ON THE CLASSIFICATION OF ORIENTED VECTOR BUNDLES OVER 5-COMPLEXES

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Abstract. Necessary and sufficient conditions on a CW-complex X of dimension ≤ 5 which allow to classify 5-dimensional oriented vector bundles over X in terms of characteristic classes are presented. As a consequence, some results on the span of such vector bundles and on the existence of 2-distributions and 4-distributions with a complex structure are derived.

1. Introduction. The effort to classify vector bundles over a fixed CW-complex has a long history. The first result in this direction is the assertion that every two-dimensional oriented vector bundle is uniquely determined by its Euler class. Complete characterization of oriented vector bundles over a 4-dimensional CW-complex was given in [D–W] using the difference cocycles. In [T2] E. Thomas found conditions for a mapping $f \in [X, Y]$ to be uniquely determined by its cohomology homomorphism $f^* \in \text{Hom}\,(H^*(Y), H^*(X))$ under the assumptions that X is a suspension or Y is an H-space. He also applied the result to $Y = BO$, the classifying space for the group O, and so he obtained conditions on $H^*(X)$ under which stable vector bundles over X are determined by their Stiefel–Whitney and Pontrjagin classes. A further progress was made in [J–T] where the question how many n-dimensional vector bundles over a CW-complex of the same dimension are determined by a stable vector bundle ξ. The results are given in terms of ξ and they allow successful application for $n = 3$ and 7. Previous results concerning characterization of oriented vector bundles over low dimensional complexes were summarized and completed in [W]. Using elementary homotopy theoretic methods and relations among characteristic classes L. M. Woodward has given the classification of stable oriented vector bundles over CW-complexes of dimension ≤ 8 and the classification of n-dimensional oriented vector bundles over CW-complexes of dimension n for $n = 3, 4, 6, 7, 8$, both in terms of characteristic classes. A typical condition on a CW-complex X to admit such a classification is: $H^4(X, \mathbb{Z})$ has no element of order 4.

In dimension 5 the situation is much more complicated as it can be seen on the example of the sphere S^5. Both trivial and tangent bundle over S^5 has all characteristic classes equal to zero. Moreover, all conditions of Woodward's type are satisfied. The aim of our paper is to derive necessary and sufficient conditions on a 5-dimensional CW-complex X which make the classification of 5-dimensional

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oriented vector bundles over X in terms of characteristic classes possible. This is carried out in Section 3 using a combination of the method of Postnikov tower and the Woodward method (see [T3] and [W]).

The maximal number of linearly independent sections in a vector bundle ξ is defined to be a span of ξ. As a consequence of the classification described above we compute the span of 5-dimensional oriented vector bundles over CW-complexes of the same dimension. These results complete computations of Thomas for tangent bundles over 5-dimensional manifolds given in [T6] and also our results for the dimension 6 and 7 obtained in [–V]. Together with results on the existence of a 2-distribution and a 4-distribution with a complex structure they form the contents of Section 4.

2. Preliminaries. All vector bundles will be considered over a connected CW-complex X and will be oriented. The letter ε will stand for trivial one-dimensional vector bundle. The mapping $\beta_k : H^*(X,\mathbb{Z}_k) \to H^*(X,\mathbb{Z})$ is the Bockstein homomorphism associated with the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_k \to 0$. The mappings $i_* : H^*(X,\mathbb{Z}_2) \to H^*(X,Z_4)$ and $\rho_k : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_k)$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and reduction mod k, respectively.

An important role in our considerations plays the Pontrjagin square $\mathfrak{P}$, a cohomology operation from $H^{2k}(X,\mathbb{Z}_2)$ into $H^{4k}(X,\mathbb{Z}_4)$ satisfying the following relations

$$\mathfrak{P}\rho_2 x = \rho_4 x^2 \tag{1}$$

$$\mathfrak{P}(u+v) = \mathfrak{P}u + \mathfrak{P}v + i_*(u \cdot v) \tag{2}$$

for $x \in H^{2k}(X,\mathbb{Z})$ and $u, v \in H^{2k}(X,\mathbb{Z}_2)$. See [M–T], chapter 2.

We will use $w_j(\xi)$ for the j-th Stiefel–Whitney class of the vector bundle ξ, $p_1(\xi)$ for the first Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle $c_j(\zeta)$ denotes the j-th Chern class. The letters w_j, p_1, e will stand for characteristic classes of the universal oriented n-dimensional vector bundle over the classifying space $BSO(n)$. Our results given below are based on the following relations among the characteristic classes

$$\rho_4 p_1(\xi) = \mathfrak{P}w_2(\xi) + i_* w_4(\xi) \tag{3}$$

$$w_6(\xi) = Sq^2 w_4(\xi) + w_2(\xi) w_4(\xi) \tag{4}$$

the former being proved in [M] and [T1] and the latter being a special case of the Wu formula.

