THE COHOMOLOGY RINGS OF REAL STIEFEL MANIFOLDS WITH INTEGER COEFFICIENTS

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Abstract. The aim of this paper is the description of the integral cohomology rings of the real Stiefel manifolds $V_{n,k}$ in terms of generators and relations. The computation is carried out by using the Gysin exact sequence for the sphere bundle $S^{n-k-1} \to V_{n,k+1} \to V_{n,k}$.

1. Introduction

Let n and k be integers, $n \geq 2$ and $1 \leq k \leq n-1$. The real Stiefel manifold $V_{n,k}$ is the homogeneous space $SO(n)/SO(n-k)$. Equivalently, it is the space of all k-tuples of orthonormal vectors in $\mathbb{R}^n$. Its dimension is $\frac{1}{2}k(2n-k-1)$.

The mod 2 cohomology of $V_{n,k}$ was computed completely by Borel in [1]. In the same paper he also described the additive structure of the cohomology with integer coefficients. The multiplicative structure was well known for $k = 1$ and 2. The special case of $SO(n) = V_{n,n-1}$ was described by Pittie in [4] where he used the Serre spectral sequence for the fibration $T \to SO(n) \to SO(n)/T$ with maximal torus T in $SO(n)$.

The aim of this note is to determine the ring structure of $H^*(V_{n,k}; \mathbb{Z})$ in terms of generators and relations. Our approach based on induction on k is independent of the methods and results in [4]. At the final step of induction we use mainly the Gysin exact sequence for the sphere bundle $S^{n-k-1} \to V_{n,k+1} \to V_{n,k}$. Simultaneously, we find the explicit description of an additive basis of $H^*(V_{n,k}; \mathbb{Z})$ as a graded group.

In the next section we repeat some basic facts about $H^*(V_{n,k}; \mathbb{Z}_2)$, introduce some notation and state our main results Theorems 2.3 and 2.9. Their proofs are given in Section 3. In the last section we conclude the paper by comparing Theorem 2.3 with Pittie's result.

2. Main result

In [1] Borel proved

Proposition 2.1. *The graded cohomology ring of the Stiefel manifolds $V_{n,k}$ with $\mathbb{Z}_2$ coefficients has a simple system of generators*

$$z_{n-k}, z_{n-k+1}, \dots, z_{n-1}$$

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where $\deg z_i = i$, $Sq^j z_i = \binom{i}{j} z_{i+j}$ *mod 2 for* $n-k \le i \le n-j-1$ *and* $Sq^j z_i = 0$ *in the other cases.*

Using Borel's result we can state the following description of $H^*(V_{n,k};\mathbb{Z}_2)$ as a graded ring:

Corollary 2.2. *The graded cohomology ring of the Stiefel manifolds* $V_{n,k}$ *with* $\mathbb{Z}_2$ *coefficients is*

$$H^*(V_{n,k};\mathbb{Z}_2) \cong \mathbb{Z}_2[z_{n-k}, z_{n-k+1}, \dots, z_{n-1}]/\mathcal{J}_{n,k}\,,$$

where $\mathcal{J}_{n,k}$ *is the ideal generated by the relations*

$$\begin{aligned} z_i^2 &= z_{2i} && \text{for } 2i \le n-1,\\ z_i^2 &= 0 && \text{for } 2i \ge n. \end{aligned}$$

Now we prepare notation for the statement of our main result. Consider the set

$$M_{n,k} = \{i \in \mathbb{Z}; n-k \le 2i-1 \le n-2\}\,.$$

Let I be a nonempty subset of $M_{n,k}$ and write

$$z_I = \prod_{i\in I} z_{2i-1}, \quad z_\varnothing = 1\,.$$

It is convenient to put

$$z_j = 0 \quad \text{for } j \notin \{n-k, n-k+1, \dots, n-2, n-1\}\,.$$

Then for a nonempty set I which is not a subset of $M_{n,k} \cup \{n/2\}$

$$z_I = \prod_{i\in I} z_{2i-1} = 0\,.$$

Denote by δ the Bockstein homomorphism associated with the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \xrightarrow{\rho_2} \mathbb{Z}_2 \to 0$ where ρ_2 is the reduction mod 2. According to the previous definitions we have $\delta z_I = 0$ whenever I is empty or not a subset of $M_{n,k} \cup \{n/2\}$. Write $D(I,J)$ for the symmetric difference of the sets I and J. Finally, let Λ stand for an exterior graded algebra over $\mathbb{Z}$, i.e., the factor of a graded free $\mathbb{Z}$-algebra over its generators modulo the relations $x \cdot y = (-1)^{\deg x \deg y} y \cdot x$.

To shorten our notation, from now on unless otherwise stated, we will use $H^*(X)$ for the cohomology groups or rings with integer coefficients.

Theorem 2.3. *Let* $1 \le k \le n-1$. *In* $H^*(V_{n,k})$ *there are classes* y_i *for* $i \in M_{n,k}$, u_{n-k} *and* v_{n-1} *of degrees* $4i-1$, $n-k$ *and* $n-1$, *respectively, such that the graded cohomology ring of the Stiefel manifold* $V_{n,k}$ *with integer coefficients is*

$$H^*(V_{n,k}) \cong \Lambda(\delta z_I, y_i, u_{n-k}, v_{n-1})/\mathcal{I}_{n,k}\,,$$

where I *ranges over all nonempty subsets of* $M_{n,k}$, i *ranges over all the elements of* $M_{n,k}$ *and* $\mathcal{I}_{n,k}$ *is an ideal generated by the relations (1) – (16) in which the set* $I \subseteq M_{n,k}$ *is nonempty, the set* $J \subseteq M_{n,k}$ *can be empty and in the relations (7) – (11) we use the convention that* $\delta z_K = 0$ *whenever the set of integers* K *is empty or not a subset of* $M_{n,k} \cup \{n/2\}$ *and*

$$\delta z_{K\cup\{n/2\}} = \delta z_K v_{n-1}$$

for n even.

The list of relations is the following:

$$y_i^2 = \delta z_{8i-3} + \delta z_{4i-3}\delta z_{4i-1} \quad \text{for } i \le \tfrac{n+1}{8}, \tag{1}$$

$$y_i^2 = \delta z_{4i-3}\delta z_{4i-1} \quad \text{for } \tfrac{n+2}{8} \le i \le \tfrac{n-1}{4}, \tag{2}$$

$$y_i^2 = 0 \quad \text{for } i \ge \tfrac{n}{4}, \tag{3}$$

$$2\delta z_I = 0, \tag{4}$$

$$(\delta z_{2i-1})^2 = \delta z_{4i-1} \quad \text{for } i \le \tfrac{n-1}{4}, \tag{5}$$

$$(\delta z_{2i-1})^2 = 0 \quad \text{for } i \ge \tfrac{n}{4}, \tag{6}$$

$$\delta z_I \delta z_J = \sum_{i\in I} \delta z_{2i-1}\delta z_{D(I-\{i\},J)} \prod_{j\in(I-\{i\})\cap J} \delta z_{4j-3}, \tag{7}$$

$$\delta z_I y_j = \delta z_{I\cup\{2j\}} + \delta z_{I-\{j\}}\delta z_{4j-3}\delta z_{2j-1} \quad \text{for } j \in I, 2j \notin I, \tag{8}$$

