G-STRUCTURES ON SPHERES

MARTIN ČADEK AND MICHAEL CRABB

Abstract. A generalization of classical theorems on the existence of sections of real, complex and quaternionic Stiefel manifolds over spheres is proved. We obtain a complete list of Lie group homomorphisms $\rho : G \to G_n$, where G_n is one of the groups $SO(n)$, $SU(n)$, $Sp(n)$ and G is one of the groups $SO(k)$, $SU(k)$, $Sp(k)$, which reduce the structure group G_n in the fibre bundle $G_n \to G_{n+1} \to G_{n+1}/G_n$.

1. Introduction

Consider the fibrations

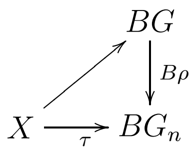
$$SO(n) \to SO(n+1) \to SO(n+1)/SO(n) \;=\; S^n\,, \tag{1.1}$$

$$SU(n) \to SU(n+1) \to SU(n+1)/SU(n) \;=\; S^{2n+1}\,, \tag{1.2}$$

$$Sp(n) \to Sp(n+1) \to Sp(n+1)/Sp(n) \;=\; S^{4n+3}\,. \tag{1.3}$$

To deal with the three cases we shall write $G_n = SO(n)$, $SU(n)$ or $Sp(n)$ and $d = 1$, 2 or 4 as appropriate. These principal bundles reduce the structure group $O(d(n+1)-1)$ of the tangent bundle of the sphere $G_{n+1}/G_n = S^{d(n+1)-1}$ to the subgroup G_n.

Let G be a Lie group and $\rho : G \to G_n$ a homomorphism. We say that the structure group G_n of a principal fibre bundle τ over a CW-complex X can be reduced to (G, ρ) if the classifying map of this fibration $\tau : X \to BG_n$ can be factored (up to homotopy) through $B\rho : BG \to BG_n$.



This paper deals with the problem of determining those groups G and homomorphisms ρ to which the structure group G_n in the fibrations above can be reduced. The problem has been solved in many interesting special cases. Considering standard inclusions $\rho : G = G_k \hookrightarrow G_n$ we get the famous problem on sections of Stiefel manifolds over spheres resolved in [1], [3], [5] and [23]. The other standard inclusions $SU(k) \hookrightarrow SO(n)$, $Sp(k) \hookrightarrow SO(n)$ and $Sp(k) \hookrightarrow SU(n)$ are dealt with in [12], [18] and [19], respectively. In these cases the question was to find a minimal standard subgroup to which G_n can be reduced.

In [15] Leonard asked an opposite question: find all maximal proper subgroups to which G_n can be reduced. He solved it in the cases when G is a reducible maximal

Date: October 4th, 2005.
Research of the first author supported by the grant MSM 0021622409 of the Czech Ministry of Education.

subgroup of G_n. Moreover, he proved that G_n cannot be reduced to any proper subgroup (G, ρ) if

(1) n is even and $G_n = SO(n)$ or $SU(n)$, unless $G_n = SO(6)$ and $G = SU(3)$;
(2) $n \not\equiv 11 \mod 12$ and $G_n = Sp(n)$;
(3) G is a nonsimple irreducible maximal proper subgroup of G_n.

Using similar methods these results were improved in [20]. Nevertheless, the cases when G is a simple Lie group and $\rho : G \to G_n$ is an irreducible representation have remained unanswered. As a consequence of our main result we will show that if G is one of the classical Lie groups $SO(k)$, $SU(k)$, $Sp(k)$ such a reduction is impossible except for the case $SU(3) \hookrightarrow SO(6)$ (and the obvious case $G_n = G$).

The paper is organized as follows. The main results are described in the next section. Homotopy theoretical results needed in their proofs are contained in Section 3. The proofs themselves appear in Section 4, based on statements on dimensions of real representations of classical Lie groups. The computations proving these statements are carried out in the following section. In an appendix we give precise self-contained proofs of several more or less known results which we need and which may be of independent interest; we also discuss possible generalizations of our main results.

2. Main results

To state our main result we recall several definitions and theorems. For any prime p let ν_p stand for the p-adic valuation.

The Hurwitz-Radon number $a(r)$ is the power of 2 given by

$$\nu_2(a(r)) = \#\{i \mid 1 \le i \le r-1 \quad \text{and } i \equiv 0, 1, 2, 4 \text{ mod } 8\}.$$

For $n+1 = (2l+1)2^{\beta+4\gamma}$ with $\beta \in \{0, 1, 2, 3\}$ put $j(n) = 2^\beta + 8\gamma$. A well known result of Adams ([1]) says that the structure group $SO(n)$ in the fibration (1.1) can be reduced to the standard subgroup $SO(k)$ if and only if $k \ge n - j(n) + 1$. This condition can be expressed in terms of the Hurwitz-Radon numbers as

$$n + 1 \equiv 0 \mod a(n+1-k).$$

The complex James number $b(r)$ is the positive integer with

$$\nu_p(b(r)) = \begin{cases} \max\{i + \nu_p(i) \mid 1 \le i \le (r-1)/(p-1)\} & \text{if } r \ge p, \\ 0 & \text{if } r < p, \end{cases}$$

for all primes p.

Similarly, the quaternionic James number $c(r)$ is the positive integer determined by

$$\nu_2(c(r)) = \max\{2r-1,\, 2i + \nu_2(i) \mid 1 \le i \le r-1\},$$
$$\nu_p(c(r)) = \nu_p(b(2r)) \quad \text{for all odd primes } p.$$

In the papers [3], [5], [12], [18], [19] and [23] the problem of when the structure group G_n in one of the fibrations (1.1) – (1.3) can be reduced to a group $G = SU(k)$ or $Sp(k)$ via a standard inclusion $G \hookrightarrow G_n$ was solved and the results were expressed in terms of complex and quaternionic James numbers in a way similar to the result quoted above. The following theorem can be regarded as a generalization of these results.

Theorem 2.1. *Let G_n be one of the groups $SO(n)$, $SU(n)$ or $Sp(n)$ and let G be one of the groups $SO(k)$ with $k \geq 4$, $SU(k)$ with $k \geq 2$ or $Sp(k)$ with $k \geq 2$. Suppose that the dimension of the sphere G_{n+1}/G_n is at least 8 and that $\dim G < \dim G_n$. Then the structure group G_n of the principal fibre bundle*

$$G_n \to G_{n+1} \to G_{n+1}/G_n \tag{2.1}$$

can be reduced to G via a homomorphism $\rho : G \to G_n$ if and only if one of the following cases occurs.

- (A) *$G_n = SO(n)$, $G = SO(k)$, $n = m - 1$, $m \equiv 0 \mod a(m-k)$ and, up to conjugation, ρ is the standard inclusion $SO(k) \hookrightarrow SO(n)$.*
- (B) *$G_n = SO(n)$, $G = SU(k)$, $n = 2m-1$, $m \equiv 0 \mod 2^{\nu_2(b(m-k))}$ and, up to conjugation, ρ is the composition of the standard inclusions $SU(k) \hookrightarrow SO(2k) \hookrightarrow SO(n)$ or the composition $SU(4) \to SO(8) \times SO(6) \hookrightarrow SO(15)$ where the first homomorphism is given on the first factor by the standard inclusion and on the second factor by the double covering $SU(4) \cong Spin(6) \to SO(6)$.*
- (C) *$G_n = SO(n)$, $G = Sp(k)$, $n = 4m-1$, $m \equiv 0 \mod 2^{\nu_2(c(m-k))}$ and, up to conjugation, ρ is the composition of the standard inclusions $Sp(k) \hookrightarrow SO(4k) \hookrightarrow SO(n)$ or the exterior square $Sp(3) \to SO(15)$.*
- (D) *$G_n = SU(n)$, $G = SU(k)$, $n = m - 1$, $m \equiv 0 \mod b(m-k)$ and, up to conjugation, ρ is the standard inclusion $SU(k) \hookrightarrow SU(n)$.*
- (E) *$G_n = SU(n)$, $G = Sp(k)$, $n = 2m-1$, $m \equiv 0 \mod c(m-k)$ and, up to conjugation, ρ is the composition of the standard inclusions $Sp(k) \hookrightarrow SU(2k) \hookrightarrow SU(n)$.*
- (F) *$G_n = Sp(n)$, $G = Sp(k)$, $n = m - 1$, $m \equiv 0 \mod c(m-k)$ and, up to conjugation, ρ is the standard inclusion $Sp(k) \hookrightarrow Sp(n)$.*

As a consequence of Theorem 2.1 we get a partial answer to Leonard's question from [15].

Corollary 2.2. *Under the assumptions of Theorem* 2.1 *there is no irreducible representation $\rho : G \to G_n$ such that the structure group G_n of the principal fibre bundle* (2.1) *can be reduced to (G, ρ).*

Remark 2.3. As for $G = Spin(k)$ or an exceptional simple Lie group, using Proposition 3.1 and similar statements for $SU(n)$ and $Sp(n)$ one can easily prove that the structure group G_n of (2.1) cannot be reduced to G via any homomorphism ρ with the possible exception of finitely many cases in which a certain dimension condition (the same as or similar to that in Proposition 3.1) is not satisfied.

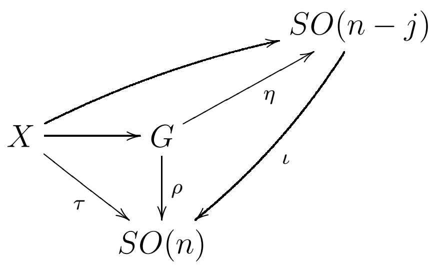
Remark 2.4. In [6] and [24] it was shown that every stably parallelizable manifold of dimension n either is parallelizable or has the same span as S^n. This suggests the possibility of extending Theorem 2.1 for $G_n = SO(n)$ from the case of the tangent bundle over S^n to the case of a stably trivial, but non-trivial, n-dimensional vector bundle over a stably parallelizable n-manifold. We say a little more about this question in the final section.

3. Auxiliary results

In this section we will summarize the results which will be needed for the proofs of Theorem 2.1 and Corollary 2.2 in the next section. In what follows we will not distinguish between maps and their homotopy classes.

Proposition 3.1. *Let τ be a principal $SO(n)$-bundle over the suspension ΣX of a pointed finite complex X. Suppose that the structure group $SO(n)$ of τ can be reduced to (G, ρ), where $\rho : G \to SO(n)$ is a homomorphism from a Lie group G of dimension less than $n-j$, $1 \le j < n$. Then the structure group of τ can be reduced to the standard subgroup $SO(n-j)$ of $SO(n)$.*

Proof. We denote the classifying map $X \to SO(n)$ of a principal fibre bundle τ by the same letter. Suppose that $SO(n)$ structure can be reduced to (G, ρ). Since the standard inclusion $\iota : SO(n-j) \to SO(n)$ is an $(n-j-1)$-equivalence and $\dim G \le n-j-1$, the map $\rho : G \to SO(n)$ can be factored as a composition of a map $\eta : G \to SO(n-j)$ and ι. Then τ can be factored through the standard inclusion ι as shown by the following diagram.



