Variational principle and half-linear oscillation criteria

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We consider the half-linear second order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1, \tag{1}$$

with continuous functions r, c and $r(t) > 0$. Together with this equation, regarded as perturbation of (1), we investigate the equation

$$\big[(r(t) + \hat{r}(t))\Phi(x')\big]' + [c(t) + \hat{c}(t)]\Phi(x) = 0. \tag{2}$$

It is known (see [4]) that the linear oscillation theory extends almost verbatim to (1), in particular, equation (1) can be classifies as oscillatory (nonoscillatory) according to whether its solutions have (do not have) a sequence of zero points tending to ∞. The variational technique is based on the following statement which relates nonoscillation of (1) to the positivity of its energy functional.

Proposition 1. *Equation* (1) *is nonoscillatory if and only if there exists $T \in \mathbb{R}$ such that*

$$\mathcal{F}(y; T, \infty) = \int_T^\infty \big[r(t)|y'|^p - c(t)|y|^p\big]\, dt > 0$$

for every $0 \not\equiv y \in W_0^{1,p}(T, \infty)$.

Consequently, we see that a positive (negative) perturbation of the coefficient c "contributes" to oscillation (nonoscillation) of (1), while perturbations of the coefficient r have an opposite effect.

The investigation of the influence of perturbations of the function c is the classical topic of the half-linear oscillation theory and the following statement from 1984 ([5]) is one of the first deeper results along this line.

Theorem 1. *If $\int^\infty d(t)t^{p-1}\, dt = \infty$, then the equation*

$$\big(\Phi(x')\big)' + \left[\frac{\gamma_p}{t^p} + d(t)\right]\Phi(x) = 0, \quad \gamma_p := \left(\frac{p-1}{p}\right)^p, \tag{3}$$

is oscillatory.

In this statement, equation (3) is viewed as a perturbation of the Euler equation

$$\big(\Phi(x')\big)' + \gamma_p t^{-p}\Phi(x) = 0. \tag{4}$$

A natural question is why just the power t^{p-1} appears by the function d in the oscillation condition. The answer is hidden in the concept of the *principal solution* of nonoscillatory equation (1) which was introduced in 1988 by D. Mirzov and independently by Elbert and Kusano ten years later, see [4] for details. Recall that a nonoscillatory solution h of (1) is said to be principal if its logarithmic derivative h'/h is less than logarithmic derivative x'/x of any linearly independent solution. Note that $h(t) = t^{\frac{p-1}{p}}$ is the principal solution of (4), so $h^p(t) = t^{p-1}$.

Theorem 1 is a special case of the following statement proved in [4].

Theorem 2. *Suppose that* (2) *is nonoscillatory and* h *is its principal solution. If*

$$\int^{\infty} \hat{c}(t) h^p(t)\, dt = \infty \tag{5}$$

then equation (2) *with* $\hat{r}(t) \equiv 0$ *is oscillatory.*

The last statement of this contribution, achieved jointly with S. Fišnarová and presented in [1], is motivated by [6] where equations (1), (2) are consider in the linear case $p = 2$ and, in contrast to the previous statements, a perturbation is allowed also in the term involving derivative.

Theorem 3. *Let* h *be the same as in Theorem 2. If the functions* $r, \hat{r}$ *are differentiable,* $(\hat{r}/r)' \le 0$ *and* (5) *holds, then* (2) *is oscillatory.*

Finally note that all previous statements are proved using the variational method and that a natural complement of [1] is the paper [2] where (2) is investigated using the Riccati technique.

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