The Eilenberg–MacLane space with n-th homotopy group G will be denoted $K(G,n)$ and ι_n will stand for the fundamental class in $H^n(K(G,n),G)$. Writing the fundamental class it will be always clear which group G we have in mind.

In the proof of Theorem 1 we will need suspension. Being defined for every fibration $F \xrightarrow{j} E \xrightarrow{p} B$, it is a natural mapping from a subgroup of $H^{k+1}(B)$ into $H^k(F)/\mathrm{im}\, j^*$ which commutes with the Steenrod squares and i_* (see [M–T]).

We say that $x \in H^*(X,\mathbb{Z})$ is an element of order k $(k = 2,3,4,\dots)$ if and only if $x \neq 0$ and k is the least positive integer such that $kx = 0$ (if it exists). Some results will involve the following hypotheses:

Condition (A). *$H^4(X,\mathbb{Z})$ has no element of order 4.*

Condition (B). *$Sq^2H^3(X,\mathbb{Z}_2) = H^5(X,\mathbb{Z}_2)$.*

Remark. An important example of CW-complex which satisfies Condition (B) is a 5-dimensional oriented smooth manifold M with $w_2(M) \neq 0$. The Poincar duality and the fact that the second Wu class is equal to $w_2(M)$ yields

$$Sq^2H^3(M,\mathbb{Z}_2) = w_2(M)H^3(M,\mathbb{Z}_2) = H^5(M,\mathbb{Z}_2)\,.$$

3. Classification theorem. Let X be a connected CW-complex of dimension ≤ 5. Our problem consists in finding conditions on X such that for every $a \in H^2(X,\mathbb{Z}_2)$, $b \in H^4(X,\mathbb{Z}_2)$, $c \in H^4(X,\mathbb{Z})$ there is at most one oriented 5-dimensional vector bundle ξ with $w_2(\xi) = a$, $w_4(\xi) = b$, $p_1(\xi) = c$. A necessary and sufficient condition on a, b, c for the existence of such a vector bundle derived in [W] is given by the relation $\rho_4 c = \mathfrak{P}a + i_* b$ (see (3)). Up to homotopy there is just one mapping $f : X \to K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4)$ such that $f^*(\iota_2\otimes 1\otimes 1) = a$, $f^*(1\otimes\iota_4\otimes 1) = b$, $f^*(1\otimes 1\otimes \iota_4) = c$. Similarly, w_2, w_4, p_1, the cohomology classes of $BSO(5)$, determine a mapping $\alpha : BSO(5) \to K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4)$ which can be considered to be a fibration. Now the problem desribed above can be formulated as a problem of lifting: *when every mapping $f : X \to K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4)$ has at most one lifting $\xi : X \to BSO(5)$ in the fibration α.*

$$\begin{array}{ccc} & & BSO(5) \\ & \nearrow & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow{\;f\;} & K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4) \end{array}$$

To solve this problem we will construct a Postnikov tower for the fibration $\alpha : BSO(5) \to K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4)$. Put $K = K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)\times K(\mathbb{Z},4)$ and denote the fibre of α by V. Let us recall that $\pi_k(BSO(5)) \cong 0$ for $k = 1,3$, $\pi_k(BSO(5)) \cong \mathbb{Z}_2$ for $k = 2,5$ and $\pi_4(BSO(5)) \cong \mathbb{Z}$. Considering the characteristic classes as mappings from $BSO(5)$ into appropriate Eilenberg–MacLane spaces, we get $w_{2*} = id : \pi_2(BSO(5)) \to \mathbb{Z}_2$, $w_{4*} = \rho_2 : \pi_4(BSO(5)) \to \mathbb{Z}_2$ and $p_{1*} : \pi_4(BSO(5)) \to \mathbb{Z}$ is a multiplication by 2. See [W]. From the long exact homotopy sequence we compute: $\pi_1(V) \cong \pi_2(V) \cong 0$, $\pi_3(V) \cong \mathbb{Z}_4$, $\pi_4(V) \cong 0$, and $\pi_5(V) = \mathbb{Z}_2$. The first invariant in the Postnikov tower is the transgression of a fundamental class in $H^3(V,\mathbb{Z}_4)$. It is a generator of $\ker\alpha^* \subset H^4(K,\mathbb{Z}_4)$. Hence it is equal to

$$\rho_4(1\otimes 1\otimes \iota_4) - \mathfrak{P}\iota_2\otimes 1\otimes 1 - 1\otimes i_*\iota_4\otimes 1.$$