□

Lemma 3.2. *Let $1 \le k < n$ and let $n \ge 9$ be odd. Consider a homomorphism $\rho : G_k \to O(n)$. Suppose that there is a map $\rho'' : BG_\infty \to BO$ such that $\rho'' \circ \iota_k \simeq \iota_n \circ B\rho$. A choice of homotopy induces a diagram of fibrations:*

$$\begin{array}{ccc} G_\infty/G_k & \xrightarrow{\rho'} & O/O(n) \\ \downarrow & & \downarrow \\ BG_k & \xrightarrow{B\rho} & BO(n) \\ \downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \iota_n} \\ BG_\infty & \xrightarrow{\rho''} & BO \end{array}$$

Let $\xi \in \pi_n(BG_k)$ be a homotopy class such that $\tau = B\rho \circ \xi \in \pi_n(BO(n))$ classifies the fibration (1.1)*. Then there is an element $\xi' \in \pi_n(BG_k)$ such that $B\rho \circ \xi' = \tau$ and $\iota_k \circ \xi' = 0$.*

Proof. We deal separately with the three cases (a) $G_k = SO(k)$, (b) $G_k = SU(k)$, (c) $G_k = Sp(k)$.
Case (a). We consider the only non-trivial case: $n \equiv 1 \bmod 8$. Let α denote the generator of $\pi_n(BO) = \mathbb{Z}/2$. Then $\pi_n(BO(n)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is generated by τ and a

class β such that

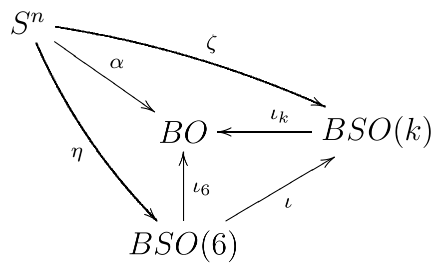
$$\iota_n \beta = \alpha, \quad \iota_n \tau = 0$$

in $\pi_n(BO)$.

We deal first with the case that $k \geq 6$. Suppose that $\xi \in \pi_n(BSO(k))$ satisfies

$$B\rho \circ \xi = \tau \quad \text{and} \quad \iota_k \xi = \alpha.$$

According to [11] (see also the Appendix, Proposition 6.2) $\alpha \in \pi_n(BO)$ can be factored as a composition $\iota_6 \eta$ where $\eta \in \pi_n(BSO(6))$ and $\iota_6 : BSO(6) \to BO$ is the standard inclusion. This gives the diagram



in which $\iota : BSO(6) \to BSO(k)$ is the standard inclusion and $\zeta = \iota\eta$.

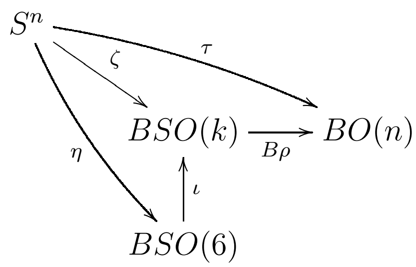
If we show that $B\rho \circ \zeta = 0 \in \pi_n(BO(n))$, then $\xi' = \xi + \zeta$ will satisfy the required conditions, since

$$\begin{aligned} B\rho \circ \xi' &= B\rho \circ (\xi + \zeta) = B\rho \circ \xi + B\rho \circ \zeta = \tau + 0 = \tau, \\ \iota_k \xi' &= \iota_k(\xi + \zeta) = \alpha + \alpha = 0. \end{aligned}$$

Write $B\rho \circ \zeta = b\beta + c\tau$ with $b, c \in \mathbb{Z}/2$. Then

$$b\alpha = \iota_n(b\beta + c\tau + \tau) = \iota_n B\rho(\zeta + \xi) = \rho'' \iota_k(\zeta + \xi) = \rho''(\alpha + \alpha) = 0,$$

which implies that $b = 0$. Suppose that $B\rho \circ \zeta = \tau$. Then $\tau = (B\rho) \circ \iota \circ \eta$ in the diagram:



Consequently, the structure group $O(n)$ of τ can be reduced to $SO(6)$. Since

$$\dim SO(6) = 15 < n - j(n), \quad \text{for } n \equiv 1 \mod 8, \ n \geq 17,$$

it follows from Proposition 3.1 that $n = 9$. However, the only possible homomorphism $SO(6) \to O(9)$ is (up to conjugation) the standard inclusion. In this case $\rho'' = \mathrm{id}$, which leads to the contradiction:

$$\alpha = \rho'' \iota_6 \eta = \rho'' \iota_k \iota \eta = \iota_n (B\rho) \iota \eta = \iota_n \tau = 0.$$

Finally, if $k < 6$, then according to [11], $(\iota_k)_* : \pi_n(BO(k)) \to \pi_n(BO)$ always vanishes and we may take $\xi' = \xi$.

Case (b). In the complex case the assertion is trivial, because $\pi_n(BSU) = 0$.

Case (c). The only non-trivial case occurs for $n \equiv 5 \bmod 8$. In the Appendix, Proposition 6.4 we show that the generator of $\pi_{8k+4}(Sp) = \mathbb{Z}/2$ lifts to an element in $\pi_{8k+4}(Sp(1))$. Using this fact the proof proceeds as in (a). $\square$

In the proof of Theorem 2.1 we will need some properties of stunted projective and quasiprojective spaces. We will describe them for the real, complex and quaternionic cases together. Denote by $\mathbb{F}$ one of the fields $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and put $d = 1$, 2 or 4, respectively, for its real dimension. $G(\mathbb{F}^n)$ will stand for the group $O(n)$, $U(n)$ or $Sp(n)$ according to the chosen field $\mathbb{F}$. Let us recall that G_n denotes one of the groups $SO(n)$, $SU(n)$ or $Sp(n)$ and that

$$G_n/G_k = G(\mathbb{F}^n)/G(\mathbb{F}^k) \quad \text{for } 1 \le k \le n.$$

P_n will stand for the projective space $P(\mathbb{R}^n)$, $P(\mathbb{C}^n)$ or $P(\mathbb{H}^n)$ and Q_n for the corresponding quasiprojective space $Q(\mathbb{R}^n)$, $Q(\mathbb{C}^n)$ or $Q(\mathbb{H}^n)$. This space is a Thom space P_n^ζ where ζ is a certain real vector bundle over P_n of dimension $d-1$. Denote the Hopf bundle over P_r by H and write $\mathbb{F}$ for trivial real vector bundle with fibre $\mathbb{F}$. The real tangent bundle to P_r has the property

$$\tau(P_r) \oplus \zeta \oplus \mathbb{R} = rH^*$$

where $H^* = \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F})$ is the dual vector bundle to H. Let $t(r)$ denote the order of the bundle $H - \mathbb{F}$ in $\widetilde{J}(P_r)$, that is, $t(r)$ is the least integer ≥ 1 such that the sphere bundle of $t(r)H$ is stably fibre homotopy trivial. Classical computations in [1], [3], [5] and [23] showed that $t(r) = a(r)$, $b(r)$ and $c(r)$ in the real, complex and quaternionic cases, respectively.

Considering the reflection maps

$$\phi : Q_\infty/Q_k \to G(\mathbb{F}^\infty)/G(\mathbb{F}^k)$$

and writing a_i, b_i, c_i for generators of $\widetilde{H}_i(Q(\mathbb{R}^\infty); \mathbb{Z}/2) = (\mathbb{Z}/2)a_i$, $\widetilde{H}_{2i+1}(Q(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}b_i$, $\widetilde{H}_{4i+3}(Q(\mathbb{H}^\infty); \mathbb{Z}) = \mathbb{Z}c_i$, respectively, we have inclusions (for unreduced homology)

$$\phi_* : H_*(Q(\mathbb{R}^\infty)/Q(\mathbb{R}^k); \mathbb{Z}/2) = \mathbb{Z}/2 \oplus \bigoplus_{i=k}^{\infty} \mathbb{Z}/2\ a_i \longrightarrow H_*(O/O(k); \mathbb{Z}/2) =$$
$$\mathbb{Z}/2[a_0, a_1, a_2, \dots]/(a_0 = 1, a_1 = 0, \cdots, a_{k-1} = 0, a_k^2 = 0, \cdots, a_i^2 = 0, \cdots)$$

where $H_*(O/O(k); \mathbb{Z}/2)$ is a module over

$$H_*(O; \mathbb{Z}/2) = \mathbb{Z}/2[a_0, a_1, a_2, \dots]/(a_0^2 = 1, a_1^2 = 0, \cdots),$$

and for $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$

$$\phi_* : H_*(Q(\mathbb{F}^\infty)/Q(\mathbb{F}^k); \mathbb{Z}) = \mathbb{Z} \oplus \bigoplus_{i=k}^{\infty} \mathbb{Z}\ e_i \longrightarrow H_*(G(\mathbb{F}^\infty)/G(\mathbb{F}^k); \mathbb{Z}) =$$
$$\mathbb{Z}[e_0, e_1, e_2, \dots]/(e_0 = 0, \cdots, e_{k-1} = 0, e_k^2 = 0, \cdots, e_i^2 = 0, \cdots)$$

where e_i stands for b_i or c_i, respectively, and $H_*(G(\mathbb{F}^\infty)/G(\mathbb{F}^k); \mathbb{Z})$ is a module over

$$H_*(G(\mathbb{F}^\infty); \mathbb{Z}) = \mathbb{Z}[e_0, e_1, \dots]/(e_0^2 = 0, \cdots).$$

Since $SO = O/O(1)$ and $SU = U/U(1)$, the formulas above describe also the homology of SO and SU.

Proposition 3.3. *Let n be odd and $k \geq 1$. Consider a map $f : G_\infty/G_k \to O/O(n)$ which fits into a commutative diagram*

$$\begin{array}{ccc} G_\infty & \xrightarrow{\tilde f} & O \\ \downarrow & & \downarrow \\ G_\infty/G_k & \xrightarrow{f} & O/O(n) \end{array}$$

in which $\tilde f$ is an H-map. Suppose that

$$f_* : \pi_n(G_\infty/G_k) \to \pi_n(O/O(n)) = \mathbb{Z}/2 \tag{3.1}$$

is onto. Then $n+1$ is divisible by d, say $n = dm - 1$, and $\nu_2(m) \geq \nu_2(t(m-k))$.

The same is true if we replace G_∞ and G_k by $G(\mathbb{F}^\infty)$ and $G(\mathbb{F}^k)$.

Proof. The homomorphism

$$f_* : H_n(G_\infty/G_k; \mathbb{Z}/2) \to H_n(O/O(n); \mathbb{Z}/2) = (\mathbb{Z}/2)a_n$$

maps decomposable elements to 0, since f_* lifts to a ring homomorphism. So it follows at once that $n = dm-1$ and that there is an element $x \in \pi_n(G_\infty/G_k)$ whose Hurewicz image is equal to a_{m-1}, b_{m-1} or c_{m-1} modulo 2 and products. Hence the projection

$$G_\infty/G_k \to G_\infty/G_{m-1}$$

maps x to a generator of the group $\pi_n(G_\infty/G_{m-1})_{(2)}$, which is $\mathbb{Z}/2$ in the orthogonal case, $\mathbb{Z}_{(2)}$ in the unitary and symplectic cases. (The lower index (2) means localization at the prime 2.)

Now recall that there are stable maps

$$\theta : G_\infty/G_k \to Q_\infty/Q_k$$

splitting the reflection maps $\phi : Q_\infty/Q_k \to G_\infty/G_k$. (See, for example, [7] for the construction of θ and [14] for a description of ϕ.) These maps are compatible with the projections $G_\infty/G_k \to G_\infty/G_l$ and $Q_\infty/Q_k \to Q_\infty/Q_l$ for $k \leq l$.

We shall use the letter ω for stable homotopy: the symbols $\widetilde{\omega}_i$ and $\widetilde{\omega}^i$ will stand for reduced stable homotopy and cohomotopy groups, respectively.

Thus $\theta(x)$ gives a class in $\widetilde{\omega}_n(Q_\infty/Q_k)$ that maps to an odd multiple of the generator of $\widetilde{\omega}_n(Q_\infty/Q_{m-1}) = \mathbb{Z}/2$, $\mathbb{Z}$ or $\mathbb{Z}$, in the three cases.

The remainder of the proof is an essentially classical computation using the stable Adams operation ψ^3 in 2-local real K-theory, KO. One may either proceed directly (as we shall show below for the orthogonal case) or dualize as follows.

