A note on asymptotic stability conditions for a three-term linear difference equation

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The problem of the asymptotic stability of the linear difference equation

$$y_{n+1} + a_0 y_n + a_1 y_{n-1} + \cdots + a_k y_{n-k} = 0, \qquad n = 0, 1, 2, \dots \tag{1}$$

is theoretically solved by the Schur-Cohn criterion (see, e.g. [2]) which provides necessary and sufficient conditions for its asymptotic stability. However, this criterion is unsuitable to give explicit stability conditions depending on a_i and k. The formulation of such conditions turns out possible only in some particular cases of (1). We recall the famous result of Kuruklis [4] concerning the equation

$$y_{n+1} - \alpha y_n + \beta y_{n-k} = 0, \qquad n = 0, 1, 2, \dots . \tag{2}$$

Theorem 1. *Let $\alpha > 0$, β be arbitrary reals and $k \geq 1$ an integer. The equation (2) is asymptotically stable if and only if $|\alpha| < (k+1)/k$, and*

$$|\alpha| - 1 < \beta < (\alpha^2 + 1 - 2|\alpha| \cos\phi)^{1/2} \qquad \text{for } k \text{ odd}$$

$$|\beta - \alpha| < 1 \quad \text{and} \quad |\beta| < (\alpha^2 + 1 - 2|\alpha| \cos\phi)^{1/2} \qquad \text{for } k \text{ even},$$

where ϕ is the solution in $(0, \pi/(k+1))$ of $\sin(k\theta)/\sin[(k+1)\theta] = 1/|\alpha|$.

Another proof technique of this result can be found in [6]. The equation (2) in the advanced case was considered by Győri and Ladas [3]. Later, the unification of these results was performed by Dannan [1]. Considering linear difference equations with more than three terms, in the literature we can find asymptotic stability conditions, which are either sufficient (but not necessary) or they are necessary and sufficient only under some other additional assumptions on equation's coefficients. For these and other related results we refer to [3], [5], [7], [8].

The investigation of explicit conditions for the asymptotic stability of (2) is usually connected with searching for proof techniques not utilizing the Schur-Cohn criterion. Our approach is opposite. We suggest the method originating from this criterion and formulate its alternative version which seems to be quite effective. Before doing this, we introduce the auxiliary difference equation

$$z_{\ell+2} - (1 + \alpha^2 - \beta^2) z_{\ell+1} + \alpha^2 z_\ell = 0 \tag{3}$$

and four couples of initial conditions

$$z_1 = 1 + \beta, \qquad z_2 = 1 + \alpha^2\beta + \beta - \beta^2 - \beta^3, \tag{4}$$

$$z_1 = 1 - \beta, \qquad z_2 = 1 - \alpha^2\beta - \beta - \beta^2 + \beta^3, \tag{5}$$

$$z_1 = 1 + \alpha\beta - \beta^2, \qquad z_2 = (1 + \alpha^2 - \beta^2) z_1 - \alpha^2, \tag{6}$$

$$z_1 = 1 - \alpha\beta - \beta^2, \qquad z_2 = (1 + \alpha^2 - \beta^2) z_1 - \alpha^2. \tag{7}$$

Lemma 1. *Let k be a positive odd integer. Then the equation (2) is asymptotically stable if and only if*

$$|\alpha| < 1 + \beta$$

and the solutions z_ℓ of the initial value problems (3), (4) and (3), (5) are positive for all $\ell = 1, 2, \dots, (k+1)/2$.

Lemma 2. *Let k be a positive even integer. Then the equation (2) is asymptotically stable if and only if*

$$|\beta - \alpha| < 1$$

and the solutions z_ℓ of the initial value problems (3), (6) and (3), (7) are positive for all $\ell = 1, 2, \dots, k/2$.

Using these assertions we can derive various types of explicit necessary and sufficient conditions for the asymptotic stability of (2) (among them those formulated in Theorem 1). Moreover, our method seems to be applicable also to four-term higher order linear difference equations. To our knowledge, an explicit form of necessary and sufficient conditions for the asymptotic stability of such equations has not been derived yet.

Acknowledgement

The research was supported by the grant # 201/08/0469 of the Czech Grant Agency.

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