THE COHOMOLOGY OF $G_{m,2}$ WITH TWISTED INTEGER COEFFICIENTS

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Abstract. Let $H^*(G_{m,2};\mathbb{Z}^t)$ be the graded cohomology group of the Grassmann manifold $G_{m,2}$ with twisted integer coefficients. Then $H^*(G_{m,2};\mathbb{Z}) \oplus H^*(G_{m,2};\mathbb{Z}^t)$ has a structure of a $\mathbb{Z}\oplus\mathbb{Z}_2$-graded ring. In the paper this ring is described in terms of generators and relations.

1. Introduction

The Grassmann manifold $G_{m,2}$ is the space of all nonoriented 2-dimensional subspaces in $\mathbb{R}^m$. Its double and simultaneously simply connected covering $\tilde{G}_{m,2}$ is the space of all oriented 2-dimensional subspaces in $\mathbb{R}^m$. Examining the existence of 2-dimensional subbundles in m-dimensional vector bundles we need to know cohomology classes of $G_{m,2}$ both with trivial and nontrivial local systems of integer coefficients. The cohomology ring of general $G_{m,k}$ with $\mathbb{Z}_2$-coefficients has been found by A. Borel in [B]. The cohomology ring $H^*(\tilde{G}_{m,2};\mathbb{Z})$ up to the dimension m has been known to E. Thomas ([T]). Its complete description can be found in [V].

Since $\pi_1(G_{m,2}) = \mathbb{Z}_2$ we have only two possible nonequivalent local systems of integer coefficients – the trivial one, denoted by $\mathbb{Z}$ and the nontrivial, which we will call twisted and denote by $\mathbb{Z}^t$. The purpose of this note is to describe all the cohomology classes with trivial and twisted integer coefficients and relations among them. However, since the cup product changes twisted and trivial coefficients, it is advantageous to consider cohomology classes with trivial and twisted integer coefficients together as a $\mathbb{Z}\oplus\mathbb{Z}_2$-graded ring

$$h^*(G_{m,2}) = H^*(G_{m,2};\mathbb{Z}) \oplus H^*(G_{m,2};\mathbb{Z}^t).$$

We will determine this ring via its generators and relations. The main instrument for it will be the Gysin exact sequence. Similar description has been recently obtained for the classifying spaces $BO(k)$ (which can be considered as "stable" Grassmannians) in [C].

1991 *Mathematics Subject Classification.* 55R40.

Key words and phrases. Singular cohomology with local coefficients, Grassmann manifold.
Research supported by the grant 201/96/0310 of the Grant Agency of the Czech Republic.

2. Preliminaries and main results

We shall use the definition of a local system of groups and singular cohomology groups with local coefficients from [Sp], exercises F in Chapter I and J in Chapter V.

Let $b_0 \in B$ be a distinguished point. Let us assume that B is arcwise connected and that $\pi_1(B; b_0) = \mathbb{Z}_2$. We shall consider local systems of groups $\mathcal{G}$ over B such that for each $b \in B$ there is $G_b \cong \mathbb{Z}$. It is well known that, up to the isomorphism, there are only two such systems. A simple one and a nonsimple one. Without any loss of generality we may assume that the simple local system is the trivial local system arising when assigning to each point $b \in B$ the group of integers $\mathbb{Z}$ and defining the corresponding isomorphisms among these groups as identity isomorphisms. For this reason we shall denote the simple local system by $\mathbb{Z}$ and the nonsimple one by $\mathbb{Z}^t$.

Having taken for the simple local system the trivial one, we have in each group of this system a canonical generator determined by the integer 1. Then we easily obtain canonical isomorphisms of local systems of groups $\mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z}$, $\mathbb{Z}^t \otimes \mathbb{Z} \cong \mathbb{Z}^t$, $\mathbb{Z} \otimes \mathbb{Z}^t \cong \mathbb{Z}^t$ and $\mathbb{Z}^t \otimes \mathbb{Z}^t \cong \mathbb{Z}$. Using them we can easily endow the the local system of rings $\mathbb{Z} \oplus \mathbb{Z}^t$ with a structure of $\mathbb{Z}_2$-graded ring.

There is only one local system of groups $\mathcal{H}$ over B such that for each $b \in B$ there is $H_b \cong \mathbb{Z}_2$, and this system is simple. We may again assume that this system is the trivial one. It can be easily seen that this local system is in fact a local system of rings.

There are obvious homomorphisms (reductions mod 2) of local systems of groups $\mathbb{Z} \to \mathbb{Z}_2$ and $\mathbb{Z}^t \to \mathbb{Z}_2$, which induce a homomorphism of local systems of groups

$$\rho_2 : \mathbb{Z} \oplus \mathbb{Z}^t \to \mathbb{Z}_2.$$

It is easy to verify that ρ_2 is also a homomorphism of local systems of rings.

Let us notice first that the cohomology of B with coefficients in $\mathbb{Z} \oplus \mathbb{Z}^t$ inherits the $\mathbb{Z}_2$-graded ring structure of the local system of coefficients. That is to say that

$$H^*(B; \mathbb{Z} \oplus \mathbb{Z}^t) = H^*(B; \mathbb{Z}) \oplus H^*(B; \mathbb{Z}^t)$$

is a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded ring.

A 0-cocycle with values in $\mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}^t$ assigning to each point b the element 1 in the group over b determines an element of $H^0(B; \mathbb{Z})$, which we shall denote again by 1. Obviously, this is the unit element in the ring $H^*(B; \mathbb{Z} \oplus \mathbb{Z}^t)$.

We define a 0-cocycle with values in $\mathbb{Z}_2$ by assigning to each point $b \in B$ the element 1 in the group over b. We shall denote the corresponding element of $H^0(B; \mathbb{Z}_2)$ again by 1. It is the unit element in $H^*(B; \mathbb{Z}_2)$.