Let E_1 be the first stage of the Postnikov tower and let the new mappings be denoted according to the diagram.

$$\begin{array}{ccccccc} \bar{F}_1 & \longrightarrow & V & \xrightarrow{\;\bar{\beta}_1\;} & K(\mathbb{Z}_4,3) & & \\ & & \downarrow & & \downarrow{\scriptstyle i_1} & & \\ F_1 & \longrightarrow & BSO(5) & \xrightarrow{\;\beta_1\;} & E_1 & & \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi_1} & & \\ & & K & = & K & \xrightarrow{\;\rho_4(1\otimes1\otimes\iota_4)-\mathfrak{P}\iota_2\otimes1\otimes1-1\otimes i_*\iota_4\otimes1\;} & K(\mathbb{Z}_4,4) \end{array}$$

Consider $\beta_1 : BSO(5) \to E_1$ as a fibration with a fibre F_1. This fibre is homotopy equivalent to the homotopy fibre $\bar{F}_1$ of the mapping $\bar{\beta}_1$ (see [T3]). Hence computing the homotopy groups of $\bar{F}_1$ we get that F_1 is 4-connected and $\pi_5(F_1) \cong \mathbb{Z}_2$. Consequently, β_1 is a 5-equivalence.

The next invariant $\varphi \in H^6(E_1, \mathbb{Z}_2)$ is the transgression of the generator of $H^5(F_1, \mathbb{Z}_2)$ in the Serre exact sequence for the fibration β_1. E_1 is also the first stage in the Postnikov tower for the fibration $\hat{\alpha} : BSO(6) \to K$ determined by w_2, w_4 and p_1. The mapping $\hat{\beta}_1 : BSO(6) \to E_1$ in this Postnikov tower is a 6-equivalence (since $\pi_5(BSO(6)) \cong 0$). Using the Serre exact sequence for the fibration $\hat{\beta}_1$, we get that $\hat{\beta}_1^*$ is an isomorphism between $H^6(E_1, \mathbb{Z}_2)$ and $H^6(BSO(6), \mathbb{Z}_2)$. The latter group has generators w_2^3, w_3^2, w_2w_4 and $Sq^2w_4 (= w_6 + w_2w_4)$. Hence the generators of $H^6(E_1, \mathbb{Z}_2)$ are $\pi_1^*(\iota_2^3 \otimes 1 \otimes 1)$, $\pi_1^*((Sq^1\iota_2)^2 \otimes 1 \otimes 1)$, $\pi_1^*(\iota_2 \otimes \iota_4 \otimes 1)$, $\pi_1^*(1 \otimes Sq^2\iota_4 \otimes 1)$. The mapping $\beta_1^* : H^6(E_1, \mathbb{Z}_2) \to H^6(BSO(5), \mathbb{Z}_2)$ maps them into w_2^3, w_3^2, w_2w_4 and $Sq^2w_4 = w_2w_4$, respectively. Consequently, using the Serre exact sequence for the fibration β_1 we get $\varphi = \pi_1^*(\iota_2 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2\iota_4 \otimes 1)$. So we can build the second stage E_2 of our Postnikov tower.

$$\begin{array}{ccccccc}
\bar{F}_2 & \longrightarrow & F_1 & \xrightarrow{\bar{\beta}_2} & K(\mathbb{Z}_2, 5) & & \\
 & & \downarrow & & \downarrow{\scriptstyle i_2} & & \\
F_2 & \longrightarrow & BSO(5) & \xrightarrow{\beta_2} & E_2 & & \\
 & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \pi_2} & & \\
 & & E_1 & = & E_1 & \xrightarrow{\pi_1^*(\iota_2 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2\iota_4 \otimes 1)} & K(\mathbb{Z}_2, 6)
\end{array}$$

Let the denotation of new mappings accords with the diagram. We can consider β_2 to be a fibration with a fibre F_2. Similarly as for the first stage, we can compute the homotopy groups of F_2. So we get that F_2 is 5-connected and β_2 is a 6-equivalence.