By connectivity, the map $\widetilde{\omega}_n(Q_m/Q_k) \to \widetilde{\omega}_n(Q_\infty/Q_k)$ is surjective. So there is an element $y \in \widetilde{\omega}_n(Q_m/Q_k)_{(2)}$ that maps to a generator of $\widetilde{\omega}_n(Q_m/Q_{m-1})_{(2)} = \mathbb{Z}_{(2)}$.

Now Q_m/Q_k is the Thom space of the bundle $kH^* \oplus \zeta$ over $P_{m-k} = P(\mathbb{F}^{m-k})$. Its stable dual is, according to [4], the Thom space of the virtual bundle $\mathbb{R} - mH^*$ over the same space P_{m-k}. It follows that the restriction map

$$\widetilde{\omega}^0(P_{m-k}^{m(\mathbb{F}-H^*)})_{(2)} \to \widetilde{\omega}^0(P_{m-(m-1)}^{m(\mathbb{F}-H^*)})_{(2)} = \widetilde{\omega}^0(S^0)_{(2)} = \mathbb{Z}_{(2)}$$

is surjective. The Hurewicz image of y in K-theory gives under duality a class in

$$\widetilde{KO}^0(P_{m-k}^{m(\mathbb{F}-H^*)})_{(2)}$$

that is fixed by ψ^3 and restricts to a generator of $\widetilde{KO}^0(S^0)_{(2)} = \mathbb{Z}_{(2)}$.

Alternatively, this shows that the vector bundle mH^* over P_{m-k} is stably fibre homotopy trivial at the prime 2 (see [4], the proof of Proposition 2.8 and [18], the proof of Theorem 2.2) and leads to the equivalent condition that $m(\mathbb{F}-H^*) \in KO^0(P_{m-k})_{(2)}$ lies in the image of $\psi^3 - 1$. (See, for example, [10], Theorem 5.1 modified to KO.)

The proof is completed by calculations of the Adams operation ψ^3. For the unitary and symplectic cases we refer to [3] and [23]. For the orthogonal case we outline a proof below, which is perhaps more direct than the classical calculation. □

We write kO for connective real K-theory.

Proposition 3.4. *Let n be odd and $1 \le k \le n$. Suppose that there is an element $y \in \widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^k))_{(2)}$ which is fixed by the Adams operation ψ^3 and maps to the generator of $\widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^n))_{(2)} = \mathbb{Z}/2$. Then*

$$n + 1 \equiv 0 \mod a(n+1-k)$$

or, equivalently, $k \ge n - j(n) + 1$.

Proof. We deal first with the case that $n+1$ is divisible by 8. Using connectivity and duality we may make the identifications

$$\begin{aligned}\widetilde{KO}^0(P(\mathbb{R}^{n-k+2})^{(n+1)\mathbb{R}-(n+2)H}) &= \widetilde{KO}_n(P(\mathbb{R}^{n+2})/P(\mathbb{R}^k)) \\ &= \widetilde{kO}_n(P(\mathbb{R}^{n+2})/P(\mathbb{R}^k)) = \widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^k)).\end{aligned}$$

The first group is isomorphic, by Bott periodicity, to $\widetilde{KO}^0(P(\mathbb{R}^{n-k+2})^{-H})$. Standard computations of the K-groups of real projective spaces give that $\widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^k))_{(2)}$ is cyclic of order $2a(n+1-k)$ and that ψ^3 acts as multiplication by $3^{(n+1)/2}$.

A generator is fixed by ψ^3 if and only if

$$3^{(n+1)/2} - 1 \equiv 0 \mod 2a(n+1-k).$$

Since $\nu_2(3^{(n+1)/2} - 1) = \nu_2(n+1) + 1$, the result follows in this case.

The case $n+1 \equiv 4 \mod 8$ is similar. We have $\widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^{n-4}))_{(2)} = \mathbb{Z}/16$ and ψ^3 acts as $3^{(n+1)/2}$. So $k > n-4$.

Finally, for $n+1 \equiv 2 \mod 4$, we have $\widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^{n-2}))_{(2)} = \mathbb{Z}/2$ and the projection map to $\widetilde{kO}_n(P(\mathbb{R}^\infty)/P(\mathbb{R}^n))_{(2)} = \mathbb{Z}/2$ is zero. So $k > n-2$.

(It is traditional to use mod 2 homology and Steenrod operations for the last two steps, but kO-theory provides a uniform proof.) □

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$. Suppose that the field $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$ is a vector space over $\mathbb{K}$. Put again $t(r) = b(r)$ for $\mathbb{F} = \mathbb{C}$ and $t(r) = c(r)$ if $\mathbb{F} = \mathbb{H}$. Let n be odd for $\mathbb{K} = \mathbb{C}$ and $n \equiv 3 \mod 4$ for $\mathbb{K} = \mathbb{H}$. The following statement may be established by the same method as Proposition 3.3.

Proposition 3.5. *Consider a map $f : G(\mathbb{F}^\infty)/G(\mathbb{F}^k) \to G(\mathbb{K}^\infty)/G(\mathbb{K}^n)$ which lifts to an H-map $\tilde{f} : G(\mathbb{F}^\infty) \to G(\mathbb{K}^\infty)$. If*

$$f_* : \pi_n(G(\mathbb{F}^\infty)/G(\mathbb{F}^k)) \to \pi_n(G(\mathbb{K}^\infty)/G(\mathbb{K}^n)) \cong \mathbb{Z},$$

is onto, then $n = m \dim_{\mathbb{K}} \mathbb{F} - 1$ and $m \equiv 0 \mod t(m-k)$.

Remark 3.6. If f in the statement of Proposition 3.3 is the standard inclusion then the condition on $n = dm-1$ is not only necessary but also sufficient for $f_* : \pi_n(G_\infty/G_k) \to \pi_n(O/O(n))$ to be onto. This can be shown by reversing the proof, since the conditions on n and k ensure that we are in stable range. The same applies to Proposition 3.5.

4. Proofs

Let λ^i stand for the representation given by the i-th exterior power and let $^-$ denote complex conjugation.

Proof of Theorem 2.1. According to [15] the structure group G_n in (2.1) cannot be reduced to any proper subgroup for even $n \geq 8$. For odd $n \geq 9$ we will examine different G_n and G separately.

A. Let $k \leq n$. Consider a homomorphism $\rho : SO(k) \to SO(n)$ which reduces the structure group $SO(n)$ of the fibre bundle (1.1). First, suppose that the class of the representation ρ in $RO(SO(k))$ is a polynomial in exterior powers.

Lemma 4.1. *Let $\rho : G = SO(k) \to O(n)$ be a homomorphism which extends to a homomorphism $O(k) \to O(n)$. Then there is a map $\rho'' : BSO \to BO$ such that $\rho'' \circ \iota_k \simeq \iota_n \circ B\rho$, where $\iota_k : BSO(k) \to BSO$ and $\iota_n : BO(n) \to BO$ are the standard inclusions.*

Proof. The real representation ring $RO(O(k))$ is generated by the exterior powers λ^i of the basic representation. Given ρ there is a polynomial p in exterior powers such that

$$p(\lambda^1, \lambda^2, \dots, \lambda^k)(\xi - k) = \rho(\xi) - n$$

for any vector bundle ξ of dimension k. This polynomial defines ρ''. To show the existence of such a polynomial it is enough to consider the special case $\rho = \lambda^i$, for which we may take p to be equal to $\sum_{j=0}^{i-1} \binom{k}{j} \lambda^{i-j}$. □

The situation is described by the commutative diagram (of fibres)

$$\begin{array}{ccc}
SO/SO(k) & \xrightarrow{\rho'} & O/O(n) \\
\downarrow & & \downarrow \\
BSO(k) & \xrightarrow{B\rho} & BO(n) \\
\downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \iota_n} \\
BSO & \xrightarrow{\rho''} & BO
\end{array}$$

Suppose that the classifying map $\tau : S^n \to BO(n)$ for the fibration (1.1) can be written as a composition $\tau = B\rho \circ \xi$ with $\xi : S^n \to BSO(k)$. According to Lemma 3.2 we can suppose that $\iota_k \circ \xi$ is zero in $\pi_n(BSO)$. Hence both τ and ξ can be lifted to a non-trivial element $t \in \pi_n(O/O(n)) \cong \mathbb{Z}/2$ and $x \in \pi_n(SO/SO(k))$, respectively, such that $t = \rho' \circ x$. Hence the map ρ' satisfies the assumptions of Proposition 3.3. Consequently, $n+1 \equiv 0 \mod a(n+1-k)$, which is equivalent to

$$k \geq n+1-j(n).$$

Since $n \geq 9$, the inequality above yields $k \geq 8$.

The homomorphism ρ is a sum of irreducible representations. If it were different from the standard inclusion, then, by the Weyl Dimension Formula (see (i) of Proposition 5.1 and Remark 5.2 in the next section) and the inequality above, its dimension would be at least

$$\min\{\dim 2\lambda^1, \dim \lambda^2\} = 2k \geq 2(n-j(n)+1) > n, \quad \text{for all } n \geq 9,$$

which is a contradiction.

Now suppose that the class of $\rho : SO(k) \to SO(n)$ in $RO(SO(k))$ is not a polynomial in exterior powers. In this case we can use the following lemma. Its proof is based on the Weyl Dimension Formula and is postponed to the next section.

Lemma 4.2. *Let $k \geq 5$. If the class of a representation $\rho : SO(k) \to SO(m)$ in $RO(SO(k))$ is not a polynomial in exterior powers, then for all $n \geq m$*

$$\dim SO(k) < n - j(n).$$

According to this lemma and Proposition 3.1 the reduction of $SO(n)$ in (1.1) to $(SO(k), \rho)$ is impossible.

B1. Let $n \geq 9$ be odd, $k \geq 2$.

Lemma 4.3. *Let $\rho : G = SU(k) \to O(n)$ be a homomorphism whose class in $R(SU(k))$ (after complexification) is of the form $q(\lambda^1, \dots, \lambda^{[k/2]}, \overline{\lambda^1}, \dots, \overline{\lambda^{[k/2]}})$ where q is a polynomial such that*

$$q(x_1, \dots, x_{[k/2]}, y_1, \dots, y_{[k/2]}) = q(y_1, \dots, y_{[k/2]}, x_1, \dots, x_{[k/2]}). \tag{4.1}$$

Then there is a map $\rho'' : BSU \to BO$ such that $\rho'' \circ \iota_k \simeq \iota_n \circ B\rho$, where $\iota_k : BSU(k) \to BU$ and $\iota_n : BO(n) \to BO$ are the standard inclusions.

Proof. If the complexification of $\rho : SU(k) \to U(n)$ has the form described above, then an extension $BSU \to BU$ can be constructed as in the proof of Lemma 4.1 using a polynomial p of the same form. This extension can be factored through the complexification $BSO \to BSU$. The reason is that any complex vector bundle of the form $\eta \oplus \overline{\eta}$ is the complexification of the realification of η and any complex vector bundle of the form $\eta \otimes \overline{\eta}$ is the complexification of the real Lie algebra bundle of skew-adjoint endomorphisms of η. □

Consider a representation $\rho : SU(k) \to O(n)$ which reduces the structure group $SO(n)$ of (1.1).

