OBSTRUCTION THEORY ON 8-MANIFOLDS

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ABSTRACT. This paper gives a uniform, self-contained, and fairly direct approach to a variety of obstruction-theoretic problems on 8-manifolds. We give necessary and sufficient cohomological criteria for the existence of complex and quaternionic structures on 8-dimensional vector bundles and for the reduction of the structure group of such bundles to U(3) by the homomorphism $U(3) \to O(8)$ given by the Lie algebra representation of PU(3).

1. Introduction

Let M be a connected, closed, smooth, 8-dimensional, spin^c manifold and let ξ be an 8-dimensional oriented real vector bundle over M admitting a spin^c-structure. We consider, for various compact Lie groups G and homomorphisms $\rho : G \to SO(8)$, the problem of reducing the structure group of ξ from SO(8), via ρ, to G. In each case we shall obtain, in terms of the cohomology of M and cohomology characteristic classes of ξ and M, necessary and sufficient conditions for the reduction. The homomorphisms ρ that we examine are:

(i) the standard inclusion $U(4) \subseteq SO(8)$;
(ii) the homomorphism $U(3) \to SO(8)$ determined by the adjoint representation of the quotient PU(3) of U(3) by its centre;
(iii) the compositions of the standard inclusions $Sp(2) \times_{\{\pm 1\}} U(1) \subseteq U(4) \subseteq SO(8)$;
(iv) the standard inclusion $Sp(2) \times_{\{\pm 1\}} Sp(1) \subseteq SO(8)$;
(v) the composition of the defining map and the standard inclusion $Spin^c(k) \to SO(k) \subseteq SO(8)$ for $k = 6, 5, 4, 3$.

The results concerning almost complex structures on M (case (i) for $\xi = \tau M$, the tangent bundle of M) were obtained, using other methods, by Heaps [15] and by Thomas [17]; see also Geiges and Müller [14]. In [4], [5], [6], [8] Čadek and Vanžura dealt with (i) for spin vector bundles and with special cases of (iii) and (iv). The result in case (ii) on reduction to U(3) through the adjoint representation of PU(3) was obtained by Crabb in response to a question of N.J. Hitchin and F. Witt; see [18]. Conditions for existence of a 4-field, (v), are also due to Čadek and Vanžura [7].

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The results of this paper apply to a broader class of vector bundles than was previously considered. In some cases we need make no further restriction on ξ; for the more difficult ones we need only additional assumptions on $w_2(\xi)$ and $w_2(M)$. The derivation of the more delicate results involves the use of real KO-theory and twisted quaternionic K-theory, which may be of independent interest.

We fix a spin^c structure on M, with characteristic class $c \in H^2(M;\mathbb{Z})$ (lifting $w_2(M)$), and an orientation for the vector bundle ξ.

It is, at first sight, somewhat surprising that one should be able to obtain purely cohomological criteria for reduction of the structure group. However, there are certain special features of the problem that make its solution tractable.

1.1. Two oriented 8-dimensional real vector bundles ξ and ξ' over M such that $[\xi] = [\xi'] \in K\mathrm{O}^0(M)$ are isomorphic as oriented bundles if and only if $e(\xi)[M] = e(\xi')[M] \in \mathbb{Z}$.

This reduces the obstruction theory to a KO-theoretic, stable problem.

1.2. The spin^c structure for M allows us to split off the 'top cell' in real and complex K-theory. Let $B \subseteq M$ be an embedded open disc of dimension 8, that is, a tubular neighbourhood of a point. Then we have short exact sequences:

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z} = K\mathrm{O}^0(M, M-B) & \underset{\longleftarrow - -}{\longrightarrow} & K\mathrm{O}^0(M) & \rightarrow & K\mathrm{O}^0(M-B) & \rightarrow & 0 \\ & & \mathrm{id}\ \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & \mathbb{Z} = K^0(M, M-B) & \underset{\longleftarrow - -}{\longrightarrow} & K^0(M) & \rightarrow & K^0(M-B) & \rightarrow & 0 \end{array}$$

split by the spin^c index:

$$\mathrm{ind} : K^0(M) \rightarrow \mathbb{Z}.$$

Note that $K\mathrm{O}^1(M, M-B)$ and $K^1(M, M-B)$ are zero. Thus the splitting of $K^0(M)$ induces a splitting of $K\mathrm{O}^0(M)$ as a direct sum $K\mathrm{O}^0(M-B) \oplus \mathbb{Z}$, and a real vector bundle ξ over M is determined stably by its restriction to $M-B$ and the image of $[\xi] \in K\mathrm{O}^0(M)$ under the spin^c index.

This leaves us with an obstruction theory problem on the essentially 7-dimensional space $M-B$.

1.3. In a cell decomposition of M as a finite complex (in which B is an open cell) the restriction map

$$K\mathrm{O}^0(M-B) \rightarrow K\mathrm{O}^0(M^{(4)})$$

to the 4-skeleton is injective. For an argument using the Atiyah-Hirzebruch spectral sequence shows that the restriction map $K\mathrm{O}^0(M-B) \rightarrow K\mathrm{O}^0(M^{(7)})$ is an isomorphism, since $H^8(M-B;\mathbb{Z}) = 0$ and $K\mathrm{O}^1(M-B, M^{(7)}) = 0$, and the restriction $K\mathrm{O}^0(M^{(7)}) \rightarrow K\mathrm{O}^0(M^{(4)})$ is injective, because $K\mathrm{O}^0(M^{(5)}, M^{(4)}) = 0$, $K\mathrm{O}^0(M^{(6)}, M^{(5)}) = 0$ and $K\mathrm{O}^0(M^{(7)}, M^{(6)}) = 0$.

This means that we can distinguish stable bundles over $M-B$ by calculations in dimension 4 and below.

The cohomological classes which can be realized as the Chern classes of a complex vector bundle over M are described in Proposition 3.2. Conditions for the existence of a complex structure on the vector bundle ξ are derived in Section 4 and stated as Proposition 4.1. The result on the reduction of the structure group from SO(8) to

U(3) is given as Proposition 6.2. Sections 7, 8 and 10 deal with almost quaternionic structures; the main results appear as Propositions 8.2, 10.1 and 10.4. Reduction to Spin(3) and to Spin(4) is considered in Section 9.

Almost fifty years ago Hirzebruch and Hopf [16] investigated the reduction of the structure group of an oriented 4-manifold from SO(4) to SO(2) × SO(2) and to U(2). This paper follows in the tradition of their work.

2. The spin characteristic class in dimension 4

In this section we recall some fairly standard facts about the spin characteristic class in dimension 4 and the classification of spin and spin^c bundles in low dimensions. We shall write $\rho_2 : H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{F}_2)$ for reduction (mod 2).

An elementary computation shows that the Chern classes c_2 and c_3 of a complex vector bundle with $c_1 = 0$ satisfy:

$$\mathrm{Sq}^2(\rho_2 c_2) = \rho_2 c_3. \tag{2.1}$$

Let F be the homotopy fibre of

$$\mathrm{Sq}^2 \circ \rho_2 + \rho_2 : K(\mathbb{Z},4) \times K(\mathbb{Z},6) \to K(\mathbb{F}_2,6).$$

By (2.1), we can lift the map $(c_2,c_3) : B\mathrm{SU}(\infty) \to K(\mathbb{Z},4) \times K(\mathbb{Z},6)$ to a map $B\mathrm{SU}(\infty) \to F$.