Let $C = K(\mathbb{Z}_4, 4) \times K(\mathbb{Z}_2, 6)$. Up to homotopy there is just one mapping $k = (k_1, k_2) : K \to C$ given by

$$k_1^*(\iota_4) = 1 \otimes 1 \otimes \rho_4\iota_4 - \mathfrak{P}\iota_2 \otimes 1 \otimes 1 + 1 \otimes i_*\iota_4 \otimes 1$$

$$k_2^*(\iota_6) = \iota_2 \otimes \iota_4 \otimes 1 + 1 \otimes Sq^2\iota_4 \otimes 1.$$

Due to Lemma 8.1 in [T4], there is a homeomorphism $g : E_2 \to E$ where $\pi : E \to K$ is a principal fibration with the classifying map $k : K \to C$. Moreover, $\pi_1 \circ \pi_2 = \pi \circ g$ and the fibration $\beta = g \circ \beta_2 : BSO(5) \to E$ is a 6-equivalence. Hence, we can consider the following situation

$$\begin{array}{ccccc}
BSO(5) & \xrightarrow{\beta = \text{6-equiv}} & E & & \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \pi} & & \\
K & = & K & \xrightarrow{k = (k_1, k_2)} & C
\end{array}$$

which allows us to prove our main result.

Theorem 1. *Let X be a connected CW-complex of dimension ≤ 5 and suppose*

$$\gamma : [X, BSO(5)] \to H^2(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z}_2) \oplus H^4(X, \mathbb{Z})$$

is defined by $\gamma(\xi) = (w_2(\xi), w_4(\xi), p_1(\xi))$. Then

(i) *im $\gamma = \{(a, b, c) \mid \rho_4 c = \mathfrak{P} a + i_* b\}$*
(ii) *γ is injective if and only if Conditions (A) and (B) are satisfied.*

Proof. (i) follows immediately from the fact that a mapping $f : X \to K$ can be lifted in the fibration α into $BSO(5)$ if and only if $f^*(1 \otimes 1 \otimes \rho_4 \iota_4 - \mathfrak{P}\iota_2 \otimes 1 \otimes 1 - 1 \otimes i_* \iota_4 \otimes 1) = 0$. (See similar proofs in [C-V].)

(ii) Since the space E is a homotopy fibre of the mapping $k : K \to C$, the Puppe sequence

$$\Omega K \xrightarrow{\Omega k} \Omega C \xrightarrow{q} E \xrightarrow{\pi} K \xrightarrow{k} C$$

yields the exact sequence

$$\to [X, \Omega K] \xrightarrow{(\Omega k)_*} [X, \Omega C] \xrightarrow{q_*} [X, E] \xrightarrow{\pi_*} [X, K] \xrightarrow{k_*} [X, C].$$

Moreover, β being a 6-equivalence, $\beta_* : [X, BSO(5)] \to [X, E]$ is a bijection for every CW-complex of dimension ≤ 5. The following statements are equivalent:

(1) $\gamma = \alpha_* = \pi_* \circ \beta_* : [X, BSO(5)] \to [X, K]$ is injective.
(2) $\pi_* : [X, E] \to [X, K]$ is injective.
(3) $q_* = 0$
(4) $(\Omega k)_* : [X, \Omega K] \to [X, \Omega C]$ is surjective.

Hence we need to compute $(\Omega k_1)^* : H^3(K(\mathbb{Z}_4, 3), \mathbb{Z}_4) \to H^3(\Omega K, \mathbb{Z}_4)$ and $(\Omega k_2)^* : H^5(K(\mathbb{Z}_2, 5), \mathbb{Z}_2) \to H^5(\Omega K, \mathbb{Z}_2)$.

First, let us consider k_1.

$$\begin{array}{ccccc}
\Omega K & = & K(\mathbb{Z}_2,1) \times K(\mathbb{Z}_2,3) \times K(\mathbb{Z},3) & \xrightarrow{\Omega k_1} & K(\mathbb{Z}_4,3) \\
\downarrow & & \downarrow & & \downarrow \\
PK & = & PK(\mathbb{Z}_2,2) \times PK(\mathbb{Z}_2,4) \times PK(\mathbb{Z},4) & \xrightarrow{Pk_1} & PK(\mathbb{Z}_4,4) \\
\downarrow & & \downarrow & & \downarrow \\
K & = & K(\mathbb{Z}_2,2) \times K(\mathbb{Z}_2,4) \times K(\mathbb{Z},4) & \xrightarrow{k_1} & K(\mathbb{Z}_4,4)
\end{array}$$