Suppose first that ρ is of the type described in Lemma 4.3 so that we have a commutative diagram:

$$\begin{array}{ccc} SU/SU(k) & \xrightarrow{\rho'} & O/O(n) \\ \downarrow & & \downarrow \\ BSU(k) & \xrightarrow{B\rho} & BO(n) \\ {\scriptstyle \iota_k}\downarrow & & \downarrow{\scriptstyle \iota_n} \\ BSU & \xrightarrow{\rho''} & BO. \end{array}$$

The classifying map $\tau : S^n \to BO(n)$ for the fibration (1.1) can be written as a composition $\tau = \rho \circ \xi$ with $\xi : S^n \to BSU(k)$. By Lemma 3.2 τ and ξ can be lifted to $t \in \pi_n(O/O(n))$ and $x' \in \pi_n(SU/SU(k))$, respectively, such that $t = \rho' \circ x'$ and t is the generator of $\pi_n(O/O(n))$. So the map ρ' satisfies the assumptions of Proposition 3.3. Consequently, $n = 2m-1$ and $m \equiv 0 \mod 2^{\nu_2(b(m-k))}$. For the maximal integer k satisfying this condition put $j_2(n) = n + 1 - 2k$. Now the divisibility condition above is equivalent to the inequality

$$2k \geq n + 1 - j_2(n).$$

It is clear, by comparing the real and complex lifting problems, that $j_2(n) \leq j(n)$. Since $n \geq 9$, the inequality above yields $k \geq 4$.

Suppose that $k \geq 5$. If ρ, which is a sum of irreducible representations, were different from the standard inclusion, then, by the Weyl Dimension Formula (see (ii) of Proposition 5.1 and Remark 5.2) and the inequality above, its dimension would be at least 20 and greater than or equal to

$$\min\{2\dim_{\mathbb{C}}(\lambda^1 + \overline{\lambda^1}), \dim_{\mathbb{C}}(\lambda^2 + \overline{\lambda^2})\} = \min\{4k, k^2 - k\} = 4k \geq 2(n - j(n) + 1) > n.$$

This means that ρ has to be the standard inclusion in this case.

Now consider the case: $k = 4$. In the same way we get

$$8 = 2 \cdot 4 \geq n - j_2(n) + 1.$$

Consequently, $n = 9$, 11 or 15 in this case. This allows only two possibilities for ρ: $\lambda^1 + \overline{\lambda^1}$ and $\lambda^1 \cdot \overline{\lambda^1} - 1$ of dimension 8 and 15, respectively.

The latter homomorphism can be factored via a double covering as

$$SU(4) \cong Spin(6) \to SO(6) \xrightarrow{\lambda^2} SO(15).$$

However, the reduction of $SO(15)$ to $SO(6)$ was excluded in **A**.

Hence ρ is a standard inclusion corresponding to $\lambda^1 + \overline{\lambda^1}$.

B2. Consider a homomorphism $\rho : SU(k) \to O(n)$ whose class in $R(SU(k))$ is not of the form described by (4.1). In the next section we prove

Lemma 4.4. *Let $k \geq 2$. If the class of an irreducible representation $SU(k) \to O(m)$ in $R(SU(k))$ is not a polynomial in exterior powers of the form (4.1), then for all $n \geq m$*

$$\dim SU(k) < n - j(n)$$

with the just two exceptions: the double covering $SU(4) \cong Spin(6) \to SO(6)$ *which after complexification gives* $\lambda^2 : SU(4) \to SU(6)$*, and the representation* $SU(8) \to SO(70)$ *which after complexification gives* $\lambda^4 : SU(8) \to SU(70)$*.*

According to Proposition 3.1 this lemma excludes reductions to $(SU(k), \rho)$ apart from two possible exceptional cases: that after complexification one of the irreducible summands of ρ is $\lambda^2 : SU(4) \to U(6)$ and $n \leq 23$ (for $n \geq 25$ we can use Proposition 3.1) or $\lambda^4 : SU(8) \to U(70)$ with $n = 71$ (for $n \geq 73$ we can again use Proposition 3.1).

The representation ρ of $SU(4)$ containing λ^2 as a summand has to contain as a summand also $\lambda^1 + \overline{\lambda^1}$, otherwise it could be factored through $SO(6)$, which is impossible. So the representation ρ of $SU(4)$ has to be of the form

$$SU(4) \to SO(8) \times SO(6) \to SO(14) \times SO(n-14) \hookrightarrow O(n) \tag{4.2}$$

where the first homomorphism is given by the standard inclusion and the double covering and the second one contains the standard inclusion $SO(8) \times SO(6) \hookrightarrow SO(14)$ as the first component. However, according to Theorem 2.A in [15] the structure group $SO(n)$ cannot be reduced to $SO(14) \times SO(n-14)$ for $17 \leq n \leq 23$. (This can be seen by observing that a reduction to the product would imply reduction to one of the factors. Compare the proof of Lemma 4.5.)

Lemma 4.5. *The structure group* $SO(15)$ *of the principal fibre bundle* (1.1) *can be reduced to* $SU(4)$ *both through the standard inclusion and through the homomorphism of the form* (4.2)*.*

Proof. By [12] or by **B1** and Remark 3.6 the structure group $SO(15)$ of (2.1) can be reduced to $SU(4)$ using the standard inclusion. Since the tangent bundle to S^{15} is determined by the non-trivial element in $\pi_{14}\big(SO(15)\big) \cong \mathbb{Z}/2$, there is a map $\gamma : S^{14} \to SU(4)$ such that the composition with $SU(4) \hookrightarrow SO(15)$ is non-trivial. The composition of γ with any homomorphism

$$SU(4) \to SO(6) \hookrightarrow SO(15)$$

is trivial since the inclusion $SO(6) \hookrightarrow SO(15)$ induces the trivial homomorphism

$$\pi_{14}\big(SO(6)\big) \to \pi_{14}\big(SO(15)\big)$$

by **A**.

Hence the composition

$$S^{14} \xrightarrow{\gamma} SU(4) \hookrightarrow SO(8) \times SO(6) \to SO(14) \hookrightarrow SO(15)$$

is homotopic to the product (in $SO(15)$) of two maps $S^{14} \to SO(15)$, one of which is non-trivial and the other trivial. That is why this composition is non-trivial. So $\gamma : S^{14} \to SU(4)$ determines the reduction through the homomorphism (4.2). □

Lemma 4.6. *Let* $n = 71$*. Then the structure group* $SO(71)$ *in* (1.1) *cannot be reduced to* $SU(8)$ *via the homomorphism induced by* $\lambda^4 : SU(8) \to SU(70)$*.*

Proof. Suppose that a reduction does exist. Then the classifying map for (1.1) admits a factorization

$$S^{70} \xrightarrow{f} SU(8) \xrightarrow{\rho} SO(70) \hookrightarrow SO(71).$$

Denote by η the real vector bundle over $\Sigma SU(8)$ which is classified by ρ. Then the tangent bundle to S^{71} is isomorphic to $f^*(\eta)\oplus\mathbb{R}$. To prove that the factorization above is impossible it is sufficient to show that the Stiefel-Whitney class $w_{64}(\eta) = 0$. Since $\dim \Sigma SU(8) = 64$, in this case the vector bundle η would have 7 linearly independent vector fields, and, consequently, there would be 8 linearly independent vector fields on S^{71}, which is a contradiction.

Now η is the pullback, via the classifying map $\Sigma SU(8) \to BSU(8)$, of a bundle $\widetilde{\eta}$ over $BSU(8)$. The cohomology ring $H^*(BSU(8); \mathbb{Z}/2) = \mathbb{Z}/2[c_2, c_3, \dots, c_8]$ is polynomial on the mod 2 reductions of the Chern classes of the universal bundle. As the dimension of the generators is bounded by 16, $w_{64}(\widetilde{\eta})$ is a product of lower dimensional classes. Since the products in $H^*(\Sigma SU(8); \mathbb{Z})$ are trivial, it follows that $w_{64}(\eta) = 0$. □

C. Consider a representation $\rho : Sp(k) \to O(n)$ which reduces the structure group $SO(n)$ of (1.1).

Lemma 4.7. *Let $\rho : G = Sp(k) \to O(n)$ be a homomorphism. Then there is a map $\rho'' : BSp \to BO$ such that $\rho'' \circ \iota_k \simeq \iota \circ B\rho$.*

Proof. In $R(Sp(k))$ any virtual representation is self-conjugate and corresponds to a polynomial in exterior powers: λ^i is real if i is even and quaternionic if i is odd. Real virtual representations in $RO(Sp(k))$ are given by those polynomials in $R(Sp(k))$ which have even coefficients at monomials $(\lambda^1)^{j_1}, (\lambda^2)^{j_2}, \dots, (\lambda^k)^{j_k}$ with $j_1 + j_2 + \dots + j_k \equiv 1$ mod 2. The proof can be completed as in the proof of Lemma 4.3. □

The classifying map $\tau : S^n \to BO(n)$ for the fibration (1.1) can be written as a composition $\tau = \rho \circ \xi$ with $\xi : S^n \to BSp(k)$. According to Lemma 3.2 this map can be chosen in such a way that $\iota_k \circ \xi = 0 \in \pi_n(BSp)$. Hence both τ and ξ can be lifted to $t \in \pi_n(O/O(n))$ and $x' \in \pi_n(Sp/Sp(k))$, respectively, such that $t = \rho' \circ x'$ and t is the generator of $\pi_n(O/O(n))$. So the map ρ' satisfies the assumptions of Proposition 3.3. Consequently, $n = 4m - 1$ and $m \equiv 0 \mod 2^{\nu_2(c(m-k))}$. For maximal k satisfying this condition put $j_4(n) = n + 1 - 4k$. Now the divisibility condition above is equivalent to the inequality

$$4k \geq n + 1 - j_4(n).$$

Again it is clear that $j_4(n) \leq j_2(n)$. Since $n \geq 9$, the inequality above yields $k \geq 3$.

Consider $k \geq 4$. If ρ were not the standard inclusion, then by the Weyl Dimension Formula (see (iii) of Proposition 5.1 and Remark 5.2) its dimension would be at least

$$\min\{2 \dim_{\mathbb{C}} 2(\lambda^1), \dim_{\mathbb{C}}(\lambda^2 - 1)\} = \min\{8k, (2k + 1)(k - 1)\} \geq 27.$$

For $n \geq 27$ the inequality above yields

$$(2k + 1)(k - 1) > 2(k - 1)^2 \geq \frac{(n - j_4(n) - 3)^2}{8} \geq n$$

and

$$8k \geq 2(n - j_4(n) + 1) > n.$$

This means that ρ has to be the standard inclusion if $k \geq 4$.

For $k = 3$ the inequality implies that $n \leq 15$. We have two irreducible representations of $Sp(3)$ of dimension ≤ 15: the standard inclusion and the one given by the polynomial $\lambda^2 - 1$ in $R(Sp(3))$ of dimension 14.

Lemma 4.8. *Let $n = 15$. The structure group $SO(15)$ in (1.1) can be reduced to $Sp(3)$ via the homomorphism ρ induced by $Sp(3) \xrightarrow{\lambda^2} SO(15)$.*

Proof. Since $Sp/Sp(3)$ is 14-connected, $SO(15)$ can be reduced to $Sp(3)$ via ρ if and only if

$$\rho'_* : \widetilde{H}_{15}(Sp/Sp(3); \mathbb{Z}/2) \to \widetilde{H}_{15}(O/O(15); \mathbb{Z}/2)$$

is onto. We shall show that the composition

$$(4.3) \quad \widetilde{H}_{15}(Q(\mathbb{H}^4); \mathbb{Z}/2) \to \widetilde{H}_{15}(Sp(4); \mathbb{Z}/2) \to \widetilde{H}_{15}(O; \mathbb{Z}/2) \to \widetilde{H}_{15}(O/O(15); \mathbb{Z}/2) = \mathbb{Z}/2$$

induced by the reflection map $\phi : Q(\mathbb{H}^4) \to Sp(4)$ and $\rho'' : BSp(4) \hookrightarrow BSp \to BO$ is non-zero. On $BSp(4)$, ρ'' is given on a 4-dimensional $\mathbb{H}$-vector bundle ξ as the virtual bundle $\lambda^2\xi - \xi \otimes_{\mathbb{C}} \mathbb{H} + \mathbb{C}^3$ (with its real structure).

