Since

$$\lim_{n\to\infty} \frac{n}{7n+3} = \frac{1}{7} \neq 0,$$

the series diverges to $+\infty$. ∎

Example 66.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mathrm{e}^n}{n}.$$

Solution. Since

$$\lim_{n\to\infty} \frac{\mathrm{e}^n}{n} = \lim_{n\to\infty} \frac{\mathrm{e}^n}{1} = +\infty,$$

the given series diverges to $+\infty$. ∎

Theorem 4.

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$$

converges if and only if $\alpha > 1$.

Example 67.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n+4}.$$

Solution. Putting $m = n + 4$, we obtain the so-called harmonic series without the first 4 terms, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n+4} = \sum_{m=5}^{\infty} \frac{1}{m} = \sum_{m=1}^{\infty} \frac{1}{m} - 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} = +\infty - \frac{25}{12} = +\infty.$$

■

Theorem 5 (Comparison test)**.**

Let $a_n, b_n \geq 0$ and let $a_n \leq b_n$ for all large n. It holds:

(i) if $\sum_{n=N}^{\infty} b_n < +\infty$, then $\sum_{n=N}^{\infty} a_n < +\infty$;

(ii) if $\sum_{n=N}^{\infty} a_n = +\infty$, then $\sum_{n=N}^{\infty} b_n = +\infty$.

Remark. Let $0 \leq a_n \leq b_n$ for all large n. The series $\sum_{n=N}^{\infty} b_n$ is called a majorant series to series $\sum_{n=N}^{\infty} a_n$ and the series $\sum_{n=N}^{\infty} a_n$ is called a minorant series to $\sum_{n=N}^{\infty} b_n$.

Example 68.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\ln(1234+n)}{n}.$$

Solution. For any $n \in \mathbb{N}$, it holds

$$0 < \frac{1}{n} \leq \frac{\ln(1234+n)}{n}$$

and the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Hence, according to the comparison test, the studied series diverges as well. ■

Example 69.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n+1}.$$

Solution. We use Theorem 4, where

$$\sum_{n=1}^{\infty} \frac{1}{4n+1} \geq \sum_{n=1}^{\infty} \frac{1}{5n} = \frac{1}{5} \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{4n+1}$ diverges. ■

Example 70.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2}.$$

Solution. We use Theorem 4 for

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The series on the right-hand side is convergent which implies that the studied series is convergent as well. ■

Example 71.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n + 7}.$$

Solution. We use Theorem 4, where

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - 3n + 7} \leq \sum_{n=1}^{\infty} \frac{1}{\frac{n^2}{2}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty,$$

which implies the convergence of the studied series. We add that

$$n^2 - 3n + 7 \geq \frac{n^2}{2}, \qquad n \in \mathbb{N},$$

which follows from the fact that $x^2 - 6x + 14$ has no real roots. ■

Example 72.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{4n^2 + 2n + 3}{4n^2 + 5}.$$

Solution. We use Theorem 4. We can see

$$\sum_{n=1}^{\infty} \frac{4n^2 + 2n + 3}{4n^2 + 5} \geq \sum_{n=1}^{\infty} \frac{4n^2 + 5}{4n^2 + 5} = \sum_{n=1}^{\infty} 1 = +\infty.$$

The divergence of the given series also follows from the fact that $\lim_{n\to\infty} a_n = 1$. ■

Example 73.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n+2}}.$$

Solution. We have

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)\sqrt{n+2}} \leq \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} n^{-\frac{3}{2}} < +\infty.$$

We recall that $\sum_{n=1}^{\infty} n^{\beta}$ is convergent for $\beta < -1$ (see Theorem 4). ■

Example 74.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right).$$

Solution. We use Theorem 4 and

$$\sum_{n=1}^{\infty}\left(\sqrt{n^2+1}-\sqrt{n^2-1}\right)=\sum_{n=1}^{\infty}\frac{n^2+1-(n^2-1)}{\sqrt{n^2+1}+\sqrt{n^2-1}}\geq\sum_{n=1}^{\infty}\frac{2}{\sqrt{4n^2}+\sqrt{n^2}}=\frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{n}.$$

The series on the right-hand side is divergent which implies that the studied series is divergent as well. ■

Example 75.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty}\left(\sqrt{n^3+3}-\sqrt{n^3+1}\right).$$

Solution. We use Theorem 4, where

$$\sum_{n=1}^{\infty}\left(\sqrt{n^3+3}-\sqrt{n^3+1}\right)=\sum_{n=1}^{\infty}\frac{n^3+3-\left(n^3+1\right)}{\sqrt{n^3+3}+\sqrt{n^3+1}}\leq\sum_{n=1}^{\infty}\frac{2}{\sqrt{n^3}}=2\sum_{n=1}^{\infty}n^{-\frac{3}{2}},$$

which implies that the studied series is convergent. ■

Example 76.

Using the comparison test, study the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\ln n}{n}.$$

Solution. We use Theorem 4 and the estimation

$$\sum_{n=1}^{\infty}\frac{\ln n}{n}\geq\sum_{n=3}^{\infty}\frac{1}{n},$$

which imply that the studied series is divergent. ■

Example 77.

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 3}$$

is convergent.

Solution. It holds

$$\frac{1}{2^n - 3} < \frac{1}{2^{n-1}} \qquad \text{for all}\, n \geq 3,\ n \in \mathbb{N},$$

because

$$2^{n-1} < 2^n - 3 \qquad \text{if and only if} \qquad 3 < 2^n - 2^{n-1} = 2^{n-1}.$$

Hence, beginning with the third term, the given series is (term by term) less than the convergent series

$$\sum_{n=3}^{\infty} \frac{1}{2^{n-1}}.$$

Therefore, the studied series converges. ■

Example 78.

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1}{(n^3 + 10)^{1/4}}$$

is convergent.

Solution. It holds

$$\frac{1}{(n^3 + 10)^{1/4}} \geq \frac{1}{\sqrt[4]{2n^3}}, \qquad n \geq 3,\ n \in \mathbb{N}.$$

Therefore,

$$\frac{1}{(n^3 + 10)^{1/4}} \geq \frac{1}{2^{\frac{1}{4}} n^{\frac{3}{4}}}, \qquad n \geq 3,\ n \in \mathbb{N}.$$

Thus, the given series is divergent by the comparison with

$$\frac{1}{2^{\frac{1}{4}}} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{4}}}.$$

■

Theorem 6 (Limit comparison test)**.**

Let $a_n \geq 0$, $b_n > 0$ for all large n and let there exist the limit

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}.$$

It holds:

(I) if $\sum_{n=N}^{\infty} b_n < +\infty$ and $L < +\infty$, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(II) if $\sum_{n=N}^{\infty} b_n = +\infty$ and $L > 0$, then the series $\sum_{n=N}^{\infty} a_n$ diverges.

Example 79.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \arctan \frac{1}{n}.$$

Solution. We use the limit comparison test to compare the studied series with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n},$$

which is divergent. We consider the limit

$$\lim_{n\to\infty} \frac{\arctan \frac{1}{n}}{\frac{1}{n}}.$$

Via the l'Hospital rule, we obtain

$$\lim_{x\to\infty} \frac{\arctan \frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x^2}} \cdot \frac{-1}{x^2}}{\frac{-1}{x^2}} = 1.$$

Hence,

$$\lim_{n\to\infty} \frac{\arctan \frac{1}{n}}{\frac{1}{n}} = 1,$$

which implies

$$\sum_{n=1}^{\infty} \arctan \frac{1}{n} = +\infty.$$

■

Example 80.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right).$$

Solution. We use the limit comparison test to compare the studied series with the divergent harmonic series. We consider the limit

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n}}.$$

Via the l'Hospital rule, we obtain

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x^2}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1+\frac{1}{x^2}} = 0.$$

Hence,

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n}} = 0.$$

The limit comparison test failed.

We try the test again. Now, we compare the given series with the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is convergent (see Theorem 4). We have the limit

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}}.$$

Via the l'Hospital rule, we obtain

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x^2}\right)}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x^2}} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1.$$

The calculation above could be done also using

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1.$$

Since $L \in (0, \infty)$, we have the result that the studied series is convergent. ■

Theorem 7 (Ratio (d'Alembert's) test)**.**

Let $a_n > 0$ for all large n. It holds:

(i) if

$$\frac{a_{n+1}}{a_n} \leq k < 1$$

for all large n and some constant k, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(ii) if

$$\frac{a_{n+1}}{a_n} \geq 1$$

for all large n, then the series $\sum_{n=N}^{\infty} a_n$ diverges to $+\infty$.

Moreover, let there exist the limit

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

It holds:

(I) if $L < 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(II) if $L > 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges to $+\infty$.

Example 81.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

Solution. We use the ratio test. It holds

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n > 1$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \mathrm{e} > 1.$$

The series is divergent. ∎

Example 82.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}.$$

Solution. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1.$$

Hence, the series is convergent. ■

Example 83.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}.$$

Solution. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{((n+1)!)^2}}{\frac{(2n)!}{(n!)^2}} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = \lim_{n\to\infty} \frac{4n^2}{n^2} = 4 > 1.$$

Thus, the series is divergent to $+\infty$. ■

Example 84.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}.$$

Solution. We have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)^{n+1}}{(2n+2)!}}{\frac{n^n}{(2n)!}}=\lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n\,(2n+1)\,(2n+2)}$$
$$=\lim_{n\to\infty}\frac{n+1}{(2n+1)(2n+2)}\cdot\left(\frac{n+1}{n}\right)^n$$
$$=\lim_{n\to\infty}\frac{n+1}{4n^2+6n+2}\left(1+\frac{1}{n}\right)^n=0\cdot\mathrm{e}=0<1,$$

i.e., the series converges. ■

Example 85.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty}\frac{3^n\,n!}{n^n}.$$

Solution. We have

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{3^{n+1}\,(n+1)!}{(n+1)^{n+1}}}{\frac{3^n\,n!}{n^n}}=\lim_{n\to\infty}\frac{3\,(n+1)\,n^n}{(n+1)^{n+1}}=3\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$
$$=3\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n}=3\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{-1}=3\,\mathrm{e}^{-1}>1,$$

i.e., the series diverges. ■

Example 86.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{1}{7^n}.$$

Solution. It holds

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\binom{2n+2}{n+1}\frac{1}{7^{n+1}}}{\binom{2n}{n}\frac{1}{7^n}}=\lim_{n\to\infty}\frac{(2n+2)\,(2n+1)}{7\,(n+1)\,(n+1)}=\frac{4}{7}<1,$$

i.e., the series converges. ■

Example 87.

Using the ratio test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}.$$

Solution. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{(n+2)!}}{\frac{n}{(n+1)!}} = \lim_{n\to\infty} \frac{n+1}{n\,(n+2)} = \lim_{n\to\infty} \frac{n}{n^2} = 0 < 1,$$

i.e., the series converges. ■

Theorem 8 (Root (Cauchy's) test)**.**

Let $a_n \geq 0$ for all large n. It holds:

(i) if

$$\sqrt[n]{a_n} \leq k < 1$$

for all large n and some constant k, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(ii) if

$$\sqrt[n]{a_n} \geq 1$$

for all large n, then the series $\sum_{n=N}^{\infty} a_n$ diverges to $+\infty$.

Moreover, let there exist the limit

$$L = \lim_{n\to\infty} \sqrt[n]{a_n}.$$

It holds:

(I) if $L < 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges.

(II) if $L > 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges to $+\infty$.

Example 88.

Via the root test, study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{(3+\frac{1}{n})^n}.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{(3+\frac{1}{n})^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{3+\frac{1}{n}} = \frac{1}{3} < 1.$$

The series converges. ■

Example 89.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{4n+3}\right)^n.$$

Solution. We apply the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{4n+3}\right)^n} = \lim_{n\to\infty} \frac{n+1}{4n+3} = \frac{1}{4} < 1.$$

Hence, the series is convergent. ■

Example 90.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n^2+1}\right)^n.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n+1}{n^2+1} = 0 < 1.$$

Therefore, the series is convergent. ■

Example 91.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^2}{2^n}} = \frac{\left(\lim_{n\to\infty} \sqrt[n]{n}\right)^2}{2} = \frac{1}{2} < 1.$$

Hence, the series is convergent. ■

Example 92.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{3^n}{n^3}.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{3^n}{n^3}} = \frac{3}{\lim_{n\to\infty} \sqrt[n]{n^3}} = \frac{3}{1^3} = 3.$$

Thus, the series is divergent. Note that the necessary condition for the convergence (see (1.1)) is not fulfilled. ■

Example 93.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1\right)^n.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\sqrt[n]{n} - 1\right)^n} = \lim_{n\to\infty} \sqrt[n]{n} - 1 = 0.$$

Hence, the series is convergent. ■

Example 94.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(n + \sqrt{n})^n}{(2n^2 + n)^{\frac{n}{2}}}.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{(n+\sqrt{n})^n}{(2n^2+n)^{\frac{n}{2}}}} = \lim_{n\to\infty} \frac{n+\sqrt{n}}{\sqrt{2n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{2}n} = \frac{1}{\sqrt{2}} < 1.$$

The series is convergent. ■

Example 95.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\arctan^n n}{2^n}.$$

Solution. We have

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{\arctan^n n}{2^n}} = \lim_{n\to\infty} \frac{\arctan n}{2} = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} < 1.$$

Thus, the series is convergent. ■

Example 96.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2+\sin n)^n}{2^n}.$$

Solution. For infinitely many $n \in \mathbb{N}$, it is seen that $a_n > 1$ ($\sqrt[n]{a_n} > 1$). The necessary condition for the convergence (see (1.1)) is not fulfilled. Hence, the series is divergent. ■

Example 97.

Study the convergence of the series

$$\frac{1}{2^1} + \frac{1}{2^2} + \frac{3}{2^3} + \frac{1}{2^4} + \frac{5}{2^5} + \frac{1}{2^6} + \frac{7}{2^7} + \cdots$$

Solution. At first, we try to use the ratio test. Indexing from 1 as $\sum_{n=1}^{\infty} a_n$, we obtain

$$a_n = \begin{cases} \frac{n}{2^n} & \text{for } n \text{ odd;} \\ \frac{1}{2^n} & \text{for } n \text{ even.} \end{cases}$$

Therefore,

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{\frac{1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{2^n}{2^{n+1} \cdot n} = \frac{1}{2n} & \text{for } n \text{ odd;} \\ \frac{\frac{n+1}{2^{n+1}}}{\frac{1}{2^n}} = \frac{(n+1) \cdot 2^n}{2^{n+1}} = \frac{n+1}{2} & \text{for } n \text{ even.} \end{cases}$$

The ratio test failed.

It holds

$$\sqrt[n]{a_n} = \begin{cases} \sqrt[n]{\frac{n}{2^n}} = \frac{\sqrt[n]{n}}{2} & \text{for } n \text{ odd;} \\ \sqrt[n]{\frac{1}{2^n}} = \frac{1}{2} & \text{for } n \text{ even,} \end{cases}$$

which implies

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{2} < 1.$$

By the root test, we know that the series is convergent. ■

Remark. In Example 97, we have seen that there are cases when we are not able to use the ratio test, but the root test is successful. For any sequence of positive numbers a_n, it holds

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Therefore, if the limit $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists, then also the limit $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and it equals L. In this sense, the root test is stronger than the ratio test.

Example 98.

Decide about the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2+5n}{n^2}.$$

Solution. Since

$$\sum_{n=1}^{\infty} \frac{2+5n}{n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$$

and since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, the series $\sum_{n=1}^{\infty} (2+5n)/n^2$ is divergent as well. ■

Example 99.

Decide about the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3+2n^2+7\ln n}.$$

Solution. Since

$$\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3+2n^2+7\ln n} \leq \sum_{n=1}^{\infty} \frac{2n}{n^3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2}$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (see Theorem 4), the given series is convergent. ■

Example 100.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{\left(6+\frac{5}{n}\right)^n}.$$

Solution. It holds

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{\left(6+\frac{5}{n}\right)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{6+\frac{5}{n}} = \frac{1}{6}.$$

The series is convergent. ■

Example 101.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{2}{\pi} \text{ arccos } \frac{1}{n}\right)^{n^2}.$$

Solution. We have

$$\begin{aligned} L &= \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{\pi} \text{ arccos } \frac{1}{n}\right)^{n^2}} \\ &= \lim_{n\to\infty} \left(\frac{2}{\pi} \text{ arccos } \frac{1}{n}\right)^n = \lim_{n\to\infty} \mathrm{e}^{n \ln\left(\frac{2}{\pi} \text{ arccos } \frac{1}{n}\right)} \\ &= \mathrm{e}^{\lim\limits_{n\to\infty} \frac{\ln\left(\frac{2}{\pi} \text{ arccos } \frac{1}{n}\right)}{\frac{1}{n}}} = \mathrm{e}^{\lim\limits_{n\to\infty} \frac{\frac{1}{\frac{2}{\pi} \text{ arccos } \frac{1}{n}} \cdot \frac{-2}{\pi\sqrt{1-1/n^2}} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}}} = \mathrm{e}^{\frac{-2}{\pi}} < 1, \end{aligned}$$

where we use the l'Hospital rule. The series is convergent. ■

Example 102.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n-3)2^n}{n!}.$$

Solution. We have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n-1)\,2^{n+1}}{(n+1)!}}{\frac{(2n-3)\,2^n}{n!}} = \lim_{n\to\infty} \frac{(2n-1)\,2^{n+1}\,n!}{(n+1)!\,(2n-3)\,2^n}$$
$$= \lim_{n\to\infty} \frac{(2n-1)\,2}{(n+1)\,(2n-3)} = \lim_{n\to\infty} \frac{4n-2}{2n^2-n-3} = 0 < 1.$$

The series is convergent. ■

Example 103.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{9^n n!}{n^4(n+3)!}.$$

Solution. We have

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{9^{n+1}(n+1)!}{(n+1)^4(n+4)!}}{\frac{9^n n!}{n^4(n+3)!}} = \lim_{n\to\infty} \frac{9n^4}{(n+1)^3(n+4)} = 9 > 1.$$

The series is divergent to $+\infty$. ■

Theorem 9 (Integral test)**.**

Let us consider a series $\sum_{n=N}^{\infty} a_n$ and let a function f be defined on the interval $[N, \infty)$ for some $N \in \mathbb{Z}$. Let f be non-negative, non-increasing, and

$$f(n) = a_n \qquad \text{for all } n \geq N,\, n \in \mathbb{Z}.$$

Then,

$$\sum_{n=N}^{\infty} a_n \text{ converges if and only if } \int\limits_N^{\infty} f(x)\,\mathrm{d}x \text{ converges,}$$

i.e.,

$$\sum_{n=N}^{\infty} a_n = +\infty \text{ if and only if } \int\limits_N^{\infty} f(x)\,\mathrm{d}x = +\infty.$$

Example 104.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

Solution. We consider the interval $[1, \infty)$, where the function $f(x) = \frac{1}{1+x^2}$ is positive and decreasing. Hence, we compute

$$\int\limits_1^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < +\infty.$$

The integral and the series are convergent. ■

Example 105.

Study the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n}}.$$

Solution. We consider the interval $[1, \infty)$, where the function $f(x) = 1/\sqrt{x+1}$ is positive and decreasing. Hence, we compute

$$\int_1^\infty \frac{1}{\sqrt{1+x}}\,\mathrm{d}x = \left[2\sqrt{x+1}\right]_1^\infty = 2\left(\lim_{x\to\infty}\sqrt{x+1}-\sqrt{2}\right) = +\infty.$$

The integral and the series are divergent. ■

Example 106.

Via the integral test, study the convergence of the series

$$\sum_{n=2}^\infty \frac{n}{n^2+3}.$$

Solution. We consider the interval $[2,\infty)$, where the function $f(x) = x/(x^2+3)$ is positive and decreasing, which may be verified, e.g., using the derivative

$$f'(x) = \frac{x^2+3-2x^2}{(x^2+3)^2} < 0, \qquad x \geq 2.$$

We have

$$\int_2^\infty \frac{x}{x^2+3}\,\mathrm{d}x = \left|t = x^2+3,\ \mathrm{d}t = 2x\,\mathrm{d}x\right|$$

$$= \int_7^\infty \frac{1}{2t}\,\mathrm{d}t = \frac{1}{2}[\ln t]_7^\infty = \frac{1}{2}(+\infty - \ln 7) = +\infty.$$

The integral and the series are divergent. ■

Example 107.

Via the integral test, study the convergence of the series

$$\sum_{n=1}^\infty \frac{1}{n^2}\ln\left(\frac{1}{n}+1\right).$$

Solution. We consider the interval $[1,\infty)$, where the function

$$f(x) = \frac{1}{x^2}\ln\left(\frac{1}{x}+1\right)$$

is positive and decreasing. Hence,

$$\int_1^\infty \frac{1}{x^2} \ln\left(\frac{1}{x}+1\right) \mathrm{d}x = \left| y = \frac{1}{x} + 1,\ \mathrm{d}y = -\frac{1}{x^2}\,\mathrm{d}x \right| = \int_2^1 -\ln y\,\mathrm{d}y$$

$$= \int_1^2 \ln y\,\mathrm{d}y = \left[y\left(\ln y - 1\right)\right]_1^2 = \ln 4 - 1 < +\infty.$$

The series is convergent. ■

Example 108.

Verify that the series

$$\sum_{n=1}^\infty \frac{1}{n^\alpha}, \qquad \alpha \in \mathbb{R},$$

converges if and only if $\alpha > 1$; i.e., prove Theorem 4.

Solution. Evidently, if $\alpha \le 0$, then the series diverges to $+\infty$. For $\alpha > 0$, we use the integral test with the function $f(x) = x^{-\alpha}$. This function is non-negative, decreasing, and $f(n) = a_n$ for $x = n \in \mathbb{N}$.

We consider the integral

$$\int_1^\infty \frac{1}{x^\alpha}\,\mathrm{d}x.$$

We denote $F(x) = \int_1^x \frac{1}{t^\alpha}\,\mathrm{d}t$. If $\alpha \ne 1$, then

$$F(x) = \frac{1}{1-\alpha}\left[\frac{1}{t^{\alpha-1}}\right]_1^x = \frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right).$$

Hence,

$$\lim_{x\to\infty} F(x) = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1; \\ +\infty & \text{if } \alpha < 1. \end{cases}$$

If $\alpha = 1$, then $F(x) = [\ln t]_1^x = \ln x$. Thus, $\lim_{x\to\infty} F(x) = +\infty$. We add that, for $\alpha = 1$, we obtain the harmonic series $\sum_{n=1}^\infty 1/n$, which diverges to $+\infty$. Altogether,

$$\int_1^\infty \frac{1}{x^\alpha}\,\mathrm{d}x = \begin{cases} \frac{1}{\alpha-1} & \text{if } \alpha > 1; \\ +\infty & \text{if } \alpha \le 1. \end{cases}$$

We have obtained

$$\sum_{n=1}^\infty \frac{1}{n^\alpha} = \begin{cases} \text{diverges for } \alpha \le 1; \\ \text{converges for } \alpha > 1. \end{cases}$$

We remark that the ratio and root tests fail, because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^\alpha}}{\frac{1}{n^\alpha}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^\alpha = 1$$

and

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n^\alpha}} = \lim_{n\to\infty} \left(\frac{1}{\sqrt[n]{n}}\right)^\alpha = 1.$$

■

Definition 4.

Let the series $\sum_{n=1}^{\infty} a_n$ converge. For $k \in \mathbb{N}$, the value

$$R_k = \sum_{n=k+1}^{\infty} a_n$$

is called the remainder after k-th term of the series $\sum_{n=1}^{\infty} a_n$.

Theorem 10.

Let the series $\sum_{n=1}^{\infty} a_n$ be convergent and let $f\colon [1,\infty) \to \mathbb{R}$ be a non-increasing and non-negative function such that $f(n) = a_n$, $n \in \mathbb{N}$. Then, the remainder R_k of $\sum_{n=1}^{\infty} a_n$ satisfies

$$R_k \leq \int_k^\infty f(x)\,\mathrm{d}x, \qquad k \in \mathbb{N}.$$

Example 109.

Estimate the error given by the approximation of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

by the partial sum of its first thousand terms.

Solution. Via the integral test, we verify that the series converges. We use the function

$$f(x) = \frac{1}{x^2+1}, \qquad x \in [1,\infty),$$

which is positive and decreasing. Further,

$$\int_1^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = [\arctan x]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < +\infty.$$

Hence, the series converges (see also Example 104). By Theorem 10, we obtain

$$R_{1\,000} \le \int_{1\,000}^{\infty} f(x)\,\mathrm{d}x = \int_{1\,000}^{\infty} \frac{1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2} - \arctan 1\,000 < 10^{-3}.$$

■

Example 110.

To get the remainder less than 0.01, how many terms of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is necessary to sum?

Solution. We use Theorem 10 for the function $f(x) = x^{-2}$. We have

$$R_k \le \int_k^{\infty} \frac{1}{x^2}\,\mathrm{d}x = \left[\frac{-1}{x}\right]_k^{\infty} = \frac{1}{k} \le \frac{1}{100},$$

i.e., we have to consider at least 100 terms. Note that the series is convergent by Theorem 4.

■

Example 111.

With regard to a parameter $\alpha \in \mathbb{R}$, determine the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^{\alpha} n}.$$

Solution. Let $\alpha \ge 0$. Via the integral test with

$$f(x) = \frac{1}{x \ln^{\alpha} x}, \qquad x \in [2, \infty),$$

we obtain

$$\int_{e}^{\infty} \frac{dx}{x \ln^{\alpha} x} = \left| \ln x = t,\ \frac{dx}{x} = dt \right| = \int_{1}^{\infty} \frac{dt}{t^{\alpha}} = \begin{cases} \text{is convergent for} & \alpha > 1; \\ \text{is divergent for} & \alpha \leq 1. \end{cases}$$

The series is convergent for $\alpha > 1$ and divergent for $\alpha \leq 1$. Using the comparison test, comparing with the harmonic series (i.e., the case $\alpha = 0$), we get that it is divergent for negative α. ■

Example 112.