$$\delta z_I y_j = \delta z_{I-\{2j\}}\delta z_{8j-3} + \delta z_{I-\{j\}}\delta z_{4j-3}\delta z_{2j-1} \quad \text{for } j \in I, 2j \in I, \tag{9}$$

$$\delta z_I y_j = \delta z_{I\cup\{j\}}\delta z_{2j-1} + \delta z_{I\cup\{2j\}} \quad \text{for } j \notin I, 2j \notin I, \tag{10}$$

$$\delta z_I y_j = \delta z_{I\cup\{j\}}\delta z_{2j-1} + \delta z_{I-\{2j\}}\delta z_{8j-3} \quad \text{for } j \notin I, 2j \in I, \tag{11}$$

$$u_{n-k} = 0 \quad \text{for } n-k \text{ odd}, \tag{12}$$

$$u_{n-k}^2 = 0 \quad \text{for } n-k \text{ even}, \ k \le \tfrac{n}{2} \tag{13}$$

$$u_{n-k}^2 = \delta z_{2n-2k-1} \quad \text{for } n-k \text{ even}, \ k \ge \tfrac{n+1}{2} \tag{14}$$

$$v_{n-1} = 0 \quad \text{for } n \text{ odd}, \tag{15}$$

$$v_{n-1}^2 = 0 \quad \text{for } n \text{ even}. \tag{16}$$

Appendix 2.4. *Moreover, as for the reduction mod 2, we have*

$$\begin{aligned}
\rho_2 y_i &= z_{4i-1} + z_{2i-1}z_{2i} && \text{for } i \le \tfrac{n-1}{4},\\
\rho_2 y_i &= z_{2i-1}z_{2i} && \text{for } i \ge \tfrac{n}{4},\\
\rho_2 u_{n-k} &= z_{n-k} && \text{for } n-k \text{ even},\\
\rho_2 v_{n-1} &= z_{n-1} && \text{for } n \text{ even},\\
\rho_2 \delta z_I &= Sq^1 z_I\,.
\end{aligned}$$

Remark 2.5. Putting $J = \emptyset$ in (7) we get

$$\sum_{i\in I} \delta z_{2i-1}\delta z_{I-\{i\}} = 0\,. \tag{2.6}$$

Taking sets I and J disjoint in the same formula we get

$$\delta z_I \delta z_J = \sum_{i\in I} \delta z_{2i-1}\delta z_{(I\cup J)-\{i\}}\,.$$

Remark 2.7. The order of a nonzero element $x \in H^*(X)$ is the least positive integer m such that $mx = 0$. If there is no such integer, we say that x is of infinite order.

To prove the relations of Theorem 2.3 we will use the well known fact that if $\operatorname{Tor} H^*(X)$ has all nonzero elements of order 2, then $\rho_2 : \operatorname{Tor} H^*(X) \to H^*(X;\mathbb{Z}_2)$ is a monomorphism. Then the proof of relations can be carried out after reducing them mod 2. Using the fact that $\rho_2\delta = Sq^1$ and Appendix 2.4 we can apply Proposition 2.1. The proof of relation (7) needs the following well known formula

$$Sq^1 x_I Sq^1 x_J = \sum_{i\in I} Sq^1 x_i Sq^1 x_{D(I-\{i\},J)} x^2_{(I-\{i\})\cap J} \tag{2.8}$$

where I, J are finite sets of indices, $x_i \in H^*(X;\mathbb{Z}_2)$ and $x_I = \prod_{i\in I} x_i$. Its consequences are also relations in the torsion part of $H^*(BO(n))$ and $H^*(BSO(n))$ (see [2] and [3]).

To prove that the list of relations in Theorem 2.3 is complete we will need an additive basis of $H^*(V_{n,k})$ as a graded Abelian group. One is described by the following Theorem 2.9.

First, for all $S \subseteq M_{n,k}$ put

$$y_S = \prod_{k\in S} y_k.$$

Theorem 2.9. *Let S range over all subsets of $M_{n,k}$, let I, J, K range over all triples of subsets of $M_{n,k}$ which are pairwise disjoint, $I \neq \emptyset$ and $\min I < \min J$ and let $q \in \{0,1\}$ for n even, $q = 0$ for n odd and $r \in \{0,1\}$ for $n-k$ even, $r = 0$ for $n-k$ odd. Then the monomials*

$$u^r_{n-k} v^q_{n-1} y_S, \quad u^r_{n-k} v^q_{n-1} y_K \delta z_I \prod_{j\in J} \delta z_{2j-1}$$

form an additive basis of $H^*(V_{n,k})$ as a graded Abelian group.

Using Theorem 2.3 it is possible in principle to express products of two elements of the basis above as a linear combination of the others.

3. Proofs

We have the sequence of canonical projections with spheres as fibres:

$$\begin{array}{ccccccccccccc}
S^1 & & S^2 & & & & S^{n-k-1} & & S^{n-k} & & & & S^{n-2} & & \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
V_{n,n-1} & \to & V_{n,n-2} & \to & \cdots & \to & V_{n,k+1} & \longrightarrow & V_{n,k} & \to & \cdots & \to & V_{n,2} & \to & V_{n,1} = S^{n-1}
\end{array}$$

The proofs of Theorems 2.3 and 2.9 will be carried out by induction on k for n fixed. The basic tool for the inductive step is the Gysin exact sequence for the fibre bundle

$$S^{n-k-1} \longrightarrow V_{n,k+1} \xrightarrow{p} V_{n,k}. \tag{3.1}$$

It has the form

$$\begin{aligned}\longrightarrow H^{i-n+k}(V_{n,k}) \xrightarrow{\cup e} H^i(V_{n,k}) \xrightarrow{p^*} H^i(V_{n,k+1}) \\ \xrightarrow{\Delta} H^{i-n+k+1}(V_{n,k}) \xrightarrow{\cup e} H^{i+1}(V_{n,k}) \longrightarrow\end{aligned} \tag{3.2}$$

Here e denotes the Euler class and Δ is a group homomorphism with the property

$$\Delta(xp^*(y)) = \Delta(x)y$$

for any $x \in H^*(V_{n,k+1})$ and any $y \in H^*(V_{n,k})$. This formula will be used very often without any further reference.

Fibration (3.1) is the part of the following commutative diagram:

(3.3)
$$\begin{tikzcd}
S^{n-k-1} \arrow[r] \arrow[d, equal] & V_{n-k+1,2} \arrow[r] \arrow[d] & S^{n-k} \arrow[d] \\
S^{n-k-1} \arrow[r] & V_{n,k+1} \arrow[r, "p"] \arrow[d, "p_2"] & V_{n,k} \arrow[d, "p_1"] \\
 & V_{n,k-1} \arrow[r, equal] & V_{n,k-1}
\end{tikzcd}$$

where p, p_1 and $p_2 = p \circ p_1$ are canonical projections.

Let us start with the induction. For $k = 1$ we have $H^*(V_{n,1}) = H^*(S^{n-1}) = \mathbb{Z}[x]/\langle x^2 \rangle$, where $\deg x = n-1$. It agrees with Theorems 2.3 and 2.9 since $M_{n,1} = \emptyset$ and $x = v_{n-1}$ for n even and $x = u_{n-1}$ for n odd.