Every element in $H^*(K, \mathbb{Z}_4)$ is suspensive. If we denote all suspensions by σ, we get

$$\begin{aligned}
(\Omega k_1)^*(\iota_3) &= (\Omega k_1)^*(\sigma \iota_4) = \sigma(k_1^* \iota_4) = \sigma(1 \otimes 1 \otimes \rho_4 \iota_4) - \sigma(\mathfrak{P}\iota_2 \otimes 1 \otimes 1) \\
&\quad - \sigma(1 \otimes i_* \iota_4 \otimes 1) = 1 \otimes 1 \otimes \sigma(\rho_4 \iota_4) - \sigma(\mathfrak{P}\iota_2) \otimes 1 \otimes 1 - 1 \otimes \sigma(i_* \iota_4) \otimes 1
\end{aligned}$$

the last equality being a consequence of the definition of suspension and coboundary operator. In the fibration $K(\mathbb{Z}, 3) \to PK(\mathbb{Z}, 4) \to K(\mathbb{Z}, 4)$ we get

$$\sigma(\rho_4 \iota_4) = \rho_4(\sigma \iota_4) = \rho_4 \iota_3.$$

In the fibration $K(\mathbb{Z}_2,3)\to PK(\mathbb{Z}_2,4)\to K(\mathbb{Z}_2,4)$ we have

$$\sigma(i_*\iota_4)=i_*(\sigma\iota_4)=i_*\iota_3$$

and finally, in the fibration $K(\mathbb{Z}_2,1)\to PK(\mathbb{Z}_2,2)\to K(\mathbb{Z}_2,2)$ we obtain

$$\sigma(\mathfrak{P}\iota_2)=i_*\iota_1^3. \tag{5}$$

Since this fact is not generally known, we will prove it at the end of this section. As a result of these computations we get

$$(\Omega k_1)_*:[X,\Omega K]\to[X,K(\mathbb{Z}_4,3)]:(a,b,c)\mapsto\rho_4c-i_*a^3-i_*b.$$

Hence $(\Omega k_1)_*$ is surjective if and only if

$$H^3(X,\mathbb{Z}_4)=\rho_4H^3(X,\mathbb{Z})+i_*H^3(X,\mathbb{Z}_2). \tag{6}$$

We show that (6) is equivalent to the condition (A).

(A) $\Rightarrow$ (6). Let $x\in H^3(X,\mathbb{Z}_4)$, then $4\beta_4x=0$. (A) implies that $2\beta_4x=0$. Consequently, there is a $y\in H^3(X,\mathbb{Z}_2)$ such that $\beta_4x=\beta_2y=\beta_4i_*y$. That is why $\beta_4(x-i_*y)=0$, which implies $x=i_*y+\rho_4z$ for some $z\in H^3(X,\mathbb{Z})$.

(6) $\Rightarrow$ (A). Let $v\in H^4(X,\mathbb{Z})$ satisfy $4v=0$. Then $v=\beta_4x$ where $x=\rho_4z+i_*y\in H^3(X,\mathbb{Z}_4)$ so that $v=\beta_4\rho_4z+\beta_4i_*y=\beta_4i_*y=\beta_2y$. Hence $2v=0$ and v is not an element of order 4.

Now consider the mapping k_2. The computation of $(\Omega k_2)^*:H^5(K(\mathbb{Z}_2,5),\mathbb{Z}_2)\to H^5(\Omega K,\mathbb{Z}_2)$ gives

$$\begin{aligned}(\Omega k_2)^*(\iota_5)&=(\Omega k_2)^*(\sigma\iota_6)=\sigma k_2^*(\iota_6)=1\otimes\sigma(Sq^2\iota_4)\otimes1+\sigma(\iota_2\otimes\iota_4)\otimes1\\&=1\otimes Sq^2\iota_3\otimes1+\sigma(\iota_2\otimes\iota_4)\otimes1\end{aligned}$$

We are going to prove that $\sigma(\iota_2\otimes\iota_4)=0$. Consider the fibration

$$\Omega B\to PB\xrightarrow{p}B$$

where $B=K(\mathbb{Z}_2,2)\times K(\mathbb{Z}_2,4)$. Let $\hat p^*:H^6(B,\mathbb{Z}_2)\to H^6(PB,\Omega B;\mathbb{Z}_2)$ be determined by the mapping p. It is sufficient to show $\hat p^*(\iota_2\otimes\iota_4)=0$. Using the Serre spectral sequence with coefficients $\mathbb{Z}_2$ for the above fibration, we have

$$\hat p^*:H^6(B,\mathbb{Z}_2)\cong E_2^{6,0}\to E_6^{6,0}\hookrightarrow H^6(PB,\Omega B;\mathbb{Z}_2).$$

We compute $d_2:E_2^{4,1}\to E_2^{6,0}$. Since $E_2^{4,1}\cong E_2^{4,0}\otimes E_2^{0,1}$, for the generators of $E_2^{4,1}$ we obtain

$$\begin{aligned}d_2(\iota_2^2\otimes\iota_1)&=d_2(\iota_2^2)\cdot\iota_1+\iota_2^2\cdot d_2(\iota_1)=\iota_2^2\cdot\iota_2=\iota_2^3\\d_2(\iota_4\otimes\iota_1)&=d_2(\iota_4)\cdot\iota_1+\iota_4\cdot d_2\iota_1=\iota_4\cdot\iota_2\,.\end{aligned}$$

Hence $\iota_4\cdot\iota_2$ vanishes in $E_3^{6,0}$ and $\hat p^*(\iota_2\otimes\iota_4)=0$.

