A combined variational–topological approach for dispersion–managed solitons in optical fibers

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An optical soliton can be described as an electromagnetic wave which is localized in time and can propagate along an optical medium without significant distortion of its shape. In a nonlinear optical medium, this physical effect is achieved by means of a suitable balance between the chromatic dispersion and the nonlinear refractive response. From a practical point of view, the concept of soliton is crucial to implement efficient optical fiber communication systems.

From a mathematical point of view, the propagation of an optical pulse in a fiber cable with varying dispersion is governed by the equation

$$i\Psi_z - \frac{1}{2}\beta_2(z)\Psi_{tt} + \sigma(z)\left|\Psi\right|^2\Psi = iG(z)\Psi, \tag{1}$$

where Ψ is the complex–valued envelope function of the electric field, z is the longitudinal coordinate of the fiber line and t is time. The functions β_2, σ, G model respectively the dispersion, nonlinear refractive response and effective gain or loss along the fiber line. It is assumed that the optical fiber has a periodic structure so that the coefficients are periodic. To find soliton–like solutions of eq. (1) is a central problem not only in Nonlinear Optics but also for a variety of physical and biological applications.

A well–known method for the analytical study of eq. (1) is the variational approach described in full detail in [1]. Eq. (1) is rewritten in the system of ordinary differential equations

$$\begin{aligned} T' &= 4d(z)M \\ M' &= \frac{d(z)C_1}{T^3} - \frac{c(z)C_2}{T^2}, \end{aligned} \tag{2}$$

with fixed constants C_1 and C_2 (now the gain–loss power term is included in the function c). The functions $T(z)$ and $M(z)$ describe the optical pulse width and the chirp (time–dependent phase) of the breathing central part of the optical soliton. The dynamics of system (2), often known as TM–equations in the related literature, is of key importance on this field. Then the problem is reduced to find conditions for the existence of ω–periodic solutions of system (2), that is, T, M verifying $T(0) = T(\omega)$, $M(0) = M(\omega)$. Although the variational approach is approximate, it is recognized as an effective theoretical method to gain insight on the dynamics of the system.

At this point, all the theoretical results presented in the literature assume that both coefficients c, d are piecewise constants. This assumption makes possible to apply a matching technique for the respective phase planes in order to find explicit existence conditions. To have piecewise continuous coefficients is a coherent assumption in the framework of Nonlinear Optics, but the main problem is that for a large number of pieces computations become too hard to handle with.

The paper [2] solves explicitly the case of c constant (the so-called lossless case) and d composed by two pieces, but for more than two pieces only numerical results are known [3]. Our approach is of a different nature. We propose the use of a classical approach like the upper and lower functions method. This technique is very known in the qualitative analysis of second order ODEs and has been applied to equations with singularities in the recent paper [4]. We take advantage of the techniques developed there to open a new path in the study of DM–solitons in optical fibers. The main technical difficulty is that in the general case with arbitrary coefficients, the system (2) cannot be written as a second–order ODE (as in fact it is done in the particular case considered in [2]). This has forced us to develop a specific upper and lower function method for the system

$$u' = p(t)v, \tag{3}$$

$$v' = f(t,u) \tag{4}$$

with periodic boundary conditions

$$u(0) = u(\omega), \qquad v(0) = v(\omega), \tag{5}$$

where $p \in L\big([0,\omega];\mathbb{R}\big)$ and $f \in \mathrm{Kar}\,\big([0,\omega] \times D;\mathbb{R}\big)$. It can be shown that such method is the natural extension to the first–order system of known results for the second order scalar ODE, so in this sense from a mathematical point of view it is interesting by itself.

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