GENERALIZED RIEMANN-STIELTJES INTEGRAL ON TIME SCALES

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We will present a generalization of the Riemann-Stieltjes integral on time scales for deal with some aspects of discontinuous dynamic equations in which Riemann-Stieltjes integral does not work. When modeling dynamic phenomena, researchers have traditionally used model in which either continuous or discrete time are considered like for instance in the differential or difference equations. In recent years it became clear that for a more accurate description of phenomena is necessary to go beyond this dichotomy in scales time. Nowadays, several examples in which more general time scales appears, can be found in the literature for instance in Agarwal et al [1] and Gravagne et al [2] .

With the purpose of deal with this situation S.Hilger introduced in 1990 the Calculus on time scales (or on measure chains) in Hilger [3] . This kind of calculus showed the possibility to manage dynamic equations considering a very wide range of time scales T transforming in this way the differential and difference calculus into special cases of a more general one. Examples of time scales T are the real numbers, the integers, the sets having cluster points or even such as a Cantor set.

Regarding integral on time scales the literature includes, among others, the Riemann delta and nabla-integral and the Henstock-Perron-Kurzweil ones as in Peterson et al [4] .

In 2009, Mozyrska,Pawluszewicz and Torres, in Mozyrska et al [5] defined and presented some initial properties for the Riemann-Stieltjes integral (R-S) in the numerical case, opening in this way the possibility to develop the theory of Fredholm and Volterra-Stieltjes integral equations on time scales, even in Banach spaces .However, we still have some difficulties inherent to this kind of integrals when dealing with dynamic equations in which non-continuous functions may appear.

This situation call, in this way, for a more general vision on the subject and it is reflected by the necessity in to consider a more general notion of integral extending the Riemann-Stieltjes one. Such extension, on general time scales is considered as following:

Let be $\wp$ the class of all partitions of $\mathbb{I} = [a, b]_{\mathbb{T}}$.

Take a subset of $\wp$, $\hat{\wp}$, where $\hat{P} \in \hat{\wp}$ if and only if $|\hat{P}|$ is an odd number, and define the relation $\hat{Q} \succ \hat{R}$ in $\hat{\wp}$, fulfilling both conditions:

1. $\hat{Q} \supseteq \hat{R}$
2. the odd-indexed points $t_{2k+1}(k = 0, 1, 2, \ldots, n-1)$ in $\hat{R} = \{a = t_0 < t_1 < \cdots < t_{2n} = b\} \subset [a, b]_{\mathbb{T}}$ are also odd-indexed points $\tau_{2r+1}(r = 0, 1, 2, \ldots, m-1)$ in $\hat{Q} = \{a = \tau_0 < \tau_1 < \cdots < \tau_{2m} = b\} \subset [a, b]_{\mathbb{T}}$ $(m \geq n)$.

For f, g, functions the (Cauchy) sum associated to $\hat{P} \in \hat{\wp}$ of f relatively to g is the number (if finite)

$$\sigma_{\hat{P}}(f;g) = \sum_{i=0}^{\frac{|\hat{P}|-1}{2}} [g(t_{2i+2}) - g(t_{2i})] \cdot f(t_{2i+1})$$

We define the odd meshed (O-M) integral of f relatively to g, and write $OM - \int_{[a,b]_{\mathbb{T}}} \cdot\Delta_s g(s) \cdot f(s)$, the following limit (if existing):

$$\text{O-M} - \int_{[a,b]_{\mathbb{T}}} \cdot\Delta_s g(s) \cdot f(s) = \lim_{\hat{P} \in \hat{\wp}} \sigma_{\hat{P}}(f;g),$$

where, $\lim_{\hat{P} \in \hat{\wp}} \sigma_{\hat{P}}(f;g) = z$ means: for every neighborhood V of z there exists $\hat{P}_V \in \hat{\wp}$ such that for every $\hat{P} \in \hat{\wp}$ with $\hat{P} \succeq \hat{P}_V$, we must have $\sigma_{\hat{P}}(f;g) \in V$.

Comparison between the (R-S) and the (O-M) integrals

Let be $\mathbb{I} = [-1,1]_{\mathbb{T}} = \{\frac{1}{2^k}, k \in \mathbb{N}^*\} \cup \{\frac{-1}{2^k}, k \in \mathbb{N}^*\} \cup \{0\}$, and define

$$g(t) = \begin{cases} 1+t, & \text{if} \quad \{\frac{1}{2^k}, k \in \mathbb{N}^*\} \cup \{0\} \\ t, & \text{otherwise} \end{cases} \quad \text{and} \quad f(t) = \begin{cases} 1, & \text{if} \quad \{\frac{1}{2^k}, k \in \mathbb{N}^*\} \\ 0, & \text{otherwise} \end{cases} .$$

Then we have: the $(R-S)$ integral of f with respect to g there not exists and the (O-M) integral $\int_{[a,b]_{\mathbb{T}}} \cdot\Delta_s g(s) \cdot f(s) = 1$.

References

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