PODZIM

TWO YEARS IN BRNO

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Abstract. This is a report on all my activities in Brno since my arrival.

Contents

Keywords

(1) dualities [13] 2.1
(2) categorical model theory [1] 2.2
(3) representation of categories [3], 1.1.2
(4) formal category theory [12, 13]

By area of expertise: accessible and locally presentable categories, Gabriel-Ulmer duality, formal category theory and concreteness.

1. Research Activity

I am interested in applying categorical techniques to logical flavoured questions. This section is an exposition of my research, providing short comments and motivations. The reader can navigate the section via the following table.

- Category theory & categorical logic
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Date: January 13, 2019.

1.1. **Category theory & categorical logic.**

1.1.1. *Homotopical Algebra is not concrete.* This project started before my arrival in Brno. It emerged as a concrete question among a group of vague questions that I had during my master. Few months before starting my PhD I proposed this project to *Fosco Loregian*, which is the actual co-author of [3].

> In [3] we generalize Freyd's well-known result that "homotopy is not concrete", offering a general method to show that under certain assumptions on a model category $\mathscr{M}$, its homotopy category $\mathrm{ho}(\mathscr{M})$ cannot be concrete.

The main result of the paper is [3, Thm 4.8]:

> Let $\mathscr{M}$ be a pointed model category; if there exist an index $n \in \mathbb{N} \geq 1$ and a 'weak classifying object' for the functor $\pi_n : \mathscr{M} \to \mathrm{Grp}$, then $\mathrm{ho}(\mathscr{M})$ is not concrete.

By 'weak classifying object' we mean a very weak notion of Eilenberg-Mac Lane spaces that we introduced in the paper. There is no need to specify that Eilemberg-Mac Lane like constructions occur all the time in model categories. The following are all examples of weak classifying objects for a model category:

- A section for π_n;
- A faithful left adjoint for π_n;
- A full right adjoint for π_n.

These are only possible examples, the definition is much weaker than so. When the category is stable it is even easier not to be concrete.

1.1.2. *Glamim.* [3] fits in a much more general picture that I would like to develop further, maybe in the future. Before starting I should say that this project is similar, or related, to *stuff, structure and property* [15]; the reader can find a simple and direct introduction in these slides. Given a category $\mathscr{C}$, very often we are interested in:

(1) Understanding the isomorphism classes of its objects;
(2) Distinguish its morphisms.

There are two, actually very different ways of tackling the first issue. One could look for either:

(1a) A functor $\mathscr{C} \to \mathrm{Set}$ which is conservative[1];
(1b) A functor $\mathscr{C} \to \mathrm{Set}$ which is full with respect to isomorphisms;

The second problem can be faced looking for:

(2) A faithful functor $\mathscr{C} \to \mathrm{Set}$.

The reader might ask why we are looking at Set, instead of a topos or an algebraic category. I might argue that there are reasons to prefer Set, but I would be glad to investigate those possibilities too. The case of algebraic categories strongly depends on Set.

Definition 1.1. A category $\mathscr{C}$ is:

(1a) a *golem*, if there is a conservative functor $\mathscr{C} \to \mathrm{Set}$.

[1] Reflects isomorphisms

(1b) a *clay golem*, if there is a functor $\mathscr{C} \to \text{Set}$ which is full with respect to isomorphisms.
(2a) a *faithful golem*, if it is a golem and the functor is also faithful.
(2b) a *faithful clay golem*, if it is a clay golem and the functor is also faithful.

Of course in the same fashion, given a base category $\mathscr{E}$, one can introduce the notion of (faithful)(clay) $\mathscr{E}$-golem.

When a category is a (faithful)(clay) golem?

The problem of understaing when a category is a golem was answered by the bright genius of Freyd in [4, Section 5].

Theorem 1.2. [4, Cor. 5.4] A locally small category $\mathscr{K}$ is a golem.

Theorem 1.3. [4, Thm 5.1] Let $\mathscr{K}$ be a category, locally small or not. There exists a functor $U : \mathscr{K} \to \text{Classes}$ such that the following are equivalent:

- U has values in Set.
- There exists a conservative functor $F : \mathscr{K} \to \text{Set}$.

I would like to prove that when $\mathscr{K}$ is enriched over an elementary topos $\mathscr{E}$ it is an $\mathscr{E}$-golem. This would be then natural translation of [4, Cor. 5.4] in the context of enriched categories.