We have obvious short exact sequences of local systems of groups

$$0 \to \mathbb{Z} \overset{2\times}{\to} \mathbb{Z} \overset{\rho_2}{\to} \mathbb{Z}_2 \to 0, \quad 0 \to \mathbb{Z}^t \overset{2\times}{\to} \mathbb{Z}^t \overset{\rho_2}{\to} \mathbb{Z}_2 \to 0,$$

which induce the long exact sequences with corresponding Bockstein homomorphisms β and β^t

$$(*) \qquad \begin{aligned} &\ldots \overset{\beta}{\to} H^k(B; \mathbb{Z}) \overset{2\times}{\to} H^k(B; \mathbb{Z}) \overset{\rho_2}{\to} H^k(B; \mathbb{Z}_2) \overset{\beta}{\to} H^{k+1}(B; \mathbb{Z}) \overset{2\times}{\to} \ldots \\ &\ldots \overset{\beta^t}{\to} H^k(B; \mathbb{Z}^t) \overset{2\times}{\to} H^k(B; \mathbb{Z}^t) \overset{\rho_2}{\to} H^k(B; \mathbb{Z}_2) \overset{\beta^t}{\to} H^{k+1}(B; \mathbb{Z}^t) \overset{2\times}{\to} \ldots \end{aligned}$$

From the definition we obtain $H^0(B;\mathbb{Z}^t)=0$ and from the long exact sequence above we get that $\beta^t 1\neq 0$. We shall denote this element by σ.

To shorten our notation we shall denote

$$h^*(B)=H^*(B;\mathbb{Z})\oplus H^*(B;\mathbb{Z}^t).$$

Denote γ and $\gamma^\perp$ the canonical 2- and $(m-2)$-dimensional plane bundles over $G_{m,2}$. Since $\pi_1(G_{m,2})=\mathbb{Z}_2$, we can apply the notation described above to state our main result. The symbols for variables in the respective polynomial rings were choosen in such a way that they suggest the corresponding elements in the relevant cohomology ring.

Theorem A. *Let $m\geq 6$ be even. The cohomology ring*

$$h^*(G_{m,2})=H^*(G_{m,2};\mathbb{Z})\oplus H^*(G_{m,2};\mathbb{Z}^t)$$

is the polynomial ring over $\mathbb{Z}$ generated by the elements

$$\sigma,\quad e,\quad e^\perp,$$

where $\sigma=\beta^t 1\in H^1(G_{m,2};\mathbb{Z}^t)$, $e\in H^2(G_{m,2};\mathbb{Z}^t)$ is the Euler class of the canonical vector bundle γ and $e^\perp\in H^{m-2}(G_{m,2};\mathbb{Z}^t)$ is the Euler class of the vector bundle $\gamma^\perp$, modulo the ideal generated by the following relations

$$\text{(1A)}\qquad 2\sigma=0$$

$$\text{(2A)}\qquad \sigma^{m-1}=\sum_{k=0}^{[(m-6)/4]}\binom{m-2k-3}{2k+2}\sigma^{m-4k-5}e^{2k+2}$$

$$\text{(3A)}\qquad \sigma e^\perp=\sum_{k=0}^{[(m-4)/4]}\binom{m-2k-3}{2k}\sigma^{m-4k-3}e^{2k+1}\neq 0$$

$$\text{(4A)}\qquad ee^\perp=0$$

$$\text{(5A)}\qquad (e^\perp)^2=-e^{m-2}$$

$$\text{(6A)}\qquad \sigma^i e^j=0\quad \textit{if}\quad i+2j\geq 2m-3$$

$$\text{(7A)}\qquad (e^\perp)^3=0$$

Theorem B. *Let $m\geq 5$ be odd. Then the cohomology ring*

$$h^*(G_{m,2})=H^*(G_{m,2};\mathbb{Z})\oplus H^*(G_{m,2};\mathbb{Z}^t)$$

is the polynomial ring over $\mathbb{Z}$ generated by the elements

$$\sigma,\quad e,$$

where $\sigma = \beta^t 1 \in H^1(G_{m,2};\mathbb{Z}^t)$ and $e \in H^2(G_{m,2};\mathbb{Z}^t)$ is the Euler class of the canonical vector bundle γ, modulo the ideal generated by the following relations

$$2\sigma = 0 \tag{1B}$$

$$\sum_{k=0}^{[(m-3)/4]} \binom{m-2k-2}{2k} \sigma^{m-4k-2} e^{2k+1} = 0 \tag{2B}$$

$$\sigma^m = \sum_{k=0}^{[(m-5)/4]} \binom{m-2k-2}{2k+2} \sigma^{m-4k-4} e^{2k+2} \tag{3B}$$

$$\sigma^i e^j = 0 \quad if \quad i + 2j \geq 2m-3 \tag{4B}$$

3. Gysin exact sequence

The derivation of the main result is based on the knowledge of the Gysin exact sequences for cohomology groups with local coefficients.

Let n be a positive integer and let $\xi = (E, B, p)$ be an n-dimensional vector bundle over a connected CW-complex B. Denote SE the total space of the sphere bundle of ξ, E_b the fiber over $b \in B$, and $i_b : E_b \to E$ the inclusion. Then $\{H_n(E_b, SE_b)\}$ form a local system of integer coefficients over B and we will denote it by $\{\mathbb{Z}_\xi\}$. An element $t \in H^n(E, SE; p^*\{\mathbb{Z}_\xi\})$ such that $i_b^* t \in H^n(E_b, SE_b; H_n(E_b, SE_b))$ corresponds for every $b \in B$ to the identity homomorphism in $\mathrm{Hom}(H_n(E_b, SE_b), H_n(E_b, SE_b))$ is called the Thom class of the vector bundle ξ.