The inductive step from k to $k+1$ needs to distinguish the case $n-k$ odd from $n-k$ even.

The case $n-k$ odd. Assume that the description of $H^*(V_{n,k})$ is given by Theorem 2.3 and its additive basis is described by Theorem 2.9. Observe that the sets $M_{n,k}$ and $M_{n,k+1}$ are identical in this case. Therefore, to show that Theorems 2.3 and 2.9 hold also for $V_{n,k+1}$ it is sufficient to prove that the graded ring $H^*(V_{n,k+1})$ is isomorphic to

$$(H^*(V_{n,k}) \otimes \mathbb{Z}[u_{n-k-1}])/\mathcal{J} \tag{3.4}$$

where u_{n-k-1} has degree $n-k-1$ and $\mathcal{J}$ is an ideal generated by the relations

$$u_{n-k-1}^2 = 0 \qquad \text{for } k \le \frac{n-2}{2},$$
$$u_{n-k-1}^2 = \delta z_{2n-2k-3} \qquad \text{for } k \ge \frac{n-1}{2}.$$

In this case the Euler class of the sphere bundle (3.1) is zero. (For $k = 1$ the bundle corresponds to the tangent bundle over $V_{n,1} = S^{n-1}$. For $k > 1$ we have $H^{n-k}(V_{n,k}) = 0$ by induction.) So the Gysin exact sequence splits into short exact sequences

$$0 \longrightarrow H^{i+n-k}(V_{n,k}) \xrightarrow{p^*} H^{i+n-k}(V_{n,k+1}) \xrightarrow{\Delta} H^{i+1}(V_{n,k}) \longrightarrow 0.$$

The same also holds for the Gysin exact sequence mod 2.

For $i = -1$ we get isomorphisms $\Delta : H^{n-k-1}(V_{n,k+1}) \to H^0(V_{n,k})$ and $\Delta : H^{n-k-1}(V_{n,k+1}; \mathbb{Z}_2) \to H^0(V_{n,k}; \mathbb{Z}_2)$. Let $\tilde{z}_{n-k-1} \in H^{n-k-1}(V_{n,k+1}; \mathbb{Z}_2)$ be the element with the property $\Delta(\tilde{z}_{n-k-1}) = 1$. Then $\tilde{z}_{n-k-1}, \tilde{z}_{n-k} = p^*(z_{n-k}), \ldots, \tilde{z}_{n-1} = p^*(z_{n-1})$ form a simple system of generators of $H^*(V_{n,k+1})$.

Analogously, in $H^{n-k-1}(V_{n,k+1})$ there is just one element u_{n-k-1} with the property $\Delta(u_{n-k-1}) = 1$. Obviously, $\rho_2 u_{n-k-1} = \tilde{z}_{n-k-1}$.

We will compute u_{n-k-1}^2 as follows. We have

$$\Delta(u_{n-k-1}^2) \in H^{n-k-1}(V_{n,k}) = 0,$$

and hence $u_{n-k-1}^2 = p^*(a)$ for some $a \in H^{2n-2k-2}(V_{n,k})$.

If $k \leq \frac{n-2}{2}$, the last group is zero according to the inductive assumption. So is u_{n-k-1}^2.

If $k \geq \frac{n-1}{2}$, the group $H^{2n-2k-2}(V_{n,k})$ is generated by the element $\delta z_{2n-2k-3}$. Since

$$\rho_2 u_{n-k-1}^2 = \tilde{z}_{n-k-1}^2 = \tilde{z}_{2n-2k-2} = \rho_2 \delta \tilde{z}_{2n-2k-3},$$

we get the relation $u_{n-k-1}^2 = \delta \tilde{z}_{2n-2k-3}$.

To establish the isomorphism between $H^*(V_{n,k+1})$ and (3.4) we will show that every element $c \in H^*(V_{n,k+1})$ is of the form

$$c = u_{n-k-1} p^*(a) + p^*(b)$$

where $a, b \in H^*(V_{n,k})$ are uniquely determined by c.

Existence: Given $c \in H^*(V_{n,k+1})$, put $a = \Delta c$. Then

$$\Delta(c - u_{n-k-1} p^*(a)) = a - (\Delta u_{n-k-1}) a = 0.$$

Using the Gysin exact sequence we obtain that there is $b \in H^*(V_{n,k+1})$ such that

$$c - u_{n-k-1} p^*(a) = p^*(b).$$

Uniqueness: Suppose that for some a, b

$$u_{n-k-1} p^*(a) + p^*(b) = 0.$$

Applying Δ we get $a = 0$. Since p^* is a monomorphism, $b = 0$ as well.

This completes the proof of the inductive step for $n - k$ odd. □

The case $n-k$ even. Suppose that $H^*(V_{n,i})$ is described by Theorems 2.3 and 2.9 for $1 \leq i \leq k$. It is convenient to consider $V_{n,0}$ as a point with $H^*(V_{n,0}) \cong \mathbb{Z}$. Then the canonical projection $p_1 : V_{n,k} \to V_{n,k-1}$ has good sense also for $k = 1$ and according the previous part of the proof it induces a monomorphism in cohomology. We will often use the same letters for elements in cohomology of $V_{n,k-1}$ and their p_1^*-images in the cohomology of $V_{n,k}$.

Put $l = (n-k)/2$ and observe that

$$M_{n,k+1} = M_{n,k} \cup \{l\}.$$

Using the Gysin exact sequence we will find ring generators of $H^*(V_{n,k+1})$. We will prove that they are of two types: the images of the generators of $H^*(V_{n,k})$ under homomorphism $p^* : H^*(V_{n,k}) \to H^*(V_{n,k+1})$ and new generators $\tilde{y}_l$, $\delta \tilde{z}_{\{l\} \cup I}$ for $I \subseteq M_{n,k}$ where $p^* u_{n-k} = \delta z_{\{l\}}$. We will carry it out in Lemmas 3.5, 3.6, 3.7 and 3.9.

Lemma 3.5. *For $n-k$ even the sphere bundle (3.1) has the Euler class $2u_{n-k}$.*

Proof. Let us consider the diagram (3.3). The generator $u_{n-k} \in H^{n-k}(V_{n,k}) \cong \mathbb{Z}$ maps into a generator of $H^{n-k}(S^{n-k})$ and the Euler class of the sphere bundle in the second row maps into the Euler class of the sphere bundle in the first row. This class is twice the generator of $H^{n-k}(S^{n-k})$ since the fibration is associated to the tangent bundle to S^{n-k}. Hence the Euler class of (3.1) is $2u_{n-k}$. □

Now, we have the Gysin exact sequence

$$\longrightarrow H^{i-n+k}(V_{n,k}) \xrightarrow{\cup 2u_{n-k}} H^i(V_{n,k}) \xrightarrow{p^*} H^i(V_{n,k+1})$$
$$\xrightarrow{\Delta} H^{i-n+k+1}(V_{n,k}) \xrightarrow{\cup 2u_{n-k}} H^{i+1}(V_{n,k}) \longrightarrow$$

at our disposal.