Lemma 2.1. *The map $B\mathrm{SU}(\infty) \to F$ above induces an isomorphism in homotopy groups $\pi_i(B\mathrm{SU}(\infty)) \to \pi_i(F)$ for $i \le 7$ (and a surjection for $i = 8$).*

Proof. We have $\pi_4(F) = \mathbb{Z}$, $\pi_6(F) = 2\mathbb{Z}$, and $\pi_i(F) = 0$ otherwise. The Chern class $c_2 : \widetilde{K}^0(S^4) = \mathbb{Z} \to H^4(S^4;\mathbb{Z}) = \mathbb{Z}$ is an isomorphism, and $c_3 : \widetilde{K}^0(S^6) = \mathbb{Z} \to H^6(S^6;\mathbb{Z}) = \mathbb{Z}$ is multiplication by 2. □

Recall that M is a closed, connected, 8-dimensional, spin^c-manifold and that $B \subseteq M$ is an open disc. The complex K-group $K^0(M - B) = [(M-B)_+;\, \mathbb{Z} \times B\mathrm{U}(\infty)]$ splits as a direct sum

$$K^0(M-B) = \mathbb{Z} \oplus H^2(M;\mathbb{Z}) \oplus [(M-B)_+;\, B\mathrm{SU}(\infty)],$$

where the summand $\mathbb{Z} = [(M-B)_+;\, \mathbb{Z}]$, corresponding to the trivial bundles, is given by the dimension and the summand $H^2(M;\mathbb{Z}) = [(M-B)_+;\, B\mathrm{U}(1)]$, corresponding to the line bundles, is given by c_1. (The subscript + denotes a disjoint basepoint, and the brackets $[-;-]$ indicate the set of pointed homotopy classes.)

According to Lemma 2.1, the map $[(M-B)_+;\, B\mathrm{SU}(\infty)] \to [(M-B)_+;\, F]$ is surjective (and, in fact, bijective). This completes the description of the Chern classes of SU-bundles over $M - B$. (See [19] for a similar description in the case of real vector bundles.)

Proposition 2.2. *The image of the mapping*

$$(c_2,c_3) : [(M-B)_+;\, B\mathrm{SU}(\infty)] \to H^4(M;\mathbb{Z}) \oplus H^6(M;\mathbb{Z})$$

is the set $\{(u,v) \mid \mathrm{Sq}^2\rho_2(u) = \rho_2(v)\}$.

By adding (or subtracting) a complex line bundle with $c_1 = l$ one obtains the following generalization.

Corollary 2.3. *The image of the mapping*

$$(c_1, c_2, c_3) : [(M - B)_+;\ BU(\infty)] \to H^2(M;\ \mathbb{Z}) \oplus H^4(M;\ \mathbb{Z}) \oplus H^6(M;\ \mathbb{Z})$$

is the set $\{(l, u, v) \mid \mathrm{Sq}^2\rho_2(u) + \rho_2(lu) = \rho_2(v)\}$.

Lemma 2.4. *The inclusion map*

$$\mathrm{SU}(\infty) \to \mathrm{Spin}(\infty).$$

induces an isomorphism $\pi_i(B\mathrm{SU}(\infty)) \to \pi_i(B\mathrm{Spin}(\infty))$ *for* $i \le 5$.

Proof. For restriction from $\mathbb{C}$ to $\mathbb{R}$ gives an isomorphism $K^{-4}(*) = \mathbb{Z} \to K\mathrm{O}^{-4}(*) = \mathbb{Z}$; the homotopy groups are zero for $i < 4$ and for $i = 5$. □

Definition. The spin characteristic class in $H^4(B\mathrm{Spin}(\infty);\ \mathbb{Z})$ corresponding to $-c_2 \in H^4(B\mathrm{SU}(\infty);\ \mathbb{Z})$ is denoted by q_1. For an oriented real vector bundle ξ admitting a spin structure, that is, such that $w_2(\xi) = 0$, the characteristic class q_1 is, as we show below, independent of the choice of spin structure and we write it as $q_1(\xi) \in H^4(M;\ \mathbb{Z})$.