Consider the series

$$\sum_{n=1}^{\infty} \frac{n+1}{e^{n}}.$$

Does it converge?

Solution. Via the ratio test (we have positive terms), we get

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)+1}{e^{n+1}}}{\frac{n+1}{e^{n}}} = \lim_{n\to\infty} \left(\frac{n+2}{n+1} \cdot \frac{1}{e} \right) = \frac{1}{e}.$$

Since $L < 1$, the series converges.

Via the root test, we get

$$L = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n+1}{e^{n}}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+1}}{e} = \frac{1}{e}.$$

Hence, $L < 1$ and the series converges.

We use the integral test for the function $f(x) = (x+1)/e^{x}$ which is positive and decreasing on $[1, \infty)$, because

$$f'(x) = \frac{e^{x} - (x+1)\,e^{x}}{e^{2x}} = -\frac{x}{e^{x}} < 0, \qquad x \geq 1.$$

We obtain

$$\begin{aligned} \int_{1}^{\infty} (1+x)\,e^{-x}\,dx &= \left| u = x+1, u' = 1, v' = e^{-x}, v = -e^{-x} \right| \\ &= \left[-(x+1)\,e^{-x} \right]_1^{\infty} + \int_{1}^{\infty} e^{-x}\,dx \\ &= \lim_{x\to\infty} \left(-\frac{x+1}{e^{x}} \right) - \left(-2e^{-1} \right) + \left[-e^{-x} \right]_1^{\infty} \\ &= -0 + 2e^{-1} + \lim_{x\to\infty} \left(-\frac{1}{e^{x}} \right) - \left(-e^{-1} \right) = 3e^{-1} = \frac{3}{e} < +\infty. \end{aligned}$$

Hence, the series $\sum_{n=1}^{\infty} (n+1)/e^{n}$ is convergent. ■

Example 113.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2+\sin n^3}{n\,(n+5)}.$$

Solution. We use Theorem 4, where

$$\sum_{n=1}^{\infty} \frac{2+\sin n^3}{n\,(n+5)} \leq \sum_{n=1}^{\infty} \frac{3}{n^2} = 3\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Hence, the series is convergent. ■

Example 114.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2n-1}{\sqrt{2^n}}.$$

Solution. Via the root test, we have

$$\lim_{n\to\infty} \sqrt[n]{\frac{2n-1}{\sqrt{2^n}}} = \frac{\lim_{n\to\infty} \sqrt[n]{2n-1}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} < 1.$$

Hence, the series is convergent. We remark that

$$1 \leq \lim_{n\to\infty} \sqrt[n]{2n-1} \leq \lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} \sqrt[n]{2} \cdot \lim_{n\to\infty} \sqrt[n]{n} = 1 \cdot 1 = 1.$$

■

Example 115.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1!+2!+3!+\cdots+n!}{(2n)!}.$$

Solution. We use Theorem 4. We have

$$\sum_{n=3}^{\infty} \frac{1! + 2! + 3! + \cdots + n!}{(2n)!} \leq \sum_{n=3}^{\infty} \frac{n \cdot n!}{(2n)!} \leq \sum_{n=3}^{\infty} \frac{n}{(n+1)\,(n+2)\,(n+3)} \leq \sum_{n=3}^{\infty} \frac{1}{n^2}.$$

The series is convergent via the comparison test. ■

Example 116.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}.$$

Solution. We use Theorem 4. Via the comparison test, we have

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

The series is convergent.

Via the root test, we can also obtain the convergence from

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n \cdot 3^n}} = \frac{1}{3 \lim_{n\to\infty} \sqrt[n]{n}} = \frac{1}{3} < 1.$$

■

Example 117.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n.$$

Solution. The necessary condition for the convergence is not fulfilled, because

$$\begin{aligned} \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n &= \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right)^n = \lim_{n\to\infty} \left(1 - \frac{1}{n+2}\right)^{n+2} \left(1 - \frac{1}{n+2}\right)^{-2} \\ &= \lim_{m\to\infty} \left(1 - \frac{1}{m}\right)^m \left(1 - \frac{1}{m}\right)^{-2} = \mathrm{e}^{-1} \cdot 1^{-2} = \frac{1}{\mathrm{e}} \neq 0. \end{aligned}$$

Hence, the series cannot converge. ■

Example 118.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n\,(n+2)}.$$

Solution. We use Theorem 4. We have

$$\sum_{n=1}^{\infty} \frac{n+1}{n\,(n+2)} \geq \sum_{n=1}^{\infty} \frac{n}{3n^2} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

The series is divergent (via the comparison test). ■

Example 119.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \sin\,\frac{1}{n}.$$

Solution. We use Theorem 4. We have

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1.$$

The series is divergent via the limit comparison test. ■

Example 120.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \sin\,\frac{1}{n^2}.$$

Solution. We use Theorem 4 and the limit comparison test. We have

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

Thus, the series is convergent. ■

Example 121.

Determine the convergence of the series

$$\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}-n}{\ln^2 n}.$$

Solution. We have

$$a_n = \frac{\sqrt{n^2+1}-n}{\ln^2 n} = \frac{\sqrt{n^2+1}-\sqrt{n^2}}{\ln^2 n} \cdot \frac{\sqrt{n^2+1}+\sqrt{n^2}}{\sqrt{n^2+1}+\sqrt{n^2}} = \frac{1}{\left(\sqrt{n^2+1}+\sqrt{n^2}\right)\ln^2 n}.$$

We consider the series with the terms

$$b_n = \frac{1}{n \ln^2 n}, \qquad n \geq 2,\ n \in \mathbb{N}.$$

Since

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\left(\sqrt{n^2+1}+\sqrt{n^2}\right)\ln^2 n}}{\frac{1}{n\ln^2 n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} \in (0,\infty)$$

and since the series $\sum_{n=2}^{\infty} 1/(n \ln^2 n)$ converges (see Example 111), the original series is convergent as well (via the limit comparison test). ■

Example 122.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2+\cos n}{n+\ln n}.$$

Solution. We use Theorem 4. Since $n > \ln n$ for $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{2+\cos n}{n+\ln n} \geq \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}.$$

Therefore, the series is divergent via the comparison test. ■

Example 123.

Determine the convergence of the series

$$\sum_{n=1}^{\infty}\left(\frac{1+\cos\, n}{4+\cos\, n}\right)^{n}.$$

Solution. The function $y=\cos t$ has values from $[-1,1]$. For any $t\in[-1,1]$, we have

$$0\le\frac{1+t}{4+t}=1-\frac{3}{4+t}\le 1-\frac{3}{4+1}=\frac{2}{5},$$

i.e.,

$$\sum_{n=1}^{\infty}\left(\frac{1+\cos n}{4+\cos n}\right)^{n}\le\sum_{n=1}^{\infty}\left(\frac{2}{5}\right)^{n}.$$

The comparison test implies that the series is convergent. ■

Example 124.

Determine the convergence of the series

$$\sum_{n=1}^{\infty}\frac{n^5}{3^n+4^n}.$$

Solution. We begin with the comparison test

$$\sum_{n=1}^{\infty}\frac{n^5}{3^n+4^n}\le\sum_{n=1}^{\infty}\frac{n^5}{3^n}$$

and we continue with the root test

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^5}{3^n}}=\frac{\left(\lim\limits_{n\to\infty}\sqrt[n]{n}\right)^5}{3}=\frac{1}{3}<1.$$

The series converges. ■

Example 125.

Determine the convergence of the series

$$\sum_{n=1}^{\infty}\frac{1}{(n+3)\sqrt{n}}.$$

Solution. We use Theorem 4. The convergence follows from

$$\sum_{n=1}^{\infty} \frac{1}{(n+3)\sqrt{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < +\infty.$$

■

Example 126.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{1+n^2}{1+n^3} \right)^2 .$$

Solution. We use Theorem 4. The convergence follows from

$$\sum_{n=1}^{\infty} \left(\frac{1+n^2}{1+n^3} \right)^2 \leq \sum_{n=1}^{\infty} \frac{n^4+2n^2+1}{n^6} \leq 4 \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

■

Example 127.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{2n}}.$$

Solution. The necessary condition for the convergence is not fulfilled, because

$$\lim_{n\to\infty} \sqrt{\frac{n-1}{2n}} = \frac{\sqrt{2}}{2} \neq 0.$$

The series is divergent. ■

Example 128.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{(4n)!}.$$

Solution. Via the ratio test, we have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(4n+4)!}}{\frac{1}{(4n)!}} = \lim_{n\to\infty} \frac{(4n)!}{(4n+4)!}$$
$$= \lim_{n\to\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+4)} = 0 < 1.$$

The series is convergent. ■

Example 129.

Find $a > 0$ for which the series

$$\sum_{n=1}^{\infty} n^4 a^n$$

is convergent.

Solution. For $a \geq 1$, we have $n^4 a^n \geq 1$, $n \in \mathbb{N}$. The necessary condition for the convergence is not fulfilled. Hence, the series cannot converge.

For $a \in (0, 1)$, we use the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^4 a^{n+1}}{n^4 a^n} = a \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^4 = a < 1.$$

The series is convergent for $a \in (0, 1)$. Later, we will see that the series converges also for $a \in (-1, 0]$. ■

Example 130.

Find $a > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{1}{1 + a^n}$$

is convergent.

Solution. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1 + a^n} = \frac{1}{1 + \lim_{n\to\infty} a^n} = \begin{cases} 1 & \text{for } a \in (0, 1); \\ \frac{1}{2} & \text{for } a = 1; \\ 0 & \text{for } a > 1. \end{cases}$$

Applying the necessary condition for the convergence, the series cannot converge for $a \leq 1$. For $a > 1$, we have

$$\sum_{n=1}^{\infty} \frac{1}{1+a^n} \leq \sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n < +\infty.$$

Via the comparison test, we obtain the convergence for $a > 1$. ■

Example 131.

Find $a > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{a^n}{n!}$$

is convergent.

Solution. We have

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{a^{n+1}}{(n+1)!}}{\frac{a^n}{n!}} = \lim_{n\to\infty} \frac{a}{n+1} = 0 < 1.$$

The series is convergent for any $a > 0$. ■

Example 132.

Find $a > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{1+na}{\sqrt{n^2+n^6a}}$$

is convergent.

Solution. For $N \geq 1/a$, $N \in \mathbb{N}$, we have (see Theorem 4)

$$\sum_{n=N}^{\infty} \frac{1+na}{\sqrt{n^2+n^6a}} \leq \sum_{n=N}^{\infty} \frac{2na}{\sqrt{n^6a}} = 2\sqrt{a} \sum_{n=N}^{\infty} \frac{1}{n^2} < +\infty.$$

The series is convergent for any $a > 0$. ■

Theorem 11 (Raabe's test)**.**

Let $a_n > 0$ for all large n and let there exist the limit

$$L = \lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right).$$

Then, it holds:

(I) if $L > 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(II) if $L < 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges k $+\infty$.

Remark. The Raabe test is stronger than the limit ratio test (see also the following example).

Example 133.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{4^n(n!)^2}.$$

Solution. The ratio test fails, because

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{4^{n+1}[(n+1)!]^2}}{\frac{(2n)!}{4^n(n!)^2}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{4(n+1)^2} = \lim_{n\to\infty} \frac{4n^2}{4n^2} = 1.$$

Via the Raabe test, we have

$$\begin{aligned}\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) &= \lim_{n\to\infty} n\left(1 - \frac{(2n+2)(2n+1)}{4(n+1)^2}\right)\\ &= \lim_{n\to\infty} n\left(1 - \frac{2n+1}{2(n+1)}\right) = \lim_{n\to\infty} \frac{n}{2n+2} = \frac{1}{2} < 1.\end{aligned}$$

The series is divergent. ■

Example 134.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!}{\left(\sqrt{2}+1\right)\left(\sqrt{2}+2\right)\cdots\left(\sqrt{2}+n\right)}.$$

Solution. The ratio test fails, because we have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{(n+1)!\left(\sqrt{2}+1\right)\left(\sqrt{2}+2\right)\cdots\left(\sqrt{2}+n\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}+2\right)\cdots\left(\sqrt{2}+n\right)\left(\sqrt{2}+n+1\right)n!}\\ &= \lim_{n\to\infty}\frac{n+1}{\sqrt{2}+n+1}=1.\end{aligned}$$

Via the Raabe test, we have

$$\begin{aligned}L &= \lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} n\left(1-\frac{n+1}{\sqrt{2}+n+1}\right)\\ &= \lim_{n\to\infty}\frac{n\left(\sqrt{2}+n+1-n-1\right)}{\sqrt{2}+n+1} = \lim_{n\to\infty}\frac{n\sqrt{2}}{\sqrt{2}+n+1}=\sqrt{2}>1.\end{aligned}$$

The series is convergent. ■

Theorem 12 (Condensation (Cauchy's) test)**.**

Let $\{a_n\}_{n=2^N}^{\infty}$ be a non-increasing sequence of non-negative numbers. Then,

$$\sum_{n=2^N}^{\infty} a_n < +\infty \qquad \text{if and only if} \qquad \sum_{k=N}^{\infty} 2^k a_{2^k} < +\infty.$$

Example 135.

Via the condensation test, determine the convergence of the series

$$\sum_{n=2}^{\infty}\frac{1}{n\ln^2 n}.$$

Solution. Via the limit ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{a_{n+1}}{a_n} &= \lim_{n\to\infty}\frac{\frac{1}{(n+1)\ln^2(n+1)}}{\frac{1}{n\ln^2 n}} = \lim_{n\to\infty}\frac{n\ln^2 n}{(n+1)\ln^2(n+1)}\\ &= \lim_{n\to\infty}\frac{n}{n+1}\cdot\left[\lim_{n\to\infty}\frac{\ln n}{\ln(n+1)}\right]^2 = 1\cdot 1^2 = 1.\end{aligned}$$

The test failed.

Via the limit root test, we have

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{n\ln^2 n}} = \lim_{n\to\infty}\frac{1}{\sqrt[n]{n}\,\sqrt[n]{\ln^2 n}} = 1.$$

The test failed as well.

But, via the condensation test, we have (see also Theorem 4)

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k \ln^2 2^k} = \sum_{k=1}^{\infty} \frac{1}{(k \ln 2)^2} = \frac{1}{\ln^2 2} \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty.$$

The series converges. ■

Theorem 13 (Gauss' test)**.**

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and let

$$\frac{a_n}{a_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\gamma_n}{n^2}, \qquad n \in \mathbb{N},$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is bounded. Then, it holds:

(i) if $\alpha \geq 1$ and $\beta > 1$, then the series $\sum_{n=N}^{\infty} a_n$ converges;

(ii) if $\alpha \leq 1$ and $\beta \leq 1$, then the series $\sum_{n=N}^{\infty} a_n$ diverges to $+\infty$.

Example 136.

Via the Gauss test, verify the divergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^2.$$

Solution. We obtain the value β putting $\alpha = 1$. We have

$$\beta = \lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{n\to\infty} n \left(\left(\frac{\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}}{\frac{1\cdot3\cdot5\cdots(2n+1)}{2\cdot4\cdot6\cdots(2n+2)}} \right)^2 - 1 \right)$$
$$= \lim_{n\to\infty} n \left(\left(\frac{2n+2}{2n+1} \right)^2 - 1 \right) = \lim_{n\to\infty} n \left(\frac{4n^2+8n+4}{4n^2+4n+1} - 1 \right) = \lim_{n\to\infty} \frac{4n^2+3n}{4n^2+4n+1} = 1.$$

Now, we have to find a sequence $\{\gamma_n\}_{n=1}^{\infty}$ such that

$$\gamma_n = \left(\frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} \right) n^2, \qquad n \in \mathbb{N}.$$

From

$$\frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} = \frac{4n+3}{4n^2+4n+1} - \frac{1}{n} = -\frac{n+1}{n(4n^2+4n+1)}, \qquad n \in \mathbb{N},$$

it follows

$$\gamma_n = -\frac{n^2+n}{4n^2+4n+1}, \qquad n \in \mathbb{N}.$$

We have

$$\lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} -\frac{n^2+n}{4n^2+4n+1} = -\frac{1}{4}.$$

Thus, $\{\gamma_n\}_{n=1}^{\infty}$ is bounded. The given sequence diverges to $+\infty$. ■

Theorem 14 (Kummer's test)**.**

Let $\sum_{n=1}^{\infty} a_n$ be given and let $a_n > 0$ for all large n.

(i) The series $\sum_{n=1}^{\infty} a_n$ converges if and only if there exist $A > 0$, $\{p_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, and $N \in \mathbb{N}$ such that

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \geq A, \qquad n \geq N,\ n \in \mathbb{N}.$$

(ii) The series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ if and only if there exist a sequence $\{p_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ such that

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = +\infty$$

and $N \in \mathbb{N}$ such that

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} \leq 0, \qquad n \geq N,\ n \in \mathbb{N}.$$

Example 137.

Via the Kummer test, determine the convergence of the series

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{1+k\ln k}{\sqrt{2}+(k+1)\ln(k+1)}.$$

Solution. We put

$$p_n = 1 + (n+1)\ln(n+1), \qquad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} p_n \frac{a_n}{a_{n+1}} - p_{n+1} &= [1 + (n+1)\ln(n+1)] \frac{\prod\limits_{k=1}^{n} \frac{1+k\ln k}{\sqrt{2}+(k+1)\ln(k+1)}}{\prod\limits_{k=1}^{n+1} \frac{1+k\ln k}{\sqrt{2}+(k+1)\ln(k+1)}} - [1 + (n+2)\ln(n+2)] \\ &= [1 + (n+1)\ln(n+1)] \frac{1}{\frac{1+(n+1)\ln(n+1)}{\sqrt{2}+(n+2)\ln(n+2)}} - [1 + (n+2)\ln(n+2)], \end{aligned}$$

i.e.,

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} = \sqrt{2} - 1, \qquad n \in \mathbb{N}.$$

Via the Kummer test for $A = \sqrt{2} - 1 > 0$, the series is convergent. ■

§ 1.3 Series with general terms

Definition 5.

A series $\sum_{n=N}^{\infty} a_n$ is called alternating if $\operatorname{sgn} a_n = -\operatorname{sgn} a_{n+1}$ for all $n \geq N$, $n \in \mathbb{N}$.

Remark. If $a_n > 0$ for all large n, then the series

$$\sum_{n=N}^{\infty} (-1)^n a_n, \qquad \sum_{n=N}^{\infty} (-1)^{n+1} a_n$$

are alternating.

Theorem 15 (Leibniz's test)**.**

Let $\{a_n\}_{n=N}^{\infty} \subset (0, \infty)$ be a non-increasing series. If $\lim_{n\to\infty} a_n = 0$, then the alternating series $\sum_{n=N}^{\infty}(-1)^n a_n$ converges.

Example 138.

Study the convergence of the so-called Leibniz series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Solution. It is an alternating series, where

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n} = 0$$

and

$$\frac{1}{n+1} < \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Hence, it converges via the Leibniz test. ∎

Example 139.

Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{8n-3}$$

converges.

Solution. The necessary condition for the convergence is not fulfilled. Indeed,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n}{8n-3} = \frac{2}{8} = \frac{1}{4} \neq 0.$$

The series cannot converge. ■

Theorem 16.

Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. The remainder after k-th term R_k of the convergent alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ satisfies

$$|R_k| < a_{k+1}, \qquad k \in \mathbb{N}.$$

Example 140.

Find the error given by the approximation of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

by its first 99 terms.

Solution. The series is convergent via the Leibniz test. Using Theorem 16, we obtain

$$|R_{99}| < |a_{100}| = \frac{1}{100^2} = 10^{-4}.$$

■

Example 141.

Find the error given by the approximation of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3+5}$$

by its first 10 terms.

Solution. The series is convergent via the Leibniz test. Indeed, the sequence

$$\{|a_n|\}_{n=1}^{\infty} \equiv \left\{\frac{1}{n^3+5}\right\}_{n=1}^{\infty}$$

is decreasing and

$$\lim_{n\to\infty} \frac{1}{n^3+5} = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n^3+5} = 0.$$

The error is less than $|a_{11}|$. Hence, it is less than

$$\frac{1}{11^3+5} = \frac{1}{1\,336}.$$

■

Definition 6.

We say that a series $\sum_{n=N}^{\infty} a_n$ converges absolutely if the series $\sum_{n=N}^{\infty} |a_n|$ converges. We say that a series $\sum_{n=N}^{\infty} a_n$ converges conditionally if it converges and the series $\sum_{n=N}^{\infty} |a_n|$ diverges.

Theorem 17.

If a series $\sum_{n=N}^{\infty} a_n$ converges absolutely, then it converges.

Example 142.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

are divergent or convergent. If they are convergent, do they converge absolutely or conditionally?

Solution. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is conditionally convergent. It is the so-called Leibniz series. It converges via the Leibniz test, but the series of the absolute values is the harmonic series which diverges.

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

is absolutely convergent, because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

■

Example 143.

Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n^6+3n^3-n+5}}{\sqrt{n+3}}$$

converge absolutely? Give reasons for the answer.

Solution. We study the series with positive terms

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n^6+3n^3-n+5}}{\sqrt{n+3}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}}.$$

Calculating

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{\sqrt{n+3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{\sqrt[n]{n+3}}} = \frac{1}{\sqrt{1}} = 1,$$

we can see that the ratio and the root tests are useless. We apply the comparison test as follows

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+3}} \geq \sum_{n=1}^{\infty} \frac{1}{n+3} = \sum_{n=4}^{\infty} \frac{1}{n} = +\infty.$$

Hence, the series $\sum_{n=1}^{\infty} 1/\sqrt{n+3}$ diverges. Therefore, the given series does not converge absolutely.

The divergence of $\sum_{n=1}^{\infty} 1/\sqrt{n+3}$ comes from the integral test as well. The function $f(x) = 1/\sqrt{x+3}$ is positive and decreasing on $[1, \infty)$. Further,

$$\int_1^{\infty} \frac{1}{\sqrt{x+3}}\, dx = |t = x+3,\ dt = dx|$$

$$= \int_4^{\infty} t^{-\frac{1}{2}}\, dt = \left[2t^{1/2}\right]_4^{\infty} = \lim_{t\to\infty} \left(2\sqrt{t}\right) - 4 = +\infty.$$

The integral diverges. Hence, the studied series does not converge absolutely. ■

Example 144.

Is the series

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{2^n}{n^2}$$

convergent? If it converges, does it converge absolutely?

Solution. The series of the absolute values is

$$\sum_{n=1}^{\infty}\left|(-1)^n\,\frac{2^n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{2^n}{n^2}.$$

Via the root test, we have

$$L=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^2}}=\lim_{n\to\infty}\frac{2}{\left(\sqrt[n]{n}\right)^2}=\frac{2}{1^2}=2.$$

Since $L>1$, the series $\sum_{n=1}^{\infty}2^n/n^2$ diverges. The studied series does not converge absolutely. Does it converges conditionally? It is an alternating series. The sequence

$$\{|a_n|\}_{n=1}^{\infty}=\left\{\left|(-1)^n\,\frac{2^n}{n^2}\right|\right\}_{n=1}^{\infty}=\left\{\frac{2^n}{n^2}\right\}_{n=1}^{\infty}$$

is increasing for $n\geq 3$. Hence, the Leibniz test is not applicable. We can consider the necessary condition for the convergence yet. Using the l'Hospital rule, we obtain

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{2^n}{n^2}=\lim_{n\to\infty}\frac{\ln 2\cdot 2^n}{2n}=\lim_{n\to\infty}\frac{\ln^2 2\cdot 2^n}{2}=+\infty.$$

The sequence does not converge to zero. Thus, the necessary condition for the convergence is not fulfilled. The series does not converge. ■

Example 145.

Decide whether the series

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{n+1}{n(n+2)}$$

converges absolutely, conditionally, or it does not converge.

Solution. We use the Leibniz test. We compare a_n and a_{n+1} for $n\in\mathbb{N}$ via the derivative of the function

$$\begin{aligned}\left(\frac{x+1}{x^2+2x}\right)'&=\frac{(x^2+2x)-(x+1)(2x+2)}{(x^2+2x)^2}\\&=\frac{x^2+2x-2x^2-4x-2}{(x^2+2x)^2}=-\frac{x^2+2x+2}{(x^2+2x)^2}<0,\qquad x\geq 1.\end{aligned}$$

Hence, $a_n > a_{n+1}$ for $n \in \mathbb{N}$. Further, we have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n+1}{n(n+2)} = 0.$$

Therefore, the series converges via the Leibniz test.

Now, we consider

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n+1}{n(n+2)}.$$

Via the comparison test with the harmonic series, we have

$$\sum_{n=1}^{\infty} \frac{n+1}{n(n+2)} \geq \sum_{n=1}^{\infty} \frac{1}{n+2} = \sum_{n=3}^{\infty} \frac{1}{n} = +\infty.$$

The series converges conditionally. ■

Example 146.

Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$$

converges absolutely. Then, write down the partial sum s_3 of the first three terms and estimate the error of this approximation. Find the exact sum s of the series and the precise error when we replace s by s_3.

Solution. The sequence of positive numbers

$$\{|a_n|\}_{n=1}^{\infty} = \left\{\frac{1}{3^n}\right\}_{n=1}^{\infty}$$

is decreasing and

$$\lim_{n\to\infty} |a_n| = 0.$$

The alternating series converges via the Leibniz test. Next, we use the ratio test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} = \frac{1}{3} < 1.$$

Hence, it converges absolutely. (It follows also from the theory of geometric series.)

Further,

$$s_3 = \frac{1}{3} - \frac{1}{3^2} + \frac{1}{3^3} = \frac{7}{27}$$

and the estimation of the error is

$$|s - s_3| < |a_4| = \frac{1}{3^4} = \frac{1}{81}.$$

The exact sum is

$$s=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n}=-\sum_{n=1}^{\infty}\left(-\frac{1}{3}\right)^n=-\left(\frac{1}{1+\frac{1}{3}}-1\right)=\frac{1}{4}.$$

Therefore, the exact error of $s\approx s_3$ is

$$|s-s_3|=\left|\frac{1}{4}-\frac{7}{27}\right|=\frac{1}{4\cdot 27}=\frac{1}{108}.$$

■

Example 147.

