Generalized differential equations

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A solution of a classical differential equation

$$\dot{x} = f(x, t) \tag{1}$$

is a function u such that its derivative $\dot{u}$ is at every τ equal to $f(u(\tau), \tau)$, i.e. in a neighborhood of τ the linear function

$$t \to u(\tau) + f(u(\tau), \tau)\,(t - \tau)$$

is a good approximation of u. Usually f and u are $\mathbb{R}^n$-valued functions. Given $u(a) = y$ the value $u(T)$ is approximately equal to

$$y + \sum_{i=1}^{k} f(u(\tau_i), \tau_i)\,(t_i - t_{i-1})$$

for $T > a$. Here

$$a = t_0 < t < \cdots < t_k = T, \tag{2}$$

$\tau_i \in [t_{i-1}, t_i]$, τ_i being called the tag of the interval $[t_{i-1}, t_i]$ and the partition of $[a, b\,]$ into intervals $[t_{i-1}, t_i]$ is sufficiently fine. Moreover, u is a solution of the Volterra integral equation

$$u(T) = u(a) + \int_a^T f(u(t), t)\,\mathrm{d}t \tag{3}$$

and vice versa, every solution of (3) is a solution of (1) and fulfils $u(a) = y$.

A generalized ordinary differential equation (GODE)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,x = \mathrm{D}_t F(x, \tau, t) \tag{4}$$

depends on a function F of three variables and its solution is a function u such that the function

$$t \to u(\tau) + F(u(\tau), \tau, t) - F(u(\tau), \tau, \tau)$$

is a good approximation of u in a neighborhood of any τ. The value $u(T)$ is approximately equal to

$$u(a) + \sum_{i=1}^{k} [F(u(\tau_i), \tau_i, t_i) - F(u(\tau_i), \tau_i, t_{i-1})]. \tag{5}$$

In fact, the sum in (5) can be viewed as an approximation of an integral which is denoted by

$$\int_a^T \mathrm{D}_t F(u(\tau), \tau, t). \tag{6}$$

The quality of approximation depends on the interpretation of the concept that the partition of $[a, T]$ is fine.

By definition, u is a solution of (4) if it fulfils

$$u(T) = u(a) + \int_a^T \mathrm{D}_t F(u(\tau), \tau, t) \tag{7}$$

for $T > a$ which is a Volterra-type integral equation.

(i) The concept of a fine partition of $[a, b]$ admits various interpretations and two of them are crucial in this treatise.

Let $\xi > 0$, $[a, T] \subset \mathbb{R}$. A set $\{([t_{i-1}, t_i], \tau_i); i = 1, 2, \ldots, k\}$ is a $\xi-$fine partition of $[a, T]$ if (2) holds, if $t_i - t_{i-1} \leq \xi$ and $\tau_i \in [t_{i-1}, t_i]$ for $i = 1, 2, \ldots, k$.

Let δ be a positive function on $[a, T]$, i.e. $\delta : [a, T] \to \mathbb{R}^+$. A set $\{([t_{i-1}, t_i], \tau_i); i = 1, 2, \ldots, k\}$ is a δ-fine partition of $[a, T]$ if (2) holds and if

$$\tau_i - \delta(\tau_i) \leq t_{i-1} \leq \tau_i \leq t_i \leq \tau_i + \delta(\tau_i) \quad \text{for } i = 1, 2, \ldots, k.$$

These two concepts of a fine partition of $[a, T]$ are a basis for two concepts of a solution of (4).

To a classical differential equation (1) there corresponds a GODE (4) where e.g.

$$F(x, \tau, t) = f(x, t)\, \tau \quad \text{or} \quad F(x, t) = \int_a^t f(x, \tau)\, \mathrm{d}\tau$$

and u is a solution of (1) if and only if it is a solution of (4). On the other hand, there are functions F such that the solutions of (4) are nowhere differentiable.

Three classes of GODE's will be presented and their relation to classical differential equations will be discussed.