Lemma 3.6. *Let $z_{n-k}, \dots, z_{n-1}$ be a simple system of generators for $H^*(V_{n,k}; \mathbb{Z}_2)$. Then there is a simple system of generators*

$$\tilde{z}_{n-k-1} = \tilde{z}_{\{l\}}, \tilde{z}_{n-k} = p^*(z_{n-k}), \dots, \tilde{z}_{n-1} = p^* z_{n-1}$$

for $H^*(V_{n,k+1}; \mathbb{Z}_2)$ and the relations

$$\begin{aligned} p^*(z_I) &= \tilde{z}_I, \\ p^*(\delta z_I) &= \delta \tilde{z}_I, \\ p^*(u_{n-k}) &= \delta \tilde{z}_{n-k-1} = \delta \tilde{z}_{\{l\}}, \\ \Delta \delta \tilde{z}_{\{l\} \cup I} &= \delta z_I \end{aligned}$$

hold for any $I \subseteq M_{n,k}$.

Proof. In the same way as for $n-k$ odd, define $\tilde{z}_{n-k-1} = \tilde{z}_{\{l\}} \in H^{n-k-1}(V_{n,k+1}; \mathbb{Z}_2)$ by the property $\Delta \tilde{z}_{n-k-1} = 1$. Then the mod 2 Gysin exact sequence gives the simple system of generators given in the statement. The first and the second relations are immediate consequences of the definitions.

As for the third one the Gysin exact sequence yields that $H^{n-k}(V_{n,k+1}) \cong \mathbb{Z}_2$ with elements $p^* u_{n-k}$ and $\delta \tilde{z}_{n-k-1}$ different from 0. Hence they have to be equal.

The first equality follows from the Gysin exact sequence modulo 2. The second one is its direct consequence. $p^*(u_{n-k}) \in H^{n-k}(V_{n,k+1})$ is an element of order 2 and there is no other possibility than $\delta \tilde{z}_{\{l\}}$.

Using the properties of Δ we obtain

$$\begin{aligned} \rho_2(\Delta \delta \tilde{z}_{\{l\} \cup I}) &= \Delta Sq^1 \tilde{z}_{\{l\} \cup I} \\ &= \Delta(\tilde{z}_{n-k} \tilde{z}_I) + \Delta(\tilde{z}_{n-k-1} Sq^1 \tilde{z}_I) = 0 + Sq^1 z_I \Delta z_{n-k-1} \\ &= \rho_2 \delta z_I. \end{aligned}$$

Since according to the inductive assumption $H^*(V_{n,k})$ has all the nonzero torsion elements of order 2, we obtain the last relation. ☐

Put $\tilde{y}_i = p^*(y_i)$ for $i \in M_{n,k}$ and $\tilde{v}_{n-1} = p^* v_{n-1}$.

Lemma 3.7. *There is just one element $\tilde{y}_l \in H^{2n-2k-1}(V_{n,k+1})$ such that*

$$\begin{aligned} \Delta \tilde{y}_l &= u_{n-k}, \\ \rho_2 \tilde{y}_l &= \tilde{z}_{n-k-1} \tilde{z}_{n-k} && \text{for } k \le \tfrac{n}{2}, \\ \rho_2 \tilde{y}_l &= \tilde{z}_{2n-2k-1} + \tilde{z}_{n-k-1} \tilde{z}_{n-k} && \text{for } k \ge \tfrac{n+1}{2}. \end{aligned}$$

Proof. From the Gysin exact sequence we can extract the following short exact sequence

$$0 \to H^{2n-2k-1}(V_{n,k}) \xrightarrow{p^*} H^{2n-2k-1}(V_{n,k+1}) \xrightarrow{\Delta} H^{n-k}(V_{n,k}) \to 0.$$

The same also holds for $\mathbb{Z}_2$ coefficients.

First suppose $k \leq \frac{n-1}{2}$. Then $n-1 < 2n-2k-1 < 2n-2k$ and according to the induction assumption

$$H^{2n-2k-1}(V_{n,k}) \cong H^{2n-2k-1}(V_{n,k};\mathbb{Z}_2) \cong 0.$$

$H^{2n-2k-1}(V_{n,k+1}) \cong \mathbb{Z}$ is generated by an element $\tilde{y}_{\{l\}}$ which is determined by the property $\Delta\tilde{y}_{\{l\}} = u_{n-k}$. $H^{2n-2k-1}(V_{n,k+1};\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by the element $\tilde{z}_{n-k-1}\tilde{z}_{n-k}$ which satisfies $\Delta\tilde{z}_{n-k-1}\tilde{z}_{n-k} = z_{n-k}$. That is why $\rho_2\tilde{y}_{\{l\}} = \tilde{z}_{n-k-1}\tilde{z}_{n-k}$.

For $k \geq \frac{n+1}{2}$, we have $2n-2k-1 \leq n-2$. In this case $H^{2n-2k-1}(V_{n,k}) = 0$ while $H^{2n-2k-1}(V_{n,k};\mathbb{Z}_2) \cong \mathbb{Z}_2$ is generated by $z_{2n-2k-1}$. Hence $H^{2n-2k-1}(V_{n,k+1}) \cong \mathbb{Z}$ is again generated by an element $\tilde{y}_{\{l\}}$ which is determined by the property $\Delta\tilde{y}_{\{l\}} = u_{n-k}$ and $H^{2n-2k-1}(V_{n,k+1};\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is generated by the elements $\tilde{z}_{n-k-1}\tilde{z}_{n-k}$ and $\tilde{z}_{2n-2k-1}$. Since

$$\Delta(\tilde{z}_{n-k-1}\tilde{z}_{n-k}) = z_{n-k}, \quad \Delta\tilde{z}_{2n-2k-1} = 0, \tag{3.8}$$

we get $\rho_2\tilde{y}_{\{l\}} = \tilde{z}_{n-k-1}\tilde{z}_{n-k} + a\tilde{z}_{2n-2k-1}$ for some $a \in \{0,1\}$. Now,

$$0 = Sq^1\rho_2\tilde{y}_{\{l\}} = aSq^1\tilde{z}_{2n-2k-1} + Sq^1(\tilde{z}_{n-k-1}\tilde{z}_{n-k}) = (a+1)\tilde{z}_{2n-2k}$$

which implies $a = 1$.

For $k = \frac{n}{2}$, we have n even and $2n-2k-1 = n-1$. In this case $H^{2n-2k-1}(V_{n,k}) = \mathbb{Z}$ is generated by v_{n-1}. $H^{2n-2k-1}(V_{n,k+1}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $\tilde{v}_{n-1} = p^*v_{n-1}$ and by an element $\tilde{y}$ which satisfies the property $\Delta\tilde{y} = u_{n-k}$. The elements $\tilde{z}_{n-k-1}\tilde{z}_{n-k}$ and $\tilde{z}_{n-1}$ generate $H^{2n-2k-1}(V_{n,k+1};\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since (3.8) still holds, we have $\rho_2\tilde{y} = \tilde{z}_{n-k-1}\tilde{z}_{n-k} + a\tilde{z}_{n-1}$ for some $a \in \{0,1\}$. Put $y_{\{l\}} = a\tilde{v}_{n-1} + \tilde{y}$. Then $\Delta\tilde{y}_{\{l\}} = u_{n-k}$ and

$$\rho_2\tilde{y}_{\{l\}} = a\tilde{z}_{n-1} + \tilde{z}_{n-k-1}\tilde{z}_{n-k} + a\tilde{z}_{n-1} = \tilde{z}_{n-k-1}\tilde{z}_{n-k}.$$

□

Lemma 3.9. *For every element $d \in H^*(V_{n,k+1})$ there are elements $a, b_I, c \in H^*(V_{n,k-1})$ such that*

$$d = \tilde{y}_l p_2^*(a) + \sum_{I \in M_{n,k}} \delta\tilde{z}_{\{l\}\cup I} p_2^*(b_I) + p_2^*(c)$$

where $p_2 : V_{n,k+1} \to V_{n,k-1}$ is the canonical projection.