Decide whether the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}$$

converges absolutely, conditionally, or it does not converge.

Solution. It is an alternating series. We use the Leibniz test. We have

$$\lim_{n\to\infty}|a_n|=\lim_{n\to\infty}\frac{1}{(2n)!}=0$$

and

$$\left\{\frac{1}{(2n)!}\right\}_{n=1}^{\infty}$$

is a decreasing series of positive numbers. Hence, the series $\sum_{n=1}^{\infty}(-1)^n/(2n)!$ converges.

Next, we consider

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{(2n)!}\right|=\sum_{n=1}^{\infty}\frac{1}{(2n)!}.$$

Via the ratio test, we have

$$L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}}=\lim_{n\to\infty}\frac{(2n)!}{(2n+2)!}=\lim_{n\to\infty}\frac{1}{(2n+2)\,(2n+1)}=0.$$

Since $L<1$, the series converges, i.e., the original series converges absolutely.

Note that the absolute convergence implies convergence. Therefore, it suffices to state only the second part of the solution. ■

Example 148.

Decide whether the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}}$$

converges absolutely, conditionally, or it does not converge.

Solution. We denote $a_n = 1/\sqrt{n}$ for $n \in \mathbb{N}$. Then,

$$0 < a_{n+1} < a_n, \qquad n \in \mathbb{N}; \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

The convergence follows from the Leibniz test. The series of the absolute values is

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}},$$

which is divergent. Therefore, the series $\sum_{n=1}^{\infty}(-1)^n/\sqrt{n}$ converges conditionally. ■

Example 149.

Decide whether the series

$$\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{(n+1)^2}$$

converges absolutely, conditionally, or it does not converge.

Solution. We use the comparison test for the series of the absolute values

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty.$$

The series converges absolutely. ■

Example 150.

Decide whether the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n^2+4}}{n+1}$$

converges absolutely, conditionally, or it does not converge.

Solution. The necessary condition for the convergence is not fulfilled, because

$$\lim_{n\to\infty} \frac{\sqrt{n^2+4}}{n+1} = \lim_{n\to\infty} \frac{n}{n} = 1 \neq 0.$$

The series does not converge. ■

Example 151.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(\frac{1}{1567}\right)^n$$

converges absolutely, conditionally, or it does not converge.

Solution. For any $n \in \mathbb{N}$, we have

$$\left| \frac{(-1)^n}{n} \left(\frac{1}{1567}\right)^n \right| \leq \left(\frac{1}{1567}\right)^n$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{1567}\right)^n$$

is a convergent geometric series ($q = 1/1\,567 < 1$). The studied series is absolutely convergent via the comparison test. ■

Example 152.

Decide whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{5n-2}$$

converges absolutely, conditionally, or it does not converge.

Solution. It is an alternating series. We use the Leibniz test. We have

$$\{|a_n|\}_{n=1}^{\infty} = \left\{ \frac{1}{5n-2} \right\}_{n=1}^{\infty}$$

which is decreasing, because

$$\left(\frac{1}{5x-2} \right)' = \frac{-5}{(5x-2)^2} < 0, \qquad x \geq 1,$$

and, further,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{5n-2} = 0.$$

The series converges. For the absolute convergence, we use the limit comparison test

$$\lim_{n\to\infty} \frac{\frac{1}{5n-2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{5n-2} = \frac{1}{5} > 0.$$

The harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges. Hence, the studies series converges conditionally. ■

Example 153.

Decide whether the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^3+1}$$

converges absolutely, conditionally, or it does not converge.

Solution. We estimate the series of the absolute values and we obtain

$$\sum_{n=1}^{\infty} \frac{n}{n^3+1} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The series is absolutely convergent via the comparison test. ■

Example 154.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$$

converges absolutely, conditionally, or it does not converge.

Solution. The necessary condition for the convergence is not fulfilled. Indeed,

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1 \neq 0.$$

The series does not converge. ■

Example 155.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{2n+3}}$$

converges absolutely, conditionally, or it does not converge.

Solution. First, we consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+3}}$. We have

$$\sum_{n=3}^{\infty} \frac{1}{\sqrt{2n+3}} \geq \sum_{n=3}^{\infty} \frac{1}{\sqrt{3n}} = \frac{1}{\sqrt{3}} \sum_{n=3}^{\infty} \frac{1}{n^{\frac{1}{2}}}.$$

Via the comparison test with the divergent series $\sum_{n=1}^{\infty} 1/\sqrt{n}$, we obtain that the series is divergent. Hence, the studied series is not absolutely convergent.

Then, we use the Leibniz test. The positive sequence

$$\left\{ \frac{1}{\sqrt{2n+3}} \right\}_{n=1}^{\infty}$$

is decreasing and its limit is 0 as $n \to \infty$. Hence, the series $\sum_{n=1}^{\infty} (-1)^n/\sqrt{2n+3}$ converges conditionally due to the Leibniz test. ■

Example 156.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 4^n}{(n+1)!}$$

converges absolutely, conditionally, or it does not converge.

Solution. For the series of the absolute values

$$\sum_{n=1}^{\infty} \frac{4^n}{(n+1)!},$$

we use the ratio test

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{4^n} = \lim_{n\to\infty} \frac{4}{n+2} = 0 < 1.$$

The series converges absolutely. ■

Example 157.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{[5+(-1)^{n+1}]^n}$$

converges absolutely, conditionally, or it does not converge.

Solution. For the series of the absolute values

$$\sum_{n=1}^{\infty} \frac{2^n}{[5+(-1)^{n+1}]^n},$$

we use the comparison test with the geometric series

$$\sum_{n=1}^{\infty} \frac{2^n}{[5+(-1)^{n+1}]^n} \le \sum_{n=1}^{\infty} \frac{2^n}{4^n} = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

The series converges absolutely. ■

Example 158.

Find $\alpha \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^\alpha}$$

converges absolutely/conditionally.

Solution. The series is not convergent for $\alpha \le 0$, because

$$\lim_{n\to\infty} \frac{(-1)^n}{n^\alpha} \ne 0,$$

i.e., the necessary condition for the convergence is not fulfilled. Further, for $\alpha > 1$, we know that the series converges absolutely; for $\alpha \in (0,1]$, the series does not converge absolutely.

Therefore, we consider $\alpha \in (0,1]$. The sequence $\{n^{-\alpha}\}_{n=1}^{\infty}$ is decreasing and

$$\lim_{n\to\infty} \frac{1}{n^\alpha} = 0, \qquad \alpha \in (0,1].$$

Due to the Leibniz test, the studied series is convergent for any $\alpha \in (0,1]$.

Altogether, the series converges absolutely for $\alpha > 1$, it converges conditionally for $\alpha \in (0,1]$, and it does not converge for $\alpha \le 0$. ■

Example 159.

Determine the absolute and the conditional convergence of the series

$$\sum_{n=1}^{\infty} n^{\ln a}$$

with regard to $a > 0$.

Solution. It is a series with positive terms. We know that the series $\sum_{n=1}^{\infty} n^{\alpha}$ converges for $\alpha < -1$ and diverges to $+\infty$ for $\alpha \geq -1$. Hence, the given series converges absolutely for $a \in (0, 1/\mathrm{e})$ and it diverges for $a \geq 1/\mathrm{e}$. For the conditional convergence, the set of a is empty. ∎

Example 160.

Determine the absolute and the conditional convergence of the series

$$\sum_{n=1}^{\infty} \frac{a^n}{n}$$

with regard to $a \in \mathbb{R}$.

Solution. Using the ratio test, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{a^{n+1}}{n+1}\right|}{\left|\frac{a^n}{n}\right|} = |a| \lim_{n\to\infty} \frac{n}{n+1} = |a|,$$

i.e., the series converges absolutely for $|a| < 1$. Considering the necessary condition for the convergence, it cannot converge for $|a| > 1$. For $a = 1$, we have the harmonic series which is divergent. For $a = -1$, we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which converges conditionally. Altogether, the series converges absolutely for $a \in (-1, 1)$, it converges conditionally for $a = -1$, and it diverges otherwise. ∎

Example 161.

Prove that the absolute convergence of a given series $\sum_{n=1}^{\infty} a_n$ implies the convergence of the series $\sum_{n=1}^{\infty} a_n^2$.

Solution. The convergence of $\sum_{n=1}^{\infty} a_n$ implies the existence of $n_0 \in \mathbb{N}$ such that $|a_n| \leq 1$ for $n \geq n_0$. Hence,

$$\sum_{n=n_0}^{\infty} a_n^2 \leq \sum_{n=n_0}^{\infty} |a_n|.$$

The first $n_0 - 1$ terms do not affect the convergence of the series. The absolute convergence of the series on the right side guarantees the convergence of the original series.

The absolute convergence is necessary. If we suppose only the convergence of $\sum_{n=1}^{\infty} a_n$, the statement is not true. For example, the series $\sum_{n=1}^{\infty}(-1)^n\, n^{-\frac{1}{2}}$ converges conditionally and the series

$$\sum_{n=1}^{\infty}\left((-1)^n\, n^{-\frac{1}{2}}\right)^2=\sum_{n=1}^{\infty}\frac{1}{n}$$

is the divergent harmonic series. ■

Example 162.

For $a_n>0$, $n\in\mathbb{N}$, prove the following implication. If $\sum_{n=1}^{\infty} a_n$ is divergent and $s_n=\sum_{k=1}^{n} a_k$, then the series $\sum_{n=1}^{\infty} a_n/s_n$ diverges and the series $\sum_{n=1}^{\infty} a_n/s_n^2$ converges.

Solution. We denote

$$b_n=\sum_{k=1}^{n}\frac{a_k}{s_k},\qquad n\in\mathbb{N}.$$

Then, for $m>n$, $m,n\in\mathbb{N}$, we have

$$\begin{aligned} b_m-b_n&=\frac{a_{n+1}}{s_{n+1}}+\frac{a_{n+2}}{s_{n+2}}+\frac{a_{n+3}}{s_{n+3}}+\cdots+\frac{a_m}{s_m}\\ &\geq\frac{a_{n+1}}{s_m}+\frac{a_{n+2}}{s_m}+\frac{a_{n+3}}{s_m}+\cdots+\frac{a_m}{s_m}=\frac{s_m-s_n}{s_m}=1-\frac{s_n}{s_m}. \end{aligned}$$

The assumption implies $\lim_{n\to\infty} s_n=+\infty$. Therefore, for any n, we can choose m so large that $s_m\geq 2\,s_n$. Then,

$$b_m-b_n\geq\frac{1}{2}$$

and the series $\sum_{n=1}^{\infty} a_n/s_n$ diverges, because the sequence of its partial sums cannot converge.

In the second part, we use the estimation

$$\sum_{n=2}^{\infty}\frac{a_n}{s_n^2}=\sum_{n=2}^{\infty}\frac{s_n-s_{n-1}}{s_n^2}\leq\sum_{n=2}^{\infty}\frac{s_n-s_{n-1}}{s_n\,s_{n-1}}=\sum_{n=2}^{\infty}\left(\frac{1}{s_{n-1}}-\frac{1}{s_n}\right)=\frac{1}{a_1},$$

which is valid for all terms except the first one. The comparison test implies that the series $\sum_{n=1}^{\infty} a_n/s_n^2$ converges. ■

Theorem 18 (Dirichlet's test)**.**

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers. If the sequence of the partial sums of the series $\sum_{n=1}^{\infty} a_n$ is bounded and $\{b_n\}_{n=1}^{\infty}$ is non-increasing and $\lim_{n\to\infty} b_n=0$, then the series $\sum_{n=1}^{\infty} a_nb_n$ converges.

Example 163.

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}, \qquad x \in \mathbb{R},$$

converges or diverges, i.e., study the convergence of the series with regard to x.

Solution. If x is zero or any (positive or negative) integer multiple of π, then it is the convergent series of the zero terms. Next, let $x \neq k\pi$ for $k \in \mathbb{Z}$. We apply the Dirichlet test. We put

$$a_n := \sin nx, \qquad b_n := \frac{1}{n}, \qquad n \in \mathbb{N}.$$

The sequence $\{b_n\}_{n=1}^{\infty}$ is non-increasing and $\lim_{n\to\infty} b_n = 0$. Further, we have to verify that $\sum_{n=1}^{\infty} a_n$ has the bounded sequence of the partial sums.

We use the series $\sum_{n=1}^{\infty} \cos nx$. We denote its partial sums as c_n. We have

$$\begin{array}{l} s_n = \sin x + \cdots + \sin nx \;\big/ \cdot \mathrm{i} \\ c_n = \cos x + \cdots + \cos nx, \\ \hline \sum_{k=1}^{n} (\cos kx + \mathrm{i} \sin kx) = \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}kx}. \end{array}$$

Hence,

$$\begin{aligned} \sum_{k=1}^{n} \mathrm{e}^{\mathrm{i}kx} &= \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{2\mathrm{i}x} + \cdots + \mathrm{e}^{n\mathrm{i}x} = \mathrm{e}^{\mathrm{i}x} \frac{\mathrm{e}^{\mathrm{i}xn} - 1}{\mathrm{e}^{\mathrm{i}x} - 1} \\ &= \mathrm{e}^{\mathrm{i}x} \frac{\mathrm{e}^{\frac{\mathrm{i}xn}{2}} \left(\mathrm{e}^{\frac{\mathrm{i}xn}{2}} - \mathrm{e}^{-\frac{\mathrm{i}xn}{2}} \right)}{\mathrm{e}^{\frac{\mathrm{i}x}{2}} \left(\mathrm{e}^{\frac{\mathrm{i}x}{2}} - \mathrm{e}^{-\frac{\mathrm{i}x}{2}} \right)} = \left| \mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{-\mathrm{i}\alpha} = 2\mathrm{i} \sin \alpha \right| \\ &= \mathrm{e}^{\frac{\mathrm{i}x}{2}} \cdot \mathrm{e}^{\frac{\mathrm{i}xn}{2}} \cdot \frac{\sin \frac{xn}{2}}{\sin \frac{x}{2}} = \left[\cos \left(\frac{n+1}{2} x \right) + \mathrm{i} \sin \left(\frac{n+1}{2} x \right) \right] \frac{\sin \frac{xn}{2}}{\sin \frac{x}{2}}. \end{aligned}$$

The imaginary part gives

$$\sin x + \cdots + \sin nx = \sin \left(\frac{n+1}{2} x \right) \frac{\sin \frac{xn}{2}}{\sin \frac{x}{2}},$$

which leads to the fact that $\sum_{n=1}^{\infty} \sin nx$ has the bounded sequence of the partial sums. The original series converges according to the Dirichlet test. ■

Example 164.

With regard to $x \in \mathbb{R}$, decide whether the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

converges absolutely or conditionally.

Solution. The convergence follows from Example 163. Now, we study the convergence of the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin nx}{n} \right|.$$

If x is zero or any (positive or negative) integer multiple of π, then we have the absolutely convergent series of the zero terms.

Otherwise,

$$\begin{aligned}\sum_{n=1}^{\infty} \left| \frac{\sin nx}{n} \right| &= \sum_{n=1}^{\infty} \frac{|\sin nx|}{n} \geq \sum_{n=1}^{\infty} \frac{\sin^2 nx}{n} \\ &= \sum_{n=1}^{\infty} \frac{1 - \cos 2nx}{2n} = \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{\cos 2nx}{n} \right),\end{aligned}$$

where we applied

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2\sin^2 \alpha, \qquad \text{i.e.,} \qquad \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}.$$

We know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and that the series $\sum_{n=1}^{\infty} \frac{\cos 2nx}{n}$ is convergent via the Dirichlet test (it can be proved as in the Example 163, where we use the real part). Thus, there exists a finite constant A such that

$$\sum_{n=1}^{\infty} \frac{1 - \cos 2nx}{2n} = +\infty - A = +\infty.$$

The series $\sum_{n=1}^{\infty} \frac{|\sin nx|}{n}$ diverges. Hence, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ converges conditionally whenever $x \neq k\pi, k \in \mathbb{Z}$. ■

Theorem 19 (Abel's test)**.**

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers. If the series $\sum_{n=1}^{\infty} a_n$ converges and if the sequence $\{b_n\}_{n=1}^{\infty}$ is monotone and convergent, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Example 165.

Using the Abel test, verify the convergence of the series

$$\sum_{n=1}^{\infty} \frac{2^n \cos \frac{1}{n}}{(-3)^n (n+1)^2}.$$

Solution. We denote

$$a_n := \frac{2^n}{(-3)^n (n+1)^2}, \qquad b_n := \cos \frac{1}{n}, \qquad n \in \mathbb{N}.$$

To prove that the series $\sum_{n=1}^{\infty} a_n$ converges, we use the absolute convergence and the (limit) ratio test. We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{3^{n+1}(n+2)^2}}{\frac{2^n}{3^n(n+1)^2}} = \frac{2}{3} \lim_{n\to\infty} \frac{(n+1)^2}{(n+2)^2} = \frac{2}{3} < 1.$$

The sequence $\{b_n\}_{n=1}^{\infty}$ is increasing and $\lim_{n\to\infty} b_n = 1$. The assumptions of the Abel test are fulfilled. Hence, the given series converges. Note that we could apply the ratio test directly to $\sum_{n=1}^{\infty} |a_n b_n|$. ■

Example 166.

Using the Abel test, verify the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\sin n \cos \frac{1}{n}}{n}.$$

Solution. We put

$$a_n := \frac{\sin n}{n}, \qquad b_n := \cos \frac{1}{n}, \qquad n \in \mathbb{N}.$$

From Examples 163 and 165 for $x = 1$, we know that the assumptions of the Abel test are fulfilled. Therefore, the series is convergent. ■

Chapter

2 Series of functions

§ 2.1 General power series

Definition 7.

Let $\{a_n\}_{n=0}^{\infty} \subset \mathbb{R}$. The infinite series of the form

$$\sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called a power series with the center 0. More generally, the series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots$$

is called a power series with the center $x_0 \in \mathbb{R}$. The values $a_0, a_1, a_2, a_3, \ldots$ are called coefficients.

Theorem 20.

For any power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots,$$

one of the following possibilities occurs.

(i) There exists $R > 0$ such that the series converges absolutely for all $x \in \mathbb{R}$ for which $|x - x_0| < R$ and it does not converge if $|x - x_0| > R$.

(ii) The series converges absolutely for all $x \in \mathbb{R}$ (we put $R := \infty$).

(iii) The series converges only for $x = x_0$ (we put $R := 0$).

Definition 8.

The number R from Theorem 20 is called the radius of the convergence of the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. The interval, where the series converges, is called the interval (the region) of the convergence.

Theorem 21 (Cauchy–Hadamard)**.**

Let

$$L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

Then, the radius of the convergence of the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is

$$R = \begin{cases} \frac{1}{L}, & L \in (0,\infty); \\ \infty, & L = 0; \\ 0, & L = \infty. \end{cases}$$

Remark. Putting $y = x - x_0$, we transform a series with the center x_0 into a series with the center $x_0 = 0$. Therefore, it is sufficient to consider only series $\sum_{n=0}^{\infty} a_n x^n$. To obtain the radius of the convergence, it suffices to use the limit (if it exists)

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \quad \text{or} \quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L.$$

For positive a_n, it holds (see also Remark § 1.2)

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \le \liminf_{n\to\infty} \sqrt[n]{|a_n|} \le \limsup_{n\to\infty} \sqrt[n]{|a_n|} \le \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Thus, if the limit $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists, then the limit $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists as well and they are of the same value.

Example 167.

Find the center and the radius R of the convergence of the power series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} (x-1)^n.$$

Solution. The center is $x_0 = 1$. We obtain the radius from

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = \mathrm{e}.$$

Hence,

$$R = \frac{1}{\mathrm{e}}.$$

■

Example 168.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} 2^n x^{2n}.$$

Solution. We have

$$a_n = \begin{cases} 2^k, & n = 2k; \\ 0, & n = 2k-1. \end{cases}$$

Putting $n = 2k$, we get $\sqrt[n]{|a_n|} = \sqrt[2k]{2^k} = \sqrt{2}$; putting $n = 2k-1$, we get $\sqrt[n]{|a_n|} = 0$. Hence,

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt{2}, & n = 2k; \\ 0, & n = 2k-1. \end{cases}$$

Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \sqrt{2}$$

and the radius is $R = 1/\sqrt{2}$. Substituting the border points $x = \pm 1/\sqrt{2}$ into the series, we find out that the necessary condition for the convergence is not fulfilled. Altogether, the interval of the convergence is

$$I = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

■

Example 169.

Find all $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{8^n \cdot n}$$

converges.

Solution. We have

$$\sum_{n=1}^{\infty} \frac{(2x+6)^n}{8^n \cdot n} = \sum_{n=1}^{\infty} \frac{1}{4^n \cdot n} (x - (-3))^n.$$

The center is $x_0 = -3$. We calculate

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4^n\cdot n}{4^{n+1}\cdot(n+1)} = \lim_{n\to\infty}\frac{n}{4\,(n+1)} = \frac{1}{4}.$$

Hence, $R = 4$.

The border points of the interval of the convergence are $x_0 - R = -7$ and $x_0 + R = 1$. For $x = -7$, we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 8^n}{8^n\cdot n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}.$$

It is an alternating series which is convergent according to the Leibniz test (see also Example 158). For $x = 1$, we have the series

$$\sum_{n=1}^{\infty}\frac{1}{n},$$

i.e., the divergent harmonic series. Altogether, the interval of the convergence is $I = [-7, 1)$.

■

Example 170.

Find the radius of the convergence of the series

$$\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}\,x^n.$$

Solution. The center is $x_0 = 0$. We calculate

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^2}{[2(n+1)]!}\cdot\frac{(2n)!}{(n!)^2} = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)\,(2n+1)} = \lim_{n\to\infty}\frac{n^2}{4n^2} = \frac{1}{4}.$$

Hence, the radius is $R = 4$. ■

Example 171.

Determine $x \in \mathbb{R}$ for which the series

$$\sum_{n=0}^{\infty}\frac{(x+1)^n}{(n+1)!}$$

is absolutely convergent.

Solution. The center is $x_0 = -1$. From

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+2)!}\right| = \lim_{n\to\infty}\frac{1}{n+2} = 0,$$

it follows $R = \infty$, i.e., the series converges for all $x \in \mathbb{R}$ absolutely. ■

Example 172.

Find the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(x-1)^n.$$

Solution. We have $x_0 = 1$ and

$$a_n = \frac{(-1)^{n+1}}{n}, \qquad n \in \mathbb{N}.$$

We obtain

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1} = 1,$$

which implies that $R = 1$. The border points of I are $x = 0$ and $x = 2$. For $x = 0$, we get the multiple of the divergent harmonic series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{-1}{n} = -\sum_{n=1}^{\infty}\frac{1}{n} = -\infty.$$

For $x = 2$, we get the convergent Leibniz series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot 1^n = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}.$$

The interval of the convergence is $I = (0, 2]$. ■

Example 173.

Determine $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty}(n+2)\,x^{n+2}$$

converges.

Solution. The center is $x_0 = 0$. We calculate

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$

which gives $R = 1$. In the border points, we have the series

$$\sum_{n=1}^{\infty} (n+2)\,(-1)^{n+2}, \qquad \sum_{n=1}^{\infty} (n+2).$$

For none of them, the necessary condition for the convergence is fulfilled. Therefore, the series converges for $x \in (-1, 1)$. ■

Example 174.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n^3} x^n.$$

Solution. We have $x_0 = 0$ and

$$a_n = \frac{(-5)^n}{n^3}, \qquad n \in \mathbb{N}.$$

We obtain

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{(n+1)^3}}{\frac{5^n}{n^3}} = \lim_{n\to\infty} \frac{5 \cdot n^3}{(n+1)^3} = 5,$$

i.e.,

$$R = \frac{1}{5}.$$

Next, we check the border points. For $x = -1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n^3} \cdot \frac{1}{(-5)^n} = \sum_{n=1}^{\infty} \frac{1}{n^3},$$

which is a convergent series (see Theorem 4). For $x = 1/5$, we have

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n^3} \cdot \frac{1}{5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3},$$

which is a convergent series according to the Leibniz test (see Example 158). The interval of the convergence is

$$I = \left[-\frac{1}{5}, \frac{1}{5} \right].$$

■

Example 175.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{(-2)^n} x^n.$$

Solution. The center is $x_0 = 0$. We have

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{2}.$$

Hence, $R = 2$. For the border points, we have the series

$$\sum_{n=1}^{\infty} \frac{1}{(-2)^n} (-2)^n = \sum_{n=1}^{\infty} 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(-2)^n} 2^n = \sum_{n=1}^{\infty} (-1)^n.$$

For them, the necessary condition for the convergence is not fulfilled. Therefore, $I = (-2, 2)$. ■

Example 176.

Determine $x \in \mathbb{R}$ for which the series

$$\sum_{n=0}^{\infty} \frac{8^n}{(n+1)!} x^n$$

converges.

Solution. The center is $x_0 = 0$ and

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{8^n}{(n+1)!} \cdot \frac{(n+2)!}{8^{n+1}} = \lim_{n\to\infty} \frac{n+2}{8} = \infty.$$

The series converges for all $x \in \mathbb{R}$. ■

Example 177.

Determine $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty}(n^2+1)^n\,(x-3)^n$$

converges.