One could look at this project as a generalized theory of invariants, or as a generalized Whitehead theorem.

Theorem 1.4 (Whitehead)**.** The functor $\pi_\bullet : \text{ho(CW)} \to \text{Set}$ makes ho(CW) into a (not faithful) (not clay) golem.

Remark 1.5. In [3] we proved that $\text{ho}(\mathscr{M})$ cannot be a faithful golem very easily.

Remark 1.6. This project also fits in the general programme of Universal Categories which was pursued with Věra Trnková [19] and her collaborators. So far, I proved that the problem of understanding if a category is a faithful golem can be split in understanding if it is concrete and if it is a golem, so that, a locally small category that satisfies the Isbell condition ([5]) is a faithful golem. There are no results about clay golems, even if this is the most interesting notion.

Finally, one can introduce a notion of complexity in CAT, as indicated here and more recently reproposed here .

1.1.3. *Protopopoi and Grothendieck Topoi.* Yet to come.

1.1.4. *The Scott adjunction.* This is a joint work with *Simon Henry*, in [6] he presents the result and manages to use it to solve an open problem proposed by J. Rosický.

> Show that the category of uncountable sets and monomorphisms between cannot be obtained as the category of point of a topos. Or give an example of an abstract elementary class that does not arise as the category points of a topos.

We established an adjunction

$$\mathsf{S} : \mathrm{Acc}_\omega \leftrightarrows \mathrm{Topoi} : \mathsf{pt}.$$

Where Acc_ω is the category of accessible categories with directed colimits, and Topoi is the category of Groethendieck topoi. This construction is a categorification of the usual Scott topology on a directed complete poset. Simon presents the adjunction in more details in [6].

1.2. **Categorical Model Theory.**

1.2.1. *Weak saturation and weak amalgamation property.* Since the very beginning of my PhD I was involved in the research line of Wieslaw Kubiś. He is interested in Fraïssé theory.

> In [1] we study the two model-theoretic concepts of *weak saturation* and *weak amalgamation* property in the context of accessible categories. We relate these two concepts providing sufficient conditions for existence and uniqueness of weakly saturated objects of an accessible category $\mathcal{K}$. We discuss the implications of these facts in classical model theory.

In the recent past years some work has been done in the direction of moving Fraïssé theory to a more category theoretic environment with two main purposes. First, it looks like this environment is a natural habitat in which some argument become trivial and evident. Second, some results can be generalized from Fraïssé classes to well behaved categories.

Two references for this work are Rosický [17] and Kubiś [11] [10] in two very different manner. In the first one Rosický presents categorical aspects of saturation, moving existence and uniqueness of saturated objects to accessible categories. In the second Kubiś studies weakly ω-saturated objects, again providing conditions to ensure existence and uniqueness. Here we bridge these two different approaches and hypotheses to obtain a generalization of both.

A significant relaxing of the amalgamation property, called the *weak amalgamation property*, was discovered by Ivanov [7] and independently by Kechris and Rosendal [9] during their study of generic automorphisms in model theory.

The main results of [1] can be synthesized as follows:

Let $\mathscr{K}$ be a λ-accessible category with the weak amalgamation property and directed colimits, satisfying a certain smallness condition, and the joint embedding property, then:

- any object K has a map $K \to M$ where M is weakly λ-saturated.
- any two weakly λ-saturated, λ^+-presentable objects are isomorphic.
- weakly λ-saturated, λ^+-presentable is weakly λ-homogeneous.

In the paper I also include a model theoretic translation of the results achieved.

1.3. **Formal Category Theory.** I met the formal theory of categories for the first time in a series of internal seminars held in MUNI curated by Fosco Loregian. Since I consider myself an applied category theorist, I was skeptical about this topic for a long time, being a passive learner in the internal seminar. Eventually I was challenged by the charme of some applications of this beautiful theory and I became more and more interested. In [13] we gave an axiomatic treatment of accessibility and local presentability, in [12] we (hopefully) gave a contribution to the study of the presheaf contruction.

1.3.1. *Accessibility and Presentability in 2-categories.* This is the second preprint which comes out from the collaboration with *Fosco Loregian.* The motivation for this work was very concrete. Locally presentable categories are very well behaved and there where some attempts of defining the notion of *locally presentable derivator.* Fosco proposed me to work on a convincing definition of locally presentable derivator. Very soon our challenge became to define the notion of locally presentable object in a 2-category.