Proposition 1. ([Sp, Chapter V, exercise J]) *Let $\mathcal{G}$ be a local system of coefficients over B. Then there is just one Thom class $t \in H^n(E, SE; p^*\{\mathbb{Z}_\xi\})$ and the homomorphism*

$$\Phi_t : H^k(B;\mathcal{G}) \to H^{k+n}(E, SE; p^*\mathcal{G} \otimes p^*\{\mathbb{Z}_\xi\}), \quad \Phi_t(u) = p^*(u) \cup t$$

is an isomorphism.

Let $j : (E, \varnothing) \to (E, SE)$ be the inclusion. The class $e \in H^n(X; \{\mathbb{Z}_\xi\})$ such that $p^*(e) = j^*(t)$ is called the Euler class of the vector bundle ξ. Using the Thom isomorhism and the long exact sequence for the couple (E, SE) with coefficients $\mathcal{G}$ we get the long exact Gysin sequence with local coefficients:

$$\xrightarrow{p^*} H^{q-1}(SE; p^*\mathcal{G}) \xrightarrow{\Delta} H^{q-n}(B; \mathcal{G} \otimes \{\mathbb{Z}_\xi\}) \xrightarrow{\cup e} H^q(B;\mathcal{G}) \xrightarrow{p^*} H^q(SE; p^*\mathcal{G}) \xrightarrow{\Delta}$$

Let $\pi_1(B) = \mathbb{Z}_2$. Then $e(\xi) \in H^n(B;\mathbb{Z})$ if ξ is orientable, and $e(\xi) \in H^n(B;\mathbb{Z}^t)$ if ξ is nonorientable. We can apply the Gysin exact sequence to both local systems of coefficients. For ξ nonorientable we get

$$\xrightarrow{p^*} H^{q-1}(SE; p^*\mathbb{Z}^t) \xrightarrow{\Delta} H^{q-n}(B; \mathbb{Z}) \xrightarrow{\cup e} H^q(B;\mathbb{Z}^t) \xrightarrow{p^*} H^q(SE; p^*\mathbb{Z}^t) \xrightarrow{\Delta}$$

and

$$\xrightarrow{p^*} H^{q-1}(SE; p^*\mathbb{Z}) \xrightarrow{\Delta} H^{q-n}(B; \mathbb{Z}^t) \xrightarrow{\cup e} H^q(B;\mathbb{Z}) \xrightarrow{p^*} H^q(SE; p^*\mathbb{Z}) \xrightarrow{\Delta}$$

Defining

$$h^*(SE) = H^*(SE;p^*\mathbb{Z}) \oplus H^*(SE;p^*\mathbb{Z}^t)$$

with $\mathbb{Z}_2$-ring structure induced from $h^*(B)$, we can write the both Gysin sequences for both oriented and nonoriented vector bundles in the following way:

$$(**) \qquad \xrightarrow{p^*} h^{q-1}(SE) \xrightarrow{\Delta} h^{q-n}(B) \xrightarrow{\cup e} h^q(B) \xrightarrow{p^*} h^q(SE) \xrightarrow{\Delta}$$

The group homomorphism $\Delta : h^k(SE) \to h^{k-n+1}(B)$ has the following property

$$(***) \qquad \Delta(p^*a \cup b) = (-1)^{\deg a} a \cup \Delta(b) \quad \text{for } a \in h^{k-i}(B), b \in h^i(SE)$$

which plays an important role in our computations. In all cases we shall consider, a will be an element of order 2, and therefore the sign will play no role.

The Gysin exact sequence applied to various vector bundles enables us to compute succesively the cohomology ring of $G_{m,1} = \mathbb{R}P^{m-1}$, and the cohomology ring of the sphere bundle $S\gamma$ of the canonical vector bundle γ over $G_{m,2}$. Then using the Gysin sequence for this vector bundle and the knowledge of the ring $H^*(G_{m,2};\mathbb{Z}_2)$ we reveal some generators and relations in the cohomology ring $h^*(G_{m,2})$. The following proposition, which is in fact a special version of the 5-lemma, provides us with an instrument to show that discovered generators and relations describe the ring structure of $h^*(G_{m,2})$ completely.

Proposition 2. *Let $n \geq 2$. Let $p : E \to B$ be an n-dimensional vector bundle over a connected CW-complex B with the Euler class e. Let $\pi_1(B) = \mathbb{Z}_2$.*

Suppose that $\mathcal{A}^*$ and $\mathcal{B}^*$ are $\mathbb{Z} \oplus \mathbb{Z}_2$-graded rings which are trivial in negative integer degrees. Suppose next that there is a ring isomorphism

$$\varphi : \mathcal{A}^* \to h^*(SE)$$

and a ring homomorphism

$$\psi : \mathcal{B}^* \to h^*(B)$$

such that $\psi(y) = e$ for some $y \in \mathcal{B}^*$. Denote μ_y the multiplication by y from the right.

If there are group homomorphisms $\tilde{p}$ and $\tilde{\Delta}$ such that the diagram of groups

$$\begin{tikzcd}
\mathcal{A}^{k+n-1} \arrow[r, "\tilde{\Delta}"] \arrow[d, "\varphi"] & \mathcal{B}^k \arrow[r, "\mu_y"] \arrow[d, "\psi"] & \mathcal{B}^{k+n} \arrow[r, "\tilde{p}"] \arrow[d, "\psi"] & \mathcal{A}^{k+n} \arrow[r, "\tilde{\Delta}"] \arrow[d, "\varphi"] & \mathcal{B}^{k+1} \arrow[d, "\psi"] \\
h^{k+n-1}(SE) \arrow[r, "\Delta"] & h^k(B) \arrow[r, "\cup e"] & h^{k+n}(B) \arrow[r, "p^*"] & h^{k+n}(SE) \arrow[r, "\Delta"] & h^{k+1}(B)
\end{tikzcd}$$

commutes, the upper row exact is exact at the members $\mathcal{B}^{k+n}$, $\mathcal{A}^{k+n}$ and $\mu_y \circ \tilde{\Delta} = 0$, then ψ is a ring isomorphism.