Proof. From the knowledge of $H^*(V_{n,k})$ we deduce that any of its elements has the form

$$u_{n-k}p_1^*q(\delta z_J, y_i, v_{n-1}) + \sum_{I \subseteq M_{n,k}} \delta z_I p_1^* r_I(\delta z_J, y_i, v_{n-1}) + \sum_{I \subseteq} a_{I,q} p_1^* y_I, v_{n-1}^q,$$

where q and r_I are polynomials in the indicated variables, $i \in M_{n,k}$, $q \in \{0,1\}$ and J are nonempty subsets of $M_{n,k}$. While the first two terms belong to $\operatorname{Im}\Delta = \operatorname{Ker} 2u_{n-k}$, the third one does not if different from zero. Hence any $d \in H^*(V_{n,k+1})$ has the form

$$\Delta d = u_{n-k}p_1^*q(\delta z_J, y_i, v_{n-1}) + \sum_{I \in M_{n,k}} \delta z_I p_1^* r_I(\delta z_J, y_i, v_{n-1}).$$

Using Lemmas 3.6, 3.7 and the properties of Δ we have

$$\Delta(d - \tilde{y}_l p_2^* q(\delta z_J, y_i, v_{n-1}) - \sum_{I \in M_{n,k}} \delta \tilde{z}_{\{l\} \cup I} p_2^* r_I(\delta z_J, y_i, v_{n-1})) = 0.$$

Since $\operatorname{Ker} \Delta = \operatorname{Im} p^* = \operatorname{Im} p_2^*$, we obtain the statement of Lemma. □

Lemma 3.10. *Every nonzero torsion element of $H^*(V_{n,k+1})$ is of order 2.*

Proof. Suppose that d is a torsion element. We already know that it is of the form

$$d = \tilde{y}_l p_2^*(a) + \sum_{I \in M_{n,k}} \delta \tilde{z}_{\{l\} \cup I} p_2^*(b_I) + p_2^*(c).$$

Then

$$\Delta d = u_{n-k} p_1^* a + p_1^*(\sum_{I \in M_{n,k}} \delta z_I b_I)$$

is also a torsion element. Since p_1^* is a monomorphism, $a \in H^*(V_{n,k-1})$ is of order 2 or zero.

Hence the last summand $p_2^*(c)$ in the decomposition of d lies in $\operatorname{Tor} H^*(V_{n,k+1})$. It remains to prove that $2p_2^*(c) = 0$. There is an integer $m \geq 2$ such that $0 = mp_2^*(c) = p^*(mp_1^*(c))$. Since $\operatorname{Ker} p^* = \operatorname{Im}(\cup 2u_{n-k})$, there is an element $d \in H^*(V_{n,k-1})$ such that

$$p_1^*(mc) - u_{n-k} p_1^*(2d) = 0.$$

From the uniqueness of such a decomposition we get that $mc = 0$ in $H^*(V_{n,k-1})$. It implies $2c = 0$, which completes the proof. □

Lemma 3.11. *As a ring, $H^*(V_{n,k+1})$ has generators $\tilde{y}_i$, $\tilde{v}_{n-1}$, $\delta \tilde{z}_I$ where i is an element and I a nonempty subset of $M_{n,k+1}$. Moreover, these generators satisfy all the relations stated in Theorem 2.3.*

Proof. The first part has been proved by Lemma 3.9. We will show how to prove the relations of Theorem 2.3 in the case of $H^*(V_{n,k+1})$.

Relations (4) and (12) are obvious. Just one of the relations (15) and (16) holds in $H^*(V_{n,1})$, hence it has to hold in $H^*(V_{n,i})$ for all $i \geq 1$. Since $\tilde{y}_i$ are elements of odd degree, we get $2\tilde{y}_i^2 = 0$. So the relations (1)–(3), (5)–(11) are relations in $\operatorname{Tor} H^*(V_{n,k+1})$. Since all nonzero torsion elements are of order 2, it is sufficient to prove the relations mod 2 using Proposition 2.1 and Corollary 2.2. For the relations (1)–(3) and (8)–(11) we have to use Lemma 3.7 first.

The proof of (7) is based on Formula 2.8 and Corollary 2.2. □

Consider

$$\mathcal{R}_{n,m} = \Lambda(\delta z_I, u_{n-m}, v_{n-1}, y_i)/\mathcal{I}_{n,m},$$

where I, i and $\mathcal{I}_{n,m}$ are described in Theorem 2.3, as an abstract ring. According to the inductive assumptions $\mathcal{R}_{n,m}$ are isomorphic to $H^*(V_{n,m})$ for $m \leq k$. Lemma 3.11 says that there is a canonical epimorphism

$$\varphi : \mathcal{R}_{n,k+1} \to H^*(V_{n,k+1}).$$

It remains to prove that φ is an isomorphism. We will carry it out in Lemmas 3.13 and 3.14.

Recall that we still consider the case when $n-k$ is even.

Lemma 3.12. *The canonical projection $p_2 : V_{n,k+1} \to V_{n,k-1}$ induces monomorphisms*

$$p_2^* : H^*(V_{n,k-1}) \to H^*(V_{n,k+1}) \quad \textit{and} \quad p_2^* : H^*(V_{n,k-1}; \mathbb{Z}_2) \to H^*(V_{n,k+1}; \mathbb{Z}_2).$$

Proof. We have $p_2 = pp_1$. In the case of $\mathbb{Z}$ coefficients p_1^* is a monomorhism and

$$\operatorname{Im} p_1^* \cap \operatorname{Ker} p^* = \operatorname{Im} p_1^* \cap \operatorname{Im}(\cup 2u_{n-k}) = 0.$$

In the case of $\mathbb{Z}_2$ coefficients both p^* and p_1^* are monomorphisms. □

Lemma 3.13. *Let F_{k+1} be the free abelian group generated by the elements $\tilde{y}_I$, $\tilde{y}_I \tilde{v}_{n-1}$ for n even and by the elements $\tilde{y}_I$ for n odd, $I \subseteq M_{n,k+1}$. Then as groups*

$$\mathcal{R}_{n,k+1} \cong F_{k+1} \oplus \operatorname{Tor} \mathcal{R}_{n,k+1}$$

and φ restricted to F_{k+1} is a group monomorphism.