In [2], we outline a definition of accessible and presentable objects in a 2-category endowed with a Yoneda structure. We describe a framework in which is possible to instantiate a Gabriel-Ulmer duality.

The general idea stems from our understanding of the following equivalent definition of locally presentable category.

Definition 1.7. $\mathscr{A}$ is locally finitely presentable if it is a (finitely) accessibly embedded reflective subcategory of a presheaf category

$$i : \mathscr{A} \leftrightarrows \mathrm{Set}^{G^{\mathrm{op}}} : L.$$

The reason that sits at the core of this is the interplay between the Ind-completion and the presheaf construction: So we isolated the essential data: $\mathrm{Ind}(G) \subset \mathrm{Set}^{G^{\mathrm{op}}}$ and we need to axiomatized it: $S \overset{Y}{\Rightarrow} P$.

Definition 1.8. A Yoneda context is a natural transformation $S \overset{Y}{\Rightarrow} P$ where

- (S, u) is a KZ monad;
- (P, y) is a KZ monad which is also a Yoneda structure.
- Y is representably fully faithful and $Y \cong \mathrm{Lan}_{u_G} y_G$.

Definition 1.9. Given a Yoneda context $S \overset{Y}{\Rightarrow} P$, we say that $\mathscr{A}$ is Y-accessible if: $\mathscr{A} \cong S(G)$ for some G.

Definition 1.10. Given a Yoneda context $S \overset{Y}{\Rightarrow} P$, we say that $\mathscr{A}$ is Y-presentable if

- $\mathcal{A}$ is accessible;
- $i : \mathcal{A} \leftrightarrows P(G) : L$.
- i is the Kan extenstion of its restriction to the unit of S.

The last request is the translation of being *accessibly embedded*. To convince yourself that these are good definitions, observe that it is possible to recover the very classical *representation theorem* for locally presentable categories.

> The following are equivalent:
> - $\mathcal{A}$ is Y-presentable.
> - $\mathcal{A}$ is Y-accessible and P-cocomplete.

The second part of the paper focuses on establishing some kind of Gabriel-Ulmer duality.

Theorem 1.11 (Gabriel Ulmer duality)**.** There is a biequivalence of 2-categories.

$$\mathrm{Lex}^{\mathrm{op}} \leftrightarrows \mathrm{Lfp}$$

- Lex is the category of small finitely complete categories and functors preserving them.
- Lfp is the category of locally presentable categories and (finitely) accessible right adjoints.

Our definition of presentability does not imply that presheaf objects are presentable. This plays a key role in the proof of the GU duality. In order to fix this issue we should start by understating why they are presentable in Cat. Observe that: $\mathrm{Set}^{G^{\mathrm{op}}} \cong \mathrm{Ind}(\widehat{G})$. By $\widehat{G}$ we mean the free finite colimit completion of G. So, in Cat there is an envelope $G \to \widehat{G}$ that fills the gap between the Ind completion and the presheaf construction.

Definition 1.12. Given a context $S \overset{Y}{\Rightarrow} P$ a Gabriel Ulmer envelope $\widehat{(\_)}$ for Y is an addition KZ monad such that $S(\widehat{(\_)}) \cong P(\_)$.

> Let $S \overset{Y}{\Rightarrow} P$ be a Yoneda context and $\widehat{(\_)}$ GU envelope for Y. If
> - $\widehat{(\_)}$ is soaking;
> - S is climbable;
>
> then
> $$\mathrm{Alg}(\widehat{(\_)})^{\mathrm{op}} \cong \mathrm{Pres}(Y).$$

This formulation is the actual starting point of the project **Formal dualities** 2.1 that the reader can find in ongoing projects.

1.3.2. *On the unicity of formal category theories.* This project was born as a prequel to 2.1 and eventually grow enough to have its independence.