Solution. The center is $x_0 = 3$ and the coefficients are

$$a_n = (n^2+1)^n, \qquad n \in \mathbb{N}.$$

We have

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty}(n^2+1) = \infty,$$

i.e., $R = 0$; and the series converges only in the center, i.e., for $x = 3$. ■

Example 178.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{7^n}{n+1}\,x^n.$$

Solution. We have

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{7^{n+1}}{n+2}\cdot\frac{n+1}{7^n} = \lim_{n\to\infty}\frac{7(n+1)}{n+2} = 7.$$

Hence, $R = 1/7$. The center is at 0 and the border points are $-1/7$ and $1/7$. At $-1/7$, we have the series

$$\sum_{n=1}^{\infty}\frac{7^n}{n+1}\left(-\frac{1}{7}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1},$$

which converges according to the Leibniz test; at $1/7$, we have the series

$$\sum_{n=1}^{\infty}\frac{7^n}{n+1}\left(\frac{1}{7}\right)^n = \sum_{n=1}^{\infty}\frac{1}{n+1},$$

which diverges (e.g., via the substitution $m = n+1$, we obtain the harmonic series without the first term). The interval of the convergence is $I = [-1/7, 1/7)$. ■

Example 179.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{4^n}{n^6} x^n.$$

Solution. We have $x_0 = 0$ and $a_n = 4^n/n^6$, $n \in \mathbb{N}$. Further,

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{4^{n+1}}{(n+1)^6} \cdot \frac{n^6}{4^n} = \lim_{n\to\infty} \frac{4n^6}{(n+1)^6} = 4,$$

i.e.,

$$R = \frac{1}{4}.$$

The border points are $x = -1/4$ and $x = 1/4$. For $x = -1/4$, we have the series

$$\sum_{n=1}^{\infty} \frac{4^n}{n^6} \cdot \frac{(-1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^6}$$

which converges via the Leibniz test (see also Example 158). For $x = 1/4$, we have the series

$$\sum_{n=1}^{\infty} \frac{4^n}{n^6} \cdot \frac{1}{4^n} = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

which is convergent (see Theorem 4 or directly Example 108). Altogether,

$$I = \left[-\frac{1}{4}, \frac{1}{4} \right].$$

■

Example 180.

With regard to $x \in \mathbb{R}$, determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{3n+2}{n\,4^n} x^n.$$

Solution. The center is $x_0 = 0$ and the radius is $R = 4$, because

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3(n+1)+2}{(n+1)\,4^{n+1}} \cdot \frac{n\,4^n}{3n+2} = \lim_{n\to\infty} \frac{n\,(3n+5)}{4\,(n+1)\,(3n+2)} = \frac{1}{4}.$$

For $x = -4$, we have

$$\sum_{n=1}^{\infty} \frac{3n+2}{n\,4^n}\,(-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n\,(3n+2)}{n}.$$

For $x = 4$, we have

$$\sum_{n=1}^{\infty} \frac{3n+2}{n\,4^n}\,4^n = \sum_{n=1}^{\infty} \frac{3n+2}{n}.$$

For both the series, the necessary condition for the convergence is not fulfilled. Therefore, the series converges only for $x \in (-4, 4)$. ■

Example 181.

With regard to $x \in \mathbb{R}$, determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(n+3)\,x^n}{n!}.$$

Solution. The center is $x_0 = 0$ and the coefficients are $a_n = (n+3)/n!$, $n \in \mathbb{N}$. Further,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+4}{(n+1)!} \cdot \frac{n!}{n+3} = \lim_{n\to\infty} \frac{n+4}{(n+1)\cdot(n+3)} = 0.$$

Hence, the series converges for any $x \in \mathbb{R}$. ■

Example 182.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \left(\frac{x}{2} - 1\right)^n.$$

Solution. At fist, we adjust the series as follows

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \left(\frac{x}{2} - 1\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n\,\sqrt{n}}\,(x-2)^n.$$

Then, the center is $x_0 = 2$ and

$$a_n = \frac{(-1)^n}{2^n\,\sqrt{n}}, \qquad n \in \mathbb{N}.$$

We get

$$L = \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^n\,\sqrt{n}}{2^{n+1}\,\sqrt{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n+1}} = \frac{1}{2}.$$

Hence, $R = 2$. For $x = 0$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n \sqrt{n}} \cdot (-2)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This series is divergent (see Theorem 4 or Example 108). For $x = 4$, we have the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n \sqrt{n}} \cdot 2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

which converges according to the Leibniz test (see also Example 158). The interval is $I = (0, 4]$. ■

Example 183.

Determine the convergence of the power series

$$\sum_{n=3}^{\infty} (-7x)^n \ln^n n.$$

Solution. We have $x_0 = 0$ and

$$a_n = (-7)^n \ln^n n, \qquad n \geq 3,\, n \in \mathbb{N}.$$

Hence,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 7 \ln n = +\infty$$

and $R = 0$. The series converges only for $x = 0$. ■

Example 184.

Find the region of the convergence of the series

$$\sum_{n=1}^{\infty} n^n\, (2x+4)^n.$$

Solution. At fist, we adjust the series as follows

$$\sum_{n=1}^{\infty} n^n\, (2x+4)^n = \sum_{n=1}^{\infty} (2n)^n\, (x+2)^n.$$

It has the center $x_0 = -2$ and $a_n = (2n)^n$, $n \in \mathbb{N}$. Further,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} 2n = +\infty$$

and $R = 0$. The series $\sum_{n=1}^{\infty} n^n\, (2x+4)^n$ converges only for $x = -2$. ■

Example 185.

Find the interval I of the convergence of the series

$$\sum_{n=0}^{\infty} \frac{n+5}{2^n} (x-1)^n$$

with the center $x_0 = 1$.

Solution. We have

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{n+5}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n+5}}{2} = \frac{1}{2},$$

which implies $R = 2$. The series converges for $x \in (-1,3)$ and diverges for any x such that $|x-1| > 2$. At the border points, there is not fulfilled the necessary condition for the convergence. Hence, $I = (-1,3)$. ■

Example 186.

Find the interval I of the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(3x+1)^n}{n^2\, 2^n} = \frac{3x+1}{2} + \frac{(3x+1)^2}{2^2\, 2^2} + \frac{(3x+1)^3}{3^2\, 2^3} + \cdots$$

Solution. We have

$$\sum_{n=1}^{\infty} \frac{(3x+1)^n}{n^2\, 2^n} = \sum_{n=1}^{\infty} \frac{3^n\, (x+\frac{1}{3})^n}{n^2\, 2^n}.$$

The center is

$$x_0 = -\frac{1}{3}$$

and

$$a_n = \frac{1}{n^2}\left(\frac{3}{2}\right)^n, \qquad n \in \mathbb{N}.$$

We calculate

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(n+1)^2}\left(\frac{3}{2}\right)^{n+1} \cdot n^2 \left(\frac{2}{3}\right)^n = \lim_{n\to\infty} \frac{3}{2}\left(\frac{n}{n+1}\right)^2 = \frac{3}{2}.$$

Thus, $R = 2/3$. The border points are $x_0 - R = -1$ and $x_0 + R = 1/3$. At the border points, we obtain the series

$$\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2\, 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{2^n}{n^2\, 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

The first of them converges according to the Leibniz test (see Example 158) and the convergence of the second one follows from the integral test (see Example 108). The interval is $I = [-1, 1/3]$. ■

Example 187.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{3^n}\, x^n.$$

Solution. We have $x_0 = 0$ and

$$a_n = \frac{(-1)^n \arctan n}{3^n}, \qquad n \in \mathbb{N}.$$

We calculate

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{\sqrt[n]{\arctan n}}{3} = \frac{1}{3}\left(\frac{\pi}{2}\right)^0 = \frac{1}{3},$$

i.e., $R = 3$. For $x = -3$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{3^n}\, (-3)^n = \sum_{n=1}^{\infty} \arctan n.$$

This series is not convergent because the necessary condition for the convergence is not fulfilled and

$$\sum_{n=1}^{\infty} \arctan n \geq \arctan 1 + \sum_{n=2}^{\infty} 1 = +\infty.$$

For $x = 3$, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{3^n}\, 3^n = \sum_{n=1}^{\infty} (-1)^n \arctan n,$$

which is an alternating series, but the necessary condition for the convergence is not fulfilled as well. The result is the interval $(-3, 3)$. ■

Example 188.

For $0 < \alpha < 1$, find the interval I of the convergence of the series

$$\sum_{n=1}^{\infty} \alpha^{n^2}\, x^n.$$

Solution. We have $x_0 = 0$ and $a_n = \alpha^{n^2}$ for $n \in \mathbb{N}$. Further,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \alpha^n = 0.$$

It implies $R = \infty$ and $I = (-\infty, \infty)$. ■

Example 189.

For $\beta > 1$, find the interval I of the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n!}{\beta^{n^2}} x^n.$$

Solution. We have $x_0 = 0$ and

$$a_n = \frac{n!}{\beta^{n^2}}, \qquad n \in \mathbb{N}.$$

Further,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!}{\beta^{(n+1)^2}} \cdot \frac{\beta^{n^2}}{n!} = \lim_{n\to\infty} \frac{n+1}{\beta^{2n+1}} \leq \lim_{n\to\infty} \frac{n+1}{\beta^n} = 0.$$

It implies $R = \infty$ and $I = (-\infty, \infty)$. ■

Example 190.

Find the radius R and the interval I of the convergence of the power series

$$\sum_{n=1}^{\infty} \left(\frac{1}{\operatorname{arccot}(-n)} \right)^n x^n.$$

Solution. We have $x_0 = 0$ and

$$a_n = \left(\frac{1}{\operatorname{arccot}(-n)} \right)^n, \qquad n \in \mathbb{N}.$$

We calculate

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\operatorname{arccot}(-n)} = \frac{1}{\pi},$$

which implies $R = \pi$. The border points of the interval of the convergence are $-\pi, \pi$. For $x = -\pi$, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{\operatorname{arccot}(-n)} \right)^n \cdot (-\pi)^n = \sum_{n=1}^{\infty} \frac{\pi^n (-1)^n}{\operatorname{arccot}^n(-n)},$$

for which the necessary condition for the convergence is not fulfilled, because

$$\frac{\pi^n}{\operatorname{arccot}^n(-n)} \geq 1, \qquad n \in \mathbb{N}.$$

For $x = \pi$, we have

$$\sum_{n=1}^{\infty} \left(\frac{1}{\operatorname{arccot}(-n)} \right)^n \cdot \pi^n = \sum_{n=1}^{\infty} \frac{\pi^n}{\operatorname{arccot}^n(-n)},$$

for which the necessary condition for the convergence is not fulfilled as well. Moreover,

$$\sum_{n=1}^{\infty} \frac{\pi^n}{\operatorname{arccot}^n(-n)} \geq \sum_{n=1}^{\infty} \frac{\pi^n}{\pi^n} = \sum_{n=1}^{\infty} 1 = +\infty.$$

Altogether, we have $I = (-\pi, \pi)$. ■

Example 191.

Find, where the series

$$\sum_{n=1}^{\infty} \left(\frac{-1}{\operatorname{arccot} n} \right)^n (x-5)^n$$

converges.

Solution. We have $x_0 = 5$ and

$$a_n = \left(\frac{-1}{\operatorname{arccot} n} \right)^n, \qquad n \in \mathbb{N}.$$

Since

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[n]{\frac{1}{\operatorname{arccot}^n n}} = \infty,$$

we have $R = 0$. The series converges only for $x = 5$. ■

We consider a power series. If we put $a_n = 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_0 = 0$, then we obtain the series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

It is the geometric series with $a = 1$ and $q = x$. Especially, this series converges whenever $|x| < 1$ and its sum is $1/(1-x)$. It holds

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad x \in (-1, 1).$$

We add that this series diverges to $+\infty$ for $x = 1$ and it oscillates for $x = -1$.

Example 192.

In the region of the convergence, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{5^n}.$$

Solution. We have

$$\sum_{n=1}^{\infty} \frac{(x-4)^{2n}}{5^n} = \sum_{n=1}^{\infty} \frac{[(x-4)^2]^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{(x-4)^2}{5}\right)^n.$$

For x from the interval of the convergence given by $|x-4| < \sqrt{5}$, we obtain the sum

$$\sum_{n=1}^{\infty} \left(\frac{(x-4)^2}{5}\right)^n = \frac{1}{1-\frac{(x-4)^2}{5}} - 1 = \frac{5}{5-(x-4)^2} - 1 = \frac{(x-4)^2}{5-(x-4)^2}.$$

■

Theorem 22.

Let the power series $\sum_{n=0}^{\infty} a_n x^n$ have the radius of the convergence $R > 0$, where $R \in \mathbb{R}$ or $R = \infty$. Let

$$s(x) = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots, \qquad x \in (-R, R).$$

Then, the function s has derivatives of all orders on $(-R, R)$ and

$$\begin{aligned} s'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n, \qquad x \in (-R, R). \end{aligned}$$

Further, on $(-R, R)$, it holds

$$\int s(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1} + C = a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4} + \cdots + C.$$

Theorem 23.

Let the power series $\sum_{n=0}^{\infty} a_n\, x^n$ have the radius of the convergence $R > 0$ and let

$$s(x) = \sum_{n=0}^{\infty} a_n\, x^n = a_0 + a_1\, x + a_2\, x^2 + a_3\, x^3 + a_4\, x^4 + \cdots\,, \qquad x \in (-R, R).$$

Then, for any interval $[a, b] \subseteq (-R, R)$, it holds

$$\int_a^b s(x)\,\mathrm{d}x = \sum_{n=0}^{\infty} \left(a_n \int_a^b x^n\,\mathrm{d}x \right) = \sum_{n=0}^{\infty} \left(a_n\, \frac{b^{n+1}}{n+1} \right) - \sum_{n=0}^{\infty} \left(a_n\, \frac{a^{n+1}}{n+1} \right).$$

Example 193.

Using Theorem 23, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n\, 2^n} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \frac{1}{4 \cdot 2^4} + \frac{1}{5 \cdot 2^5} + \cdots$$

Solution. We have

$$\frac{1}{(n+1)\, 2^{n+1}} = \int_0^{1/2} x^n\,\mathrm{d}x, \qquad n \in \mathbb{N} \cup \{0\}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n\, 2^n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)\, 2^{n+1}} = \sum_{n=0}^{\infty} \left(\int_0^{1/2} x^n\,\mathrm{d}x \right) = \int_0^{1/2} \left(\sum_{n=0}^{\infty} x^n \right) \mathrm{d}x$$
$$= \int_0^{1/2} \frac{1}{1-x}\,\mathrm{d}x = -\ln\left(1 - \frac{1}{2}\right) - (-\ln 1) = -\ln\frac{1}{2} = \ln 2.$$

∎

Example 194.

Find the sum of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}, \qquad x \in (-1, 1),$$

and use it to find the sum of the number series

$$\sum_{n=1}^{\infty} \frac{1}{n\, 3^n}.$$

Solution. For $x \in (-1, 1)$, we have

$$\int_0^x s^{n-1} \mathrm{d}s = \left[\frac{s^n}{n}\right]_0^x = \frac{x^n}{n}, \qquad n \in \mathbb{N},$$

and

$$\int_0^x \left(\sum_{n=1}^{\infty} s^n\right) \mathrm{d}s = \sum_{n=1}^{\infty} \left(\int_0^x s^n \,\mathrm{d}s\right), \qquad x \in (-1, 1).$$

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{x^n}{n} &= \sum_{n=1}^{\infty} \int_0^x s^{n-1} \,\mathrm{d}s = \int_0^x \left(\sum_{n=1}^{\infty} s^{n-1} \,\mathrm{d}s\right) = \int_0^x \frac{1}{1-s} \,\mathrm{d}s \\ &= \left[-\ln|1-s|\right]_0^x = -\ln|1-x| + \ln 1 = -\ln(1-x), \qquad x \in (-1, 1).\end{aligned}$$

Putting $x = 1/3$, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n\, 3^n} = -\ln\left(1 - \frac{1}{3}\right) = -\ln\frac{2}{3} = \ln\frac{3}{2}.$$

■

Example 195.

For $|x| < 1$, find the sum of the series

$$\sum_{n=1}^{\infty} n\,(n+1)\, x^n.$$

Solution. We use the sum of a geometric series. We have

$$\begin{aligned}\sum_{n=1}^{\infty} n\,(n+1)\,x^n &= \left(\sum_{n=1}^{\infty} n\,x^{n+1}\right)' = \left(x^2 \sum_{n=1}^{\infty} n\,x^{n-1}\right)' \\ &= \left(x^2\left(\sum_{n=1}^{\infty} x^n\right)'\right)' = \left(x^2\left(\frac{x}{1-x}\right)'\right)' \\ &= \left(\frac{x^2}{(1-x)^2}\right)' = \frac{2x(1-x)^2+2x^2(1-x)}{(1-x)^4} = \frac{2x}{(1-x)^3}.\end{aligned}$$

■

Example 196.

For $x \in (-1,1)$, find the sum of the series

$$\sum_{n=1}^{\infty} n\,(-1)^{n-1} x^{3n-1}.$$

Solution. We consider the function

$$s(x) = \sum_{n=1}^{\infty} n\,(-1)^{n-1} x^{3n-1}, \qquad x \in (-1,1).$$

Since

$$\int_0^x t^{3n-1}\,\mathrm{d}t = \frac{1}{3n}\left[t^{3n}\right]_0^x = \frac{1}{3n} x^{3n}, \qquad x \in (-1,1),$$

we have

$$\int_0^x s(t)\,\mathrm{d}t = \int_0^x \sum_{n=1}^{\infty} n\,(-1)^{n-1} t^{3n-1}\,\mathrm{d}t = \sum_{n=1}^{\infty} n\,(-1)^{n-1} \int_0^x t^{3n-1}\,\mathrm{d}t = \frac{1}{3}\sum_{n=1}^{\infty} (-1)^{n-1}\,x^{3n}$$

for $x \in (-1,1)$. We have the geometric series with $q = -x^3$, with the region of the convergence $(-1,1)$, and with the sum

$$\frac{x^3}{1+x^3}.$$

It holds

$$\int_0^x s(t)\,\mathrm{d}t = \frac{1}{3}\cdot\frac{x^3}{1+x^3}, \qquad x \in (-1,1).$$

Hence, we obtain

$$s(x) = \left(\frac{1}{3}\cdot\frac{x^3}{1+x^3}\right)' = \frac{x^2}{(1+x^3)^2}, \qquad x \in (-1,1).$$

Therefore,

$$\sum_{n=1}^{\infty} n\,(-1)^{n-1} x^{3n-1} = \frac{x^2}{(1+x^3)^2}, \qquad x \in (-1,1).$$

■

Example 197.

For $|x| < 1$, find the sum of the series

$$\sum_{n=1}^{\infty} n\,x^n.$$

Solution. We denote

$$s(x) = \sum_{n=1}^{\infty} n\,x^n, \qquad x \in (-1,1).$$

We proceed for $x \in (-1,1) \setminus \{0\}$. We divide s by x and we obtain

$$\frac{1}{x}\,s(x) = \sum_{n=1}^{\infty} n\,x^{n-1}.$$

Then,

$$\int \frac{1}{x}\,s(x)\,\mathrm{d}x = \int \sum_{n=1}^{\infty} n\,x^{n-1}\,\mathrm{d}x = \sum_{n=1}^{\infty} x^n + C.$$

It holds

$$\begin{aligned}\int \frac{1}{x}\,s(x)\,\mathrm{d}x &= x\sum_{n=1}^{\infty} x^{n-1} + C = |m = n-1| \\ &= x\sum_{m=0}^{\infty} x^m + C = x\,\frac{1}{1-x} + C, \qquad |x| < 1, x \neq 0.\end{aligned}$$

To obtain $s(x)$, we differentiate

$$\frac{1}{x}\,s(x) = \left[\frac{x}{1-x} + C\right]' = \frac{1}{(1-x)^2}, \qquad |x| < 1, x \neq 0,$$

$$s(x) = \frac{x}{(1-x)^2}, \qquad |x| < 1.$$

We supposed that $x \neq 0$. But, from the continuity of the functions in the equality, it follows that they are equal to each other at this point as well. ■

Example 198.

For $|x| < 1$, find the sum of the series

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

Solution. The differentiation leads to the geometric series

$$\sum_{n=1}^{\infty} x^{2n-2} = 1 + x^2 + x^4 + \cdots$$

with the quotient $q = x^2$. This series converges on $(-1, 1)$ and has the sum

$$\frac{1}{1-x^2}.$$

Denoting the sum of the original series by s, we have

$$s'(x) = \frac{1}{1-x^2}, \qquad x \in (-1, 1).$$

The radius is unchanged for the original series and

$$s(x) = \int \frac{1}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\frac{1+x}{1-x} + C, \qquad x \in (-1, 1).$$

We have to find C. Putting $x = 0$, we obtain $s(0) = 0$ which implies $C = 0$. Therefore,

$$s(x) = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad x \in (-1, 1).$$

■

Example 199.

Find the sum of the series

$$\sum_{n=1}^{\infty} n^2\, x^{n-1}$$

for $|x| < 1$.

Solution. We have

$$\sum_{n=1}^{\infty} n^2\, x^{n-1} = \left(\sum_{n=1}^{\infty} nx^n\right)' = \left(x\sum_{n=1}^{\infty} nx^{n-1}\right)' = \left(x\left(\sum_{n=1}^{\infty} x^n\right)'\right)'$$

$$= \left(x\left(\frac{x}{1-x}\right)'\right)' = \left(\frac{x}{(1-x)^2}\right)' = \frac{(1-x)^2 + 2x(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}.$$

■

Example 200.

For $x \in (-4/3, 2)$, find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(1-3x)^{n+1}}{(n+1)\,5^n}.$$

Solution. We intend to find the function s for which

$$s(x) = \sum_{n=0}^{\infty} \frac{(1-3x)^{n+1}}{(n+1)\,5^n}, \qquad x \in \left(-\frac{4}{3}, 2\right).$$

For $x \in (-4/3, 2)$, we obtain

$$s'(x) = \sum_{n=0}^{\infty} \left(\frac{(1-3x)^{n+1}}{(n+1)\,5^n}\right)' = \sum_{n=0}^{\infty} \frac{(1-3x)^n\,(-3)}{5^n} = \sum_{n=0}^{\infty} -3\left(\frac{1-3x}{5}\right)^n.$$

It is the geometric series with $a = -3$ and $q = (1-3x)/5$. Moreover, it converges for $x \in (-4/3, 2)$ with the sum

$$s'(x) = \frac{-3}{1 - \frac{1-3x}{5}} = -\frac{15}{4+3x}.$$

The integration leads to

$$s(x) = \int -\frac{15}{4+3x}\,\mathrm{d}x = -5\ln(4+3x) + C, \qquad x \in \left(-\frac{4}{3}, 2\right).$$

We have to find C. Putting $x = 1/3$, we obtain

$$0 = -5\ln 5 + C, \qquad \text{i.e.,} \qquad C = 5\ln 5.$$

Hence, for $x \in (-4/3, 2)$, we have

$$s(x) = \sum_{n=0}^{\infty} \frac{(1-3x)^{n+1}}{(n+1)\,5^n} = -5\ln(4+3x) + 5\ln 5 = 5\,(-\ln(4+3x) + \ln 5) = 5\ln\frac{5}{4+3x}.$$

■

Example 201.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{x^{4n+1}}{4n+1}$$

for $x \in (-1, 1)$.

Solution. We denote

$$s(x)=\sum_{n=1}^{\infty}\frac{x^{4n+1}}{4n+1},\qquad x\in(-1,1).$$

Then,

$$s'(x)=\sum_{n=1}^{\infty}x^{4n}=\frac{x^4}{1-x^4},\qquad x\in(-1,1).$$

It implies

$$\begin{aligned}s(x)&=\int\frac{x^4}{1-x^4}\,\mathrm{d}x\\&=\int\left(-1+\frac{1}{1-x^4}\right)\mathrm{d}x\\&=\int\left(-1+\frac{1}{2}\frac{1}{1-x^2}+\frac{1}{2}\frac{1}{1+x^2}\right)\mathrm{d}x\\&=-x+\frac{1}{4}\ln\frac{1+x}{1-x}+\frac{1}{2}\arctan x+C.\end{aligned}$$

The choice $x=0$ leads to $C=0$. Therefore,

$$\sum_{n=1}^{\infty}\frac{x^{4n+1}}{4n+1}=-x+\frac{1}{4}\ln\frac{1+x}{1-x}+\frac{1}{2}\arctan x,\qquad x\in(-1,1).$$

■

Example 202.

For $|x|<1$, find the sum of the series

$$\sum_{n=0}^{\infty}(-1)^n\,(8n+1)\,x^{8n}.$$

Solution. The series is close to the geometric series with the quotient $q=-x^8$, i.e.,

$$\sum_{n=0}^{\infty}\left(-x^8\right)^n.$$

For $|x|<1$, we have

$$s(x)=\sum_{n=0}^{\infty}\left(-x^8\right)^n=\frac{1}{1+x^8}.$$

Multiplying the both sides by x, we get

$$\sum_{n=0}^{\infty}(-1)^n\,x^{8n+1}=\frac{x}{1+x^8},\qquad |x|<1.$$

We differentiate. The result is

$$\sum_{n=0}^{\infty}(-1)^n(8n+1)\,x^{8n}=\frac{\left(1+x^8\right)-x\left(8x^7\right)}{\left(1+x^8\right)^2}=\frac{1-7x^8}{\left(x^8+1\right)^2},\qquad |x|<1.$$

■

Example 203.

For $|x|<1$, find the sum of the series

$$\sum_{n=1}^{\infty}n(n+2)x^n.$$

Solution. We know that

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x},\qquad x\in(-1,1).$$

Thus,

$$\sum_{n=1}^{\infty}n\,x^{n-1}=\left(\frac{1}{1-x}\right)',\qquad x\in(-1,1),$$

$$\sum_{n=1}^{\infty}n\,x^{n+2}=x^3\cdot\left(\frac{1}{1-x}\right)',\qquad x\in(-1,1),$$

$$\sum_{n=1}^{\infty}n(n+2)\,x^{n+1}=\left(x^3\cdot\left(\frac{1}{1-x}\right)'\right)',\qquad x\in(-1,1).$$

Therefore, for $|x|<1$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}n(n+2)\,x^n&=\frac{1}{x}\left(x^3\cdot\left(\frac{1}{1-x}\right)'\right)'=\frac{1}{x}\left(x^3\cdot\frac{1}{(1-x)^2}\right)'\\&=\frac{1}{x}\,\frac{3x^2(1-x)^2+2x^3(1-x)}{(1-x)^4}=\frac{3x(1-x)+2x^2}{(1-x)^3}=\frac{3x-x^2}{(1-x)^3}.\end{aligned}$$

■

Example 204.

