Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations

S. M. Afonso, E. M. Bonotto, M. Federson and L. P. Gimenes

São Paulo, Brazil

In this work, we give sufficient conditions for the uniform boundedness of solutions of a class of retarded functional differential equations with impulse effects acting on variable times (we write impulsive RFDEs, for short). We employ the theory of generalized ordinary differential equations (we write generalized ODEs, for short) to obtain our result.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with norm $|\cdot|$ and $\mathbb{R}_+ = [0, +\infty)$.

Let $[a, b] \subset \mathbb{R}$. We denote by $G^-([a, b], \mathbb{R}^n)$ the space of left continuous regulated functions $f : [a, b] \to \mathbb{R}^n$ equipped with the usual supremum norm.

Let $t_0 \geq 0$ and $r > 0$. Given a function $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, we consider $y_t \in G^-([-r, 0], \mathbb{R}^n)$ defined, as usual, by $y_t(\theta) = y(t + \theta)$ for $\theta \in [-r, 0]$ and $t \in [t_0, +\infty)$. We consider the impulsive RFDE

$$\begin{cases} \dot{y}(t) = f(y_t, t), & t \neq \tau_k(y(t)), \quad t \geq t_0, \\ \Delta y(t) = I_k(y(t)), & t = \tau_k(y(t)), \quad k = 0, 1, 2, \ldots, \end{cases} \tag{1}$$

subject to the initial condition

$$y_{t_0} = \phi, \tag{2}$$

where $\phi \in G^-([-r, 0], \mathbb{R}^n)$. We assume that f maps each pair $(\varphi, t) \in G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty)$ to $\mathbb{R}^n$, for $k = 0, 1, 2, \ldots$, I_k maps $\mathbb{R}^n$ to itself and τ_k maps $\mathbb{R}^n$ to $(t_0, +\infty)$. Moreover

$$\Delta y(t) := y(t+) - y(t-) = y(t+) - y(t),$$

for any $t \geq t_0$.

We assume that $f : G^-([-r, 0], \mathbb{R}^n) \times [t_0, +\infty) \to \mathbb{R}^n$ is such that for every $y \in G^-([t_0 - r, +\infty), \mathbb{R}^n)$, $t \mapsto f(y_t, t)$ is locally Lebesgue integrable on $t \in [t_0, +\infty)$.

We require that the indefinite integral of f and the operators I_k, for $k = 0, 1, 2, \ldots$, satisfy Carathéodory and Lipschitz-type properties. See [2].

We assume that the surfaces τ_k, $k = 0, 1, 2, \ldots$, are continuous functions, $\tau_0(x) \equiv t_0 < \tau_1(x) < \tau_2(x) < \ldots$, for each $x \in \mathbb{R}^n$, and $\tau_k(x) \to +\infty$ as $k \to +\infty$ uniformly on $x \in \mathbb{R}^n$. We also assume that the integral curves of system (1)-(2) meet successively each hypersurface $S_1, S_2, \ldots$ a finite number of times, where $S_k = \{(t, x) \in [t_0, +\infty) \times \mathbb{R}^n : t = \tau_k(x)\}$, for $k = 1, 2, \ldots$.

In [2], M. Federson and Š. Schwabik proved that RFDEs subject to pre-assigned moments of impulse effects satisfying the conditions above can be identified with a certain class of generalized ODEs. With some adaptations, it is possible to show that RFDEs subject to variable moments of impulse effects satisfying the conditions above can be identified with a certain class of generalized ODEs too.

Let $y : [t_0 - r, +\infty) \to \mathbb{R}^n$ be the solution of the initial value problem (1)-(2). We write $y(t) = y(t, t_0, \phi)$.

The solution $y(t) = y(t, t_0, \phi)$ of system (1)-(2) is said to be *uniformly bounded*, if for every $\alpha > 0$, there exists $M = M(\alpha) > 0$ such that if $\|\phi\| < \alpha$, then $|y(t)| < M$, for every $t \geq t_0$.

By means of Lyapunov functionals and because impulsive RFDEs can be regarded as generalized ODEs, we obtain the following result.

Theorem 1. *Consider system* (1)-(2). *Let* $U : [t_0, +\infty) \times G^-([-r, 0], \mathbb{R}^n) \to \mathbb{R}$ *be left continuous on* $(t_0, +\infty)$ *and assume that* U *satisfies the following conditions:*

(i) $U(t, 0) = 0$, $t \in [t_0, +\infty)$*;*

(ii) For each $a > 0$*, there is a constant* $K_a > 0$ *such that*

$$|U(t, \psi) - U(t, \overline{\psi})| \leq K_a \|\psi - \overline{\psi}\|, \quad t \in [t_0, +\infty), \ \ \psi, \overline{\psi} \in \mathcal{B}_a;$$

(iii) There is a monotone increasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$*, such that* $b(0) = 0$*,* $b(s) \to +\infty$ *as* $s \to +\infty$ *and*

$$U(t, \psi) \geq b(\|\psi\|),$$

for all $t \geq t_0$ *and for all* $\psi \in G^-([-r, 0], \mathbb{R}^n)$*;*

(iv) Given $t \geq t_0$ *and* $\psi \in G^-([-r, 0], \mathbb{R}^n)$*, for the solution* $y : [t - r, +\infty) \to \mathbb{R}^n$ *of* (1) *satisfying the initial condition* $y_t = y_t(t, \psi) = \psi$*, the inequality*

$$D^+U(t, \psi) \leq 0$$

holds.

Then the solution $y(t) = y(t, t_0, \phi)$ *of system* (1)-(2) *is uniformly bounded.*

The proof of the previous theorem is presented in [1], Theorem 5.4.

Acknowledgement

The research was supported by FAPESP.

References

[1] AFONSO, S. M., BONOTTO, E. M., FEDERSON, M., GIMENES L. P. - Boundedness of solutions of functional differential equations with variable impulses via generalized ordinary differential equations. *Submitted*.

[2] FEDERSON, M., SCHWABIK, Š. - Generalized ODEs approach to impulsive retarded differential equations. *Differential and Integral Equations*, **19(11)**, 1201-1234, 2006.