Proof. The first statement is obvious from the definition of $\mathcal{R}_{n,k+1}$. Further, any element of F_{k+1} has the form

$$\tilde{y}_l a + b$$

where $a, b \in F_{k-1}$. Suppose that the φ-image of this element in $H^*(V_{n,k+1})$ is zero:

$$\tilde{y}_l p_2^*(a) + p_2^*(b) = 0.$$

Application of Δ yields $u_{n-k} p_1^*(a) = 0$. From our knowledge of $H^*(V_{n,k})$ we get $a = 0$. Now, $p_2^*(b) = 0$ and by Lemma 3.12 also $b = 0$, which completes the proof. □

To simplify our notation we will abandon the tildes and we will not distinguish formally elements of $\mathcal{R}_{n,k-1}$ from their images in $\mathcal{R}_{n,k+1}$ and elements of $H^*(V_{n,k-1})$ from their p_2^*-images in $H^*(V_{n,k+1})$.

Let us consider couples (n, m) with $n \geq m \geq 0$. For any I, J, $K \subseteq M_{n,m}$ we put

$$Z_{I,J,K,q} = v_{n-1}^q y_K \delta z_I \prod_{j \in J} \delta z_{2j-1},$$

where $q \in \{0, 1\}$ for n even and $q = 0$ for n odd.

Definition. The monomials $Z_{I,J,K,q}$ with I, J and K pairwise disjoint, $I \neq \emptyset$ and $\min I < \min J$ will be called admissible or (n, m)-admissible if we want to emphasize that $I, J, K \subseteq M_{n,m}$. The triples of sets which are indices of admissible monomials will be also called admissible.

Since $\operatorname{Tor} \mathcal{R}_{n,k+1}$ and $\operatorname{Tor} H^*(V_{n,k+1})$ have only elements of order 2, they can be considered as $\mathbb{Z}_2$-vector spaces. To prove that φ restricted to $\operatorname{Tor} \mathcal{R}_{n,k+1}$ is an isomorphism we will show that the $(n, k+1)$-admissible monomials in $\mathcal{R}_{n,k+1}$ have the following properties:

(a) They generate $\operatorname{Tor} \mathcal{R}_{n,k+1}$ as a $\mathbb{Z}_2$-vector space.
(b) Their φ-images are linearly independent in $\operatorname{Tor} H^*(V_{n,k+1})$.

Lemma 3.14. *Let $n - m$ be odd. Then (n, m)-admissible monomials generate* $\operatorname{Tor} \mathcal{R}_{n,m}$ *as a $\mathbb{Z}_2$-vector space.*

Proof. If $M_{n,m} = \emptyset$, then $\operatorname{Tor}\mathcal{R}_{n,m} = 0$. For $M_{n,m} \neq \emptyset$ the relations (1) – (3), (5) – (7), (15) and (16) imply that the monomials $Z_{I,J,K,q}$, where I, J and K range over all subsets of $M_{n,m}$, $I \neq \emptyset$, generate $\operatorname{Tor}\mathcal{R}_{n,m}$. We will successively reduce this set of generators to the set of admissible monomials.

First, we prove by induction that for all $r \in M_{n,m} \cup \{\min M_{n,m} - 1\}$ the set

$$\{Z_{I,J,K,q};\ I \neq \emptyset,\ \min(I \cap J) > r,\ \min(I \cap K) > r,\ \min(J \cap K) > r\}$$

generates $\operatorname{Tor}\mathcal{R}_{n,m}$. The first step of induction for $r = \min M_{n,m} - 1$ has been already done. Suppose that the statement is true for $r-1$ and prove it for r using the relations from the definition of $\mathcal{I}_{n,m}$. For the purposes of the proof we enlarge the definition of $Z_{I,J,K,q}$ for all finite subsets I of integers using the convention that $Z_{I,J,K,q} = 0$ whenever I is empty or not a subset of $M_{n,m} \cup \{\frac{n}{2}\}$ and $Z_{I\cup\{n/2\},J,K,q} = v_{n-1} Z_{I,J,K,q}$ for n even and $I \subseteq M_{n,m}$. (This is compatible with the conventions introduced in Section 2.)

(1) Suppose that $\min(I \cap J) = \min(I \cap K) = \min(J \cap K) = r$. If $2r \notin I$, then according to (10)

$$\begin{aligned}
Z_{I,J,K,q} &= v_{n-1}^q y_K \Big(\prod_{j\in J-\{r\}} \delta z_{2j-1}\Big) \delta z_I \delta z_{2r-1} \\
&= v_{n-1}^q y_K \Big(\prod_{j\in J-\{r\}} \delta z_{2j-1}\Big)(\delta z_{I-\{r\}} y_r + \delta z_{(I-\{r\})\cup\{2r\}}) \\
&= Z_{I-\{r\},J-\{r\},K-\{r\},q} y_r^2 + Z_{(I-\{r\})\cup\{2r\},J-\{r\},K,q}.
\end{aligned}$$

If $2r \in I$, then according to (11)

$$Z_{I,J,K,q} = Z_{I-\{r\},J-\{r\},K-\{r\},q} y_r^2 + Z_{I-\{r,2r\},J-\{r\},K,q} \delta z_{8r-3}.$$

Using (1) or (2) or (3), our conventions from Theorem 2.3 and (5) or (6) if necessary, we get that $Z_{I,J,K,q}$ is a sum of monomials $Z_{I',J',K',q}$ where $\min(I' \cap J') > r$, $\min(I' \cap K') > r$ and $\min(J' \cap K') > r$.

(2) Consider indices (I, J, K) with $\min(I \cap J) > r$, $\min(I \cap K) > r$ and $\min(J \cap K) = r$. If $2r \notin I$ then according to (8) and (7)

$$\begin{aligned}
Z_{I,J,K,q} &= v_{n-1}^q \delta z_I \Big(\prod_{j\in J-\{r\}} \delta z_{2j-1}\Big) y_{K-\{r\}} y_r \delta z_{2r-1} \\
&= v_{n-1}^q \Big(\prod_{j\in J-\{r\}} \delta z_{2j-1}\Big) y_{K-\{r\}} \delta z_I \delta z_{\{r,2r\}} \\
&= v_{n-1}^q \Big(\prod_{j\in J-\{r\}} \delta z_{2j-1}\Big) y_{K-\{r\}} (\delta z_{2r-1} \delta z_{I\cup\{2r\}} + \delta z_{4r-1} \delta z_{I\cup\{r\}}) \\
&= Z_{I\cup\{2r\},J,K-\{r\},q} + Z_{I\cup\{r\},J-\{r\},K-\{r\},q} \delta z_{4r-3}.
\end{aligned}$$

Notice that $4r - 3 \geq r$ with equality only for $r = 1$. If $2r \in I$, then according to (8) and (7) similarly

$$Z_{I,J,K,q} = Z_{I-\{2r\},J,K-\{r\},q} \delta z_{8r-3} + Z_{I\cup\{r\},J-\{r\},K-\{r\},q} \delta z_{4r-1}.$$

So in both cases $Z_{I,J,K,q}$ is a sum of monomials $Z_{I',J',K',q}$ where $\min(I'\cap J')\geq r$, $\min(I'\cap K')>r$ and $\min(J'\cap K')>r$.