> In [12] we prove an equivalence between cocomplete Yoneda structures and certain proarrow equipments on a 2-category $\mathscr{K}$. In order to do this, we recognize the presheaf construction of a cocomplete Yoneda structure as a relative, lax idempotent monad sending each admissible 1-cell $f : A \to B$ to an adjunction $\boldsymbol{P}_! f \dashv \boldsymbol{P}^* f$. Each cocomplete Yoneda structure on $\mathscr{K}$ arises in this way from a 'relative lax idempotent monad with enough adjoint 1-cells', whose domain generates the ideal of admissibles, and the Kleisli category of such a monad equips its domain with proarrows. We call these structures *yosegi*. Quite often, the presheaf construction associated to a yosegi generates a *ambidextrous* Yoneda structure; in such a setting there exists a fully formal version of Isbell duality.

The original intention of the paper was to present a formal understanding of Morita Theory, we are still after this quest. As a matter of facts, the paper organizes the existing literature axiomatizing the relevant features of the presheaf construction, unveiling an (un)expected unity. Morita theory, and its connection to Hyperdoctrines was my main motivation during the whole process of writing, hopefully this will appear in 2.1.

2. Ongoing projects

2.1. **Formal Dualities.** This is the natural prosecution of [2] and is a joint work with *Fosco Loregian*.

2.2. **Categoricity in Locally presentable categories.** This problem is motivated by a very classical results in model theory: Morley's categoricity theorem. The general idea is relatively simple, image that,

> Given the category of models of a first order theory T, it happens that there is precisely one model in cardinality λ, what can we say about T?

This question was deeply incarnated by the study of Abstract Elementary Classes, where is called *Shelah categoricity conjecture*. Recently Rosický, Beke and Lieberman discovered the deep connection that relates AEC with accessible categories. The problem of understanding categoricity in an accessible category is even more complicated than the classical problem of AEC. My hope is that the very special case of locally presentable categories might be easier to attack with category theoretic techniques. So far I described what happens when the locally presentable category is the category of modules over a ring and when is a Grothendieck topos.

2.3. **Locally presentable categories enriched over metric spaces.** This is a very broad project, it might be a good idea to split between the more achievable and the more conjectural part.

ϵ-colimits were introduced in [16]. It is meaningful to talk about ϵ-colimits for any category enriched over Met_∞. We proved the following:

- ϵ-pushout are weighted.
- ϵ-colimits are weighted.
- ϵ-colimits are preserved by left adjoints.
- Cocomplete categories with copowers have ϵ-colimits.
- Described the weight for ϵ-coequalizers.
- Described the weight for ϵ-directed colimits. This description is very unsatisfactory but I do believe that is impossible to get something better. In fact, to weight the colimit one has to change the shape of the diagram.

We still have some very important open questions about ϵ-colimits.

> Given the definition of ϵ-finitely presented object, is it true that his hom functor takes directed colimits into ϵ-directed colimits?

Even thought the study of ϵ-colimits might be interesting per se, this is just the beginning of this project. The general idea is that the category of metric spaces is $\aleph_1$-accessible because finite metric spaces are not finitely presentable. On the other hand they are, in a way, *approximately* finitely presentable. This definition is given in terms of ϵ-colimits in [16].

Describing a complete theory of ϵ and *approximate* colimits should lead to the definition of *locally approximate presentable categories* in such a way that Met_∞ will be locally approximate finitely presentable.

3. Seminars

3.1. **In conferences.**

- Weak saturation and weak amalgamation property, PSSL101, Leeds 2017;
- Weak saturation and weak amalgamation property, ECI workshop, Telc 2017;
- An axiomatic approach to Gabriel-Ulmer duality, CT2018, Ponta delgada 2018.
- Faithful and conservative functors, ECI workshop, Trest 2018;

3.2. **Invited.**

- On the concreteness of certain categories, Università di Pisa, 2017.
- A geometric perspective on the Yoneda lemma, Università di Pisa, 2017.
- The shape of water: Homotopical Algebra and set functors, MPI, Bonn 2018.
- Morita theories: a 2-dimensional approach to Stone-like dualities, Università di Milano 2018.

3.3. **In MUNI.**

3.3.1. *Algebra Seminars.*

- The importance of being a subobject;
- Material and Structural foundations [18];
- Accessibility and presentability in 2-categories [2].

3.3.2. *Hott internal seminar.*

- A brief introduction to type theory [14];
- Joyal's Tribes [8];
- Univalence Axiom [20].

3.3.3. $(\infty, 1)$*-categories internal seminar.*

- What is an ∞-category.

4. Travels

4.1. **Invitations.**

(1) Invited guest in Max Plank institut to collaborate with F. Loregian.