So we conclude that

$$(\Omega k_2)_*[X,\Omega K] \to [X, K(\mathbb{Z}_2,5)] : (a,b,c) \mapsto Sq^2 b$$

and its surjectivity is given directly by Condition (B).

It remains to prove the relation (5). Consider the Serre spectral sequence for the fibration $K(\mathbb{Z}_2,1) \to PK(\mathbb{Z}_2,2) \to K(\mathbb{Z}_2,2)$ with coefficients $\mathbb{Z}_4$. For shortening we will again denote this fibration $\Omega B \longrightarrow PB \xrightarrow{p} B$. It is not difficult to show that $H^4(B,\mathbb{Z}_4) \cong \mathbb{Z}_4$ with the generator $\mathfrak{P}\iota_2$ and $H^3(\Omega B,\mathbb{Z}_4) \cong \mathbb{Z}_2$ with the generator $i_*\iota_1^3$. The coboundary operator in the long exact sequence for the couple $(PB,\Omega B)$ is an isomorphism, hence it is sufficient to prove that $\hat{p}^*(\mathfrak{P}\iota_2) \neq 0$, $\hat{p}^* : H^4(B,\mathbb{Z}_4) \to H^4(PB,\Omega B;\mathbb{Z}_4)$ being induced by p. Since

$$E_4^{4,0} \cong H^4(B,\mathbb{Z}_4)/\ker \hat{p}^*\,,$$

it is sufficient to show that $E_4^{4,0} \neq 0$. We have

$$\begin{aligned} E_2^{2,1} &\cong H^2(B,H^1(\Omega B,\mathbb{Z}_4)) \cong \mathbb{Z}_2 \cong E_2^{2,0} \otimes E_2^{0,1} \\ E_2^{4,0} &\cong H^4(B,H^0(\Omega B,\mathbb{Z}_4)) \cong \mathbb{Z}_4\,. \end{aligned}$$

Moreover, $d_2 : E_2^{2,1} \to E_2^{4,0}$ is injective because

$$\begin{aligned} d_2(i_*\iota_2 \otimes i_*\iota_1) &= d_2(i_*\iota_2)\cdot i_*\iota_1 + i_*\iota_2 \cdot d_2(i_*\iota_1) = \\ &= i_*\iota_2 \cdot \tau(i_*\iota_1) = i_*\iota_2^2 \end{aligned}$$

where τ is a transgression. Hence $E_3^{4,0} \cong \mathbb{Z}_2$. Further, $E_3^{1,2} \cong 0$, $E_3^{7,-2} \cong 0$ and consequently, $E_4^{4,0} \cong \mathbb{Z}_2$.

4. Span and the existence of distributions. In this section we compute the span of oriented 5-dimensional vector bundles over a 5-dimensional CW-complex satisfying Conditions (A) and (B) of Theorem 1. Under the same conditions we find all oriented 5-dimensional vector bundles which admit a 2-distribution, i. e. an oriented 2-dimensional subbundle, and all oriented 5-dimensional vector bundles which admit a 4-distribution endowed with a complex structure, i. e. a complex 2-dimensional subbundle. For these purposes we need

Theorem 2. *Let X be a connected CW-complex of dimension ≤ 5 and let $W \in H^2(X,\mathbb{Z}_2)$, $P \in H^4(X,\mathbb{Z})$. Then there exists an oriented 3-dimensional vector bundle ξ over X with*

$$w_2(\xi) = W \qquad , \qquad p_1(\xi) = P$$

if and only if

$$\rho_4 P = \mathfrak{P} W\,.$$

Proof is very similar to the proof of the first part of Theorem 1. See also [W].