The composition (4.3) is given by a cohomology class $w \in \widetilde{H}^{15}(Q(\mathbb{H}^4); \mathbb{Z}/2)$. To compute the class it is convenient to lift from $Q(\mathbb{H}^4)$, which is the Thom space of the 3-dimensional Lie algebra bundle ζ over the quaternionic projective space $P(\mathbb{H}^4)$, to the sphere bundle $S(\mathbb{R} \oplus \zeta)$. The map

$$S(\mathbb{R} \oplus \zeta) \to P(\mathbb{H}^4)^{\zeta} = Q(\mathbb{H}^4) \to Sp(4)$$

determines a 4-dimensional $\mathbb{H}$-vector bundle ξ over $S^1 \times S(\mathbb{R} \oplus \zeta)$. The class w lifts to

$$w_{16}(\rho''(\xi)) \in H^{16}(S^1 \times S(\mathbb{R} \oplus \zeta); \mathbb{Z}/2).$$

Now we can write $H^*(S^1; \mathbb{Z}/2) = \mathbb{Z}/2[t]/(t^2)$, $H^*(P(\mathbb{H}^4); \mathbb{Z}/2) = \mathbb{Z}/2[x]/(x^4)$, where $x = w_4(H)$ is the mod 2 Euler class of the quaternionic Hopf bundle H, and $H^*(S(\mathbb{R} \oplus \zeta); \mathbb{Z}/2) = \mathbb{Z}/2[x, y]/(x^4, y^2)$, where y is the 3-dimensional class corresponding to the Thom class of ζ. The vector bundle ξ, constructed using the reflection map, is a direct sum $\eta \oplus H^{\perp}$, where $H^{\perp}$ is the orthogonal complement of H in the trivial bundle $\mathbb{H}^4$ over $P(\mathbb{H}^4)$ and η is the quaternionic line bundle obtained by twisting H. It follows that

$$\rho''(\xi) = \lambda^2\eta + \eta \otimes_{\mathbb{C}} H^{\perp} + \lambda^2 H^{\perp} - \eta \otimes_{\mathbb{C}} \mathbb{H} - H^{\perp} \otimes_{\mathbb{C}} \mathbb{H} + \mathbb{C}^3.$$

Using the triviality of the bundles $\lambda^2\eta$ and $H \oplus H^{\perp}$, one obtains

$$\rho''(\xi) = (\eta - H) \otimes_{\mathbb{C}} \mathbb{H}^3 + (H - \eta) \otimes_{\mathbb{C}} H + \mathbb{C}^{15}.$$

It is understood here that each complex bundle which we have written down has a real structure. We have to compute the Stiefel-Whitney class w_{16} of the virtual real vector bundle so defined. This will be done by calculating the total Stiefel-Whitney classes of the various constituents: $\eta \otimes \mathbb{H}$, $H \otimes \mathbb{H}$, $H \otimes H$ and $\eta \otimes H$.

To compute the Stiefel-Whitney classes involving η it is enough to consider the restrictions to the subspaces $S(\mathbb{R} \oplus \zeta)$, where η coincides with H, and $S^1 \times S^3$, where S^3 is the fibre of $S(\mathbb{R} \oplus \zeta)$ at a point in $P(\mathbb{H}^4)$. The second restriction gives us the generator of $\pi_3(Sp(1)) = \pi_4(BSp(1))$, which determines a 4-dimensional real vector bundle over S^4 with non-zero Stiefel-Whitney class. (One can, for example, think of this vector bundle as the Hopf bundle over $S^4 = P(\mathbb{H}^2)$.)

One finds that:

$$w(\eta \otimes \mathbb{H}) = 1 + x + ty, \quad w(H \otimes \mathbb{H}) = 1 + x, \quad w(H \otimes H) = 1, \quad w(\eta \otimes H) = 1 + ty.$$

Hence $w(\rho''(\xi)) = (1+x+ty)^3(1+x)^{-3}(1+ty)^{-1} = 1 + (x+x^2+x^3)ty$. This verifies that $w_{16} = x^3ty$ is non-zero, as claimed. □

D, **E**, **F**. Now suppose $G_n = SU(n)$ or $Sp(n)$. Put $m = n+1$ and $d = 2$ or 4 in the complex or quaternionic case, respectively. Consider the diagram

$$\begin{array}{ccccc} G_n & \longrightarrow & SO(dm-d) & \longrightarrow & SO(dm-1) \\ \downarrow & & & & \downarrow \\ G_{n+1} & & \longrightarrow & & SO(dm) \\ \downarrow & & & & \downarrow \\ S^{dm-1} & & \xrightarrow{=} & & S^{dm-1} \end{array}$$

If G_n in (2.1) can be reduced to G via $\rho : G \to G_n$, then $SO(dm-1)$ in (1.1) can be reduced to G via

$$G \xrightarrow{\rho} G_n \hookrightarrow SO(dm-1). \tag{4.4}$$

According to the previous steps this composition has to be a standard inclusion or the composition $SU(4) \to SO(8) \times SO(6) \hookrightarrow SO(15)$ described in **B1** or $\lambda^2 : Sp(3) \to SO(15)$ from Lemma 4.8. However, the last two homomorphisms cannot be factored through $SU(7)$ or $Sp(4)$. The inclusion (4.4) has to satisfy the inequality

$$\begin{aligned} k &\geq dm - j(dm-1) && \text{for } G = SO(k), \\ 2k &\geq dm - j_2(dm-1) && \text{for } G = SU(k), \\ 4k &\geq dm - j_4(dm-1) && \text{for } G = Sp(k), \end{aligned}$$

which implies that the cases $G = SO(k)$ with $G_n = SU(n)$ or $Sp(n)$ and $G = SU(k)$ with $G_n = Sp(n)$ cannot occur, since $k > n$. In the remaining cases ρ is forced to be a standard inclusion, again for dimensional reasons. Then Proposition 3.5 gives the divisibility conditions in (D), (E) and (F) of Theorem 2.1. and Remark 3.6 says that these conditions are sufficient for the existence of reductions. □

5. The Weyl Dimension Formula

The estimates of dimension of real irreducible representations used in the previous section are based on the Weyl Dimension Formula. Here we show how it is used. In particular, we deduce Lemmas 4.2 and 4.4 from the following propositions:

Proposition 5.1. *Let $\rho : G_k \to SO(m)$ be an irreducible, non-trivial, non-standard real representation.*

(i) $G_k = SO(k)$, $k \geq 7$. *Then* $m \geq k(k-1)/2$.
(ii) $G_k = SU(k)$, $k \geq 5$. *Then* $m \geq k(k-1)$.
(iii) $G_k = Sp(k)$, $k \geq 3$. *Then* $m \geq k(2k-1) - 1$.

Remark 5.2. The bounds on m are achieved by (i) λ^2, (ii) the underlying real representation of λ^2, (iii) the real representation λ^2 modulo the one-dimensional trivial summand given by the defining symplectic form.

Proposition 5.3. *Let $\rho : G_k \to SO(m)$ be a real representation.*

(i) $G_k = SO(k)$. *If $[\rho] \in RO(G_k)$ is not a polynomial in the exterior powers λ^j, $j \geq 1$, then $k \equiv 0$ mod 4 and $m \geq \frac{1}{2}\binom{k/2}{k/4}$ $(= \dim \lambda_+^{k/2})$.*

(ii) $G_k = SU(k)$. *If $[\rho] \in RO(G_k)$ is not of the type described in Lemma 4.3 then k is even and $m \geq \binom{k}{k/2}$ (the dimension of $\lambda^{k/2}$ with its real structure).*

We start by recalling necessary prerequisities from the representation theory of Lie groups. Let G be a simple Lie group. Denote by $\omega_1, \omega_2, \dots, \omega_l$ its fundamental weights. Then any complex irreducible representation is determined by its dominant weight

$$\omega = m_1\omega_1 + m_2\omega_2 + \cdots + m_l\omega_l, \quad m_i \in \mathbb{N}.$$

The complex vector space on which it acts will be denoted by $V(\omega) = V(m_1, m_2, \dots, m_l)$. Put $\delta = \sum_{i=1}^{l} \omega_i$. The Weyl Dimension Formula gives the complex dimension of $V(\omega)$

$$\dim_{\mathbb{C}} V(\omega) = \prod_{\beta > 0} \frac{\langle \beta, \omega + \delta \rangle}{\langle \beta, \delta \rangle},$$

where β goes through all the positive roots. Treating separately all the types of classical simple Lie groups we always get that if $m_i' \geq m_i$ for all i, then

$$\dim_{\mathbb{C}} V(m_1', m_2', \dots, m_l') \geq \dim_{\mathbb{C}} V(m_1, m_2, \dots, m_l). \tag{5.1}$$

We will need to compute dimensions of real irreducible representations. According to Cartan's Theorem ([13], page 366) there are two kinds of real irreducible representations: those the complexification of which is irreducible, and those the complexification of which is a sum of an irreducible complex representation with its complex conjugate representation. So the dimensions of real irreducible representations can be computed from the knowledge of dimensions of complex irreducible representations.

Proof of (i) *of Propositions* 5.1 *and* 5.3. First, consider representations of $Spin(k)$ or $SO(k)$ with $k = 2l + 1 \geq 7$. Denote by $\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$, $1 \leq i, j \leq l$, $i \neq j$, the roots of the corresponding Lie algebra $\mathfrak{so}(2l+1)$. Its positive roots are $\varepsilon_i \pm \varepsilon_j$ for $1 \leq i < j \leq l$, and ε_i for $1 \leq i \leq l$. The fundamental weights are

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2, \dots, \omega_{l-1} = \varepsilon_1 + \cdots + \varepsilon_{l-1},\ \omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l).$$

A complex representation with dominant weight $\omega = \sum_{i=1}^{l} m_i\omega_i$ acting on $V(\omega)$ is the complexification of a real representation $\rho : SO(2l+1) \to SO(m)$ if and only if m_l is even. Such a representation in $RO(SO(2l+1))$ is described by a polynomial in exterior powers. (ω_j corresponds to λ^j for $1 \leq j \leq l-1$ and $2\omega_l$ corresponds to λ^l.)

Put $\omega + \delta = \sum_{i=1}^{l} g_i\varepsilon_i$. Then the Weyl Dimension Formula reads as

$$\dim_{\mathbb{C}} V(\omega) = \prod_{1 \leq i < j \leq l} \frac{(g_i - g_j)(g_i + g_j)}{(j-i)(2l+1-i-j)} \prod_{1 \leq i \leq l} \frac{2g_i}{2l - 2i + 1}.$$

From this formula we can see immediately that the inequality (5.1) holds. We compute the dimensions of several irreducible representations:

$$\dim_{\mathbb{C}} V(\omega_j) = \binom{2l+1}{j}, \quad 1 \le j \le l-1,$$
$$\dim_{\mathbb{C}} V(2\omega_l) = \binom{2l+1}{l}, \quad \dim_{\mathbb{C}} V(2\omega_1) = (2l+3)l.$$

Let ρ be a real irreducible representation $SO(2l+1) \to SO(m)$ with a dominant weight ω different from ω_1. For $l \ge 3$, using the inequality (5.1), we get that its dimension $m = \dim_{\mathbb{C}} V(\omega)$ is at least

$$\min\{\dim_{\mathbb{C}} V(2\omega_1), \dim_{\mathbb{C}} V(\omega_2), \ldots, \dim_{\mathbb{C}} V(\omega_{l-1}), \dim_{\mathbb{C}} V(2\omega_l)\} = (2l+1)l.$$

Now consider representations of $Spin(k)$ or $SO(k)$ with $k = 2l \ge 8$. For $l \ge 4$ the roots of the corresponding Lie algebra $\mathfrak{so}(2l)$ are $\varepsilon_i \pm \varepsilon_j$, $\pm\varepsilon_i$, $1 \le i, j \le l$, $i \ne j$. The positive roots are $\varepsilon_i \pm \varepsilon_j$ for $1 \le i < j \le l$, and ε_i for $1 \le i \le l$. The fundamental weights are

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2, \ldots, \omega_{l-2} = \varepsilon_1 + \cdots + \varepsilon_{l-2},$$
$$\omega_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l),$$
$$\omega_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l).$$

A complex representation with dominant weight $\omega = \sum_{i=1}^{l} m_i\omega_i$ acting on $V(\omega)$ is a representation of $SO(2l)$ if and only if $m_{l-1} + m_l$ is even.