Proof. Use induction and a little bit modified 5-lemma.

4. Preparatory results

Consider the canonical line bundle γ_1 over $G_{m,1} = \mathbb{R}P^{m-1}$, the canonical $(m-1)$-dimensional vector bundle $\gamma_1^\perp$ over the same space, and the canonical 2-dimensional vector bundle $\gamma_2 = \gamma$ over $G_{m,2}$. Their sphere bundles are the following fibrations

$$\mathbb{Z}_2 \longrightarrow S^{m-1} \xrightarrow{\pi_1} \mathbb{R}P^{m-1} \tag{8}$$

$$S^{m-2} \xrightarrow{i} S\gamma \xrightarrow{\pi} \mathbb{R}P^{m-1} \tag{9}$$

$$S^1 \longrightarrow S\gamma \xrightarrow{p} G_{m,2} \tag{10}$$

where π in the second fibration is defined in the following way: For every 2-dimensional vector subspace α in $\mathbb{R}^m$ and $v \in \alpha$, $\|v\| = 1$ there is just one 1-dimensional subspace β in α, $\beta \perp v$. Put

$$\pi(\alpha, v) = \beta.$$

One can easily verify that it is really the sphere bundle corresponding to $\gamma_1^\perp$ over $\mathbb{R}P^{m-1}$.

Another geometric fact, we shall need later, joins the fibrations (9) and (10) in the following way:

Lemma 1. *$i^*p^*\gamma^\perp$ is the tangent vector bundle of S^{m-2}.*

Proof follows from definitions of p and i.

Lemma 2. *Let $m \geq 2$. The cohomology ring*

$$h^*(\mathbb{R}P^{m-1}) \cong \mathbb{Z}[\sigma, A]/\langle 2\sigma, \sigma^m, A^2, \sigma A\rangle,$$

where $\sigma = \beta^t 1 \in H^1(\mathbb{R}P^{m-1}; \mathbb{Z}^t)$, $A \in H^{m-1}(\mathbb{R}P^{m-1}; \mathbb{Z})$ if m is even and $A \in H^{m-1}(\mathbb{R}P^{m-1}; \mathbb{Z}^t)$ if m is odd. In both cases $\pi_1^*(A)$ is twice the generator of $H^{m-1}(S^{m-1}; \mathbb{Z})$.

Proof. The first Stiefel-Whitney class $w_1(\gamma_1)$ is the generator of the cohomology ring $H^*(\mathbb{R}P^{m-1}; \mathbb{Z}_2)$. Utilizing the naturality of the Gysin sequence for the fibration (8) with respect to the coefficients, one can show that reduction mod 2 $\rho_2 : H^1(\mathbb{R}P^{m-1}; \mathbb{Z}^t) \to H^1(\mathbb{R}P^{m-1}; \mathbb{Z}_2)$ is an isomorphism and $H^1(\mathbb{R}P^{m-1}; \mathbb{Z}^t)$ has the generator σ which is the Euler class of γ_1. Now, the Gysin sequence for γ_1 yields the assertion of the lemma.

Proposition 3A. *If $m \geq 6$ is even, then*

$$h^*(S\gamma) \cong \mathbb{Z}[\sigma, b, a]/\langle 2\sigma, \sigma^{m-1}, \sigma b, b^2, a^2\rangle,$$

where $\sigma = \beta^t 1 \in H^1(S\gamma; \mathbb{Z}^t)$, $b \in H^{m-2}(S\gamma; \mathbb{Z}^t)$ is the Euler class of the vector bundle $p^*\gamma^\perp$ and $a \in H^{m-1}(S\gamma; \mathbb{Z})$.

Sketch of the proof. Consider the fibration (9). Since $w_{m-1}(\gamma_1^\perp) = w_1^{m-1}(\gamma_1)$ and $\rho_2(\sigma) = w_1(\gamma_1)$, the Euler class for the Gysin sequence corresponding to (9) is σ^{m-1}.

Now, the Gysin sequence yields subsequently the existence of generators $\sigma \in H^1(S\gamma;\mathbb{Z}^t)$, $b \in H^{m-2}(S\gamma;\mathbb{Z}^t)$ and $a \in H^{m-1}(S\gamma;\mathbb{Z})$ of $h^*(S\gamma)$ such that in this sequence

$$\Delta b = 2, \quad \Delta a = \sigma, \quad \pi^* A = 2a, \quad \Delta(ba) = A,$$

and relations

$$\sigma^{m-1} = 0, \quad \sigma b = 0, \quad b^2 = 0.$$

The other relations are obvious from the fact that $S\gamma$ is a manifold of dimension $2m-3$.

$i^*p^*\gamma_2^\perp$ is a tangent bundle to S^{m-2}, so its Euler class is $2g$ with a generator $g \in H^{m-2}(S^{m-2};\mathbb{Z})$. From the Serre exact sequence for the fibration (9) follows that $i^*b = \pm 2g$. (See [MC].) Hence we can choose b in such a way that $i^*b = 2g$. Consequently, this b is the Euler class od $p^*\gamma^\perp$.

Proposition 3B. *If $m \geq 5$ is odd, then*

$$h^*(S\gamma) \cong \mathbb{Z}[\sigma,\eta,\theta]/\langle 2\sigma, \sigma^{m-1} - 2\eta, \eta^2, \theta^2, \eta\theta\rangle,$$

where $\sigma \in H^1(S\gamma;\mathbb{Z}^t)$, $\eta \in H^{m-1}(S\gamma;\mathbb{Z})$, and $\theta \in H^{2m-3}(S\gamma;\mathbb{Z})$.