Using the sum of an appropriate power series, find the sum of the number series

$$\sum_{n=1}^{\infty}\frac{n}{2^n}.$$

Solution. For $x \in (-1, 1)$, the sum of the power series

$$\sum_{n=1}^{\infty} n\, x^n$$

is

$$\begin{aligned}\sum_{n=1}^{\infty} n\, x^n &= x \cdot \sum_{n=1}^{\infty} n\, x^{n-1} = x \cdot \left(\sum_{n=1}^{\infty} n \int x^{n-1}\,\mathrm{d}x\right)' \\ &= x \cdot \left(\sum_{n=1}^{\infty} x^n\right)' = x \cdot \left(\frac{x}{1-x}\right)' = x \cdot \frac{1-x+x}{(1-x)^2} = \frac{x}{(1-x)^2}.\end{aligned}$$

Putting $x = 1/2$, we get

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \frac{1}{2} \cdot \frac{4}{1} = 2.$$

■

Theorem 24 (Ábel)**.**

Let the power series $\sum_{n=0}^{\infty} a_n\, x^n$ have the radius of the convergence $R \in (0, \infty)$ and let

$$s(x) = \sum_{n=0}^{\infty} a_n\, x^n, \qquad x \in (-R, R).$$

Then, the sum s of this series is a continuous function on $(-R, R)$.

(i) If the given series is convergent at $x = R$, then the sum s is continuous from left at $x = R$, i.e.,

$$\sum_{n=0}^{\infty} a_n\, R^n = \lim_{x \to R^-} s(x).$$

(ii) If the given series is convergent at $x = -R$, then the sum s is continuous from right at $x = -R$, i.e.,

$$\sum_{n=0}^{\infty} a_n\, (-R)^n = \lim_{x \to -R^+} s(x).$$

Example 205.

Find the sum of the Leibniz series

$$\sum_{n=1}^{\infty} (-1)^{n-1}\, \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots,$$

which is known to be convergent (via the Leibniz test).

Solution. We consider

$$s(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

and the geometric series (the derivative)

$$\sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1} = \sum_{m=0}^{\infty} (-1)^m x^m.$$

This geometric series converges for $(-1, 1)$ with the sum

$$\frac{1}{1+x}.$$

The original series converges for $(-1, 1)$ with the sum

$$s(x) = \int \frac{1}{1+x}\, \mathrm{d}x = \ln(x+1) + C.$$

Putting $x = 0$, we obtain

$$s(0) = 0 = \ln(0+1) + C, \qquad \text{i.e.,} \qquad C = 0.$$

The sum is $s(x) = \ln(x+1)$. Moreover, the series converges also for $x = 1$. Hence, the sum s is continuous from left at this point. Therefore,

$$s(1) = \lim_{x \to 1^-} \ln(x+1) = \ln 2,$$

which is the sum of the Leibniz series. ■

§ 2.2 Taylor series

Definition 9.

Let a function f have derivatives of all orders at x_0. Then, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x-x_0)^3 + \frac{f^{(4)}(x_0)}{4!}(x-x_0)^4 + \cdots$$

is called the Taylor series of f; i.e., the Taylor series is the power series with the center x_0 and the coefficients

$$a_n = \frac{f^{(n)}(x_0)}{n!}, \qquad n \in \mathbb{N} \cup \{0\}.$$

For $x_0 = 0$, it is called the Maclaurin series.

Example 206.

Find the Taylor series of $f(x) = \mathrm{e}^x$ at $x_0 = 0$ and determine its convergence with regard to $x \in \mathbb{R}$.

Solution. We calculate

$$\begin{aligned} f(x) &= \mathrm{e}^x, \\ f'(x) &= \mathrm{e}^x, \\ &\vdots \\ f^{(n)}(x) &= \mathrm{e}^x \end{aligned}$$

and (consider $x = 0$)

$$\begin{aligned} f(0) &= 1, \\ f'(0) &= 1, \\ &\vdots \\ f^{(n)}(0) &= 1. \end{aligned}$$

The Taylor series of $f(x) = \mathrm{e}^x$ at $x_0 = 0$ is

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n.$$

To determine its convergence, we use the quotient test

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}\,x^{n+1}}{\frac{1}{n!}\,x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

The series converges for all $x \in \mathbb{R}$. ■

Example 207.

Find the Taylor series of $f(x) = 1/x$ with the center $x_0 = 2$.

Solution. For $x \neq 0$, we have

$$\begin{aligned} f'(x) &= -x^{-2}, \\ f''(x) &= 2\,x^{-3}, \\ f'''(x) &= -6\,x^{-4}, \\ f^{(4)}(x) &= 24\,x^{-5}, \\ &\vdots \\ f^{(n)}(x) &= (-1)^n\,n!\,x^{-(n+1)}. \end{aligned}$$

Hence, for $x_0 = 2$,

$$\begin{aligned} f(2) &= \frac{1}{2} = \frac{0!}{2^1}, \\ f'(2) &= -\frac{1}{4} = -\frac{1!}{2^2}, \\ f''(2) &= \frac{2}{8} = \frac{2!}{2^3}, \\ f'''(2) &= -\frac{6}{16} = -\frac{3!}{2^4}, \\ f^{(4)}(2) &= \frac{24}{32} = \frac{4!}{2^5}, \\ &\vdots \\ f^{(n)}(2) &= \frac{(-1)^n\,n!}{2^{n+1}}. \end{aligned}$$

The coefficients a_n are

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}, \qquad n \in \mathbb{N} \cup \{0\},$$

and the Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}\,(x-2)^n.$$

■

Example 208.

Find the Taylor series of $f(x) = 3^x$ with the center $x_0 = 0$.

Solution. We have

$$\begin{aligned} f(x) &= 3^x, \\ f'(x) &= 3^x \ln 3, \\ f''(x) &= 3^x \ln^2 3, \\ &\vdots \\ f^{(n)}(x) &= 3^x \ln^n 3. \end{aligned}$$

Hence,

$$\begin{aligned} f(0) &= 1, \\ f'(0) &= \ln 3, \\ f''(0) &= \ln^2 3, \\ &\vdots \\ f^{(n)}(0) &= \ln^n \, 3. \end{aligned}$$

The resulting series is

$$1 + \frac{\ln 3}{1!}\,x + \frac{\ln^2 3}{2!}\,x^2 + \frac{\ln^3 3}{3!}\,x^3 + \cdots + \frac{\ln^n 3}{n!}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{\ln^n 3}{n!}\,x^n.$$

■

Example 209.

Find the Taylor series of $f(x) = 2x^2 + 4x - 3$ with the center $x_0 = 1$.

Solution. For $x \in \mathbb{R}$, we have $f'(x) = 4x + 4$, $f''(x) = 4$, and $f^{(n)}(x) = 0$ for $n > 2$. Hence, $f(1) = 3$, $f'(1) = 8$, $f''(1) = 4$ and the Taylor series is $3 + 8(x-1) + 2(x-1)^2$. ■

Example 210.

For the function

$$f(x) := \begin{cases} \mathrm{e}^{-\frac{1}{x^2}}, & x \neq 0; \\ 0, & x = 0, \end{cases}$$

find the Maclaurin series.

Solution. The function has all derivatives at all points of the real axis and all derivatives are equal to 0 at $x_0 = 0$, which follows from

$$\lim_{x\to 0} P\left(\frac{1}{x}\right) \mathrm{e}^{-\frac{1}{x^2}} = 0,$$

where P is an arbitrary polynomial. The Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\, x^n = 0 + 0\cdot x + \frac{0}{2}\, x^2 + \frac{0}{3!}\, x^3 + \cdots \equiv 0.$$

The Maclaurin series is the zero series which is different from the original function at any non-zero point. ■

Example 211.

Find the power series with the center at 0, whose sum is

$$\frac{1}{x^2 - x - 12}$$

for $x \in (-3, 3)$.

Solution. From

$$\frac{1}{x^2 - x - 12} = \frac{1}{(x-4)(x+3)} = \frac{1}{7}\left(\frac{1}{x-4} - \frac{1}{x+3}\right)$$

and from

$$\frac{1}{x-4} = -\frac{\frac{1}{4}}{1 - \frac{x}{4}} = -\frac{1}{4}\left(1 + \frac{x}{4} + \frac{x^2}{4^2} + \cdots + \frac{x^n}{4^n} + \cdots\right),$$

$$\frac{1}{x+3} = \frac{\frac{1}{3}}{1 - \left(-\frac{x}{3}\right)} = \frac{1}{3}\left(1 - \frac{x}{3} + \frac{x^2}{3^2} + \cdots + \frac{(-x)^n}{3^n} + \cdots\right),$$

we have

$$\frac{1}{x^2 - x - 12} = -\frac{1}{28}\sum_{n=0}^{\infty}\frac{x^n}{4^n} - \frac{1}{21}\sum_{n=0}^{\infty}\frac{(-x)^n}{3^n} = -\sum_{n=0}^{\infty}\left(\frac{1}{28\cdot 4^n} + \frac{(-1)^n}{21\cdot 3^n}\right) x^n$$

for any $x \in (-3, 3)$. ■

It is often more effective to use known series of elementary functions than to compute the Taylor series via the definition (the derivatives). In the upcoming examples, there is used the following fact. If one can express a function as the sum of a power series, then the series is uniquely determined and it is the Taylor series.

Theorem 25.

It holds

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad x \in \mathbb{R}, \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \qquad x \in \mathbb{R}, \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \qquad x \in \mathbb{R}, \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad x \in (-1,1), \\
\ln(1+x) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad x \in (-1,1], \\
(c+x)^K &= \sum_{n=0}^{\infty} \binom{K}{n} c^{K-n} x^n \\
&= \sum_{n=0}^{\infty} \frac{K(K-1)\cdots(K-n+1)}{n!} c^{K-n} x^n, \qquad x \in (-c,c),
\end{aligned}$$

where $c > 0$ and $K \in \mathbb{R}$. The last series is called the binomial series.

Example 212.

Using the expansion of e^x, find the Maclaurin series of $f(x) = x^2 e^x$.

Solution. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for any $x \in \mathbb{R}$. Therefore,

$$f(x) = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}, \qquad x \in \mathbb{R}.$$

■

Example 213.

Find the Maclaurin series of $f(x) = e^{x^2}$.

Solution. We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Hence,

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}.$$

■

Example 214.

For

$$f(x) = \sum_{n=0}^{\infty} 2^n x^n, \qquad x \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

find $f^{(33)}(0)$.

Solution. The n-th coefficient of the series is

$$a_n = \frac{f^{(n)}(0)}{n!}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Thus,

$$f^{(33)}(0) = 33!\, a_{33} = 33!\, 2^{33}.$$

■

Example 215.

Using the binomial series find the Taylor series of $f(x) = \sqrt{1+x^2}$ at 0.

Solution. We use the binomial series with $c = 1$, $K = 1/2$, and x^2. Then,

$$\sqrt{1+x^2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^{2n} = \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} x^{2n}.$$

The coefficients are

$$a_{2n} = \binom{1/2}{n}, \qquad a_{2n+1} = 0, \qquad n \in \mathbb{N} \cup \{0\}.$$

■

Example 216.

Find the Maclaurin series of $f(x) = \sinh x$.

Solution. The function is defined by

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}, \qquad x \in \mathbb{R}.$$

Hence,

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{2} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{1}{2} \frac{(-1)^n x^n}{n!},$$

which implies

$$a_{2n-2} = 0, \qquad a_{2n-1} = \frac{1}{(2n-1)!}, \qquad n \in \mathbb{N}.$$

Therefore, the series is

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

■

Example 217.

Find the Maclaurin series of $f(x) = \cosh x$.

Solution. The function is defined by

$$\cosh x = \frac{\mathrm{e}^x + \mathrm{e}^{-x}}{2}, \qquad x \in \mathbb{R}.$$

Hence,

$$f(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \frac{(-x)^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{2n!} x^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^n, \qquad x \in \mathbb{R}.$$

If we write it as one series, then we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{1 + (-1)^n}{2n!} x^n, \qquad x \in \mathbb{R}.$$

We have

$$a_{2n+1} = 0, \qquad a_{2n} = \frac{1}{(2n)!}, \qquad n \in \mathbb{N} \cup \{0\},$$

and, therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \qquad x \in \mathbb{R}.$$

■

Example 218.

Find the Maclaurin series of $f(x) = \sin^2 x$.

Solution. We have

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

Thus, we can use the expansion of $\cos x$ with the argument $2x$. We obtain

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = -\frac{1}{2} \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n}}{(2n)!} x^{2n}.$$

■

Example 219.

Find the Maclaurin series whose sum is

$$x^3 \cos x$$

for any $x \in \mathbb{R}$.

Solution. We use the expansion of the cosine function. For $x \in \mathbb{R}$, we have

$$x^3 \cos x = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n+3}.$$

■

Example 220.

Find the Maclaurin series whose sum is

$$1 + x^2 \sin x$$

for any $x \in \mathbb{R}$.

Solution. We use the expansion of the sine function. For $x \in \mathbb{R}$, we have

$$1 + x^2 \sin x = 1 + x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+3}.$$

■

Example 221.

Find the Maclaurin series whose sum is equal to the function

$$(x+2)^2$$

for any $x \in \mathbb{R}$.

Solution. Immediately, we obtain

$$(x+2)^2 = x^2 + 4x + 4.$$

■

Example 222.

Find the Maclaurin series whose sum is

$$\sin(5x)$$

for any $x \in \mathbb{R}$.

Solution. For $x \in \mathbb{R}$, it holds

$$\sin(5x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (5x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n+1}}{(2n+1)!} x^{2n+1}.$$

■

Example 223.

Find the Maclaurin series whose sum is equal to the function

$$\sin^2 x - \cos^2 x$$

for any $x \in \mathbb{R}$.

Solution. We use the expansion of the cosine function. For $x \in \mathbb{R}$, we have

$$\sin^2 x - \cos^2 x = -\cos(2x) = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (2x)^{2n} = -\sum_{n=0}^{\infty} \frac{(-4)^n}{(2n)!} x^{2n}.$$

■

Example 224.

Find the Maclaurin series whose sum is equal to the function

$$\cos\left(x^4\right)$$

for any $x \in \mathbb{R}$.

Solution. For $x \in \mathbb{R}$, we can see

$$\cos\left(x^4\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(x^4\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{8n}.$$

■

Example 225.

Find the Maclaurin series whose sum is equal to the function

$$2\sin\left(-x^2\right)$$

for any $x \in \mathbb{R}$.

Solution. For $x \in \mathbb{R}$, we have

$$2\sin\left(-x^2\right) = -2\sin\left(x^2\right) = -2\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(x^2\right)^{2n+1} = 2\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{4n+2}.$$

■

Example 226.

Find the Maclaurin series whose sum is equal to the function

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x$$

for any $x \in \mathbb{R}$.

Solution. For $x \in \mathbb{R}$, it holds

$$1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = -\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

■

Example 227.

Write down first terms of the Maclaurin series of $f(x) = e^x \sin x$.

Solution. We immediately obtain

$$f(x) = e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$
$$\times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots\right)$$
$$= x + x^2 + x^3\left(\frac{1}{2!} - \frac{1}{3!}\right) + x^4\left(-\frac{1}{3!} + \frac{1}{3!}\right) + \cdots$$

■

Example 228.

Expand the function

$$f(x) = \frac{1}{(1+x)^2}$$

into the Taylor series with the center at 0.

Solution. Since

$$\frac{1}{(1+x)^2} = -\left(\frac{1}{1+x}\right)', \qquad x \neq -1,$$

and since (for $|x| < 1$)

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n,$$

we obtain (for $|x| < 1$)

$$\frac{1}{(1+x)^2} = -\left(\sum_{n=0}^{\infty} (-1)^n x^n\right)',$$

$$\frac{1}{(1+x)^2} = -\sum_{n=1}^{\infty} (-1)^n n\, x^{n-1},$$

$$\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^{n-1} n\, x^{n-1} = \sum_{n=0}^{\infty} (-1)^n (n+1)\, x^n.$$

The expansion is

$$\sum_{n=0}^{\infty} (-1)^n (n+1)\, x^n.$$

■

Example 229.

Using an appropriate function and its expansion, find the Taylor series of

$$f(x) = x \sin x$$

with the center at 0.

Solution. It holds

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}, \qquad x \in \mathbb{R}.$$

Therefore,

$$x \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+2}, \qquad x \in \mathbb{R}.$$

■

Example 230.

Using an appropriate function and its expansion, find the Taylor series of

$$f(x) = \frac{1}{(1+x)^3}$$

with the center at 0.

Solution. We know that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n\, x^n, \qquad |x| < 1.$$

For $|x| < 1$, we obtain

$$-\frac{1}{(1+x)^2} = \sum_{n=1}^{\infty} (-1)^n\, n\, x^{n-1},$$

$$\frac{2}{(1+x)^3} = \sum_{n=2}^{\infty} (-1)^n\, n\,(n-1)\, x^{n-2},$$

$$\frac{1}{(1+x)^3} = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n+1}\,(n+1)\, n\, x^{n-1}.$$

■

Example 231.

Using the Maclaurin series, find the indefinite integral

$$\int \frac{\sin x}{x}\,\mathrm{d}x.$$

Solution. It holds

$$\begin{aligned}\int \frac{\sin x}{x}\,\mathrm{d}x &= \int \left(1 - \frac{1}{3!}\,x^2 + \frac{1}{5!}\,x^4 - \frac{1}{7!}\,x^6 + \cdots\right)\mathrm{d}x \\ &= x - \frac{1}{3!}\,\frac{x^3}{3} + \frac{1}{5!}\,\frac{x^5}{5} - \frac{1}{7!}\,\frac{x^7}{7} + \cdots + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(2n+1)}\,x^{2n+1} + C, \qquad x \in \mathbb{R}.\end{aligned}$$

Note that the result is the so-called higher transcendental function. ■

Example 232.

Using the Maclaurin series, find the indefinite integral

$$\int \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Solution. It holds

$$\begin{aligned}\int \mathrm{e}^{-x^2}\,\mathrm{d}x &= \int \left(1 - x^2 + \frac{1}{2!}\,x^4 - \frac{1}{3!}\,x^6 + \frac{1}{4!}\,x^8 - \frac{1}{5!}\,x^{10} + \cdots\right)\mathrm{d}x \\ &= x - \frac{x^3}{3} + \frac{1}{2!}\,\frac{x^5}{5} - \frac{1}{3!}\,\frac{x^7}{7} + \frac{1}{4!}\,\frac{x^9}{9} - \frac{1}{5!}\,\frac{x^{11}}{11} + \cdots + C \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)}\,x^{2n+1} + C, \qquad x \in \mathbb{R}.\end{aligned}$$

Note that the result is the so-called higher transcendental function. ■

Example 233.

Find the Taylor expansion of $f(x) = \ln(1-x)$ at $x_0 = 0$.

Solution. It holds

$$f'(x) = -\frac{1}{1-x}, \qquad x \in (-1,1),$$

and we know that

$$\frac{1}{1-s} = \sum_{n=0}^{\infty} s^n, \qquad s \in (-1,1).$$

Integrating from 0 to x, we obtain

$$\int_0^x \frac{1}{1-s}\,\mathrm{d}s = \sum_{n=0}^{\infty} \int_0^x s^n\,\mathrm{d}s$$

for any $x \in (-1,1)$. This implies

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\,x^{n+1} = \sum_{m=1}^{\infty} \frac{1}{m}\,x^m, \qquad x \in (-1,1).$$

Hence,

$$\ln(1-x) = -\sum_{m=1}^{\infty} \frac{1}{m}\,x^m, \qquad x \in (-1,1).$$

According to Theorem 24, it holds on $[-1,1)$, i.e.,

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n}\,x^n, \qquad x \in [-1,1).$$

Putting $y = -x$, we obtain the Maclaurin expansion

$$\ln(1+y) = -\sum_{n=1}^{\infty} \frac{1}{n}\,(-y)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\,y^n, \qquad y \in (-1,1],$$

which is a known formula that one could use to obtain the result quicker (it suffices to put $y = -z$). ■

Example 234.

Using the solution of Example 233, find the sum f of the series

$$f(x) := \sum_{n=0}^{\infty} \frac{(x+1)^n}{n+1}$$

for $x \in [-2,0)$.

Solution. According to the solution of Example 233, we have

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n}\, x^n, \qquad x \in [-1, 1).$$

Putting $t = x + 1$, we obtain

$$f(t-1) = \sum_{n=0}^{\infty} \frac{t^n}{n+1}, \qquad t \in [-1, 1).$$

Further, we proceed

$$t\, f(t-1) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{t^n}{n} = -\ln(1-t), \qquad t \in [-1, 1).$$

Returning to x, we have

$$(x+1)\, f(x) = -\ln(-x), \qquad x \in [-2, 0),$$

i.e.,

$$f(x) = -\frac{\ln(-x)}{x+1}, \qquad x \in [-2, 0),\ x \neq -1.$$

Using the original series, we find out that $f(-1) = 1$. ■

Example 235.

Without using the solution of Example 233, find the sum f of the series

$$f(x) := \sum_{n=0}^{\infty} \frac{(x+1)^n}{n+1}$$

for $x \in [-2, 0)$.

Solution. We multiply the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^n}{n+1}, \qquad x \in [-2, 0),$$

by $x + 1$, we differentiate, and simplify. For $x \in (-2, 0)$, we obtain

$$(x+1)\, f(x) = \sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{n+1},$$

$$[(x+1)\, f(x)]' = \left(\sum_{n=0}^{\infty} \frac{(x+1)^{n+1}}{n+1} \right)' = \sum_{n=0}^{\infty} \left(\frac{(x+1)^{n+1}}{n+1} \right)',$$

$$[(x+1)\,f(x)]' = \sum_{n=0}^{\infty}(x+1)^n = \frac{1}{1-(x+1)},$$

$$[(x+1)\,f(x)]' = -\frac{1}{x}.$$

Hence,

$$(x+1)\,f(x) = -\int \frac{\mathrm{d}x}{x} = -\ln|x| + C, \qquad x \in (-2,0).$$

Putting $x=-1$, we obtain $C=0$. Using Theorem 24, we get

$$f(x) = -\frac{\ln(-x)}{x+1}, \qquad x \in [-2,0),\ x \neq -1.$$

Using the original series, we find out that $f(-1)=1$. ■

Example 236.

Find the Maclaurin series of $f(x) = \arcsin x$.

Solution. We use the binomial expansion to get

$$\begin{aligned}
(\arcsin x)' &= \frac{1}{\sqrt{1-x^2}} = \left| t = x^2,\ (1-t)^{-1/2} = (1+(-t))^{-1/2} \right| \\
&= \binom{\frac{-1}{2}}{0} + \binom{\frac{-1}{2}}{1}(-t) + \cdots + \binom{\frac{-1}{2}}{n}(-t)^n + \cdots \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{2n-1}{2})}{n!}\,t^n \\
&= 1 + \sum_{n=1}^{\infty}(-1)^n(-1)^n \frac{1\cdot 3\cdots(2n-1)}{2^n n!}\,t^n \\
&= 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}\,t^n = 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}\,x^{2n}, \qquad x \in (-1,1).
\end{aligned}$$

Thus,

$$\arcsin x = \int_0^x 1 + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}\,s^{2n}\,\mathrm{d}s = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!(2n+1)}x^{2n+1}$$

for $x \in (-1,1)$.

Alternatively, for $x \in (-1,1)$, we know that

$$\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}, \qquad \sin(\arcsin x) = x.$$

Therefore,

$$x = \arcsin x - \frac{\arcsin^3 x}{3!} + \frac{\arcsin^5 x}{5!} + \cdots$$

If

$$\arcsin x = a_1 x + a_3 x^3 + a_5 x^5 + \cdots ,$$

then

$$x = \left(a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right) - \frac{1}{3!}\left(a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right)^3 + \frac{1}{5!}\left(a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right)^5 + \cdots$$

We obtain

$$\begin{aligned} x^1 : &\qquad 1 = a_1, \\ x^3 : &\qquad 0 = a_3 - \frac{a_1^3}{3!}, \qquad \text{i.e.,} \qquad a_3 = \frac{1}{6}, \\ x^5 : &\qquad 0 = a_5 - \frac{3}{3!}\, a_1^2 a_3 + \frac{a_1}{5!}, \qquad \text{i.e.,} \qquad a_5 = \frac{3}{40}, \\ &\qquad \vdots \end{aligned}$$

■

Example 237.

Write down first terms of the Maclaurin series of $f(x) = \tan x$.

Solution. It holds

$$f(x) = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots} = c_1 x + c_3 x^3 + c_5 x^5 + \cdots$$

Multiplying by the cosine function, we obtain

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) \cdot \left(c_1 x + c_3 x^3 + c_5 x^5 + \cdots\right).$$

Hence,

$$\begin{aligned} x^1 : &\qquad 1 = c_1, \\ x^3 : &\qquad -\frac{1}{3!} = c_3 - \frac{c_1}{2!} = c_3 - \frac{1}{2}, \qquad \text{i.e.,} \qquad c_3 = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}, \\ x^5 : &\qquad \frac{1}{5!} = \frac{c_1}{4!} - \frac{c_3}{2!} + c_5, \qquad \text{i.e.,} \qquad c_5 = \frac{2}{15}, \\ &\qquad \vdots \end{aligned}$$

and

$$f(x) = x + \frac{1}{3}\, x^3 + \frac{2}{15}\, x^5 + \cdots$$

■

Example 238.

Find the Taylor series of

$$f(x) = \frac{2x+3}{4x-2}$$

with the center at $x_0 = 1$.