If l is even, $V(\omega)$ is always the complexification of an irreducible representation $\rho : SO(2l) \to SO(m)$. In this case $RO(SO(2l))$ is generated by λ^j, $1 \le j \le l-1$, λ^l_+, λ^l_-, where $\lambda^l_- + \lambda^l_+ = \lambda^l$. Here ω_j is the dominant weight of the complexification of λ^j for $1 \le j \le l-2$, $\omega_{l-1} + \omega_l$ is the dominant weight of the complexification of λ^{l-1}, and $2\omega_{l-1}$, $2\omega_l$ are the dominant weights of complexifications of λ^l_-, λ^l_+, respectively.

If l is odd, $RO(SO(2l))$ is a polynomial ring in exterior powers. ω_j corresponds to the complexification of λ^j for $1 \le j \le l-2$ and $\omega_{l-1} + \omega_l$ corresponds to the complexification of λ^{l-1}, while the complexification of λ^l is the sum of complex irreducible representations with dominant weights $2\omega_{l-1}$ and $2\omega_l$.

Let $\omega = \sum_{i=1}^{l} m_i\omega_i$ be the dominant weight of a complex representation acting on $V(\omega)$. Put $\omega + \delta = \sum_{i=1}^{l} g_i\varepsilon_i$. The Weyl Dimension Formula now gives

$$\dim_{\mathbb{C}} V(\omega) = \prod_{1 \le i < j \le l} \frac{(g_i - g_j)(g_i + g_j)}{(j-i)(2l-i-j)}.$$

From the formula one readily verifies the inequality (5.1) and computes the dimensions:

$$\begin{aligned}
\dim_{\mathbb{C}} V(\omega_j) &= \binom{2l}{j}, \quad 1 \le j \le l-2,\\
\dim_{\mathbb{C}} V(\omega_{l-1}+\omega_l) &= \binom{2l}{l-1},\\
\dim_{\mathbb{C}} V(2\omega_{l-1}) = \dim_{\mathbb{C}} V(2\omega_l) &= \frac{1}{2}\binom{2l}{l},\\
\dim_{\mathbb{C}} V(2\omega_1) &= (2l-1)(l+1).
\end{aligned}$$

Let ρ be a real irreducible representation $SO(2l) \to SO(m)$ the complexification of which has a dominant weight ω different from ω_1. Then according to (5.1) for $l \ge 4$ its dimension is either $\dim_{\mathbb{C}} V(\omega)$ or $2\dim_{\mathbb{C}} V(\omega)$ and greater than or equal to

$$\min\{\dim_{\mathbb{C}} V(2\omega_1), \dim_{\mathbb{C}} V(\omega_2), \dots, \dim_{\mathbb{C}} V(\omega_{l-1}+\omega_l), \dim_{\mathbb{C}} V(2\omega_{l-1})\} = l(2l-1).$$

If the class of an irreducible representation $\rho : SO(k) \to SO(m)$ is not a polynomial in exterior powers in $RO(SO(k))$, then $k = 2l \equiv 0 \bmod 4$ and the complexification of ρ has a dominant weight $\omega = \sum_{i=1}^{l} m_i\omega_i$ with $m_{l-1} \ge 2$ or $m_l \ge 2$. By (5.1) its dimension is at least

$$\dim_{\mathbb{C}} V(2\omega_{l-1}) = \dim_{\mathbb{C}} V(2\omega_l) = \frac{1}{2}\binom{2l}{l}.$$

□

For $l \ge 4$ and $n \ge \frac{1}{2}\binom{2l}{l}$ we get

$$\dim SO(2l) = l(2l-1) < n - j(n),$$

which proves Lemma 4.2.

Proof of (ii) *of Propositions* 5.1 *and* 5.3. Consider $SU(k)$ with $k \ge 2$. In standard notation $\varepsilon_i - \varepsilon_j$, $1 \le i, j \le k$, $i \ne j$, are the roots of the Lie algebra $\mathfrak{su}(k)$. Its positive roots are $\varepsilon_i - \varepsilon_j$ for $1 \le i < j \le k$, and the fundamental weights are

$$\omega_1 = \varepsilon_1,\ \omega_2 = \varepsilon_1 + \varepsilon_2, \dots, \omega_{k-1} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{k-1}.$$

Let $\omega = \sum_{i=1}^{k-1} m_i\omega_i$ be the dominant weight of a complex irreducible representation which acts on a complex vector space $V(\omega)$. The only complex irreducible representations which are complexifications of real irreducible representations are those satisfying

$$m_i = m_{k-i}, \quad \text{for } 1 \le i \le k-1,$$

with the exception of the case that $k \equiv 2 \bmod 4$ and $m_{k/2}$ is odd. See [22], Theorem E, page 140. The remaining complex irreducible representations $V(\omega)$ determine real irreducible representations ρ with $\dim_{\mathbb{R}} \rho = 2\dim_{\mathbb{C}} V(\omega)$.

The complex exterior power λ^i has dominant weight ω_i and its conjugate representation is λ^{k-i}. Hence the representations described in $R(SU(k))$ by polynomials (4.1) are complexifications of real ones.

Put $\omega+\delta=\sum_{i=1}^{k-1} g_i\varepsilon_i$. In this case the Weyl Dimension Formula reads as

$$\dim_{\mathbb{C}} V(\omega)=\prod_{1\le i<j\le k-1}\frac{(g_i-g_j)}{(j-i)}\prod_{1\le i\le k-1}\frac{g_i}{k-i}.$$

Again we can check that the inequality (5.1) holds. Specific dimensions are:

$$\begin{aligned}\dim_{\mathbb{C}} V(\omega_j)&=\binom{k}{j},\quad 1\le j\le k-1,\\ \dim_{\mathbb{C}} V(2\omega_1)&=\frac{1}{2}k(k+1)\\ \dim_{\mathbb{C}} V(\omega_1+\omega_{k-1})&=k^2-1,\\ \dim_{\mathbb{C}} V(\omega_2+\omega_{k-2})&=\frac{1}{4}k^2(k+1),\quad k\ge 5.\end{aligned}$$

For $k\ge 5$ consider a real irreducible representation of $SU(k)$ which is determined by a complex irreducible representation with a dominant weight ω different from ω_1. Its real dimension is at least

$$\min\{2\dim_{\mathbb{C}} V(2\omega_1), 2\dim_{\mathbb{C}} V(\omega_2), \dim_{\mathbb{C}} V(\omega_1+\omega_{k-1}), \dim_{\mathbb{C}} V(\omega_2+\omega_{k-2})\}=k^2-k.$$

If the class of an irreducible representation $\rho: SU(k)\to SO(m)$ (after complexification) is not a polynomial in exterior powers in $R(SU(k))$ of the form (4.1), then $k=2l\equiv 0 \bmod 4$ and the complexification of ρ has a dominant weight $\omega=\sum_{i=1}^{k-1} m_i\omega_i$ with $m_l\ge 1$. According to (5.1) its dimension is at least

$$\dim_{\mathbb{C}} V(\omega_l)=\binom{2l}{l}.$$

□

For $l\ge 6$ and $n\ge\binom{2l}{l}$ we get

$$\dim SU(2l)=4l^2-1<n-j(n).$$

For $k=4$, the dimension of $SU(4)$ is 15 and the only representation of the above form which does not satisfy the required inequality has the dominant weight ω_2. For $k=8$, the dimension of $SU(8)$ is 63 and the only representation of the above form which does not satisfy the required inequality has the dominant weight ω_4. This completes the proof of Lemma 4.4.

Proof of (iii) *of Proposition* 5.1. Consider $Sp(k)$ with $k\ge 2$. Denote by $\varepsilon_i-\varepsilon_j$, $\pm(\varepsilon_i+\varepsilon_j)$ for $1\le i,j\le k$, $i\ne j$, and $\pm 2\varepsilon_i$ for $1\le i\le k$ the roots of the Lie algebra $\mathfrak{sp}(k)$. Its positive roots are $\varepsilon_i\pm\varepsilon_j$ for $1\le i<j\le k$ and $2\varepsilon_i$. The fundamental weights are

$$\omega_1=\varepsilon_1,\ \omega_2=\varepsilon_1+\varepsilon_2,\dots,\omega_k=\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_k.$$

Let $\omega = \sum_{i=1}^{k} m_i\omega_i$ be the dominant weight of a complex irreducible representation which acts on a complex vector space $V(\omega)$. The only complex irreducible representations which are complexifications of real representations are those for which

$$\sum_{i=0}^{\frac{k-1}{2}} m_{2i+1}$$

is even. (See [22], Theorem E, page 140.) The other complex irreducible representations $V(\omega)$ determine real irreducible representations of real dimension $2\dim_{\mathbb{C}} V(\omega)$.

$R(Sp(k))$ is a polynomial ring in exterior powers which are self-conjugate. A complex representation is real if and only if it is represented by a polynomial with coefficients as specified in the proof of Lemma 4.7.

Put $\omega + \delta = \sum_{i=1}^{k} g_i\varepsilon_i$. In this case the Weyl Dimension Formula gives

$$\dim_{\mathbb{C}} V(\omega) = \prod_{1\le i<j\le k} \frac{(g_i - g_j)(g_i + g_j)}{(j-i)(2k+2-i-j)} \prod_{1\le i\le k} \frac{g_i}{k-i+1}.$$

Since the inequality (5.1) again holds, it is sufficient to compute only the dimensions:

$$\dim_{\mathbb{C}} V(\omega_j) = \binom{2k+1}{j}\frac{2k-2j+2}{2k-j+2}, \quad 1 \le j \le k,$$
$$\dim_{\mathbb{C}} V(2\omega_1) = (2k+1)k.$$

For $k \ge 3$ consider a real irreducible representation ρ which is determined by a complex irreducible representation with a dominant weight ω different from ω_1. Its dimension is at least

$$\min\{\dim_{\mathbb{C}} V(\omega_2), 2\dim_{\mathbb{C}} V(\omega_3), \dim_{\mathbb{C}} V(2\omega_1)\} = 2k^2 - k - 1.$$

□

6. Appendix

In this appendix we give a self-contained proof of the result of Davis and Mahowald ([11]) that the generator of $\pi_{8k}(SO) = \mathbb{Z}/2$, for $k \ge 1$, lifts to $\pi_{8k}(SO(6))$ and of a similar result that the generator of $\pi_{8k+4}(Sp) = \mathbb{Z}/2$ lifts to $\pi_{8k+4}(Sp(1))$. These results are used in the proof of Lemma 3.2. At the very end we develop Remark 2.4 on possible generalizations of Theorem 2.1.

Let us write C_n for the cofibre of the map $z \mapsto z^n$: $S^1 \to S^1$ (where S^1 is the space of complex numbers of modulus 1). Thus we have a cofibre sequence:

$$S^1 \xrightarrow{n} S^1 \longrightarrow C_n \longrightarrow S^2 \xrightarrow{n} S^2 \to \cdots$$

For pointed finite complexes X and Y, we write $\omega^0\{X;\, Y\}$ for the group of stable maps $X \to Y$ and $\omega^i\{X;\, Y\}$, with cohomological indexing, for $\omega^0\{X;\, S^i \wedge Y\}$. In K-theory we write in the same way $KO^0\{X;\, Y\}$ for $[\mathbf{X};\, \mathbf{Y} \wedge \mathbf{KO}]$, where $\mathbf{X}$ and $\mathbf{Y}$ are the suspension spectra of X and Y, and $\mathbf{KO}$ is the real K-theory spectrum, and $KO^i\{X;\, Y\} = KO^0\{X;\, S^i \wedge Y\}$. There is a Hurewicz map (or d-invariant)

$$\omega^i\{X;\, Y\} \to KO^i\{X;\, Y\}.$$

The groups $\omega^i\{X;\,Y\}$ and $KO^i\{X;\,Y\}$ can be identified with $\widetilde{\omega}^i(X \wedge D(Y))$, and $\widetilde{KO}^i(X \wedge D(Y))$, respectively, where $D(Y)$ is the stable dual of Y.