Corollary 1. *Let X be a connected CW-complex of dimension ≤ 5 satisfying Conditions (A) and (B). Then an oriented 5-dimensional vector bundle ξ has a 2-distribution with Euler class U if and only if*

$$\rho_2 U^2 + w_2(\xi)\rho_2 U + w_4(\xi) = 0\,. \tag{7}$$

Proof. ($\Rightarrow$) Let $\xi = \zeta \oplus \tau$ where τ is an oriented 2-dimensional vector bundle over X with the Euler class U and ζ is an oriented 3-dimensional vector bundle over X. Then

$$\begin{aligned} w_2(\xi) &= w_2(\zeta) + w_2(\tau) = w_2(\zeta) + \rho_2 U \\ w_4(\xi) &= w_2(\zeta)\cdot w_2(\tau) = w_2(\zeta)\cdot \rho_2 U \end{aligned}$$

Substituting from here into the expression $\rho_2 U^2 + w_2(\xi) + w_4(\xi)$, we get (7).

($\Leftarrow$) Let $U \in H^2(X,\mathbb{Z})$ satisfy (7). There is an oriented 2-dimensional vector bundle τ over X with the Euler class U. Put

$$W = w_2(\xi) + \rho_2 U \qquad , \qquad P = p_1(\xi) - U^2\,.$$

Then

$$\begin{aligned} \rho_4 P - \mathfrak{P} W &= \rho_4 p_1(\xi) - \rho_4 U^2 - \mathfrak{P}(w_2(\xi) + \rho_2 U) = \\ &= \rho_4 p_1(\xi) - \rho_4 U^2 - \mathfrak{P} w_2(\xi) - \mathfrak{P}\rho_2 U - i_*(w_2(\xi)\rho_2 U) = \\ &= i_*(\rho_2 U^2 + w_2(\xi)\rho_2 U + w_4(\xi)) = 0\,. \end{aligned}$$

According to Theorem 2, there is an oriented 3-dimensional vector bundle ζ over X with $w_2(\zeta) = W$ and $p_1(\zeta) = P$. We compute the characteristic classes of the vector bundle $\zeta \oplus \tau$.

$$\begin{aligned} w_2(\zeta\oplus\tau) &= w_2(\zeta) + w_2(\tau) = W + \rho_2 U = w_2(\xi) \\ w_4(\zeta\oplus\tau) &= w_2(\zeta)\cdot w_2(\tau) = W\cdot \rho_2 U = w_2(\xi)\rho_2 U + \rho_2 U^2 = \\ &= w_4(\xi) \\ p_1(\zeta\oplus\tau) &= p_1(\zeta) + p_1(\tau) = P + U^2 = p_1(\xi)\,. \end{aligned}$$

(See [W] for the aditivity of p_1 in this case.) Theorem 1 now implies that $\xi = \zeta\oplus\tau$, which completes the proof.

Remark. As far as it is known to the authors there are only two general results concerning 2-distributions in 5 or $4k+1$-dimensional vector bundles. See [T5], Theorems 1.3 and 4.1. The former deals with spin manifolds (i.e. $w_1(X) = w_2(X) = 0$) and tangent bundles and the latter requires span ≥ 2. Both examine the existence of 2-distributions with the Euler class $2U \in H^2(X,\mathbb{Z})$.

Corollary 2. *Let X be a connected CW-complex of dimension ≤ 5 and let ξ be an oriented 5-dimensional vector bundle over X.*

(1) *span $\xi \geq 1$ if and only if $e(\xi) = 0$.*

If Conditions (A) and (B) are satisfied then

(2) *span $\xi \geq 2$ if and only if $w_4(\xi) = 0$.*
(3) *span $\xi \geq 3$ if and only if $w_4(\xi) = 0$ and there is a $U \in H^2(X, \mathbb{Z})$ such that $w_2(\xi) = \rho_2 U$, $p_1(\xi) = U^2$.*
(4) *span $\xi = 5$ if and only if $w_2(\xi) = 0$, $w_4(\xi) = 0$, $p_1(\xi) = 0$.*

Proof. (1) is well known and is included only for comleteness.

(2) is the immediate consequence of Corollary 1 for $U = 0$.

(3)($\Rightarrow$) Let $\xi = \zeta \oplus 3\varepsilon$ where ζ is an oriented 2-dimensional vector bundle over X. Then $w_4(\xi) = w_4(\zeta) = 0$ and for $U = e(\zeta)$ we get $w_2(\xi) = w_2(\zeta) = \rho_2 U$, $p_1(\xi) = p_1(\zeta) = U^2$.

($\Leftarrow$) For $U \in H^2(X, \mathbb{Z})$ there is an oriented 2-dimensional vector bundle ζ over X such that $e(\zeta) = U$. Then $w_2(\zeta \oplus 3\varepsilon) = w_2(\zeta) = \rho_2 U = w_2(\xi)$, $w_4(\zeta \oplus 3\varepsilon) = w_4(\zeta) = 0 = w_4(\xi)$ and $p_1(\zeta \oplus 3\varepsilon) = p_1(\zeta) = U^2 = p_1(\xi)$. Theorem 1 implies that $\zeta \oplus 3\varepsilon = \xi$ since the characteristic classes of both vector bundles are the same.