The sketch of the proof. To apply the Gysin sequence for (9), we need to determine the Euler class of $\gamma_1^\perp$. Since

$$\rho_2 e(\gamma_1^\perp) = w_{m-1}(\gamma_1^\perp) = w_1^{m-1}(\gamma_1) \neq 0,$$

we get that

$$e(\gamma_1^\perp) = (2r+1)A \in H^{m-1}(\mathbb{R}P^{m-1};\mathbb{Z}^t)$$

for some $r \in \mathbb{Z}$. From Lemma 2 we know that $\pi_1^*(A) = 2\omega$ for some generator $\omega \in H^{m-1}(S^{m-1};\mathbb{Z})$. Simultaneously, $\pi_1^*(\gamma_1^\perp)$ is the tangent bundle to S^{m-1}. Hence

$$(2r+1)2\omega = \pi_1^*(e(\gamma_1^\perp)) = \pm 2\omega,$$

which yields $r=0$ or $r=-1$. So we may assume that we have chosen the generator $A \in H^{m-1}(\mathbb{R}P^{m-1};\mathbb{Z}^t)$ in such a way that $e(\gamma_1^\perp) = A$.

Now, we can apply the Gysin sequence for the fibration (9). It yields $h^k(S\gamma) \cong \mathbb{Z}_2$ with generator σ^k for $k \leq m-2$. Next we obtain

$$0 \longrightarrow H^{m-1}(\mathbb{R}P^{m-1};\mathbb{Z}) \cong \mathbb{Z}_2 \xrightarrow{\pi^*} H^{m-1}(S\gamma;\mathbb{Z}) \xrightarrow{\Delta} H^1(\mathbb{R}P^{m-1};\mathbb{Z}^t) \cong \mathbb{Z}_2 \longrightarrow 0.$$

From the Gysin sequence with $\mathbb{Z}_2$-coefficients we get

$$\rho_2\pi^*(\sigma^{m-1}) = \pi^*(w_1^{m-1}(\gamma_1)) = \pi^*(1 \cup w_{m-1}(\gamma_1^\perp)) = 0.$$

Consequently, $H^{m-1}(S\gamma;\mathbb{Z}) \cong \mathbb{Z}_4$ with generator η, $2\eta = \sigma^{m-1}$, $\Delta(\eta) = \sigma$.

A further application of the Gysin sequence implies the existence of a generator $\theta \in H^{2n-3}(S\gamma;\mathbb{Z})$, $\Delta(\theta) = A$, and the remaining relations of our assertion.

In the following series of lemmas we reveal subsequently the generators and relations in $h^*(G_{m,2})$. The lemmas denoted by A concern the case $m \geq 6$ even, those denoted by B describe the situation for $m \geq 5$ odd.

Let us recall that $\sigma = \beta^t 1 \in H^1(G_{m,2};\mathbb{Z}^t)$. Next let $e \in H^2(G_{m,2};\mathbb{Z}^t)$ be the Euler class of the vector bundle γ, and let $e^\perp \in H^{m-2}(G_{m,2};\mathbb{Z}^t)$ be the Euler class of its complement $\gamma^\perp$.

Lemma 3A. *Let m be even and $k \le m-3$. Then the group $h^k(G_{m,2})$ is isomorphic to the homogeneous component of degree k in the ring $\mathbb{Z}[\sigma, e]/\langle 2\sigma\rangle$, where $\deg\sigma = 1$, $\deg e = 2$.*

Proof can be carried out using the Gysin exact sequence for the fibration (10) and the knowledge of $h^*(S\gamma)$, realizing that $\Delta : h^k(S\gamma) \to h^{k-1}(G_{m,2})$ is zero for $k \le m-3$.

Lemma 3B. *Let m be odd and $k \le m-2$. Then the group $h^k(G_{m,2})$ is isomorphic to the homogeneous component of degree k in the ring $\mathbb{Z}[\sigma, e]/\langle 2\sigma\rangle$, where $\deg\sigma = 1$, $\deg e = 2$.*

Proof is the same as for the previous lemma.

Lemma 4A. *Let m be even. Then the group $h^{m-2}(G_{m,2})$ is isomorphic to the homogeneous component of degree $m-2$ in the ring $\mathbb{Z}[\sigma, e, e^\perp]/\langle 2\sigma\rangle$, where $\deg\sigma = 1$, $\deg e = 2$, $\deg e^\perp = m-2$.*

Moreover, in the Gysin sequence for (10)

$$\Delta b = 0, \text{ and } p^*(e^\perp) = b.$$

Proof follows from the Gysin sequence since $\Delta : h^{m-2}(S\gamma) \to h^{m-3}(G_{m,2})$ is again zero. The reason consists in the fact that $b \in h^{m-2}(S\gamma)$ is the Euler class of $p^*\gamma^\perp$. Cosequently, $b = p^*(e^\perp)$ and $\Delta(b) = 0$.

Lemma 4B. *Let m be odd. Then*

$$e^\perp = \sum_{k=0}^{[(m-3)/4]} \binom{m-2k-2}{2k} \sigma^{m-4k-2} e^{2k}.$$

Proof. From the knowledge of $H^*(G_{m,2};\mathbb{Z}_2)$ we get

$$\begin{aligned} w_{m-2}(\gamma^\perp) &= \sum_{k=0}^{(m-3)/2} \binom{m-k-2}{k} w_1^{m-2k-2}(\gamma) w_2^k(\gamma) = \\ &= \sum_{k=0}^{[(m-3)/4]} \binom{m-2k-2}{2k} w_1^{m-4k-2}(\gamma) w_2^{2k}(\gamma) \end{aligned}$$

since $\binom{n}{i} \equiv 0 \mod 2$ for n even and i odd.

