Generalized linear differential equations in a Banach space: Continuous dependence on a parameter

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We will present a continuous dependence result for integral equations in a Banach space X of the form

$$x(t) = \widetilde{x} + \int_a^t \mathrm{d}[A(s)]\, x(s), \quad t \in [a,b], \tag{1}$$

where $-\infty < a < b < \infty$, $\widetilde{x} \in X$ and $A\colon [a,b] \to L(X)$ has a bounded variation on $[a,b]$. The contribution is based on the joint research [2] with M. Tvrdý.

Throughout these notes X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in a Banach space X and $\|f\|_\infty = \sup_{t\in[a,b]} \|f(t)\|_X$ for $f : [a,b] \to X$. Further, $BV([a,b],X)$ denotes the set of functions valued in X of bounded variation on $[a,b]$ and $G([a,b],X)$ denotes the set of regulated functions. Recall that $BV([a,b],X) \subset G([a,b],X)$ (cf. [6]). The following estimate will be helpful later.

Lemma 1. *If $F \in G([a,b], L(X))$ and $H \in BV([a,b], L(X))$ then*

$$\sum_{t\in[a,b)} \|\Delta^+ F(t)\Delta^+ H(t)\|_{L(X)} + \sum_{t\in(a,b]} \|\Delta^- F(t)\Delta^- H(t)\|_{L(X)} \le 2\,\|F\|_\infty \operatorname{var}_a^b H\,.$$

The integrals are the abstract Kurzweil-Stieltjes integrals defined as in [6].

The result we are about to present extends that presented for the case of a finite dimensional X by Opial in [3]. The following assumption implies the existence of solution to (1) (cf. [6]) and hence it is crucial for our purposes:

$$\big[I - \Delta^- A(t)\big]^{-1} \in L(X) \quad \text{for all } \ t \in (a,b]\,. \tag{2}$$

Theorem. *Let $A, A_k \in BV([a,b], L(X))$ satisfy (2) and $\widetilde{x}$, $\widetilde{x}_k \in X$ for $k \in \mathbb{N}$. Furthermore, assume that*

$$\lim_{n\to\infty} \|A_k - A\|_\infty \left(1 + \operatorname{var}_a^b A_k\right) = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\widetilde{x}_k - \widetilde{x}\|_X, = 0. \tag{3}$$

Then (1) has a unique solution x on $[a,b]$. Moreover, for each $k \in \mathbb{N}$, the equation

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]\, x_k(s)\,, \quad t \in [a,b] \tag{4}$$

has a unique solution x_k on $[a,b]$ and $\lim_{n\to\infty} \|x_k - x\|_\infty = 0$.

Sketch of the Proof. Denote by x and x_k the solutions on $[a,b]$ of (1) and (4), respectively. For $t \in [a,b]$ and $k \in \mathbb{N}$, integrating by parts (cf. [7] and [2]) and using the substitution formula (cf. [1, Proposition II.1.9]), we get

$$x_k(t) - x(t) = \widetilde{x}_k - \widetilde{x} + \int_a^t \mathrm{d}[A](x_k - x) + [A_k(t) - A(t)]\, x_k(t) - [A_k(a) - A(a)]\, \widetilde{x}_k$$
$$- \int_a^t (A_k - A)\,\mathrm{d}[A_k] x_k - \sum_{a\le\tau<t} [\Delta^+(A_k - A)(\tau)\Delta^+ x_k(\tau)] - \sum_{a<\tau\le t} [\Delta^-(A_k - A)(\tau)\Delta^- x_k(\tau)].$$

Recalling that $\Delta^{\pm} x_k(s) = \Delta^{\pm} A_k(s)\, x_k(s)$, by Lemma 1 we have

$$\sum_{a\le\tau<t}[\Delta^+(A_k-A)(\tau)\Delta^+ x_k(\tau)] - \sum_{a<\tau\le t}[\Delta^-(A_k-A)(\tau)\Delta^- x_k(\tau)]$$
$$\le 2\|A_k-A\|_\infty \operatorname{var}_a^b A_k \|x_k\|_\infty .$$

Having this in mind and using [5, Proposition 10] we obtain

$$\|x_k(t)-x(t)\|_X \le \|\widetilde{x}_k - \widetilde{x}\|_X + \alpha_k\|x_k\|_\infty + \int_a^t \mathrm{d}[\operatorname{var}_a^s A]\, \|x_k(s)-x(s)\|_X \,,$$

where $\alpha_k := \|A_k-A\|_\infty \Big(2+3\operatorname{var}_a^b A_k\Big)$. Hence, the generalized Gronwall inequality (cf. [4]) yields $\|x_k(t)-x(t)\|_X \le \Big(\|\widetilde{x}_k-\widetilde{x}\|_X + \alpha_k\,\|x_k\|_\infty\Big)\, \exp(\operatorname{var}_a^t A)$. Since t was arbitrary, it follows that

$$\|x_k - x\|_\infty \le \Big(\|\widetilde{x}_k-\widetilde{x}\|_X + \alpha_k\|x_k\|_\infty\Big)\, \exp(\operatorname{var}_a^b A).$$

Note that α_k tends to zero if $k\to\infty$. Moreover,

$$\|x_k\|_\infty \le \|x_k-x\|_\infty + \|x\|_\infty \le \Big(\|\widetilde{x}_k-\widetilde{x}\|_X + \alpha_k\|x_k\|_\infty\Big)\, \exp(\operatorname{var}_a^b A) + \|x\|_\infty$$

which together with (3) imply that the sequence $\|x_k\|_\infty$ is bounded and this completes the proof. □

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