(4) follows immediately from Theorem 1.

Remark. Statements (3) and (4) of Corollary 2 under a little bit different conditions were already known to E. Thomas [T6]. Statement (2) under Conditions (A) and (B) is new. It deals with the cases which are not covered in [T6]. The condition $w_4(\xi) = 0$ coincides with the condition for the stable span of $4k+1$-dimensional vector bundles over a CW-comlex of the same dimension to be ≥ 2. See [Ng], Theorem 2.1.1.

Now we will investigate the existence of distributions with complex structure. The case of 2-distributions is treated in Corollary 1. Here we will deal with 4-distributions. For this purpose we need the following

Theorem 3. *Let X be a connected CW-complex of dimension ≤ 5 and let $C_1 \in H^2(X, \mathbb{Z})$, $C_2 \in H^4(X, \mathbb{Z})$. Then there exists a 2-dimensional complex vector bundle ζ over X with the Chern classes*

$$c_1(\zeta) = C_1 \quad , \quad c_2(\zeta) = C_2 \, .$$

Proof of this theorem follows the same lines as in [W].

Corollary 3. *Let X be a connected CW-complex of dimension ≤ 5 satisfying the conditions (A) and (B). Then an oriented 5-dimensional vector bundle ξ over X has a 4-distribution with a complex structure if and only if*

(i) *$e(\xi) = 0$*
(ii) *$\beta_2 w_2(\xi) = 0$*

Proof. ($\Rightarrow$) Let η be a 4-distribution in ξ with complex structure. Then obviously $e(\xi) = 0$ and $\beta_2 w_2(\xi) = \beta_2 w_2(\eta \oplus \varepsilon) = \beta_2 w_2(\eta) = \beta_2 \rho_2 c_1(\eta) = 0$.

($\Leftarrow$) We have $\beta_2 w_2(\xi) = 0$ and $\beta_2 w_4(\xi) = e(\xi) = 0$. Consequently, we can find $a_1 \in H^2(X, \mathbb{Z})$ and $a_2 \in H^4(X, \mathbb{Z})$ such that $\rho_2 a_1 = w_2(\xi)$ and $\rho_2 a_2 = w_4(\xi)$. Then

$$\rho_4(a_1^2 - 2a_2) = \mathfrak{P}\rho_2 a_1 + i_* \rho_2 a_2 = \mathfrak{P} w_2(\xi) + i_* w_4(\xi) = \rho_4 p_1(\xi)\,.$$

Hence there is a $b \in H^4(X, \mathbb{Z})$ such that $a_1^2 - 2a_2 - 4b = p_1(\xi)$. Put $C_1 = a_1$ and $C_2 = a_2 + 2b$. According to Theorem 3 there exists a complex vector bundle η over X of complex dimension 2 with

$$c_1(\eta) = C_1 \quad \text{and} \quad c_2(\eta) = C_2.$$

Let us consider now the 5-dimensional real vector bundle $\eta \oplus \varepsilon$. We get

$$w_2(\eta \oplus \varepsilon) = w_2(\eta) = \rho_2 c_1(\eta) = \rho_2 C_1 = w_2(\xi),$$
$$w_4(\eta \oplus \varepsilon) = w_4(\eta) = \rho_2 c_2(\eta) = \rho_2 C_2 = w_4(\xi),$$
$$p_1(\eta \oplus \varepsilon) = p_1(\eta) = c_1(\eta)^2 - 2c_2(\eta) = C_1^2 - 2C_2 = p_1(\xi).$$

Theorem 1 implies that $\xi = \eta \oplus \varepsilon$. This finishes the proof.

Remark. Let us recall that an f-structure on a vector bundle ξ is an endomorphism $f : \xi \to \xi$ satisfying the polynomial equation $f^3 + f = 0$ with dim ker f constant. It can be easily seen that if f is an f-structure then $\xi = \zeta \oplus \eta$ where $\zeta = \ker f$ and $\eta = \ker\,(f^2 + \mathrm{id})$. This means that on a vector bundle ξ there exists an f-structure if and only if there exists a distribution $\eta \subset \xi$ endowed with a complex structure. If ξ is an oriented 5-dimensional vector bundle over a connected CW-complex X of dimension 5, we can distinquish two cases. In the first case dim $\eta = 2$ the existence problem for an f-structure is covered by Corollary 1. The second case dim $\eta = 4$ is treated in Corollary 3.

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