# Averaging for impulsive functional differential equations: a new approach

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The purpose of the averaging method or averaging principle is to determine conditions under which the solutions of an autonomous differential system can approximate the solutions of a more complicated time varying system. The averaging method is therefore a powerful tool in studying the perturbation theory of differential equations, since it allows one to replace a time-varying small perturbation, acting on a long time interval, by a time-invariant perturbation and, in this process, only a small error is introduced.

In the present work, we consider retarded functional differential equation with impulses at variable times (we write impulsive RFDEs) and we establish an averaging principle where the averaged system is an autonomous RFDE and not an ODE. More precisely, we consider the initial value problem

$$\begin{cases} \dot{y}\left(t\right)=\varepsilon f\left(t,y_t\right), & t\neq\tau_k(y(t)), \quad t\geq 0, \\ \Delta y\left(t\right)=I_k\left(y\left(t\right)\right), & t=\tau_k(y(t)), \quad k=1,2,\ldots, \\ x_0=\phi, \end{cases} \tag{1}$$

where $\varepsilon>0$ is a small parameter, the initial function $\phi$ is a left continuous regulated function defined on $[-r,0]$, with $r>0$. We assume that for each solution $y:[-r,+\infty)\to\mathbb{R}^n$ of (1), the mapping $t\mapsto f(t,y_t)$ is Lebesgue integrable and its indefinite integral satisfies Carathéodory- and Lipschitz-type conditions. Roughly speaking, these conditions on the indefinite integral of $f$ allow the function $f$ to behave "badly", e.g., $f$ may have many discontinuities, and yet we can obtain good results, provided its indefinite integral is "well-behaved".

We assume that the impulse operators $I_k(x)$, $k=0,1,2,\ldots$ are continuous functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ and that

$$\Delta y(t)=y(t+)-y(t-)=y(t+)-y(t)$$

that is, $y$ is left continuous.

We denote by $m(\tau_k)$ the number of times at which the integral curves of system (1) meet the hypersurface $\tau_k$, $k=1,2,\ldots$. By $t_k^i$ we mean the $i$-th moment of time at which the integral curves of system (1) meet the hypersurface $\tau_k$, with $i=1,\ldots,m(\tau_k)$, and $k=1,2,\ldots$. We assume $m(\tau_k)<\infty$, $k=1,2,\ldots$.

The averaged system for problem (1) is given by

$$\begin{cases} \dot{y}=\varepsilon f_0\left(y_t\right)+\varepsilon I_0(y) \\ y_0=\phi, \end{cases} \tag{2}$$

where we assume that the following limits exist

$$f_0(\varphi)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(s,\varphi)ds \quad \text{and} \quad I_0(x)=\lim_{T\to\infty}\sum_{\substack{0\leq t_k^i<T, \\ i=1,\ldots,m(\tau_k)}} I_k(x).$$

Our averaging principle for the impulsive RFDE (1) says that, under the above conditions, given $\mu > 0$ and $L > 0$, $\|x(t) - y(t)\| < \mu$, for $t \in [0, \frac{L}{\varepsilon}]$, where $x$ is a solution of (1) and $y$ is a solution of (2).

In order to obtain this result, we adapted the theory of generalized ODEs, developed by Š. Schwabik for functions with values in $\mathbb{R}^n$ (see [9]), to the case where the functions take values in a general Banach space $X$. Because impulsive RFDEs can be regarded as generalized ODEs whose solutions are functions of locally bounded variation (see [4]), it is natural to consider $X$ as the space of regulated functions (which includes functions of locally bounded variation). We also use an averaging result for non-impulsive RFDEs borrowed from [3] to get the main theorem.

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## References

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