We shall also need the cohomology theory J^* defined to be the fibre of the stable Adams operation $\psi^3 - 1$ as a self-map of the the real KO-theory spectrum localized at 2. It is thus related to KO-theory by a long exact sequence:

$$\cdots \to KO^{*-1}_{(2)} \xrightarrow{\psi^3-1} KO^{*-1}_{(2)} \to J^* \to KO^*_{(2)} \xrightarrow{\psi^3-1} KO^*_{(2)} \to \cdots$$

For any space X and integer k, the smash product with the identity on X gives a map

$$\wedge 1_X : \mathbb{Z} \cong KO^{-8k}(*) \to KO^{-8k}\{X;\,X\}\,.$$

The image of one of the generators will be called a *Bott element.* Suppose that the rational homology of X is zero. We call a stable map $A \in \omega^{-8k}\{X;\,X\}$ an *Adams map* if its Hurewicz image in $KO^{-8k}\{X;\,X\}$ is a Bott element. (See [8] and [9].)

The proof of the result of Davis and Mahowald is based on the existence of an unstable Adams map on C_8.

Proposition 6.1. *There is an unstable Adams map*

$$A : \Sigma^{15}C_8 \to \Sigma^7 C_8.$$

More precisely, the Hurewicz map

$$[\Sigma^{15}C_8;\,\Sigma^7 C_8] \to KO^{-8}\{C_8;\,C_8\} = \mathbb{Z}/8 \oplus \mathbb{Z}/2$$

is surjective.

Proof. We start from the standard fact that $\pi_{14}(S^7) = (\mathbb{Z}/120)\xi$, where ξ stabilizes to $2\sigma \in \omega_7(*) = (\mathbb{Z}/240)\sigma$. From the diagram of exact sequences of the defining cofibration of C_8:

$$\begin{array}{ccccccc}
\xrightarrow{8} & [S^{15};\,S^7] & \longrightarrow & [\Sigma^{13}C_8;\,S^7] & \longrightarrow & \mathbb{Z}/120 = [S^{14};\,S^7] & \xrightarrow{8} \\
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow{8} & \omega_8 & \longrightarrow & \widetilde{\omega}^{-6}(C_8) & \longrightarrow & \mathbb{Z}/240 = \omega_7 & \xrightarrow{8} \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle h_J} & \\
\xrightarrow{8} & J_8 & \longrightarrow & \widetilde{J}^{-6}(C_8) & \longrightarrow & \mathbb{Z}/16 = J_7 & \xrightarrow{8} \\
 & \downarrow & & \downarrow & & \downarrow & \\
\xrightarrow{8} & \mathbb{Z}_{(2)} = (KO_8)_{(2)} & \longrightarrow & \mathbb{Z}/8 = \widetilde{KO}^{-6}(C_8)_{(2)} & \longrightarrow & 0 = (KO_7)_{(2)} & \xrightarrow{8}
\end{array}$$

we can see that there is an element $a \in [\Sigma^{13}C_8;\,S^7]$ lifting an odd multiple of ξ and mapping to the generator of $\mathbb{Z}/8 = \widetilde{KO}^{-6}(C_8)$. The reason is that the map $\widetilde{J}^{-6}(C_8) \to \widetilde{KO}^{-6}(C_8)$ is onto (since the Adams operator ψ^3 acts as multiplication by 3^4 on $(KO_8)_{(2)}$, and so $\psi^3 - 1$ as 80 on $\widetilde{KO}^{-6}(C_8)_{(2)}$) and that the Hurewicz homomorphism $h_J : \omega_7 \to J_7$ is also onto, which follows from the computation of the J-homomorphism and the e-invariant in [2], Theorem 1.6, and the relation between the e-invariant and h_J (see [10], Section 1).

The Moore space C_n is essentially self-dual. For we have a cofibration sequence:

$$D(S^1) = S^{-1} \xleftarrow{n} D(S^1) = S^{-1} \longleftarrow D(C_n) \longleftarrow D(S^2) = S^{-2} \xleftarrow{n} D(S^2) = S^{-2},$$

which allows us to identify the stable dual $D(C_n)$ with $\Sigma^{-3}C_n$. The stable duality is specified by two structure maps

$$S^3 \to C_n \wedge C_n \to S^3.$$

We choose an unstable representative

$$d : S^{15} \to C_8 \wedge S^{12} \wedge C_8$$

of the first structure map.

We shall also need the standard stable equivalence:

$$C_8 \wedge C_8 \cong \Sigma^1 C_8 \vee \Sigma^2 C_8.$$

This comes from the basic cofibration sequence by smashing with C_8:

$$S^1 \wedge C_8 \xrightarrow{8} S^1 \wedge C_8 \longrightarrow C_8 \wedge C_8 \longrightarrow S^2 \wedge C_8 \xrightarrow{8} S^2 \wedge C_8,$$

because $8 = 0 \in \omega^0\{C_8;\, C_8\}$. We choose an unstable representative:

$$p : \Sigma^{12} C_8 \wedge C_8 \to \Sigma^{13} C_8$$

of the projection onto the first factor.

Now we can define the Adams map A (modulo slight adjustment which will be described later) as the composition:

$$C_8 \wedge S^{15} \xrightarrow{1\wedge d} C_8 \wedge S^{12} \wedge C_8 \wedge C_8 \xrightarrow{p\wedge 1} S^{13} \wedge C_8 \wedge C_8 \xrightarrow{a\wedge 1} S^7 \wedge C_8.$$

We have to check that its Hurewicz image in KO-theory is a Bott element. We have

$$KO^{-8}\{C_8;\, C_8\} = \widetilde{KO}^{-5}(C_8 \wedge C_8) = \widetilde{KO}^{-6}(C_8) \oplus \widetilde{KO}^{-7}(C_8),$$
$$\omega^{-8}\{C_8;\, C_8\} = \widetilde{\omega}^{-5}(C_8 \wedge C_8) = \widetilde{\omega}^{-6}(C_8) \oplus \widetilde{\omega}^{-7}(C_8)$$

by duality and the stable splitting. From the cofibration exact sequence:

$$\xrightarrow{8} KO_9 = \mathbb{Z}/2 \to \widetilde{KO}^{-7}(C_8) \to KO_8 = \mathbb{Z} \xrightarrow{8} KO_8 = \mathbb{Z} \to \widetilde{KO}^{-6}(C_8) \to KO_7 = 0$$

we see that $KO^{-8}\{C_8;\, C_8\} = \mathbb{Z}/8 \oplus \mathbb{Z}/2$. Indeed, the same calculation determines the stable homotopy: $\omega^0\{C_8;\, C_8\} = \mathbb{Z}/8 \oplus \mathbb{Z}/2$ generated by the identity map of order 8 and an element of order 2 given by the composition:

$$h : \Sigma C_8 \to S^3 \xrightarrow{\eta} S^2 \to \Sigma C_8$$

of the maps in the cofibration sequence and the Hopf element η. The smash products of these maps with a Bott class $v \in KO_8$ give generators of $KO^{-8}\{C_8; C_8\}$.

The calculation will be made by considering the commutative diagram

$$\begin{array}{ccccc}
[S^{13} \wedge C_8,\, S^7] & \longrightarrow & \widetilde{\omega}^{-6}(C_8) & \xrightarrow{h_{KO}} & \widetilde{KO}^{-6}(C_8) \\
\downarrow & & \downarrow i_1 & & \downarrow i_1 \\
[S^{15} \wedge C_8,\, S^7 \wedge C_8] & \longrightarrow & \omega^{-8}\{C_8;\, C_8\} & \xrightarrow{h_{KO}} & KO^{-8}\{C_8;\, C_8\}
\end{array}$$

in which the first vertical map is given by the smash with identity on C_8 composed with $(p \wedge 1)^*$ and $(1 \wedge d)^*$, i_1 is the inclusion of the first summand and h_{KO} is the Hurewicz homomorphism. The element $a \in [S^{13} \wedge C_8;\, S^7]$ maps to $A \in [S^{15} \wedge C_8;\, S^7 \wedge C_8]$ and simultaneously to a generator of $\widetilde{KO}^{-6}(C_8) = \mathbb{Z}/8$ as shown above. This implies that the Hurewicz image of A is equal to $v \cdot (m + nh)$, where m is an odd integer, $n \in \mathbb{Z}/2$, and $v \in KO_8$ is a Bott class. We can certainly multiply A by an odd integer to arrange that $m = 1$. We can also modify A by the element $\Sigma^6 h$ if necessary to achieve $n = 0$, because $h^2 = 0$ (stably). These adjustments produce the required unstable Adams map on C_8. We have noted in passing that vh lies in the image of the Hurewicz homomorphism, and this completes the proof. □

Now consider the diagram:

$$\begin{array}{ccccccc}
\xrightarrow{8} \widetilde{KO}^{-1}(S^9) = \mathbb{Z}/2 & \longrightarrow & \widetilde{KO}^{-1}(\Sigma^7 C_8) & \longrightarrow & \widetilde{KO}^{-1}(S^8) = \mathbb{Z}/2 \xrightarrow{8} \\
{\scriptstyle 4}\uparrow{\scriptstyle =0} & & \uparrow & & {\scriptstyle 1}\uparrow \\
\xrightarrow{2} \widetilde{KO}^{-1}(S^9) = \mathbb{Z}/2 & \longrightarrow & \widetilde{KO}^{-1}(\Sigma^7 C_2) & \longrightarrow & \widetilde{KO}^{-1}(S^8) = \mathbb{Z}/2 \xrightarrow{2}
\end{array}$$

The generator $y \in \pi_8(O) = \widetilde{KO}^{-1}(S^8) = \mathbb{Z}/2$ has a lift, x say, a generator of $\widetilde{KO}^{-1}(\Sigma^7 C_8)$ coming from a generator of $\widetilde{KO}^{-1}(\Sigma^7 C_2) \cong \mathbb{Z}/4$. This class x gives a map $\Sigma^7 C_8 \to O$. We shall show that it lifts to an element $\tilde{x} \in [\Sigma^7 C_8; SO(6)]$.

Recall from obstruction theory that, if X is a pointed finite complex with $\dim X < 2n - 1$, the obstruction to lifting a class $x \in [X; O]$ to $[X; O(n)]$ is precisely the image of $\theta(x)$ in $\omega^0\{X; P_n^\infty\}$ (where $P_n^\infty = P(\mathbb{R}^\infty)/P(\mathbb{R}^n)$ and θ is the stable splitting used in Section 3). Since $\dim \Sigma^7 C_8 = 9 < 2 \cdot 6 - 1$, it suffices to show that the obstruction vanishes in $\omega^0\{\Sigma^7 C_8; P_6^\infty\}$. We look at the diagram in stable homotopy corresponding to the KO-theory diagram above:

$$\begin{array}{ccccccc}
\xrightarrow{8} \widetilde{\omega}_9(P_6^\infty) = \mathbb{Z}/12 & \longrightarrow & \omega^0\{\Sigma^7 C_8;\, P_6^\infty\} & \longrightarrow & \widetilde{\omega}_8(P_6^\infty) = 0 \xrightarrow{8} \\
{\scriptstyle 4}\uparrow & & \uparrow & & {\scriptstyle 1}\uparrow \\
\xrightarrow{2} \widetilde{\omega}_9(P_6^\infty) = \mathbb{Z}/12 & \longrightarrow & \omega^0\{\Sigma^7 C_2;\, P_6^\infty\} & \longrightarrow & \widetilde{\omega}_8(P_6^\infty) = 0 \xrightarrow{2}
\end{array}$$

(The calculations of the stable homotopy groups $\widetilde{\omega}_9(P_6^\infty) = \pi_9(O/O(6))$ and $\widetilde{\omega}_8(P_6^\infty) = \pi_8(O/O(6))$ can be found in [21].) From the diagram, the map $\omega^0\{\Sigma^7 C_2;\, P_6^\infty\} \to \omega^0\{\Sigma^7 C_8;\, P_6^\infty\}$ is zero, and so the obstruction to lifting x to $SO(6)$ is zero.