Solution. We use

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots, \qquad x \in (-1, 1).$$

Since

$$f(x) = \frac{1}{2} + \frac{4}{4x-2} = \frac{1}{2} + \frac{4}{4(x-1)+2} = \frac{1}{2} + \frac{2}{1+2(x-1)} = \frac{1}{2} + 2 \cdot \frac{1}{1-[-2(x-1)]},$$

we have

$$f(x) = \frac{1}{2} + 2 \cdot \sum_{n=0}^{\infty} [-2(x-1)]^n = \frac{1}{2} + \sum_{n=0}^{\infty} 2 \cdot (-2)^n \, (x-1)^n$$

for $|x-1| < 1/2$. Thus,

$$f(x) = \frac{1}{2} + \left(2 + \sum_{n=1}^{\infty} (-1)^n \, 2^{n+1} \, (x-1)^n \right) = \frac{5}{2} + \sum_{n=1}^{\infty} (-1)^n \, 2^{n+1} \, (x-1)^n,$$

whenever

$$|x-1| < \frac{1}{2}.$$

■

Example 239.

Expand the function $f(x) = x \, \mathrm{e}^{2x}$ to the Taylor series at $x_0 = 1$.

Solution. We use

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in \mathbb{R},$$

where the center is $x_0 = 0$. We write

$$f(x) = ((x-1)+1) \, \mathrm{e}^{2(x-1)+2},$$

$$f(x) = (x-1) \, \mathrm{e}^2 \cdot \mathrm{e}^{2(x-1)} + \mathrm{e}^2 \cdot \mathrm{e}^{2(x-1)},$$

and

$$
\begin{aligned}
f(x) &= (x-1)\,\mathrm{e}^2 \sum_{n=0}^{\infty} \frac{[2(x-1)]^n}{n!} + \mathrm{e}^2 \sum_{n=0}^{\infty} \frac{[2(x-1)]^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^{n+1} + \sum_{n=0}^{\infty} \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^n.
\end{aligned}
$$

Since the original Taylor expansion holds for any $y = 2(x-1)$, the above one holds for any x. Next, we rewrite it as one series. For $m = n+1$ and $x \in \mathbb{R}$, it holds

$$
\begin{aligned}
f(x) &= \sum_{m=1}^{\infty} \frac{\mathrm{e}^2\, 2^{m-1}}{(m-1)!}\,(x-1)^m + \sum_{n=0}^{\infty} \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^n \\
&= \sum_{n=1}^{\infty} \frac{\mathrm{e}^2\, 2^{n-1}}{(n-1)!}\,(x-1)^n + \sum_{n=0}^{\infty} \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} \frac{\mathrm{e}^2\, 2^{n-1}}{(n-1)!}\,(x-1)^n + \left(\sum_{n=1}^{\infty} \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^n + \frac{\mathrm{e}^2\, 2^0}{0!}\,(x-1)^0 \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{\mathrm{e}^2\, 2^{n-1}}{(n-1)!}\,(x-1)^n + \frac{\mathrm{e}^2\, 2^n}{n!}\,(x-1)^n \right) + \mathrm{e}^2 \\
&= \sum_{n=1}^{\infty} \left(\frac{\mathrm{e}^2\, 2^{n-1}}{(n-1)!} + \frac{\mathrm{e}^2\, 2^n}{n!} \right) (x-1)^n + \mathrm{e}^2 \\
&= \mathrm{e}^2 + \sum_{n=1}^{\infty} \frac{\mathrm{e}^2\, 2^{n-1}\,(n+2)}{n!}\,(x-1)^n, \qquad x \in \mathbb{R}.
\end{aligned}
$$

■

The functions, which are equal to their Taylor expansions, are called analytical. Such functions can be approximated by the partial sums of their Taylor series. The way, how to estimate errors given by such approximations, is described in the following theorem.

Theorem 26 (Taylor)**.**

Let a function f have derivatives of all orders on (a,b) and let $x_0 \in (a,b)$. Then, for any $x \in (a,b)$, there exists $c \in (a,b)$ such that

$$
\begin{aligned}
f(x) = f(x_0) + f'(x_0)\,(x-x_0) + \frac{f''(x_0)}{2}\,(x-x_0)^2 + \frac{f'''(x_0)}{3!}\,(x-x_0)^3 + \cdots \\
+ \frac{f^{(n)}(x_0)}{n!}\,(x-x_0)^n + R_n(x),
\end{aligned}
$$

where

$$
R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-x_0)^{n+1}.
$$

Example 240.

Determine x from a neighbourhood of 0 for which $\cos x$ can be replaced by the first three non-zero terms of its Maclaurin series with an error less than $0.000\,05$.

Solution. The Maclaurin series of $f(x) = \cos x$ is

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

The first three non-zero terms are

$$1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

We use the Taylor theorem. We want to have

$$|R_5(x)| < 0.000\,05.$$

Since

$$\left|f^{(6)}(c)\right| = |-\cos c| \leq 1, \qquad c \in \mathbb{R},$$

we have

$$|R_5(x)| = \frac{\left|f^{(6)}(c)\right|}{6!} |x|^6 \leq \frac{|x|^6}{720}.$$

We need

$$\frac{|x|^6}{720} < 0.000\,05,$$

i.e.,

$$|x| < \sqrt[6]{0.036} \approx 0.574\,6.$$

Note that a larger neighbourhood can be found via the error estimate based on the Leibniz test for alternating number series. ■

Example 241.

Estimate the error of the approximation of $\sqrt{\mathrm{e}}$ by the first four terms of the Maclaurin series of e^x.

Solution. The Maclaurin series of e^x is

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

We approximate $\sqrt{\mathrm{e}}$ by

$$1 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} \approx 1.645\,8.$$

The error $|R_3(x)|$ is

$$\frac{\left|f^{(4)}(c)\right|}{4!}\left(\frac{1}{2}\right)^4$$

for some c between 0 and 1/2. Since

$$f^{(4)}(x) = \mathrm{e}^x, \qquad x \in \mathbb{R},$$

the error is

$$\frac{\mathrm{e}^c}{384}, \qquad \text{where} \qquad 0 \le c \le \frac{1}{2}.$$

Further,

$$\mathrm{e}^c \le \mathrm{e}^{1/2} < 2,$$

because $\mathrm{e} < 4$. Hence, the error is less than

$$\frac{2}{384} = \frac{1}{192} \approx 0.0052.$$

■

Example 242.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}.$$

Solution. We know that

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad x \in \mathbb{R}.$$

We differentiate

$$\sum_{n=1}^{\infty} \frac{n}{n!}\, x^{n-1} = \mathrm{e}^x, \qquad x \in \mathbb{R},$$

and we multiply by x which yields

$$\sum_{n=1}^{\infty} \frac{n}{n!}\, x^{n} = \mathrm{e}^x\, x, \qquad x \in \mathbb{R}.$$

We differentiate again

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}\, x^{n-1} = \mathrm{e}^x\, x + \mathrm{e}^x, \qquad x \in \mathbb{R},$$

and we multiply by x again which gives

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}\, x^{n} = \mathrm{e}^x\, x^2 + \mathrm{e}^x\, x, \qquad x \in \mathbb{R}.$$

Now, we put $x = 1$ which leads to

$$\sum_{n=1}^{\infty} \frac{n^2}{n!} = \mathrm{e}^1 \cdot 1^2 + \mathrm{e}^1 \cdot 1 = 2\mathrm{e}.$$

■

Example 243.

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!\,5^n}.$$

Solution. According to

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R},$$

we have

$$\sum_{n=0}^{\infty} \frac{1}{n!\,5^n} = \mathrm{e}^{\frac{1}{5}}.$$

■

Example 244.

Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{5^{4n}}{n!}.$$

Solution. We use the identity

$$\mathrm{e}^{x^4} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{4n}, \qquad x \in \mathbb{R},$$

which follows from the Maclaurin expansion of e^x. Hence,

$$\sum_{n=0}^{\infty} \frac{5^{4n}}{n!} = \mathrm{e}^{5^4}.$$

We remark that

$$\sum_{n=0}^{\infty} \frac{5^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(5^4\right)^n}{n!}.$$

■

Example 245.

Find the sum of the series
$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(\frac{\pi}{4}\right)^{2n}.$$

Solution. From
$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n},\qquad x\in\mathbb{R},$$
it follows
$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(\frac{\pi}{4}\right)^{2n}=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}.$$

■

Example 246.

Find the sum of the series
$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(\frac{13\pi}{6}\right)^{2n+1}.$$

Solution. From
$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1},\qquad x\in\mathbb{R},$$
it follows
$$\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(\frac{13\pi}{6}\right)^{2n+1}=-\sin\frac{13\pi}{6}=-\sin\frac{\pi}{6}=-\frac{1}{2}.$$

■

Example 247.

Find the sum of the series
$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1}.$$

Solution. From

$$\sin x - x = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \qquad x \in \mathbb{R},$$

it follows

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) - \frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\pi}{4}.$$

■

Example 248.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Solution. From

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \qquad x \in (-1, 1],$$

it follows (for $x = 1$)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\ln 2.$$

■

Example 249.

For $x \in \mathbb{R}$, find the sum of the series

$$\sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\, 6^n}.$$

Solution. From

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \qquad x \in \mathbb{R},$$

we obtain

$$\mathrm{e}^{x^3} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{3n}, \qquad x \in \mathbb{R},$$

$$\mathrm{e}^{\frac{x^3}{6}} = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^{3n}}{6^n}, \qquad x \in \mathbb{R},$$

$$x\,\mathrm{e}^{\frac{x^3}{6}} = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{n!\,6^n}, \qquad x \in \mathbb{R}.$$

■

Example 250.

For $x \in \mathbb{R}$, find the sum of the series

$$\sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{10n}}{(2n)!}.$$

Solution. From

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,x^{2n}, \qquad x \in \mathbb{R},$$

we obtain

$$\cos\left(x^5\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,x^{10n}, \qquad x \in \mathbb{R},$$

$$-\cos\left(x^5\right) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{10n}}{(2n)!}, \qquad x \in \mathbb{R}.$$

■

Example 251.

For $x \in \mathbb{R}$, find the sum of the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!}\,(6x)^{2n+1}.$$

Solution. From

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,x^{2n+1}, \qquad x \in \mathbb{R},$$

we obtain

$$\sin(6x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}\,(6x)^{2n+1}, \qquad x \in \mathbb{R}.$$

Thus,

$$\sin(6x) - 6x = \sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n+1)!}\,(6x)^{2n+1}, \qquad x \in \mathbb{R}.$$

■

Example 252.

Using a Maclaurin series, express e^3 as the sum of an infinite number series.

Solution. From

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n, \qquad x \in \mathbb{R},$$

we have

$$e^3 = \sum_{n=0}^{\infty} \frac{3^n}{n!}.$$

■

Example 253.

Using a Maclaurin series, express $\sqrt{2}/2$ as the sum of an infinite number series.

Solution. From

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}, \qquad x \in \mathbb{R},$$

we have

$$\frac{\sqrt{2}}{2} = \sin\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{4}\right)^{2n+1}.$$

■

Example 254.

Using a Maclaurin series, express $\sin 2^\circ$ as the sum of an infinite number series.

Solution. From

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1}, \qquad x \in \mathbb{R},$$

we have

$$\sin 2^\circ = \sin\frac{\pi}{90} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{90}\right)^{2n+1}.$$

■

Example 255.

Using a Maclaurin series, express $\ln(1.1)$ as the sum of an infinite number series.

Solution. Since

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n, \qquad x \in (-1, 1],$$

it holds

$$\ln(1.1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (0.1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,10^n}.$$

■

Example 256.

Using a Maclaurin series, express $(1.2)^{-1}$ as the sum of an infinite number series.

Solution. We use

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad x \in (-1, 1).$$

Hence,

$$\frac{1}{1.2} = \sum_{n=0}^{\infty} (-0.2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n}.$$

■

Example 257.

Using a Maclaurin series, express $\sqrt[3]{11\mathrm{e}}$ as the sum of an infinite number series.

Solution. We use the extension of e^x to get

$$\sqrt[3]{11\mathrm{e}} = \sqrt[3]{11}\,\mathrm{e}^{\frac{1}{3}} = \sqrt[3]{11} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{3}\right)^n = \sqrt[3]{11} \sum_{n=0}^{\infty} \frac{1}{n!\,3^n}.$$

■

Example 258.

Using a Maclaurin series, express $e^{10} - 1 - 10$ as the sum of an infinite number series.

Solution. The extension of e^x gives

$$e^{10} - 1 - 10 = -1 - 10 + \sum_{n=0}^{\infty} \frac{1}{n!} 10^n = \sum_{n=2}^{\infty} \frac{10^n}{n!}.$$

■

Example 259.

Using a Maclaurin series, express $\cos(\pi/5)$ as the sum of an infinite number series.

Solution. From

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad x \in \mathbb{R},$$

we obtain

$$\cos \frac{\pi}{5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{5}\right)^{2n}.$$

■

Example 260.

Using a Maclaurin series, express 5 as the sum of an infinite number series.

Solution. We can use, e.g., the sum of geometric series. We put

$$5 = \frac{1}{1-q}.$$

Thus, $q = 4/5$ and

$$5 = \sum_{n=0}^{\infty} q^n = \sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n.$$

■

Example 261.

Using a Maclaurin series, express π as the sum of an infinite number series.

Solution. We use the sum of geometric series. We put

$$\pi = \frac{1}{1-q},$$

which implies

$$q = 1 - \frac{1}{\pi} = \frac{\pi - 1}{\pi}.$$

Hence,

$$\pi = \sum_{n=0}^{\infty} q^n = \sum_{n=0}^{\infty} \left(\frac{\pi-1}{\pi}\right)^n.$$

We note that this approach can be used for any number $a > 1$. ■

Example 262.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$$

Solution. We use a modification of the approach which was used by L. Euler (without a description of the correctness, similarly to L. Euler). We use the Maclaurin series of $y = \sin x$. It holds

$$\begin{aligned} f(x) = \frac{\sin x}{x} &= \frac{1}{x} \cdot \left(x - \frac{1}{3!}\,x^3 + \frac{1}{5!}\,x^5 - \frac{1}{7!}\,x^7 + \cdots\right) \\ &= 1 - \frac{1}{3!}\,x^2 + \frac{1}{5!}\,x^4 - \frac{1}{7!}\,x^6 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,x^{2n} \end{aligned}$$

for any $x \in \mathbb{R},\ x \neq 0$. The roots of f are located in $\pm\pi$, $\pm 2\pi$, $\pm 3\pi$, $\pm 4\pi$, etc. Hence, we can express f as

$$f(x) = \left(1 - \frac{x}{\pi}\right)\cdot\left(1 + \frac{x}{\pi}\right)\cdot\left(1 - \frac{x}{2\pi}\right)\cdot\left(1 + \frac{x}{2\pi}\right)\cdot\left(1 - \frac{x}{3\pi}\right)\cdot\left(1 + \frac{x}{3\pi}\right)\cdots, \qquad x \in \mathbb{R},\ x \neq 0,$$

i.e.,

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\cdot\left(1 - \frac{x^2}{4\pi^2}\right)\cdot\left(1 - \frac{x^2}{9\pi^2}\right)\cdots, \qquad x \in \mathbb{R},\ x \neq 0.$$

The coefficients of x^2 give

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \frac{1}{16\pi^2} - \cdots,$$

i.e.,

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

■

Example 263.

Using the Maclaurin series of $\ln(1+x)$, find the sum of the number series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)}.$$

Solution. We put

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} x^{n+1}, \qquad x \in (-1,1),$$

and we differentiate

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n = -\ln(1+x), \qquad x \in (-1,1).$$

Hence,

$$f(x) = \int -\ln(1+x)\,\mathrm{d}x = x - (x+1)\ln(1+x) + C, \qquad x \in (-1,1),$$

where we used the integration by parts. Putting $x = 0$, we get $C = 0$. Then, considering $x = 1$ and Theorem 24, we get the result

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = f(1) = 1 - 2\ln 2.$$

■

Example 264.

For $|x| < 1$, find the sum of the power series

$$\sum_{n=0}^{\infty} \left(1 - \frac{1}{(n+1)!}\right) x^n.$$

Solution. We can write

$$\sum_{n=0}^{\infty}\left(1-\frac{1}{(n+1)!}\right)x^n=\sum_{n=0}^{\infty}x^n-\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,x^n.$$

The first part is the geometric series

$$\sum_{n=0}^{\infty}x^n=\frac{1}{1-x},\qquad |x|<1.$$

In the second series, we put $m=n+1$, which leads to

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,x^n=\sum_{m=1}^{\infty}\frac{1}{m!}\,x^{m-1}.$$

Then, we calculate

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,x^n&=\frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n!}\,x^n=\frac{1}{x}\left(-1+1+\sum_{n=1}^{\infty}\frac{1}{n!}\,x^n\right)\\&=\frac{1}{x}\left(-1+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)=\frac{1}{x}\,(-1+\mathrm{e}^x),\end{aligned}$$

which is true for any $x\neq 0$. For $x=0$, the sum is trivial. Hence,

$$\sum_{n=0}^{\infty}\left(1-\frac{1}{(n+1)!}\right)x^n=\begin{cases}\frac{1}{1-x}+\frac{\mathrm{e}^x-1}{x}, & |x|<1, x\neq 0;\\ 0, & x=0.\end{cases}$$

■

Example 265.

Find the sum of the power series

$$\sum_{n=0}^{\infty}\frac{2n+1}{n!}\,x^{2n}$$

for any $x\in\mathbb{R}$.

Solution. From

$$\mathrm{e}^x=\sum_{n=0}^{\infty}\frac{x^n}{n!},\qquad x\in\mathbb{R},$$

it follows

$$\mathrm{e}^{x^2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{n!},\qquad x\in\mathbb{R}.$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2n+1}{n!} x^{2n} &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(x^{2n+1}\right)' = \left(x \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}\right)' \\ &= \left(x\mathrm{e}^{x^2}\right)' = \mathrm{e}^{x^2} + x\mathrm{e}^{x^2} 2x = \mathrm{e}^{x^2}\left(2x^2+1\right)\end{aligned}$$

for all $x \in \mathbb{R}$. ■

Example 266.

Find the expansion of $f(x) = \arctan x$ to the power series with the center at the origin.

Solution. For all $x \in \mathbb{R}$, it holds

$$f'(x) = \frac{1}{1+x^2}.$$

This function is the sum of the geometric series with $q = -x^2$. Hence, for $|x| < 1$, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \left(-x^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

For $x \in (-1, 1)$, it holds

$$\int_0^x \frac{1}{1+s^2}\,\mathrm{d}s = \sum_{n=0}^{\infty} (-1)^n \int_0^x s^{2n}\,\mathrm{d}s = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

We obtain

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1.$$

Using Theorem 24, we can generalize the above expression to

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \qquad |x| \leq 1.$$

■

Example 267.

Using Example 266, expand the function

$$f(x) = \int_0^x \frac{\arctan s}{s}\,\mathrm{d}s, \qquad x \in (0, 1),$$

to the power series with the center at the origin.

Solution. We know that

$$\frac{\arctan x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,x^{2n}, \qquad x \in (0,1).$$

Now, it suffices to integrate

$$\int_0^x \frac{\arctan s}{s}\,\mathrm{d}s = \int_0^x \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,s^{2n}\right)\mathrm{d}s = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\,x^{2n+1}$$

for any $x \in (0,1)$. ∎

Example 268.

Using the solution of Example 266, find the sum of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}.$$

Solution. We put

$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\,x^{2n-1} = \sum_{n=0}^{\infty}(-1)^n\,\frac{x^{2n+1}}{2n+1},$$

whenever the series on the right-hand side is convergent. We know that

$$f(x) = \arctan x, \qquad x \in [-1,1].$$

Putting $x = 1$, we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = f(1) = \arctan 1 = \frac{\pi}{4}.$$

∎

Example 269.

Using the solution of Example 266, for the function

$$f(x) = \arctan x, \qquad x \in (-1,1),$$

find $f^{(99)}(0)$.

Solution. According to the solution of Example 266, in a neighbourhood of 0, it holds

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \, \frac{x^{2n+1}}{2n+1}.$$

For the series on the right-hand side, we have

$$a_n = \frac{f^{(n)}(0)}{n!}, \qquad n \in \mathbb{N} \cup \{0\}.$$

Hence, $f^{(99)}(0) = 99! \, a_{99}$ and

$$a_{99} = (-1)^{49} \frac{1}{99} = -\frac{1}{99}.$$

Therefore,

$$f^{(99)}(0) = -\frac{99!}{99} = -98!.$$

■

Example 270.

Find the limit

$$\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt[3]{1-x}}{x}.$$

Solution. We write the both terms in the numerator as the binomial extensions of the functions

$$\sqrt{1+x}, \qquad \sqrt[3]{1-x}.$$

We obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt[3]{1-x}}{x} &= \lim_{x\to 0} \frac{1}{x} \left[\left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \right) - \left(1 - \frac{1}{3}x - \frac{1}{9}x^2 + \cdots \right) \right] \\ &= \lim_{x\to 0} \frac{1}{x} \left(\frac{5}{6}x - \frac{1}{72}x^2 + \cdots \right) = \lim_{x\to 0} \left(\frac{5}{6} - \frac{1}{72}x + \cdots \right) = \frac{5}{6}.\end{aligned}$$

■

Example 271.

Find the limit

$$\lim_{x\to 0} \frac{\mathrm{e}^x \sin x - x(1+x)}{x^3}.$$

Solution. We use the Maclaurin extensions of e^x and $\sin x$. It holds

$$\begin{aligned}\mathrm{e}^x \sin x &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\\ &= \left(x + x^2 + \frac{1}{3}\,x^3 - \frac{1}{30}\,x^5 + \cdots\right).\end{aligned}$$

See also Example 227. The limit equals

$$\begin{aligned}\lim_{x\to 0} \frac{\mathrm{e}^x \sin x - x(1+x)}{x^3} &= \lim_{x\to 0} \frac{(x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots) - x(1+x)}{x^3}\\ &= \lim_{x\to 0}\left(\frac{1}{3} - \frac{1}{30}x^2 + \cdots\right) = \frac{1}{3}.\end{aligned}$$

■

Example 272.

Using the notion of power series, solve the differential equation $y'' + y = 0$.

Solution. We intend to find the solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

Hence,

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

From the equation, we get

$$\begin{aligned}y'' + y &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n\\ &= (a_0 + 2\cdot 1\cdot a_2) + x\,(a_1 + 3\cdot 2\cdot a_3)\\ &+ x^2\,(a_2 + 4\cdot 3\cdot a_4) + \cdots + x^{n-2}\,(a_{n-2} + n(n-1)a_n) + \cdots = 0.\end{aligned}$$

We can see that

$$a_n = -\frac{a_{n-2}}{n(n-1)}, \qquad n \ge 2,\ n \in \mathbb{N}.$$

Putting $a_0 = 0, a_1 = 1$, we obtain

$$a_{2n} = 0, \qquad a_{2n+1} = \frac{(-1)^n}{(2n+1)!}, \qquad n \in \mathbb{N},$$

i.e.,

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{2n+1} = \sin x.$$

Putting $a_0 = 1, a_1 = 0$, analogously, we obtain $y = \cos x$. The general solution is

$$y = a_1 \sin x + a_0 \cos x, \qquad a_1, a_0 \in \mathbb{R}.$$

■

Example 273.

Find the general solution of the differential equation

$$y'' + xy = 0$$

in the form of the linear combination of power series.

Solution. We intend to find the solution in the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Thus,

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}.$$

We obtain

$$\begin{aligned} y'' + xy &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= 2a_2 x^0 + x^1 (3 \cdot 2 \cdot a_3 + a_0) \\ &+ x^2 (4 \cdot 3 \cdot a_4 + a_1) + \cdots + x^{n-2} (n(n-1) a_n + a_{n-3}) + \cdots = 0, \end{aligned}$$

which implies

$$\begin{aligned} a_2 &= 0, \\ a_3 &= -\frac{a_0}{3 \cdot 2}, \\ a_4 &= -\frac{a_1}{4 \cdot 3}, \\ &\vdots \\ a_n &= -\frac{a_{n-3}}{n(n-1)}, \\ &\vdots \end{aligned}$$

Therefore,

$$a_3 = \frac{-a_0}{3 \cdot 2}, \qquad a_6 = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \qquad a_9 = \frac{-a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}, \qquad \ldots,$$

$$a_4 = \frac{-a_1}{4 \cdot 3}, \qquad a_7 = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, \qquad a_{10} = \frac{-a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}, \qquad \dots,$$
$$a_5 = a_8 = \dots = 0.$$

Altogether,

$$\begin{aligned} y &= a_0 + a_1 x - \frac{a_0}{3 \cdot 2} x^3 - \frac{a_1}{4 \cdot 3} x^4 + \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2} x^6 + \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots \\ &= a_0 \Bigg(\underbrace{1 - \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \cdots}_{y_1} \Bigg) + a_1 \Bigg(\underbrace{x - \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \cdots}_{y_2} \Bigg). \end{aligned}$$

■

§ 2.3 Fourier series and general series

Definition 10.

Let f be integrable on $[-\pi, \pi]$. The series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,\mathrm{d}x, \qquad n \in \mathbb{N} \cup \{0\},$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,\mathrm{d}x, \qquad n \in \mathbb{N},$$

is called the Fourier expansion (Fourier series) of f. We write $f \sim F$.

Remark. Let f be integrable on $[-\pi, \pi]$.

(i) If f is even, then its Fourier expansion is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

where

$$a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos(nx)\,\mathrm{d}x, \qquad n \in \mathbb{N} \cup \{0\}.$$

(ii) If f is odd, then its Fourier expansion is

$$\sum_{n=1}^{\infty} b_n \sin(nx),$$

where

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx)\,\mathrm{d}x, \qquad n \in \mathbb{N}.$$

Definition 11.