Now, Lemma 3B says that $H^{m-2}(G_{m,2};\mathbb{Z}^t)$ consists only of elements of order 2. That is why $\rho_2 : H^{m-2}(G_{m,2};\mathbb{Z}^t) \to H^{m-2}(G_{m,2};\mathbb{Z}_2)$ is a monomorhism and it yields the desired formula for $e^\perp$.

Lemma 5A. *Let m be even. Then*

$$e \cup e^\perp = 0,$$

and in the Gysin sequence for the fibration (10)

$$\Delta(a) = e^\perp.$$

Proof. The first formula is obvious. The knowledge of $h^{m-1}(S\gamma)$ and $h^{m-2}(G_{m,2})$ and the Gysin exact sequence implies $\Delta(a) = \pm e^\perp$. If necessary we can change the sign of the generator a, and we get $\Delta(a) = e^\perp$.

Lemma 5B. *Let m be odd. Then*

$$e \cup e^{\perp} = 0,$$

and in the Gysin sequence for the fibration (10)

$$\Delta(\eta) = e^{\perp}.$$

Proof follows the same lines as that of Lemma 5A.

Lemma 6A. *Let m be even. Then $h^{m-1}(G_{m,2}) \cong \mathbb{Z}_2^{(m-2)/2}$ with generators*

$$\sigma^{m-3}e, \sigma^{m-5}e^2, \dots, \sigma e^{(m-2)/2}.$$

Proof. Use again the Gysin sequence. $h^{m-1}(S\gamma) = \mathbb{Z}$ with generator a, $\Delta(a) = e^{\perp}$. Hence $p^* : h^{m-1}(G_{m,2}) \to h^{m-1}(S\gamma)$ is zero and $\cup e : h^{m-3}(G_{m,2}) \to h^{m-1}(G_{m,2})$ is an isomorphism.

Lemma 6B. *Let m be odd. Then the group $h^{m-1}(G_{m,2})$ is isomorphic to the homogeneous component of degree $m-1$ in the ring $\mathbb{Z}[\sigma, e]/\langle 2\sigma \rangle$.*

Proof. $h^{m-1}(S\gamma) \cong \mathbb{Z}_4$ with the generator η, $\Delta(\eta) = e^{\perp}$. Hence the image of $p^* : h^{m-1}(G_{m,2}) \to h^{m-1}(S\gamma)$ is $\mathbb{Z}_2$ with generator $\sigma^{m-1} = 2\eta$. The rest follows from the Gysin sequence.

Lemma 7A. *For m even*

$$\sigma^{m-1} = \sum_{k=0}^{[(m-6)/4]} \binom{m-2k-3}{2k+2} \sigma^{m-4k-5} e^{2k+2}.$$

Proof. In $H^*(G_{m,2}; \mathbb{Z}_2)$ we can express $w_{m-2}(\gamma^{\perp})$ and $w_{m-3}(\gamma^{\perp})$ as polynomials in $w_1 = w_1(\gamma)$ and $w_2 = w_2(\gamma)$. Substituting in the relation

$$w_1(\gamma) w_{m-2}(\gamma^{\perp}) + w_2(\gamma) w_{m-3}(\gamma^{\perp}) = 0,$$

we get

$$\begin{aligned} w_1^{m-1} &= \sum_{k=0}^{(m-4)/2} \binom{m-k-2}{k+1} w_1^{m-2k-3} w_2^{k+1} = \\ &= \sum_{k=0}^{[(m-6)/4]} \binom{m-2k-3}{2k+2} w_1^{m-4k-5} w_2^{2k+2} \end{aligned}$$

since $\binom{n}{i} \equiv 0 \mod 2$ for n even, i odd. Now the injectivity of ρ_2 on $H^{m-1}(G_{m,2}; \mathbb{Z}^t)$ yields the formula for σ^{m-1}.

Lemma 8A. *For m even*

$$\sigma e^{\perp} = \sum_{k=0}^{[(m-4)/4]} \binom{m-2k-3}{2k} \sigma^{m-4k-3} e^{2k+1} \neq 0.$$

Proof. We have

$$\rho_2(\sigma e^{\perp}) = \rho_2(\sigma)\rho_2(e^{\perp}) = w_1 w_{m-2}(\gamma^{\perp}) = w_2 w_{m-3}(\gamma^{\perp}) =$$
$$= \sum_{k=0}^{(m-4)/2} \binom{m-k-3}{k} w_1^{m-2k-3} w_2^{k+1} = \sum_{k=0}^{[(m-4)/4]} \binom{m-2k-3}{2k} w_1^{m-4k-3} w_2^{2k+1}.$$

The formula for $\sigma e^{\perp}$ follows from injectivity of ρ_2 on $H^{m-1}(G_{m,2};\mathbb{Z})$.

Lemma 8B. *For m odd*

$$\sum_{k=0}^{[(m-3)/4]} \binom{m-2k-2}{2k} \sigma^{m-4k-2} e^{2k+1} = 0$$

$$\sigma^m = \sum_{k=0}^{[(m-5)/4]} \binom{m-2k-2}{2k+2} \sigma^{m-4k-4} e^{2k+2}$$

Proof. The first formula is an immediate consequence of lemmas 4B and 5B. From the relation

$$w_1(\gamma)w_{m-2}(\gamma^{\perp}) + w_2(\gamma)w_{m-3}(\gamma^{\perp}) = 0,$$

when multiplying by $w_1(\gamma)$, we obtain

$$w_1^m = \sum_{k=0}^{(m-3)/2} \binom{m-k-2}{k+1} w_1^{m-2k-2} w_2^{k+1}$$
$$= \sum_{k=0}^{[(m-5)/4]} \binom{m-2k-2}{2k+2} w_1^{m-4k-4} w_2^{2k+2}$$

since $\binom{n}{i} \equiv 0 \mod 2$ for n even and i odd.