(3) Now consider a triple (I,J,K) with $\min(I\cap J)=r$, $\min(I\cap K)>r$ and $\min(J\cap K)>r$. If $2r\notin I$, then according to (10)

$$Z_{I,J,K,q}=Z_{I-\{r\},J-\{r\},K\cup\{r\},q}+Z_{(I-\{r\})\cup\{2r\},J-\{r\},K,q}.$$

If $2r\in I$, then according to (11)

$$Z_{I,J,K,q}=Z_{I-\{r\},J-\{r\},K\cup\{r\},q}+Z_{I-\{r,2r\},J-\{r\},K,q}\delta z_{8r-3}.$$

Using our conventions and (5) or (6) if necessary, we obtain $Z_{I,J,K,q}$ as a sum of monomials $Z_{I',J',K',q}$ where $\min(I'\cap J')>r$, $\min(I'\cap K')>r$ and $\min(J'\cap K')>r$.

(4) Consider indices (I,J,K) with $\min(I\cap J)>r$, $\min(I\cap K)=r$ and $\min(J\cap K)>r$. If $2r\notin I$, then according to (8)

$$Z_{I,J,K,q}=Z_{I\cup\{2r\},J,K-\{r\},q}+Z_{I-\{r\},J\cup\{r\},K-\{r\},q}\delta z_{4r-3}.$$

If $2r\in I$, then according to (9)

$$Z_{I,J,K,q}=Z_{I-\{2r\},J,K-\{r\},q}\delta z_{8r-3}+Z_{I-\{r\},J\cup\{r\},K-\{r\},q}\delta z_{4r-3}.$$

Using our conventions and (5) or (6) if necessary, we can write $Z_{I,J,K,q}$ as a sum of monomials $Z_{I',J',K',q}$ where $\min(I'\cap J')>r$, $\min(I'\cap K')>r$ and $\min(J'\cap K')>r$.

We have proved that the monomials $Z_{I,J,K,q}$ with $I\neq\emptyset$ and I,J,K pairwise disjoint generate $\operatorname{Tor}\mathcal{R}_{n,m}$. Consider such a triple of indices with $\min J=r<\min I$. Then using (2.6) we have

$$\delta z_{2r-1}\delta z_I=\sum_{i\in I}\delta z_{2i-1}\delta z_{(I\cup\{r\})-\{i\}},$$

which yields

$$Z_{I,J,K,q}=\sum_{i\in I}Z_{(I\cup\{r\})-\{i\},(J\cup\{i\})-\{r\},K,q},$$

where all the indices (I',J',K') on the right hand side are admissible. □

For $m=1$ or for $m=2$ and n even there are no admissible triples since $M_{n,m}=\emptyset$. For $m=2$ and n odd there is just one admissible monomial $Z_{M_{n,2},\emptyset,\emptyset,0}=\delta z_{n-2}$ since $M_{n,2}=\{\frac{n-1}{2}\}$. A similar situation occurs when $m=3$ and n is even. Here there are two monomials since $q=0,1$. In both cases the admissible monomials are linearly independent in $\operatorname{Tor}H^*(V_{n,m})$.

Lemma 3.15. *Let $n-k$ be even. Suppose that $(n,k-1)$-admissible monomials are linearly independent in $\operatorname{Tor}\mathcal{R}_{n,k-1}\cong\operatorname{Tor}H^*(V_{n,k-1})$. Then the φ-images of $(n,k+1)$-admissible monomials are linearly independent in $\operatorname{Tor}H^*(V_{n,k+1})$.*

Proof. Recall that $M_{n,k+1}=M_{n,k-1}\cup\{l\}$. First, we prove that if I,J,K range over all $(n,k-1)$-admissible indices, then the elements

$$\varphi(Z_{I,J,K,q}),\ \varphi(Z_{I\cup\{l\},J,K,q}),\ \varphi(Z_{I,J\cup\{l\},K,q}),\ \varphi(Z_{I,J,K\cup\{l\},q}) \tag{3.16}$$

are linearly independent in $\operatorname{Tor} H^*(V_{n,k+1})$. Suppose that

$$\sum_{I,J,K,q} a_{I,J,K,q}\varphi(Z_{I,J,K,q}) + \sum_{I,J,K,q} b_{I,J,K,q}\varphi(Z_{I\cup\{l\},J,K,q}) \\ + \sum_{I,J,K,q} c_{I,J,K,q}\varphi(Z_{I,J\cup\{l\},K,q}) + \sum_{I,J,K,q} d_{I,J,K,q}\varphi(Z_{I,J,K\cup\{l\},q}) = 0.$$

Applying $\Delta : H^*(V_{n,k+1}) \to H^*(V_{n,k})$ we get

$$\sum_{I,J,K,q} b_{I,J,K,q}p_1^*(Z_{I,J,K,q}) + u_{n-k}(\sum_{I,J,K,q} d_{I,J,K,q}p_1^*(Z_{I,J,K,q})) = 0$$

in $H^*(V_{n,k})$. From the description of this ring (see (3.4)) we obtain

$$\sum_{I,J,K,q} b_{I,J,K,q}Z_{I,J,K,q} = 0, \qquad \sum_{I,J,K,q} d_{I,J,K,q}Z_{I,J,K,q} = 0.$$

Linear independence of $(n, k-1)$-admissible monomials implies that all $b_{I,J,K,q} = d_{I,J,K,q} = 0$. Now, reduce the remaining terms mod 2

$$\sum_{I,J,K,q} a_{I,J,K,q}\rho_2\varphi(Z_{I,J,K,q}) + z_{2l}(\sum_{I,J,K,q} c_{I,J,K,q}\rho_2\varphi(Z_{I,J,K,q})) = 0.$$

Since $z_{2l-1}, z_{2l}, \ldots, z_{n-1}$ form a simple system of generators in $H^*(V_{n,k+1};\mathbb{Z}_2)$ and p_2^* is a monomorphism, we have

$$\sum_{I,J,K,q} a_{I,J,K,q}\rho_2 Z_{I,J,K,q} = 0, \qquad \sum_{I,J,K,q} c_{I,J,K,q}\rho_2 Z_{I,J\cup\{l\},K,q} = 0$$

in $H^*(V_{n,k-1};\mathbb{Z}_2)$. Now, $\rho_2 : \operatorname{Tor} H^*(V_{n,k-1}) \to H^*(V_{n,k-1};\mathbb{Z}_2)$ is a monomorphism, so as a consequence of linear independence of $Z_{I,J,K,q}$ in $\operatorname{Tor} H^*(V_{n,k-1})$, we get $a_{I,J,K,q} = c_{I,J,K,q} = 0$.