We can now use iterates of the Adams map A to lift the images of x under Bott periodicity to $SO(6)$ as $(A^{k-1})^*(\tilde{x})$:

$$\begin{array}{ccccc}
[\Sigma^7 C_8,\, SO(6)] & \xrightarrow{A^*} & [\Sigma^{15} C_8,\, SO(6)] & \xrightarrow{(\Sigma^8 A)^*} & \cdots \\
\downarrow & & \downarrow & & \\
\widetilde{K}O^{-1}(\Sigma^7 C_8) & \xrightarrow[\cong]{v} & \widetilde{K}O^{-1}(\Sigma^{15} C_8) & \xrightarrow[\cong]{v} & \cdots \\
\downarrow & & \downarrow & & \\
\widetilde{K}O^{-1}(S^8) & \xrightarrow[\cong]{v} & \widetilde{K}O^{-1}(S^{16}) & \xrightarrow[\cong]{v} & \cdots
\end{array}$$

(where v is the Bott periodicity class). Each vertical composition

$$[\Sigma^{8k-1} C_8,\, SO(6)] \to \widetilde{K}O^{-1}(\Sigma^{8k-1} C_8) \to \widetilde{K}O^{-1}(S^{8k}) = \mathbb{Z}/2$$

is surjective (since the first one is) and factors through

$$[\Sigma^{8k-1} C_8,\, SO(6)] \to [S^{8k},\, SO(6)] \to \widetilde{K}O^{-1}(S^{8k}) = \mathbb{Z}/2.$$

We have thus established:

Proposition 6.2 (Davis, Mahowald)**.** *For $k \geq 1$, the generator of $\pi_{8k}(O) = \widetilde{K}O^{-1}(S^{8k}) = \mathbb{Z}/2$ lifts to an element (with order a divisor of 8) in $\pi_{8k}(SO(6))$.*

Corollary 6.3. *The generator of $\pi_{8k+1}(O) = \widetilde{K}O^{-1}(S^{8k+1}) = \mathbb{Z}/2$ lifts to an element of order 2 in $\pi_{8k+1}(SO(6))$.*

Proof. A lift is obtained by composing with the Hopf element $S^{8k+1} \to S^{8k}$, which has order 2. □

Proposition 6.4. *The generator of $\pi_{8k+4}(Sp) = \widetilde{K}Sp^{-1}(S^{8k+4}) = \mathbb{Z}/2$ lifts to an element in $\pi_{8k+4}(Sp(1))$ for $k \geq 1$.*

Proof. According to [16] and [17], p. 261, the homomorphism

$$\pi_{12}(Sp(1)) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \to \pi_{12}(Sp) = \mathbb{Z}/2$$

induced by the standard inclusion is an epimorphism.

By the Bockstein sequence any element of $\pi_{12}\big(Sp(1)\big)$ can be factored through $\Sigma^{11} C_8$. Hence the composition in the first column of the following diagram is surjective:

$$\begin{array}{ccccc}
[\Sigma^{11} C_8,\, Sp(1)] & \xrightarrow{(\Sigma^4 A)^*} & [\Sigma^{19} C_8,\, Sp(1)] & \xrightarrow{(\Sigma^{12} A)^*} & \cdots \\
\downarrow & & \downarrow & & \\
\widetilde{K}Sp^{-1}(\Sigma^{11} C_8) & \xrightarrow[\cong]{v} & \widetilde{K}Sp^{-1}(\Sigma^{19} C_8) & \xrightarrow[\cong]{v} & \cdots \\
\downarrow & & \downarrow & & \\
\widetilde{K}Sp^{-1}(S^{12}) & \xrightarrow[\cong]{v} & \widetilde{K}Sp^{1}(S^{20}) & \xrightarrow[\cong]{v} & \cdots
\end{array}$$

Since A induces a Bott isomorphism in KO-theory, it induces an isomorphism in KSp-theory as well. Consequently, all vertical compositions

$$\left[\Sigma^{8k+3}C_8,\, Sp(1)\right] \to \left[\Sigma^{8k+3}C_8,\, Sp\right] \to \left[S^{8k+4}, Sp\right]$$

are surjective and factor through

$$\left[\Sigma^{8k+3}C_8,\, Sp(1)\right] \to \left[S^{8k+4},\, Sp(1)\right] \to \left[S^{8k+4},\, Sp\right] .$$

$\square$

Remark 6.5. Let M be a connected, stably parallelizable, closed manifold of odd dimension $n \geq 9$. There is, up to isomorphism, exactly one stably trivial, but not trivial, n-dimensional vector bundle τ over M. It may be described as the pullback of the tangent bundle of S^n by a map $f : M \to S^n$ of degree one (collapsing the complement of an open disc).

In [24] Thomas showed that the structure group of τ can be reduced to $SO(k)$ by the standard inclusion $SO(k) \hookrightarrow SO(n)$ if and only if $m \equiv 0 \bmod a(m-k)$, where $n = m-1$ as in Theorem 2.1(A). (The special case in which τ is the tangent bundle of M was considered in [6].) One might ask whether Theorem 2.1 admits a generalization to such a bundle τ over M.

Consider a homomorphism $\rho : G_k \to O(n)$ for which there is a map $\rho'' : BG_\infty \to BO$ such that $\rho'' \circ \iota_k \simeq \iota_n \circ B\rho$:

$$\begin{array}{ccc} BG_k & \xrightarrow{B\rho} & BO(n) \\ \downarrow{\scriptstyle \iota_k} & & \downarrow{\scriptstyle \iota_n} \\ BG_\infty & \xrightarrow{\rho''} & BO \end{array}$$

and suppose that $\tau : M \to BO(n)$ can be factored through $B\rho$ by a map $\xi : M \to BG_k$ such that $\tau = B\rho \circ \xi$. Now if, as in Lemma 3.2, there is a map ξ' such that $B\rho \circ \xi' = \tau$ and $\iota_k \circ \xi' : M \to BG_\infty$ is trivial, then we may argue as in the proof of (A), (B) and (C) of Theorem 2.1. Since ξ' can be lifted to $\hat{\xi} : M \to G_\infty/G_k$ and a stable parallelization of M gives a stable splitting $h : S^n \to M$ of the map $f : M \to S^n$, we may take $x = \hat{\zeta} \circ h \in \widetilde{\omega}_n(G_\infty/G_k)$ satisfying the assumptions of Proposition 3.3. Consequently, $\rho : G_k \to G_\infty$ is described by (A), (B) and (C) of Theorem 2.1.

In the case that $\rho : SO(k) \to O(n)$ is the standard inclusion we may take ρ'' to be identity, $\xi = \xi'$ satisfies the required condition and we reprove the result of [24].

There are other special cases where the existence of ξ' is guaranteed: for example if $G_k = SU(k)$ and the map $K^o(M) \to KO^o(M)$ is injective, as happens when M is a product of spheres of the form $S^{4i_1} \times \cdots \times S^{4i_l} \times S^{2j-1}$.

Acknowledgement. Work on this paper started when the first author visited the Department of Mathematical Sciences of the University of Aberdeen and was completed when the second author visited the Department of Algebra and Geometry of the Masaryk University of Brno. The financial support of the London Mathematical Society and the hospitality of both Departments are gratefully acknowledged. The authors would also like to thank Mamoru Mimura for drawing their attention to this problem.

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.
[2] J. F. Adams, *On the groups J(X), IV*, Topology **5** (1966), 21–71.
[3] J. F. Adams, G. Walker *On complex Stiefel manifolds*, Proc. Cambridge Philos. Soc. **61** (1965), 81–103.
[4] M. F. Atiyah, *Thom complexes*, Proc. London Math. Soc. (3) **11** (1961), 291–310.
[5] M. F. Atiyah, J. A. Todd, *On complex Stiefel manifolds*, Proc. Cambridge Philos. Soc. **56** (1966), 342–353.
[6] G. E. Bredon, A. Kosinski, *Vector fields on π-manifolds*, Ann. of Math. **84** (1966), 85–90.
[7] M. C. Crabb, *On the stable splitting of $U(n)$ and $\Omega U(n)$*, in Algebraic Topology Barcelona 1986 (ed. J. Aquadé and R. Kane), Lecture Notes in Math. **1298** (1987), 35–53.
[8] M. C. Crabb, K. Knapp, *Adams periodicity in stable homotopy*, Topology **24** (1985), 475–486.
[9] M. C. Crabb, K. Knapp, *Central Adams operators*, Manuscripta Math. **60** (1988), 131–137.
[10] M. C. Crabb, K. Knapp, *Applications of non-connective $Im(J)$-theory*, Handbook of Algebraic Topology (ed. I. M. James), North-Holland, Amsterdam (1988), 465–503.
[11] D. Davis, M. Mahowald, *The $SO(n)$-of-origin*, Forum Math. **1** (1988), 239–250.
[12] I. Dibag, *Almost-complex substructures on the sphere*, Proc. Amer. Math. Soc. **61** (1976), 361–366.
[13] M. Goto, F. D. Grosshans, *Semisimple Lie algebras*, Marcel Dekker, Inc., New York, Lecture Notes in Pure and Applied Mathematics, **38** (1978).
[14] I. M. James, *Spaces associated with Stiefel manifolds*, Proc. London Math. Soc. **8**, (1958), 507–535.
[15] P. Leonard, *G-structures on spheres*, Trans. Amer. Math. Soc. **157** (1971), 311–327.
[16] M. Mimura, H. Toda, *The homotopy groups of $SU(3)$, $SU(4)$* and $Sp(2)$, J. Math. Kyoto Univ. **2** (1964), 217–250.
[17] M. Mimura, H. Toda, *Homotopy groups of symplectic groups*, J. Math. Kyoto Univ. **3** (1964), 251–273.
[18] T. Önder, *On the quaternionic James numbers and almost-quaternionic substructures on sphere*, Proc. Amer. Math. Soc. **86** (1982), 535–540.
[19] T. Önder, *Almost quaternionic substructure on the canonical C^{n-1}-bundle over S^{2n-1}*, J. London Math. Soc. **28** (1983), 435–442.
[20] Y. Ozaki, *G-structures on spheres*, Math. J. Okayama. Univ. **33** (1991), 189–200.
[21] G. F. Paechter, *The groups $\pi_r(V_{n,m})$ (I)*, Quart. J. Math. Oxford (2), **7** (1956), 249–268.
[22] H. Samelson, *Notes on Lie Algebras*, Springer, Berlin (1990).
[23] F. Sigrist, U. Suter, *Cross-sections of symplectic Stiefel manifolds*, Trans. Amer. Math. Soc. **184** (1973), 247–259.
[24] E. Thomas, *Cross-sections of stably equivalent vector bundles*, Quart. J. Math. Oxford **17** (1966), 53–57.

Department of Algebra and Geometry, Masaryk University, Janáčkovo nám. 2a, 602 00 Brno, Czech Republic
E-mail address: cadek@math.muni.cz

Department of Mathematical Sciences, University of Aberdeen, Aberdeen AB24 3UE, U.K.
E-mail address: m.crabb@maths.abdn.ac.uk