RESONANT AND NON-RESONANT NON-LOCAL FOURTH ORDER BOUNDARY VALUE PROBLEMS

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We are interested in the existence of positive solutions of some non-local boundary value problems (BVPs) for equations of the form

$$u^{(4)}(t) - \omega^4 u(t) = f(t, u(t)), \text{ a.e. } t \in (0, 1), \tag{1}$$

for some constant $\omega \in (0, \pi)$, subject to the following non-local boundary conditions (BCs)

$$u(0) = \beta_1[u],\ u''(0) + \beta_2[u] = 0,\ u(1) = \beta_3[u],\ u''(1) + \beta_4[u] = 0, \tag{2}$$

where each $\beta_i[u]$ is a linear functional on $C[0, 1]$, that is, is given by a Riemann-Stieltjes integral

$$\beta_i[u] = \int_0^1 u(s)\, dB_i(s).$$

Since some of the β_i can be zero, while others are not, this covers many BCs in one. A distinguishing feature of our work is that each B_i is a function of bounded variation, that is, dB_i is a *signed* measure. Some kind of positivity on the functionals β_i is needed in order to have positive solutions, but we do not suppose that $\beta_i[u] \geq 0$ for all $u \geq 0$.

For (1) we can consider cases where $f(t, u)$ is not positive for all positive u but is such that $f(t, u) + k^4 u \geq 0$ for $u \geq 0$ for some constant $k \in (0, \omega)$. One useful motivation is that the original problem (1) with the BCs (2) may be at resonance, that is, $\lambda = 0$ is an eigenvalue of the linear problem $u^{(4)} - \omega^4 u = \lambda u$ with the given BCs. In such a case we can consider the equivalent problem, which is of the same type as the original one,

$$u^{(4)}(t) - \widetilde{\omega}^4 u(t) = \widetilde{f}(t, u(t)),$$

where $\widetilde{\omega}^4 := \omega^4 - k^4$, $\widetilde{f}(t, u) := f(t, u) + k^4 u$, with the same BCs. We will show that, under natural conditions, this perturbed problem is non-resonant. In order to study the existence of positive solutions for (1)-(2) we use the method developed by Webb and Infante in [1] and [2].

References

[1] J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. London Math. Soc.*, (2) **74** (2006), 673–693.

[2] J. R. L. Webb and G. Infante, Non-local boundary value problems of arbitrary order, *J. London Math. Soc.*, (2) **79** (2009), 238–258.