Let f be integrable on $[0, \pi]$. The even extension of f to $[-\pi, \pi]$ is the function

$$f_s(x) = \begin{cases} f(x), & x \in [0, \pi]; \\ f(-x), & x \in [-\pi, 0). \end{cases}$$

The Fourier expansion of f_s is called the cosine Fourier series of f. The odd extension of f to $[-\pi, \pi]$ is the function

$$f_\ell(x) = \begin{cases} f(x), & x \in (0, \pi]; \\ 0, & x = 0; \\ -f(-x), & x \in [-\pi, 0). \end{cases}$$

The Fourier expansion of f_ℓ is called the sine Fourier series of f.

Theorem 27 (Dirichlet)**.**

Let f be partially continuous and partially monotone on $[-\pi, \pi]$. It holds:

(i) if f is continuous at $x_0 \in (-\pi, \pi)$, then the sum of its Fourier series is $f(x_0)$ at this point;

(ii) if f has a jump discontinuity at $x_0 \in (-\pi, \pi)$, then the sum of its Fourier series is

$$\frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x) \right)$$

at this point;

(iii) at the border points $\pm\pi$, the sum of the Fourier series is

$$\frac{1}{2}\left(\lim_{x \to -\pi^+} f(x) + \lim_{x \to \pi^-} f(x) \right).$$

Remark. The statement of the Dirichlet theorem is also true if f is partially continuous and has a partially continuous derivative.

Example 274.

Find the Fourier series of $f(x) = x$, $x \in [-\pi, \pi]$. Determine the sum of this series on $\mathbb{R}$.

Solution. Directly (e.g., using the integration by parts), we obtain

$$a_0 = 0, \qquad a_n = 0, \qquad b_n = (-1)^{n+1}\frac{2}{n}, \qquad n \in \mathbb{N}.$$

Note that $a_n = 0$ follows from the fact that f is odd. We have

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin(nx)\,\mathrm{d}x = \left| \begin{array}{ll} u = x & u' = 1 \\ v' = \sin(nx) & v = \frac{-\cos(nx)}{n} \end{array} \right|$$

$$= \frac{2}{\pi}\left[\frac{-x\cos(nx)}{n}\right]_0^\pi + \underbrace{\frac{2}{\pi}\int_0^\pi \frac{\cos(nx)}{n}\,\mathrm{d}x}_{=0} = \frac{2}{\pi n}[-\pi\cos(n\pi)] = \frac{2}{n}(-1)^{n+1}.$$

The Fourier series is

$$2\left(\sin x - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \frac{\sin(4x)}{4} + \cdots\right), \qquad x \in (-\pi, \pi).$$

Using the continuity of the derivative of f on $(-\pi, \pi)$, the series converges to f on $(-\pi, \pi)$. At the border points, the values are $F(\pi) = F(-\pi) = 0$. The sum (on $\mathbb{R}$) is given by the periodic extension. ■

Example 275.

Find the sine and the cosine Fourier series of $f(x) = \cos x$, $x \in [0, \pi]$.

Solution. To get the cosine Fourier series, for $x \in [-\pi, 0)$, we put $f(x) = f(-x) = \cos(-x) = \cos x$. Then, $b_n = 0$ for $n \in \mathbb{N}$. Further,

$$a_0 = \frac{2}{\pi}\int_0^\pi \cos x\,\mathrm{d}x = 0,$$

$$a_1 = \frac{2}{\pi}\int_0^\pi \cos^2 x\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi 1 + \cos(2x)\,\mathrm{d}x = 1,$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi \cos x\,\cos(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^\pi \cos[(n+1)x] + \cos[(n-1)x]\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\sin[(n+1)x]}{n+1} + \frac{\sin[(n-1)x]}{n-1}\right]_0^\pi = 0, \qquad n \in \mathbb{N} \setminus \{1\}. \end{aligned}$$

Of course, we have obtained $\cos_s x \sim \cos x$ which follows from the uniqueness of the Fourier series as well.

Next, we find the sine Fourier series. We put $f(0) = 0$ and, for $x \in [-\pi, 0)$, we define $f(x) = -\cos x$. We obtain that $a_n = 0$, $n \in \mathbb{N} \cup \{0\}$. For $n \neq 1$, $n \in \mathbb{N}$, we have

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^\pi \cos x \, \sin(nx) \, \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\pi \sin[(n+1)x] + \sin[(n-1)x] \, \mathrm{d}x \\
&= \frac{1}{\pi} \left[-\frac{\cos[(n+1)x]}{n+1} - \frac{\cos[(n-1)x]}{n-1} \right]_0^\pi \\
&= \frac{1}{\pi} \left(\frac{(-1)^n + 1}{n+1} + \frac{(-1)^n + 1}{n-1} \right) \\
&= \frac{2n[(-1)^n + 1]}{\pi(n^2 - 1)} = \begin{cases} 0, & n \text{ is odd, } n \neq 1; \\ \frac{4n}{\pi(n^2-1)}, & n \text{ is even.} \end{cases}
\end{aligned}$$

Finally,

$$b_1 = \frac{2}{\pi} \int_0^\pi \cos x \, \sin x \, \mathrm{d}x = \frac{1}{\pi} \int_0^\pi \sin 2x \, \mathrm{d}x = 0.$$

Altogether, we have

$$\cos_\ell x \sim \sum_{n=1}^{\infty} \frac{8n}{\pi\,(4n^2 - 1)} \sin(2nx).$$

■

Example 276.

For the function $f(x) = |x|$ on $[-\pi, \pi]$, find the Fourier series and sketch the graph of its sum.

Solution. The function is even. Hence, $b_n = 0$ for all $n \in \mathbb{N}$. Further, we calculate

$$a_0 = \frac{2}{\pi} \int_0^\pi x \, \mathrm{d}x = \pi$$

and

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) \, \mathrm{d}x = \frac{2}{\pi n^2} \left[(-1)^n - 1 \right] = \begin{cases} 0, & n \text{ is even;} \\ -\frac{4}{\pi n^2}, & n \text{ is odd.} \end{cases}$$

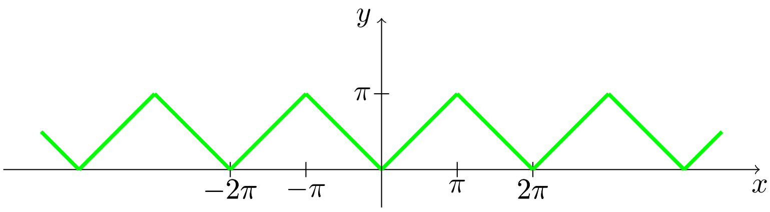
Thus, it holds

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}.$$

According to the Dirichlet theorem, we have

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}, \qquad x \in [-\pi, \pi].$$

The sum of the Fourier series has the period 2π, see Picture 2.1. ■



Pic. 2.1: Sum of Fourier series of $f(x) = |x|$, $x \in [-\pi, \pi]$

Example 277.

Expand the function $f(x) = \mathrm{id}(x) = x$, $x \in [0, \pi]$, into the cosine Fourier series.

Solution. The even extension of the given function is $|x|$. Hence, the rest is similar to Example 276. The result is

$$\mathrm{id}_s \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos[(2n-1)x]}{(2n-1)^2}.$$

■

Example 278.

Using the solution of Example 276, find the sum of the series

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Solution. We consider $x = 0$ in Example 276 (and the Dirichlet theorem), which leads to

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},$$

i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

■

Example 279.

Write the Fourier series F of $f(x) = x^2$ on $[-\pi, \pi]$.

Solution. The function f is even. Hence, all b_n are zero. We calculate

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, \mathrm{d}x = \frac{2}{\pi} \cdot \frac{\pi^3}{3} = \frac{2}{3}\pi^2$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) \, \mathrm{d}x = \frac{2}{\pi} \underbrace{\left[x^2 \frac{\sin(nx)}{n} \right]_0^{\pi}}_{=0} - \frac{4}{\pi n} \int_0^{\pi} x \sin(nx) \, \mathrm{d}x \\ &= -\frac{4}{\pi n} \left[-x \frac{\cos(nx)}{n} \right]_0^{\pi} - \frac{4}{\pi n^2} \underbrace{\int_0^{\pi} \cos(nx) \, \mathrm{d}x}_{=0} = \frac{4}{\pi n} \pi \frac{(-1)^n}{n} = \frac{4}{n^2}(-1)^n. \end{aligned}$$

We have obtained the series

$$F(x) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

■

Example 280.

Using the solution of Example 279, find the sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Solution. We emphasize that we have found the sum of this series in Example 262 (via the original approach of L. Euler). Now, we use the Dirichlet theorem for the Fourier series from Example 279 with $x = \pi$, which gives

$$\pi^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

■

Example 281.

Using the solution of Example 279, find the sum of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

Solution. We use the Dirichlet theorem for $x = 0$ in Example 279, which leads to

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

i.e.,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

■

Example 282.

Find the Fourier series of $f(x) = \cos^4 x$, $x \in [-\pi, \pi]$.

Solution. It holds

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos^4 x \cos(nx)\,\mathrm{d}x = \begin{cases} a_0 = \frac{3}{4}; \\ a_2 = \frac{1}{2}; \\ a_4 = \frac{1}{8}, \end{cases}$$

all other a_n and all b_n are equal to 0. Alternatively, we can use the direct calculation

$$\begin{aligned} \cos^4 x = (\cos^2 x)^2 &= \left(\frac{1+\cos(2x)}{2}\right)^2 \\ &= \frac{1}{4} + \frac{1}{2}\cos(2x) + \frac{1}{4}\cdot\frac{1+\cos(4x)}{2} = \frac{3}{8} + \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x). \end{aligned}$$

■

Definition 12.

Let f be integrable on $[a,b]$. Its Fourier expansion is

$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right),$$

where

$$a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,\mathrm{d}x,\qquad n\in\mathbb{N}\cup\{0\},$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,\mathrm{d}x,\qquad n\in\mathbb{N}.$$

We write $f\sim F$.

Remark. Analogously, we can introduce the sine and the cosine Fourier series for an integrable function f on the general interval $[a,b]$.

Remark. We add that the corresponding generalization of the Dirichlet theorem for a general interval $[a,b]$ holds unchanged (as well as the statement in Remark § 2.3).

Example 283.

Find the Fourier series of $f(x)=\operatorname{sgn}x$, $x\in(-1,1)$. Determine the sum F of this Fourier series on $\mathbb{R}$.

Solution. It is an odd function. Hence,

$$a_0=0,\qquad a_n=0,\qquad n\in\mathbb{N}.$$

For b_n, $n\in\mathbb{N}$, we have

$$b_n=\int_{-1}^{1}\operatorname{sgn}x\sin(n\pi x)\,\mathrm{d}x=2\int_0^1\sin(n\pi x)\,\mathrm{d}x=2\left[\frac{-\cos(n\pi x)}{n\pi}\right]_0^1$$

$$=\frac{2}{n\pi}\left[(-1)^{n+1}-(-1)\right]=\frac{2}{n\pi}\left[1-(-1)^n\right].$$

The Fourier series is

$$\frac{4}{\pi}\left(\sin(\pi x)+\frac{\sin(3\pi x)}{3}+\frac{\sin(5\pi x)}{5}+\cdots\right).$$

Since f,f' are continuous for $x\in(-1,1)\setminus\{0\}$, the Fourier series converges to the original function. We calculate that $F(1)=F(-1)=F(0)=0$. The sum F (on $\mathbb{R}$) is given by the periodic extension of the values from $[-1,1]$. ■

Example 284.

Find the Fourier series of $f(x) = \mathrm{e}^x$ on $[0, 2\pi]$.

Solution. We calculate

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} \mathrm{e}^x \,\mathrm{d}x = \frac{1}{\pi} \left[\mathrm{e}^x\right]_0^{2\pi} = \frac{1}{\pi} \left(\mathrm{e}^{2\pi} - 1\right).$$

Let $n \in \mathbb{N}$. Via the integration by parts, we get

$$\begin{aligned}\int \mathrm{e}^x \cos(nx) \,\mathrm{d}x &= \mathrm{e}^x \cos(nx) + \int n\,\mathrm{e}^x \sin(nx) \,\mathrm{d}x \\ &= \mathrm{e}^x \cos(nx) + n \left[\mathrm{e}^x \sin(nx) - n \int \cos(nx)\mathrm{e}^x \,\mathrm{d}x\right],\end{aligned}$$

which gives

$$\int \mathrm{e}^x \cos(nx) \,\mathrm{d}x = \frac{\mathrm{e}^x \left(\cos(nx) + n \sin(nx)\right)}{n^2 + 1} + C.$$

Using the integration by parts again, we have

$$\begin{aligned}\int \mathrm{e}^x \sin(nx) \,\mathrm{d}x &= \mathrm{e}^x \sin(nx) - \int n\,\mathrm{e}^x \cos(nx) \,\mathrm{d}x \\ &= \mathrm{e}^x \sin(nx) - n \left[\mathrm{e}^x \cos(nx) + n \int \sin(nx)\mathrm{e}^x \,\mathrm{d}x\right].\end{aligned}$$

Hence,

$$\int \mathrm{e}^x \sin(nx) \,\mathrm{d}x = \frac{\mathrm{e}^x \left(\sin(nx) - n \cos(nx)\right)}{n^2 + 1} + C.$$

Therefore,

$$a_n = \frac{1}{\pi} \left[\frac{\mathrm{e}^x(\cos(nx) + n \sin(nx))}{n^2 + 1}\right]_0^{2\pi} = \frac{1}{\pi} \frac{\mathrm{e}^{2\pi} - 1}{n^2 + 1}, \qquad n \in \mathbb{N},$$

and

$$b_n = \frac{1}{\pi} \left[\frac{\mathrm{e}^x(\sin(nx) - n \cos(nx))}{n^2 + 1}\right]_0^{2\pi} = -\frac{1}{\pi} \left(\mathrm{e}^{2\pi} - 1\right) \frac{n}{n^2 + 1}, \qquad n \in \mathbb{N}.$$

Altogether,

$$\mathrm{e}^x \sim \frac{\mathrm{e}^{2\pi} - 1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2 + 1} \cos(nx) - \frac{n}{n^2 + 1} \sin(nx)\right)\right].$$

■

Example 285.

Using the solution of Example 284, find the sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

Solution. According to the solution of Example 284 and the Dirichlet theorem, the sum of the Fourier series

$$\frac{e^{2\pi}-1}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{n^2+1} \cos(nx) - \frac{n}{n^2+1} \sin(nx) \right) \right]$$

equals e^x for $x \in (0, 2\pi)$ and, at the border points, it equals $\left(e^{2\pi}+1\right)/2$, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi\left(e^{2\pi}+1\right)}{2\left(e^{2\pi}-1\right)} - \frac{1}{2} = \frac{\pi+1+e^{2\pi}(\pi-1)}{2\left(e^{2\pi}-1\right)}.$$

■

Example 286.

Consider $f(x) = x - \pi$, $x \in [0, 2\pi]$, and find its Fourier series. Sketch the graph of its sum on $\mathbb{R}$.

Solution. We have

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} (x-\pi)\, dx = \frac{1}{\pi} \left[\frac{1}{2}(x-\pi)^2 \right]_0^{2\pi} = 0.$$

Via the integration by parts, we obtain

$$\begin{aligned} a_n &= \frac{2}{2\pi} \int_0^{2\pi} (x-\pi) \cos(nx)\, dx \\ &= \frac{1}{\pi} \left(\left[(x-\pi) \frac{1}{n} \sin(nx) \right]_0^{2\pi} - \int_0^{2\pi} \frac{1}{n} \sin(nx)\, dx \right) \\ &= \frac{1}{n\pi} \left[\pi \sin(2n\pi) + \pi \sin 0 \right] + \frac{1}{n\pi} \left[\frac{1}{n} \cos(nx) \right]_0^{2\pi} \\ &= 0 + \frac{1}{n^2\pi} \left[\cos(2n\pi) - \cos 0 \right] = \frac{1}{n^2\pi} [1-1] = 0, \qquad n \in \mathbb{N}, \end{aligned}$$

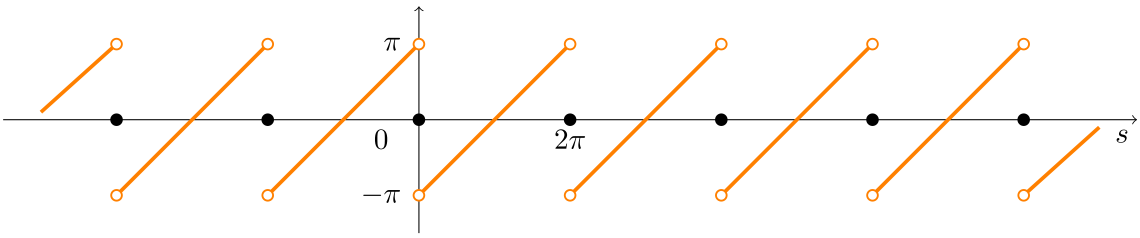
and

$$\begin{aligned} b_n &= \frac{2}{2\pi} \int_0^{2\pi} (x-\pi)\,\sin(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left(\left[(x-\pi)\,\frac{-1}{n}\,\cos(nx)\right]_0^{2\pi} + \int_0^{2\pi} \frac{1}{n}\,\cos(nx)\,\mathrm{d}x\right) \\ &= \frac{-1}{n\pi}\left[\pi\cos(2n\pi) + \pi\cos 0\right] + \frac{1}{n\pi}\left[\frac{1}{n}\,\sin(nx)\right]_0^{2\pi} \\ &= -\frac{2}{n} + \frac{1}{n^2\pi}[\sin(2n\pi) - \sin 0] = -\frac{2}{n} + 0 = -\frac{2}{n}, \qquad n \in \mathbb{N}. \end{aligned}$$

Hence,

$$f(x) \sim -\sum_{n=1}^{\infty} \frac{2}{n}\,\sin(nx).$$

Then, to sketch the graph of the sum, we use the Dirichlet theorem, see Picture 2.2.



Pic. 2.2: Sum of Fourier series of $f(x) = x - \pi$, $x \in [0, 2\pi]$

■

Example 287.

Consider the function $f(x) = x - \pi$ on $[0, 2\pi]$ and find its sine Fourier series.

Solution. The sine Fourier series can be obtained by the odd extension of f to $[-2\pi, 0)$. Then,

$a_0 = 0$ and $a_n = 0$, $n \in \mathbb{N}$. We calculate

$$\begin{aligned}
b_n &= \frac{2}{2\pi} \int_0^{2\pi} (x-\pi) \sin\left(\frac{1}{2}nx\right) \mathrm{d}x \\
&= \frac{1}{\pi} \left(\left[(x-\pi) \frac{-2}{n} \cos\left(\frac{1}{2}nx\right) \right]_0^{2\pi} + \int_0^{2\pi} \frac{2}{n} \cos\left(\frac{1}{2}nx\right) \mathrm{d}x \right) \\
&= \frac{-2}{n\pi} \left[\pi \cos(n\pi) + \pi \cos 0 \right] + \frac{2}{n\pi} \left[\frac{2}{n} \sin\left(\frac{1}{2}nx\right) \right]_0^{2\pi} \\
&= -\frac{2}{n} \left[(-1)^n + 1 \right] + \frac{4}{n^2\pi} \left[\sin(n\pi) - \sin 0 \right] \\
&= -\frac{2}{n} \left[1 + (-1)^n \right], \qquad n \in \mathbb{N}.
\end{aligned}$$

Hence,

$$f_\ell \sim -\sum_{n=1}^{\infty} \frac{2}{n} \left[1 + (-1)^n \right] \sin\left(\frac{1}{2}nx\right).$$

We add that $1 + (-1)^n$ equals 0 for odd n and it equals 2 for even n. We can rewrite the result into

$$f_\ell \sim -\sum_{n=1}^{\infty} \frac{2}{2n} \cdot 2 \sin\left(\frac{1}{2} 2nx\right) = -\sum_{n=1}^{\infty} \frac{2}{n} \sin(nx).$$

Of course, we have obtained the same series as in Example 286. ■

Example 288.

Consider the function $f(x) = x - \pi$ on $[0, 2\pi]$ and find its cosine Fourier series. Then, sketch the graph of the sum of the series.

Solution. To obtain an even function, we extend f to $[-2\pi, 0)$. All b_n are zero and we have to calculate a_0 and a_n for $n \in \mathbb{N}$. We have

$$a_0 = \frac{2}{2\pi} \int_0^{2\pi} (x-\pi) \, \mathrm{d}x = 0$$

and

$$\begin{aligned}
a_n &= \frac{2}{2\pi} \int_0^{2\pi} (x-\pi) \cos\left(\frac{1}{2}nx\right) \mathrm{d}x \\
&= \frac{1}{\pi} \left(\left[(x-\pi) \frac{2}{n} \sin\left(\frac{1}{2}nx\right) \right]_0^{2\pi} - \int_0^{2\pi} \frac{2}{n} \sin\left(\frac{1}{2}nx\right) \mathrm{d}x \right) \\
&= 0 + \frac{2}{n\pi} \left[\frac{2}{n} \cos\left(\frac{1}{2}nx\right) \right]_0^{2\pi} = \frac{4}{n^2\pi} [\cos(n\pi) - \cos 0] = \frac{4}{n^2\pi} \left[(-1)^n - 1 \right], \qquad n \in \mathbb{N}.
\end{aligned}$$

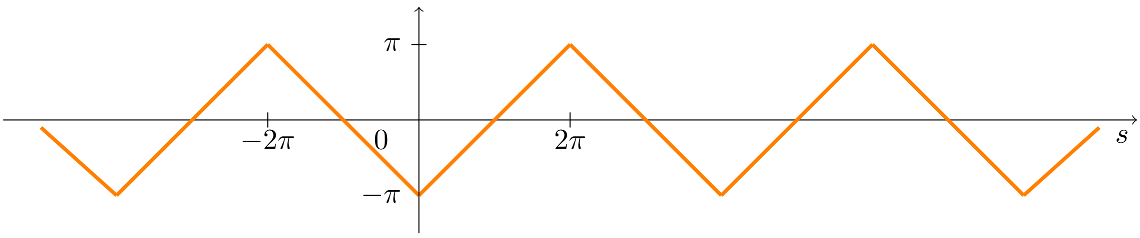
Hence,

$$f_s \sim \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \left[(-1)^n - 1\right] \cos\left(\frac{1}{2}nx\right).$$

The term $(-1)^n - 1$ equals zero for even n and it equals -2 for odd n. Therefore, we obtain

$$f_s \sim -\sum_{n=0}^{\infty} \frac{8}{(2n+1)^2\pi} \cos\left(\frac{2n+1}{2}x\right).$$

To get the sum of this series, we consider the continuity of f (and its derivative). According to the Dirichlet theorem, the cosine Fourier series converges to the periodic extension, see Picture 2.3. ■



Pic. 2.3: Sum of cosine Fourier series of $f(x) = x - \pi$, $x \in [0, 2\pi]$

Example 289.

Consider the function $f(x) = x$ on $[-1, 1]$. Find its Fourier series and sketch the graph of its sum.

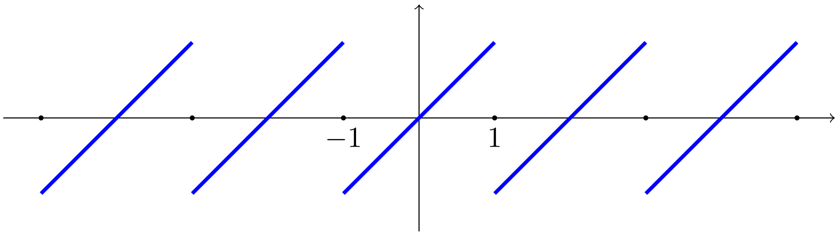
Solution. Since f is odd, we have $a_n = 0$ for $n \in \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, we calculate

$$\begin{aligned} b_n &= 2\int_0^1 x\sin(n\pi x)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\left[x\cos(n\pi x)\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,\mathrm{d}x \\ &= -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1 = \frac{2}{n\pi}(-1)^{n-1}. \end{aligned}$$

Altogether,

$$x \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(n\pi x).$$

Using the Dirichlet theorem, we easily find the sum, see Picture 2.4. ■



Pic. 2.4: Sum of Fourier series of $f(x) = x$, $x \in [-1, 1]$

Definition 13.

We say that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges to a function f uniformly on $M \subseteq \mathbb{R}$ if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that the inequality

$$|f_n(x) - f(x)| < \varepsilon$$

holds for all integers $n \geq N$ and for all $x \in M$. We say that a series of functions converges uniformly on a given interval if the sequence of its partial sums converges uniformly on this interval.

For a better understanding of the uniform convergence of series, we treat also the (uniform) convergence of sequences of functions.

Example 290.

Consider the sequence of the functions $f_n(x) = \arctan(nx)$, where $x \in \mathbb{R}$. Decide whether it converges uniformly.

Solution. The sequence of $f_n(x) = \arctan(nx)$, $x \in \mathbb{R}$, satisfies

$$\lim_{n\to\infty} f_n(x) = \frac{\pi}{2} \operatorname{sgn} x, \qquad x \in \mathbb{R}.$$

For arbitrary $n \in \mathbb{N}$, we have

$$\sup_{x\in\mathbb{R}} \left| \frac{\pi}{2} \operatorname{sgn} x - \arctan(nx) \right| = \frac{\pi}{2}.$$

The convergence is not uniform. ■

Example 291.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{\sin\left(n^2 x\right)}{n^2}$$

converges uniformly with respect to $x \in \mathbb{R}$.

Solution. We denote

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin\left(n^2 x\right)}{n^2}, \qquad x \in \mathbb{R},$$

and

$$f_n(x) = \sum_{k=1}^{n} \frac{\sin\left(k^2 x\right)}{k^2}, \qquad x \in \mathbb{R},\, n \in \mathbb{N}.$$

It holds

$$|f(x) - f_n(x)| = \left| \sum_{k=n+1}^{\infty} \frac{\sin\left(k^2 x\right)}{k^2} \right| \leq \sum_{k=n+1}^{\infty} \left| \frac{\sin\left(k^2 x\right)}{k^2} \right| \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2}, \qquad n \in \mathbb{N}.$$

Since $\sum_{n=1}^{\infty} 1/n^2$ converges (see Theorem 4), the above estimation implies that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{R}$ (especially, the definition of f is correct). ■

Example 292.