All the $(n, k+1)$-admissible monomials form the following list

$$Z_{I,J,K,q},\ Z_{I\cup\{l\},J,K,q},\ ,Z_{S\cup\{l\},T,K,q},\ Z_{I,J,K\cup\{l\},q}, \tag{3.17}$$

where (I, J, K) are $(n, k-1)$-admissible and $S, T, K \subseteq M_{n,k-1}$ are pairwise disjoint, $T \neq \emptyset$ and $\min T = r < \min S$. According to (2.6)

$$Z_{S\cup\{l\},T,K,q} = Z_{S\cup\{r\},(T-\{r\})\cup\{l\},K,q} + \sum_{i\in S} Z_{(S-\{i\})\cup\{l,r\},(T-\{r\})\cup\{i\},K,q}.$$

Here the triples $S\cup\{r\}, (T-\{r\})\cup\{l\}, K$ and $(S-\{i\})\cup\{l,r\}, (T-\{r\})\cup\{i\}, K$ are $(n, k-1)$-admissible. Hence $\varphi(Z_{S\cup\{l\},T,K,q})$ can be expressed as a sum of $\varphi(Z_{S\cup\{r\},(T-\{r\})\cup\{l\},K,q})$ and elements of the form $\varphi(Z_{I\cup\{l\},J,K,q})$ with $(n, k-1)$-admissible indices (I, J, K). Moreover, the assignment

$$(S, T, K) \mapsto (S \cup \{\min T\}, T - \{\min T\}, K)$$

is one-to-one from the set of the triples of pairwise disjoint subsets of $M_{n,k-1}$ with $\min T < \min S$ onto the set of $(n, k-1)$ admissible triples (I, J, K). So the linear independence of elements in the list (3.16) implies that the φ-images of $(n, k+1)$-admissible monomials which form the list (3.17) are also linearly independent. $\square$

Theorem 2.9 for $V_{n,k+1}$ is now an immediate consequence of Lemmas 3.13, 3.14 and 3.15.

4. Comparison with Pittie's result

If we apply Theorem 2.3 for $k = n-1$, we obtain the description of the ring $H^*(SO(n);\mathbb{Z})$. In [4] H. V. Pittie derived the description of the ring structure of $H^*(SO(n);\mathbb{Z})$ using the method based on the existence of a maximal torus in $SO(n)$. Comparing his result [4, 7.5] with Theorem 2.3 for $k = n-1$, it is seen at the first sight that his set of ring generators is smaller than ours. Consequently, some of his relations are different and more complicated. In this section we will outline how to reduce the set of the ring generators in Theorem 2.3 and how to get new relations if $k = n-1$ to obtain the result from [4].

For $j \in M_{n,n-1}$ we define

$$\begin{aligned} \alpha_j &= \min\{d; 2^{d+1}j \geq n\} = \max\{d; 2^{d-1}j \in M_{n,n-1}\}, \\ a_j &= 2^{\alpha_j}. \end{aligned}$$

From Corollary 2.2 we get that

$$\begin{aligned} z_{2j}^{a_j-1} &\neq 0, \\ z_{2j}^{a_j} &= z_{2^{\alpha_j+1}j} = 0. \end{aligned}$$

Similarly as in [4] we put

$$k_{ij} = a_j - \sum_{r=0}^{i} 2^r.$$

Finally, put $L_n = \{i \in M_{n,n-1};\ i \text{ odd}\}$.

Proposition 4.1. (Compare with [4, 7.5]) *In $H^*(SO(n))$ there are classes y_i for $i \in M_{n,n-1}$, and v_{n-1} of degrees $4i-1$ and $n-1$, respectively, such that the graded cohomology ring of $SO(n)$ with integer coefficients is*

$$H^*(SO(n)) \cong \Lambda(\delta z_I, \delta z_{2i-1}, y_i, v_{n-1})/\mathcal{K}_n,$$

where I ranges over all the subsets of L_n with at least two elements, i ranges over all the elements of $M_{n,n-1}$ and $\mathcal{K}_n$ is an ideal generated by the relations (1) – (7), (15), (16) of Theorem 2.3 and by the relations (a) – (e) in which the set $I \subseteq L_n$ has at least two elements and the set $J \subseteq M_{n,n-1}$ is either empty or contains only one element or $J \subseteq L_n$.

The list of new relations is the following:

(a) $2\delta z_{2i-1} = 0,$

(b) $\displaystyle\sum_{i=0}^{\alpha_j-1} y_{2^i j}(\delta z_{2j-1})^{k_{ij}} = 0,$ $\quad j \in M_{n,k}$

(c) $\displaystyle\delta z_I \sum_{i=0}^{\alpha_j-1} y_{2^i j}(\delta z_{2j-1})^{k_{ij}-1} + \delta z_{I-\{j\}}\delta z_{4j-3}(\delta z_{2j-1})^{a_j-1} = 0,$ $\quad j \in I, j \leq \frac{n+1}{4},$

(d) $\delta z_I y_j = 0,$ $\quad j \in I, j \geq \frac{n+2}{4},$

(e) $\displaystyle\delta z_{I-\{j\}} \sum_{i=0}^{\alpha_j-1} y_{2^i j}(\delta z_{2j-1})^{k_{ij}-1} + \delta z_I(\delta z_{2j-1})^{a_j-1} = 0,$ $\quad j \in I,\ j$ *odd.*

The reduction of elements y_i and v_{n-1} mod 2 is given in Appendix 2.4 to Theorem 2.3.

Remark 4.2. The relation (7) stands for equivalent relations

(f) $\displaystyle\delta z_I \delta z_J = \delta z_{I-J}\delta z_{J-I} \prod_{j\in I\cap J} \delta_{4j-3} + \delta z_{I\cap J}\delta z_{I\cup J}$ $\quad$ for $I \cap J \neq \varnothing$

(g) $\displaystyle\delta z_I \delta z_J = \sum_{i\in I} \delta z_{2i-1}\delta z_{(I\cup J)-\{i\}}$ $\quad$ for $I \cap J = \varnothing$

from [4, 7.5].

Outline of the proof of Proposition 4.1. First, we will show how to reduce the set of the ring generators from Theorem 2.3 to the set described in Proposition 4.1. Consider a set $K \subseteq M_{n,n-1}$ with at least two elements and with the biggest even element $2j$. Put $I = K - \{2j\}$. If $j \in K$, from relation (8) we get

$$\delta z_K = \delta z_I y_j + \delta z_{I-\{j\}}\delta z_{4j-3}\delta z_{2j-1},$$

while if $j \notin K$, relation (10) implies

$$\delta z_K = \delta z_I y_j + \delta z_{I\cup\{j\}}\delta z_{2j-1},$$

since $j \in M_{n,n-1}$. (And this is just the reason why we cannot perform this reduction for any k.) All the even elements of the sets I, $I - \{j\}$ and $I \cup \{j\}$ are less than $2j$. Hence repeating this procedure we obtain δz_K as a polynomial in the generators described by Proposition 4.1.

Now, on an example we will outline how to derive the relations (b), (c), (d) and (e) from the relations (8), (10), (5), and (6). Consider $j \in M_{n,n-1}$, $4j < n \leq 8j$, j odd, and $I \subseteq L_n$ which does not contain j. Then $\alpha_j = 2$ and $\delta z_{I\cup\{4j\}} = 0$ according to our conventions. Using twice the relation (10) and then the relation (5) we get

$$\begin{aligned}\delta z_I y_{2j} &= \delta z_{I\cup\{2j\}}\delta z_{4j-1} + \delta z_{I\cup\{4j\}} = (\delta z_I y_j + \delta z_{I\cup\{j\}}\delta z_{2j-1})\delta z_{4j-1}\\ &= \delta z_I y_j(\delta z_{2j-1})^2 + \delta z_{I\cup\{j\}}(\delta z_{2j-1})^3,\end{aligned}$$

which gives just the relation (e) for I and j.

To show that the list of relations in Proposition 4.1 is complete, we would have to find a basis of Tor $H^*(SO(n))$ in terms of monomials of the ring generators. Here we cannot use Theorem 2.3 directly and so we will not carry it out since a different proof has been given in [4].

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