4.2. **Conferences.**

4.2.1. *Organized: PSSL103.* Between the 6th and the 8th of April we hosted the 103$^{\text{rd}}$ edition of the PSSL. I was one of the most active organizers. We (John Borke, Mike Lieberman, Fosco Loregian and Simon Henry) were a very energetic team and we were very pleased of the final result.

4.2.2. *Attended.*

(1) Logic Colloquium, University of Stockholm. 2017
(2) PSSL101, University of Leeds. 2017
(3) ECI Workshop, Telc. 2017
(4) PSSL103, Masaryk University. 2018
(5) Toposes in Como, University of Como. 2018
(6) CT2018, University of Azores. 2018
(7) Accessible categories and their connections, University of Leeds. 2018
(8) PSSL104, University of Amsterdam. 2018
(9) ECI Workshop, Trest. 2018

5. Teaching Activity

5.1. **Rings and Modules '17.** During the second semester of my first year I was the teaching assistant of Rings and Modules, this was a lot of fun and led to the production of 10 short notes that one can find on my website.

5.2. **Topics in category Theory '18.** During the first semester of my second year I was given the opportunity of being one of the two lecturer of the course Topics in category theory.

TiCT was a seminar reading at MU. The goal is to give students a glance in category theory. The course had two lecturers: John Bourke and myself. The course was structured in 12 classes. Every student that intended to receive credits from the course was expected to give a talk.

5.3. **Topics in category Theory '19.** During the first semester of my third year some student pushed for a second edition of Topics in category theory.

TiCT had to happen again.

5.4. **Topology '19.**

References

[1] I. Di Liberti. Weak saturation and weak amalgamation property. *ArXiv e-prints*, Aug. 2017.
[2] I. Di Liberti and F. Loregian. Accessibility and presentability in 2-categories. *ArXiv e-prints*, Apr. 2018.
[3] I. Di Liberti and F. Loregian. Homotopical algebra is not concrete. *Journal of Homotopy and Related Structures*, Feb 2018.
[4] P. J. Freyd. On the concreteness of certain categories. In *Symposia Mathematica*, volume 4, pages 431–456, 1969.
[5] P. J. Freyd. Concreteness. *Journal of Pure and Applied Algebra*, 3(2):171–191, 1973.
[6] S. Henry. An abstract elementary class non-axiomatizable in $L_{(,)}$. *arXiv e-prints*, page arXiv:1812.00652, Dec. 2018.
[7] A. A. Ivanov. Generic expansions of ω-categorical structures and semantics of generalized quantifiers. *The Journal of Symbolic Logic*, 64(2):775–789, 1999.
[8] A. Joyal. Notes on Clans and Tribes. *ArXiv e-prints*, Oct. 2017.
[9] A. S. Kechris and C. Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proceedings of the London Mathematical Society*, 94(2):302–350, 2007.
[10] W. Kubiś. Weak Fraisse categories. *ArXiv e-prints*, Dec. 2017.
[11] W. Kubiś. Fraïssé sequences: category-theoretic approach to universal homogeneous structures. *Annals of Pure and Applied Logic*, 165(11):1755 – 1811, 2014.

[12] I. D. Liberti and F. Loregian. On the unicity of formal category theories, 2019.

[13] M. Makkai and A. M. Pitts. Some results on locally finitely presentable categories. *Transactions of the American Mathematical Society*, 299(2):473–496, 1987.

[14] P. Martin-Löf. *Intuitionistic type theory*, volume 1 of *Studies in Proof Theory. Lecture Notes*. Bibliopolis, Naples, 1984. Notes by Giovanni Sambin.

[15] nLab authors. stuff, structure, property. `http://ncatlab.org/nlab/show/stuff%2C%20structure%2C%20property`, July 2018. Revision 50.

[16] J. Rosický and W. Tholen. Approximate injectivity. *Applied Categorical Structures*, Dec 2017.

[17] J. Rosický. Accessible categories, saturation and categoricity. *The Journal of Symbolic Logic*, 62(3):891–901, 1997.

[18] M. A. Shulman. Stack semantics and the comparison of material and structural set theories. *ArXiv e-prints*, Apr. 2010.

[19] V. Trnková. Universal categories. *Commentationes Mathematicae Universitatis Carolinae*, 007(2):143–206, 1966.

[20] T. Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. `https://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.