Decide whether, for the sequence of the functions

$$f_n(x) = n^2 x \left(1 - x^2\right)^n, \qquad x \in [0,1],$$

the equality

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x$$

is true.

Solution. One can easily verify that

$$\lim_{n\to\infty} f_n(x) = 0, \qquad x \in [0,1].$$

The calculation

$$\int_0^1 x\left(1 - x^2\right)^n\,\mathrm{d}x = \frac{1}{2n+2}, \qquad n \in \mathbb{N},$$

gives

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{n^2}{2n+2} \to +\infty.$$

Hence,

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = 0 \neq +\infty = \lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x.$$

■

Example 293.

For $n \in \mathbb{N}$, consider the sequence of the functions

$$f_n(x) = \frac{1}{n}\arctan x^n, \qquad x \in \mathbb{R}.$$

Does f_n' converge on $\mathbb{R}$?

Solution. One can easily verify that f_n converges to 0 on $\mathbb{R}$. Moreover, one can prove that it converges uniformly. Evaluating $f_n'(x)$ at $x = 1$, we obtain

$$f_n'(1) = \left.\frac{x^{n-1}}{1+x^{2n}}\right|_{x=1} = \frac{1}{2}.$$

Hence, f_n' does not converge to the zero function. For $x = -1$, the limit $\lim_{n\to\infty} f_n'(-1)$ does not exist. Hence, it is not true that f_n' converges on $\mathbb{R}$. In more details, $f_n'(x)$ converges to 0 for $x \in \mathbb{R} \setminus \{-1, 1\}$. For $x \in (-1, 1)$, we have the estimation $|f_n'(x)| \le |x|^{n-1} \to 0$ and, for $|x| > 1$, it holds $|f_n'(x)| \le 1/|x|^{n+1} \to 0$. ■

Example 294.

Consider the series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\,\frac{x^2+n}{n^2}.$$

Determine its conditional and absolute convergence for $x \in \mathbb{R}$ and its uniform convergence for $x \in [-M, M]$, where $M > 0$.

Solution. Since

$$\frac{1}{n} \le \frac{x^2+n}{n^2} = \left|(-1)^{n+1}\,\frac{x^2+n}{n^2}\right|, \qquad x \in \mathbb{R},\ n \in \mathbb{N},$$

the series cannot converge absolutely at any point $x \in \mathbb{R}$. The convergence follows from the Leibniz test for any $x \in \mathbb{R}$. For the uniform convergence of an alternating series in the form $\sum_{n=1}^{\infty}(-1)^{n+1}a_n(x)$ on $[-M, M]$, it suffices that $\{a_n\}_{n=1}^{\infty}$ converges uniformly to 0. In this case, we have

$$\left|(-1)^{n+1}\frac{x^2+n}{n^2}\right| \leq \frac{M^2}{n^2} + \frac{1}{n} \leq \frac{M^2+1}{n}, \qquad n \in \mathbb{N}.$$

Hence, for $x \in [-M, M]$, the series converges uniformly. ■

Example 295.

Determine the uniform convergence of the series $\sum_{n=1}^{\infty} f_n$, where

$$f_n(x) = (1-x)x^n, \qquad x \in [0, 1].$$

Solution. For $x \in [0, 1)$, we have

$$x = \sum_{n=1}^{\infty}(1-x)x^n = (1-x)\sum_{n=1}^{\infty} x^n = \frac{(1-x)x}{1-x}.$$

Therefore, $\sum_{n=1}^{\infty} f_n(x) = x$ for $x \in [0, 1)$ and $\sum_{n=1}^{\infty} f_n(1) = 0$. The convergence is not uniform, because

$$\left|x - \sum_{k=1}^{n}(1-x)x^k\right| = \left|x - x\frac{(1-x)(1-x^n)}{1-x}\right| = x^{n+1}$$

and $\{x^{n+1}\}_{n=1}^{\infty}$ does not converge uniformly on $[0, 1]$. ■

Example 296.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the functions

$$f_n(x) = \frac{\sin\left(n^2 x\right)}{n}$$

give the sequence $\{f_n\}_{n=1}^{\infty}$. Does this sequence converge uniformly on $\mathbb{R}$?

Solution. It holds

$$\lim_{n\to\infty} f_n(x) = 0, \qquad x \in \mathbb{R}.$$

The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$, because

$$|f_n(x) - 0| \leq \frac{1}{n}, \qquad x \in \mathbb{R},\ n \in \mathbb{N}.$$

■

Example 297.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, the functions

$$f_n(x) = \sin \frac{x}{n^2}$$

give the sequence $\{f_n\}_{n=1}^{\infty}$. Does this sequence converge uniformly on $\mathbb{R}$?

Solution. It holds

$$\lim_{n\to\infty} f_n(x) = 0, \qquad x \in \mathbb{R}.$$

The sequence $\{f_n\}_{n=1}^{\infty}$ does not converge uniformly on $\mathbb{R}$, because

$$\left| f_n\left(n^2 \frac{\pi}{2}\right) - 0 \right| = 1, \qquad n \in \mathbb{N}.$$

■

Example 298.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, consider the functions

$$f_n(x) = \frac{n}{1 + n^2 x^2}.$$

Determine the (uniform) convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$.

Solution. The functions are continuous on $\mathbb{R}$. For f_n, it holds

$$f_n(x) \to 0, \qquad x \in \mathbb{R} \setminus \{0\},$$

and

$$f_n(0) \to +\infty.$$

Hence, the sequence $\{f_n\}_{n=1}^{\infty}$ does not converge on $\mathbb{R}$. ■

Example 299.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, consider the functions

$$f_n(x) = \frac{nx}{1 + n^2 x^2}.$$

Determine the (uniform) convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$.

Solution. The functions are continuous on $\mathbb{R}$. The sequence $\{f_n\}_{n=1}^{\infty}$ converges to 0 on $\mathbb{R}$. This convergence is not uniform, because

$$\left|f_n\left(\frac{1}{n}\right) - 0\right| = \frac{\frac{n}{n}}{1+\frac{n^2}{n^2}} = \frac{1}{2}, \qquad n \in \mathbb{N},$$

i.e.,

$$\sup_{x\in\mathbb{R}} |f_n(x) - 0| \geq \frac{1}{2} \neq 0, \qquad n \in \mathbb{N}.$$

■

Example 300.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, consider the functions

$$f_n(x) = \frac{x}{1+n^2x^2}.$$

Determine the (uniform) convergence of the sequence $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$.

Solution. The functions are continuous on $\mathbb{R}$. They are odd and they converge to 0 on $\mathbb{R}$. It holds

$$f_n'(x) = \frac{1+n^2x^2-2xn^2x}{(1+n^2x^2)^2} = \frac{1-x^2n^2}{(1+n^2x^2)^2}, \qquad n \in \mathbb{N},\ x \in \mathbb{R}.$$

Further, f_n' has only one zero on $(0,\infty)$ (at $x = 1/n$). We have

$$\sup_{x\in\mathbb{R}} |f_n(x) - 0| = \sup_{x\in\mathbb{R}} |f_n(x)| = f_n\left(\frac{1}{n}\right) = \frac{1}{2n} \to 0.$$

We have obtained a uniform estimate with regard to x, i.e., the uniform convergence of $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}$. ■

Theorem 28 (Weierstrass test)**.**

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on an interval I. Let $|f_n(x)| \leq a_n$ for some sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ and for all $x \in I$ and $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} a_n$ converges, then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on I.

Example 301.

Decide whether the series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+x^2}$$

converges uniformly.

Solution. Since it holds

$$\left|\frac{\cos(nx)}{n^2+x^2}\right| \le \frac{1}{n^2+x^2} \le \frac{1}{n^2}$$

for any $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$ and since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent (see Theorem 4), the uniform convergence follows from the Weierstrass test. ■

Example 302.

Let us define the functions f_n for $n \in \mathbb{N}$ on $[0,1]$ as follows. For $x \in [0,1] \setminus (1/(n+1), 1/n)$, we define $f_n(x) = 0$, at the center of the interval $[1/(n+1), 1/n]$, we define the value of f_n as $1/n$, and let f_n be the linear function on the remaining subintervals. Does the series $\sum_{n=1}^{\infty} f_n$ converge uniformly on $[0,1]$?

Solution. It is seen that the series $\sum_{n=1}^{\infty} f_n$ converges on $[0,1]$, because, at any $x \in [0,1]$, at most one function f_n is non-zero. It holds

$$\sup_{x\in[0,1]} \left| \sum_{n=1}^{\infty} f_n(x) - \sum_{n=1}^{m} f_n(x) \right| = \frac{1}{m+1}, \qquad m \in \mathbb{N}.$$

Hence, the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[0,1]$ (although the Weierstrass test is not applicable). ■

Example 303.

On $[0,1]$, determine the uniform convergence of the series $\sum_{n=1}^{\infty} f_n$, where

$$f_n(x) = \frac{x^n}{n} - \frac{x^{n+1}}{n+1}, \qquad n \in \mathbb{N}.$$

Solution. Since it holds

$$\begin{aligned} \left| x - \sum_{k=1}^{n} \left(\frac{x^k}{k} - \frac{x^{k+1}}{k+1} \right) \right| &= \left| x - \left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^3}{3} + \cdots + \frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right) \right| \\ &= \left| \frac{x^{n+1}}{n+1} \right| = \frac{x^{n+1}}{n+1} \le \frac{1}{n+1} \to 0 \end{aligned}$$

on $[0,1]$, the series converges to x uniformly. ■

Chapter

3 Unsolved problems

Problem 1.

Consider a fast-growing miraculous tree, whose height will be increased by the half of its current height on the first day. The next day, its height increases by the third of its current height. The third day, its height increases by the quarter of its current height, etc. Compute how many times its height increases in 6 days. Will its height exceed any value?

Result. Four times. Yes. △

Problem 2.

Find the sum of the series

$$\sum_{n=2}^{\infty} \frac{(-1)^{n^2} + 2^n}{6^n}.$$

Result. 4/21. △

Problem 3.

Consider the function $p(n) = k \cdot (0.4)^n$ for $n = 1, 2, \ldots$ Determine the value of k so that the function p would be a probability function of a discrete random variable.

Result. For the probability function, it holds $\sum_{n=1}^{\infty} p(n) = 1$. Therefore, $k = 3/2$. △

Problem 4.

The so-called geometric distribution describes the number of failures before the first success in a sequence, also known as the waiting for the first success. It is denoted as $X \sim Ge(p)$, $p \in (0,1)$. The probability function is

$$p(n) = (1-p)^n p, \qquad n \in \mathbb{N}_0 := \{0, 1, \dots\}.$$

Calculate the expected value $E(X)$ and the variance $D(X)$ of the random variable which follows the geometric distribution.
Note that the expected value is given by

$$E(X) = \sum_{n \in \mathbb{N}_0} n \cdot p(n)$$

and the variance is given by

$$D(X) = E\left(X^2\right) - E^2(X),$$

where

$$E\left(X^2\right) = \sum_{n \in \mathbb{N}_0} n^2 \cdot p(n).$$

Result.

$$E(X) = \frac{1-p}{p}, \qquad D(X) = \frac{1-p}{p^2}.$$

△

Problem 5.

The so-called Poisson distribution ($X \sim Po(\lambda)$) describes the number of success in a given time (or space) interval. The probability function is

$$p(n) = \frac{\lambda^n}{n!}\,\mathrm{e}^{-\lambda}, \qquad n \in \mathbb{N}_0 := \{0, 1, \dots\}, \lambda > 0.$$

While correcting a book, it was found that there is eight typos on average per 20 pages, i.e., $\lambda = 8$. What is the probability that there is more than 10 typos on 20 pages? Next, calculate the expected value and the variance of this distribution.
(The formulas of the expected value and the variance are given in Problem 4.)

Result. The probability is

$$P(X > 10) = \sum_{n=11}^{\infty} p(n) \doteq 0.184\,114,$$

the expected value is

$$E(X) = \sum_{n\in\mathbb{N}_0} n \cdot p(n) = 8,$$

and the variance is $D(X) = 8$. △

Problem 6.

Prove that the Poisson distribution $Po(\lambda)$ (described in Problem 5) has the expected value λ and its variance is λ for any $\lambda > 0$.
(The formulas of the expected value and the variance are given in Problem 4.)

Result. It suffices to use the formulas for the expected value and for the variance. △

Problem 7.

Determine the expected value $E(X)$ and the variance $D(X)$ of the random variable which follows the so-called Borel distribution with the probability function

$$p(n) = \frac{1}{n!}\,\mathrm{e}^{-\mu n}(\mu n)^{n-1}, \qquad n \in \mathbb{N},\ \mu \in [0, 1).$$

(The formulas of the expected value and the variance are given in Problem 4.)

Result. It suffices to use the fact that p is a probability function, i.e., $\sum_{n=1}^{\infty} p(n) = 1$. Then, one can calculate

$$E(X) = \frac{1}{1-\mu}, \qquad D(X) = \frac{\mu}{(1-\mu)^3}.$$

△

Problem 8.

Using an appropriate infinite geometric series, solve (in $\mathbb{R}$) the equation

$$1 - \tan x + \tan^2 x - \tan^3 x + \cdots = \frac{\tan 2x}{1 + \tan 2x}.$$

Result. $x = \pi/6 + k\pi$, $x = 5\pi/6 + k\pi$, $k \in \mathbb{Z}$. △

Problem 9.

Consider a square whose sides have the length of 25 (units). Consider the circle which is inscribed in the given square and the square which is inscribed in this circle. In this way, continue to infinity. Determine the sum of the areas of all these squares.

Result. 1 250. △

Problem 10.

Determine the number of all people who would have been on Earth in 5 000 years if the following applies. At the beginning, there was one pair. Each couple brought just two girls and two boys into the world, at their age of 25–35 years. All people always live to the age of 75 (unaware genetic degeneration).

Result. $6 \cdot 2^{199} \approx 4.8208 \cdot 10^{60}$. △

Problem 11.

Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n^2 + 2n}.$$

Result. 1/4. △

Problem 12.

If you know that

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots,$$

find the sum of the series

$$\left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \cdots$$

Result. $(3 \ln 2)/2$. △

Problem 13.

Decide whether the series

$$\sum_{n=1}^{\infty} \sqrt[n]{\frac{1}{10^{10}}}$$

converges.

Result. The series does not converge (the necessary condition for the convergence is not fulfilled). △

Problem 14.

Is the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$$

convergent?

Result. Yes (consider the limit comparison test and Theorem 4 for $\alpha = 3/2$ or also the integral test). △

Problem 15.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \left(3^{\frac{1}{n}} - 1\right).$$

Result. The series diverges to $+\infty$ (use the limit comparison test with the $(\ln 3)$-multiple of the harmonic series). △

Problem 16.

Is the series

$$\frac{4}{2} + \frac{4 \cdot 7}{2 \cdot 6} + \frac{4 \cdot 7 \cdot 10}{2 \cdot 6 \cdot 10} + \cdots$$

convergent?

Result. Yes, applying the ratio test. △

Problem 17.

Decide whether the series

$$\frac{1\,000}{1} + \frac{1\,000 \cdot 1\,001}{1 \cdot 3} + \frac{1\,000 \cdot 1\,001 \cdot 1\,002}{1 \cdot 3 \cdot 5} + \cdots$$

converges.

Result. Yes, using the ratio test. △

Problem 18.

Prove the ratio (the d'Alembert) test.

Result. It suffices to consider a convergent geometric series, the comparison test, and the necessary condition for the convergence. △

Problem 19.

Using a suitable test, decide on the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^{n^2}}{(n+1)^{n \cdot n}}.$$

Result. The series converges via the root test. △

Problem 20.

Determine the convergence of the series

$$\sum_{n=2}^{\infty} \left(\frac{n-1}{n+1}\right)^{n^2-n}.$$

Result. The series converges via the root test. △

Problem 21.

Prove the root (the Cauchy) test.

Result. It suffices to consider a convergent geometric series, the comparison test, and the necessary condition for the convergence. △

Problem 22.

How many terms do you have to take to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

with the remainder less than 0.005?

Result. At least 10 terms (11 via the integral estimation). △

Problem 23.

Estimate the error given by the approximation of the series

$$\sum_{n=1}^{\infty} \frac{1}{4n^2}$$

using the first 10 terms.

Result. The error is less than 0.025 (via the integral estimation). △

Problem 24.

Applying appropriate criteria (tests), decide on the convergence of the series

$$\frac{\alpha}{\beta} + \frac{\alpha(\alpha+\gamma)}{\beta(\beta+\gamma)} + \frac{\alpha(\alpha+\gamma)(\alpha+2\gamma)}{\beta(\beta+\gamma)(\beta+2\gamma)} + \cdots$$

with regard to parameters $\alpha, \beta, \gamma > 0$.

Result. The series converges if and only if $\beta - \alpha > \gamma$ (via, e.g., the Raabe test for $\beta - \alpha \neq \gamma$ and the limit comparison test with the harmonic series for $\beta - \alpha = \gamma$). △

Problem 25.

Determine the convergence of the series

$$\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} \right)^{\alpha} = \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \right)^{\alpha}$$

with regard to a real parameter α.

Result. The series converges if and only if $\alpha > 2$ (via, e.g., the Raabe test for $\alpha \neq 2$ and the Gauss test for $\alpha = 2$, see Example 136). △

Problem 26.

With regard to parameters $\alpha > 0$, $\beta > 0$, determine the convergence of the series

$$\sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\alpha + k \ln k}{\beta + (k+1) \ln (k+1)}$$

via the Kummer test.

Result. For $\beta > \alpha$, the series converges; for $\beta \leq \alpha$, it diverges to $+\infty$. △

Problem 27.

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and let there exist positive numbers p_n for $n \in \mathbb{N}$ such that the limit

$$K = \lim_{n \to \infty} \left(p_n \frac{a_n}{a_{n+1}} - p_{n+1} \right)$$

exists and it is finite. Decide, whether the following statements are true.
- If $K > 0$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $K < 0$ and if $\sum_{n=1}^{\infty} \frac{1}{p_n} = +\infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

Result. The both statements are true. It is the so-called limit Kummer test, which follows from the Kummer test directly. We remark that, putting $p_n = 1$ for all $n \in \mathbb{N}$, we obtain the limit ratio test and that, putting $p_n = n$ for all $n \in \mathbb{N}$, we obtain an equivalent version of the Raabe test. △

Problem 28.

Find the unique real constant B such that the following statement (the so-called DeMorgan–Bertrand test) holds. Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms and let the numbers p_n, $n \in \mathbb{N}$, be such that

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{p_n}{n \ln n}, \qquad n \in \mathbb{N}.$$

Then, the following implications are valid.
- If $\liminf_{n\to\infty} p_n > B$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $\limsup_{n\to\infty} p_n < B$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Result. $B = 1$ (consider, e.g., the Gauss test). △

Problem 29.

Choose the unique real constant A so that the following statement (the so-called Ermak test) holds. Let f be a non-increasing and positive function on $[1, \infty)$ such that $a_n = f(n)$, $n \in \mathbb{N}$, and let there exist the limit

$$M = \lim_{x\to\infty} \frac{e^x f(e^x)}{f(x)}.$$

Then, the following implications are valid.
- If $M < A$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $M > A$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Result. $A = 1$. It suffices to consider, e.g., the series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}.$$

See Theorem 4 (Example 108). △

Problem 30.

Using the Ermak test (see the previous problem), for positive α, study the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^\alpha n}.$$

Result. The series converges for $\alpha > 1$ and diverges to $+\infty$ for $\alpha \leq 1$ (see also Example 111). △

Problem 31.

Mention an example of two divergent series such that their sum (term by term) is a non-zero convergent series.

Result. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n}; \qquad \sum_{n=1}^{\infty} \frac{1-n}{n^2}.$$

△

Problem 32.

Determine whether the series

$$\sum_{n=2}^{\infty} (-1)^n \ln\left(1 + \frac{1}{n \ln n}\right)$$

converges. If so, is the convergence conditional or absolute?

Result. The series converges conditionally (consider, e.g., the Leibniz test and the limit comparison test together with Example 111). △

Problem 33.

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2\sqrt{n} + (-1)^n)^3}$$

converges absolutely, conditionally, or it does not converge.

Result. The series converges absolutely. △

Problem 34.

Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n+\sqrt[4]{n^8+7n^5+n}}{\sqrt[5]{n^{11}+5n^5+2}+\sqrt{n^4+4n^3+n}}$$

converges absolutely, conditionally, diverges to $+\infty$, diverges to $-\infty$, or it oscillates.

Result. The series converges conditionally. △

Problem 35.

Find $\alpha \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\mathrm{e}^{\alpha n}}{n}$$

converges absolutely.

Result. $\alpha < 0$ (consider, e.g., the ratio test and Theorem 4). △

Problem 36.

Find $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n\, n\,(x-2)^{2n}}{4^n\sqrt{n}}$$

converges absolutely.

Result. $x \in (0,4)$. △

Problem 37.

Find all $x \in \mathbb{R}$ for which the series

$$\sum_{n=1}^{\infty}\frac{\left((n+5)^4\,(x-5)^2\,3^{2n+1}\right)^n}{7^{n+1}\arctan\left(n^2+8\right)}$$

converges.

Result. $x = 5$. △

Problem 38.

Study the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{5^{n-1} n^5}{n!} (8x+6)^{3n}.$$

Result. The series converges for any $x \in \mathbb{R}$. △

Problem 39.

Find the interval of the convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{4^{n-1}} \left(1 - \frac{x}{2}\right)^n.$$

Result. $(-6, 10)$. △

Problem 40.

For $p \in \{0, 1, 2\}$, consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} (2x-4)^n.$$

Find the interval I of the convergence of this power series with regard to $p \in \{0, 1, 2\}$.

Result. $I = (3/2, 5/2)$ for $p = 0$, $I = [3/2, 5/2)$ for $p = 1$, and $I = [3/2, 5/2]$ for $p = 2$. △

Problem 41.

For $x \in (-1, 1)$, find the sum of the series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{n+1}}{n(n+1)}.$$

Result. $(x+1)\ln(x+1)-x$. △

Problem 42.

For $x \neq 1$, find the sum of the power series

$$\sum_{n=1}^{\infty} 9^n \frac{(x-1)^{2n}}{(n+1)!}.$$

Result.

$$\frac{e^{9(x-1)^2}-1-9(x-1)^2}{9(x-1)^2}.$$

△

Problem 43.

Using an appropriate power series, find the sum of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{n^2}{2^{n-1}}.$$

Result. $-4/27$. △

Problem 44.

Express the function (with the variable x)

$$\int_0^x \frac{\sqrt[4]{1+s^4}-1}{s^2}\,\mathrm{d}s$$

as the power series with the center at the origin.

Result.

$$\frac{x^3}{4\cdot 3}-\frac{3x^7}{32\cdot 7}+\frac{21x^{11}}{384\cdot 11}-\cdots$$

△

Problem 45.

Consider the function

$$f(x) = x + (x+1)\sin\left(\frac{\pi}{2}x\right), \qquad x \in \mathbb{R}.$$

Express this function as its Taylor series at $x_0 = -1$.

Result.

$$-1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^{2n}}{4^n (2n)!} (x+1)^{2n+1}.$$

△

Problem 46.

How many non-zero terms of the Maclaurin series of $\ln(1+x)$ do you have to take to approximate $\ln 2$ with an error less than 10^{-5}?

Result. 100 000. △

Problem 47.

How many non-zero terms of the Maclaurin series of

$$\ln \frac{1+x}{1-x}$$

do you have to take to approximate $\ln 2$ with an error less than 10^{-5}?

Result. 5. △

Problem 48.

Express the number

$$\int_0^{\frac{1}{2}} \frac{\mathrm{d}x}{1+x^4}$$

with an error less than 10^{-5}.

Result.

$$\frac{1}{2} - \frac{1}{5}\left(\frac{1}{2}\right)^5 + \frac{1}{9}\left(\frac{1}{2}\right)^9.$$

△

Problem 49.

Evaluate the integral

$$\int_0^1 \ln x \cdot \ln(1-x)\,\mathrm{d}x.$$

Result. $2 - \pi^2/6$. △

Problem 50.

Using power series, solve the initial problem

$$y'' - 2xy' - 2y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Result. $y(x) = \mathrm{e}^{x^2}$. △

Problem 51.

Find the Fourier series for

$$f(x) = x^2, \qquad x \in [-1, 1].$$

Result.

$$\frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n \frac{4}{n^2\pi^2} \cos(n\pi x).$$

△

Problem 52.

Using the result of the previous problem, find the sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

Result. $\pi^2/6$, $-\pi^2/12$. △

Problem 53.

Find the coefficients of the Fourier series for

$$f(x) = \begin{cases} -1, & x \in [-1, 0); \\ x, & x \in [0, 1] \end{cases}$$

on $[-1, 1]$.

Result.

$$a_0 = -\frac{1}{2}, \qquad a_n = \frac{1}{n^2\pi^2}\left((-1)^n - 1\right), \qquad n \in \mathbb{N},$$

$$b_n = \frac{1}{n\pi}\left(1 - 2(-1)^n\right), \qquad n \in \mathbb{N}.$$

△

Problem 54.

Consider the series

$$\sum_{n=1}^{\infty} \frac{x^2}{1+x^2}\left(\frac{1}{1+x^2}\right)^n, \qquad x \in \mathbb{R}.$$

Study the absolute and uniform convergence on $\mathbb{R}$.

Result. The series converges absolutely, not uniformly. △

Problem 55.

Let $\{s_n\}_{n=1}^{\infty}$ be an injective sequence of all rational numbers from $(0, 1)$ and let

$$H(x) := \sum_{n=1}^{\infty} \frac{|x - s_n|}{2^n}, \qquad x \in (0, 1).$$

Determine $x \in (0, 1)$, for which the derivatives $H'(x)$ and $H''(x)$ exist.

Result. $H'(x)$ exists whenever x is not rational; $H''(x)$ does not exist for any $x \in (0, 1)$. △

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