Now let us consider the Gysin sequence

$$0 = H^{m-1}(S\gamma;\mathbb{Z}^t) \xrightarrow{\Delta} H^{m-2}(G_{m,2};\mathbb{Z}) \xrightarrow{\cup e} H^m(G_{m,2};\mathbb{Z}^t) \xrightarrow{p^*}$$
$$\xrightarrow{p^*} H^m(S\gamma;\mathbb{Z}^t) \xrightarrow{\Delta} H^{m-1}(G_{m,2};\mathbb{Z})$$

Obviously, $\cup e$ is a monomorphism. Further $H^m(S\gamma;\mathbb{Z}^t) = \mathbb{Z}_2$ with generator $\sigma\eta$. Because $\Delta(\sigma\eta) = -\sigma e^{\perp} \neq 0$, we can see that $p^* = 0$, and $\cup e$ is also an epimorphism. This shows that $2H^m(G_{m,2};\mathbb{Z}^t) = 0$. Consequently $\rho_2 : H^m(G_{m,2};\mathbb{Z}^t) \to H^m(G_{m,2};\mathbb{Z}^t)$ is an isomorphism. The second formula is now obvious.

Lemma 9. *For m even or odd, in the Gysin exact sequence for (10)*

$$\Delta : h^{2m-3}(S\gamma) \to h^{2m-4}(G_{m,2})$$

is an isomorphism.

Proof follows immediately from dimension reasons.

Lemma 10A. *Let m be even. Every element of $h^*(G_{m,2})$ can be expressed as a polynomial in σ, e and $e^\perp$.*

Proof. First, from the previous lemma $H^{2m-4}(G_{m,2}; \mathbb{Z}^t) \cong \mathbb{Z}_2$ with generator $\Delta(\sigma^{m-2}a) = \sigma^{m-2}e^\perp \neq 0$.

The statement of Lemma 10A has been already proved for $h^k(G_{m,2})$ with $k \leq m-1$. Suppose it is true for all $k < l$, $l \geq m$. Let $x \in h^l(G_{m,2})$. Since

$$\sigma b = 0 \quad \text{and} \quad \Delta(\sigma^{l-m+1}a) = \sigma^{l-m+1}e^\perp \neq 0,$$

we get that $p^*(x) = 0$, which implies that $x = ye$ for some $y \in h^{l-2}(G_{m,2})$.

Lemma 10B. *Let m be odd. Every element of $h^*(G_{m,2})$ can be expressed as a polynomial in σ and e.*

Proof. From Lemma 9 we get that $h^{2m-4}(G_{m,2}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ with generators

$$\Delta(\sigma^{m-2}\eta) = \sigma^{m-2}e^\perp \neq 0 \quad \text{and} \quad \Delta(\theta) \neq 0.$$

The rest follows the same lines as the proof of the previous lemma.

Lemma 11A. *For m even*

$$e^{\perp 2} = -e^{m-2}.$$

Proof. ¿From Lemma 9 we obtain that the generator of $H^{2m-4}(G_{m,2}; \mathbb{Z}) \cong \mathbb{Z}$ is

$$\Delta(ba) = e^\perp \Delta(a) = \pm e^{\perp 2}.$$

Simultaneously, $p^*(e^{\perp 2}) = b^2 = 0$. That is why $e^{\perp 2} \in \mathrm{im}(\cup e)$. From the point of view of Lemma 10A, the only possibility is $e^{\perp 2} = \pm e^{m-2}$.

In order to find the right sign we shall use the double covering $\tilde{\pi} : \tilde{G}_{m,2} \to G_{m,2}$, where $\tilde{G}_{m,2}$ denote the Grassmann manifold of oriented planes. We denote $\tilde{e} = \tilde{\pi}^* e$. According to [V], Theorem 5, we have $\tilde{\pi}^* e^\perp = \tilde{e}^{(m-2)/2} + 2f$, where $f^2 = 0$. Then

$$\tilde{\pi}^* e^{\perp 2} + \tilde{e}^{(m-2)} = (\tilde{\pi}^* e^\perp - \tilde{e}^{(m-2)/2})^2 = 4f^2 = 0,$$

which implies that $e^{\perp 2} = -e^{m-2}$.

Lemma 11B. *Let m be odd. Then $H^{2m-4}(G_{m,2}; \mathbb{Z}^t) \cong \mathbb{Z}$ with generator e^{m-2}. Moreover, there is $\Delta(\theta) = e^{m-2}$.*

Proof. From Lemma 9 we have $H^{2m-4}(G_{m,2}; \mathbb{Z}^t) \cong \mathbb{Z}$ with generator $\Delta(\theta)$. Since $H^{2m-4}(S\gamma); \mathbb{Z}^t) = 0$, $\Delta(\theta) \in \mathrm{Im} \cup e$ and the only possibility (due to Lemma 10B) is $\Delta(\theta) = \pm e^{m-2}$. Changing again, if necessary, the sign of θ, we get $\Delta(\theta) = e^{m-2}$.

5. Proof of Theorems

In this section we use the previous results to verify the assumptions of Proposition 2 and to prove Theorems A and B.

First, we shall treat the case $m \geq 6$ even. In Proposition 2 consider the 2-dimensional vector bundle γ over $G_{m,2}$ and put

$$\mathcal{A}^* = \mathbb{Z}[s, u, t]/\langle 2s, s^{m-1}, su, u^2, t^2\rangle,$$

where $\deg s = (1, 1)$, $\deg u = (m-2, 1)$ and $\deg t = (m-1, 0)$,

$$\mathcal{B}^* = \mathbb{Z}[x, y, z]/\mathcal{I}$$

where $\deg x = (1, 1)$, $\deg y = (2, 1)$, $\deg z = (m-2, 1)$ and $\mathcal{I}$ is the ideal generated by the relations (1A) – (7A) of Theorem A with σ, e and $e^{\perp}$ substituted by x, y, and z, respectively.

There is a unique ring homomorphism $\tilde{p} : \mathcal{B}^* \to \mathcal{A}^*$ such that

$$\tilde{p}(x) = s, \quad \tilde{p}(y) = 0, \quad \tilde{p}(z) = u.$$

$\mathcal{A}^*$, considered as a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded group, has generators

$$s^i \text{ for } 0 \leq i \leq m-2, \quad s^i t \text{ for } 0 \leq i \leq m-2, \quad u, \quad ut.$$

Obviously, there exists a homomorphism of bidegree $(-1, 1)$ $\tilde{\Delta} : \mathcal{A}^* \to \mathcal{B}^*$ such that

$$\tilde{\Delta}(s^i) = 0, \quad \tilde{\Delta}(s^i t) = x^i z, \quad \tilde{\Delta}(u) = 0, \quad \tilde{\Delta}(ut) = z^2.$$

According to Lemmas 3A, 4A, 5A, 6A, 7A and 8A the ring homomorphisms $\varphi : \mathcal{A}^* \to h^*(S\gamma)$ and $\psi : \mathcal{B}^* \to h^*(G_{m,2})$ such that

$$\varphi(s) = \sigma, \quad \varphi(u) = b, \quad \psi(t) = a,$$
$$\psi(x) = \sigma, \quad \psi(y) = e, \quad \psi(z) = e^{\perp}$$

are well defined. Proposition 3A ensures us that φ is an isomorphism. According to Lemma 4A, 5A, the properties of p^* and the property (***) of Δ the diagram of Proposition 2 commutes.

It suffices to show the assumptions of Proposition 2 concerning the upper row of the diagram. From definitions it is easy to see that $\tilde{\Delta} \circ \tilde{p} = 0$, $\tilde{p} \circ \mu_y = 0$ and $\mu_y \circ \tilde{\Delta} = 0$.

Next, $\operatorname{Ker} \tilde{\Delta}$ is generated by elements $s^i u^j = \tilde{p}(x^i z^j)$ hence $\operatorname{Ker} \tilde{\Delta} = \operatorname{Im} \tilde{p}$. $\operatorname{Ker} \tilde{p}$ is generated by elements $x^i y^j z^k$ for $j \neq 0$. Consequently, $x^i y^j z^k = \mu_y(x^i y^{j-1} z^k)$ and $\operatorname{Ker} \tilde{p} = \operatorname{Im} \mu_y$.

Now, Proposition 2 implies that ψ is an isomorphism, which completes the proof of Theorem A.

Now we shall deal with $m \geq 5$ odd. In Proposition 2 consider again the 2-dimensional vector bundle γ over $G_{m,2}$ and put

$$\mathcal{A}^* = \mathbb{Z}[s, u, t]/\langle 2s, s^{m-1} - 2u, u^2, t^2, st, ut\rangle,$$

where $\deg s = (1,1)$, $\deg u = (m-1,0)$ and $\deg t = (2m-3,0)$,

$$\mathcal{B}^* = \mathbb{Z}[x,y]/\mathcal{I}$$

where $\deg x = (1,1)$, $\deg y = (2,1)$, and $\mathcal{I}$ is the ideal generated by the relations (1B) – (4B) of Theorem B with σ and e substituted by x and y, respectively. There is a unique ring homomorphism $\tilde{p} : \mathcal{B}^* \to \mathcal{A}^*$ such that

$$\tilde{p}(x) = s, \quad \tilde{p}(y) = 0.$$

$\mathcal{A}^*$, considered as a $\mathbb{Z} \oplus \mathbb{Z}_2$-graded group, has generators

$$s^i \text{ for } 0 \le i \le m-2, \quad s^i u \text{ for } 0 \le i \le m-2, \quad t.$$

Obviously, there exists a homomorphism of bidegree $(-1,1)$ $\tilde{\Delta} : \mathcal{A}^* \to \mathcal{B}^*$ such that

$$\tilde{\Delta}(s^i) = 0, \quad \tilde{\Delta}(s^i u) = \sum_{k=0}^{[(m-3)/4]} \binom{m-2k-2}{2k} x^{m-4k-2} y^{2k} \quad \tilde{\Delta}(t) = y^{m-2}.$$

Further, according to Lemmas 3B, 4B and 5B, the ring homomorphisms $\varphi : \mathcal{A}^* \to h^*(S\gamma)$ and $\psi : \mathcal{B}^* \to h^*(G_{m,2})$ such that

$$\varphi(s) = \sigma, \quad \varphi(u) = \eta, \quad \varphi(t) = \theta,$$
$$\psi(x) = \sigma, \quad \psi(y) = e$$

are well defined. Due to Proposition 3B, φ is an isomorphism. According to Lemma 5B and Lemma 11B, the properties of p^* and the property (***) of Δ the diagram of Proposition 2 commutes in this case, as well.

It remains to prove the assumptions of Proposition 2 concerning the upper row of the diagram. From definitions it is easy to see that $\tilde{\Delta} \circ \tilde{p} = 0$, $\tilde{p} \circ \mu_y = 0$ and $\mu_y \circ \tilde{\Delta} = 0$.

Further, $\operatorname{Ker} \tilde{\Delta}$ is generated by elements $s^i = \tilde{p}(x^i)$ hence $\operatorname{Ker} \tilde{\Delta} = \operatorname{Im} \tilde{p}$, and $\operatorname{Ker} \tilde{p}$ is generated by elements $x^i y^j$ for $j \neq 0$. Consequently, $x^i y^j = \mu_y(x^i y^{j-1})$ and $\operatorname{Ker} \tilde{p} = \operatorname{Im} \mu_y$.

Now, Proposition 2 yields that ψ is an isomorphism.

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