To see the independence we may argue as follows. Over the 5-skeleton there is a complex vector bundle ζ with $c_1(\zeta) = 0$ that is isomorphic to $\xi \mid M^{(5)}$ as a real bundle. We must verify that $c_2(\zeta)$ does not depend on the choice of an SU-structure. If ζ' is another such bundle, then $\zeta - \zeta'$ is stably trivial as a real bundle and so, considered as a map $M^{(5)} \to B\mathrm{SU}$, lifts to SO/SU. We have to show that $c_2(\zeta - \zeta') = 0$. But $\rho_2 : H^4(\mathrm{SO/SU};\ \mathbb{Z}) \to H^4(\mathrm{SO/SU};\ \mathbb{F}_2) = \mathbb{F}_2$ is an isomorphism and $\rho_2(c_2(\zeta - \zeta')) = w_4(\zeta - \zeta') = 0$..

From the definition it is immediate that, for a spin bundle ξ, we have $2q_1(\xi) = p_1(\xi)$ and $\rho_2(q_1(\xi)) = w_4(\xi)$. Note also that q_1 is additive: $q_1(\xi \oplus \xi') = q_1(\xi) + q_1(\xi')$ for two bundles ξ and ξ' admitting spin structures. A complex vector bundle ζ admits a spin structure if and only if $c_1(\zeta)$ is divisible by 2, say equal to $2m$ and then $q_1(\zeta) = 2m^2 - c_2(\zeta)$ (which does not depend on the choice of m).

We next extend this definition formally to a characteristic class for spin^c-bundles. Let ξ now be a real vector bundle admitting a spin^c structure. Thus $w_2(\xi)$ is the reduction (mod 2) $\rho_2(l)$ of an integral class $l \in H^2(M;\ \mathbb{Z})$.

Definition. For an orientable vector bundle ξ and class $l \in H^2(M;\ \mathbb{Z})$ such that $\rho_2(l) = w_2(\xi)$ we define

$$q_1(\xi;\ l) = q_1(\xi - \lambda) \in H^4(M;\ \mathbb{Z}),$$

where λ is a complex line bundle with $c_1(\lambda) = l$.

It is elementary to verify that:

$$2q_1(\xi;\ l) = p_1(\xi) - l^2; \qquad \rho_2(q_1(\xi;\ l)) = w_4(\xi);$$
$$q_1(\xi;\ l + 2m) = q_1(\xi;\ l) - 2lm - 2m^2 \text{ for } m \in H^2(M;\ \mathbb{Z}),$$

and, if ζ is a complex bundle with $c_1(\zeta) = l$, then $q_1(\zeta;\ l) = -c_2(\zeta)$.

This invariant is enough to distinguish oriented vector bundles over $M - B$.

Proposition 2.5. *Consider two oriented (stable) vector bundles* ξ *and* ξ' *over* $M - B$ *with* $w_2(\xi) = w_2(\xi') = \rho_2(l)$, *where* $l \in H^2(M;\ \mathbb{Z})$. *Then* ξ *and* ξ' *are stably isomorphic if and only if* $q_1(\xi;\ l) = q_1(\xi';\ l)$.

Proof. For $\xi - \xi'$ is a spin bundle with $q_1(\xi - \xi') = 0$. So $\xi - \xi'$ is trivial over $M^{(4)}$, and hence, by (1.3), over $M - B$. □

Remark 2.6. With the benefit of hindsight it is now easy to rederive the main results of Hirzebruch and Hopf [16]. Let η be an oriented 4-dimensional real vector bundle with $w_2(\eta)$ the reduction of an integral class over a closed, connected, oriented 4-manifold N. Then η is isomorphic to a direct sum $\mu_1 \oplus \mu_2$ of oriented 2-dimensional bundles with $e(\mu_i) = m_i$ if and only if $w_2(\eta) = \rho_2(l)$, where $l = m_1 + m_2$, $e(\eta) = m_1 m_2$ and $p_1(\eta) = m_1^2 + m_2^2$ $(= l^2 - 2m_1 m_2)$. For we have only to compare the classes $q_1(\eta;\, l)$ with $q_1(\mu_1 \oplus \mu_2;\, l)$ and $e(\eta)$ with $e(\mu_1 \oplus \mu_2)$. Similarly, η has a complex structure (compatible with its given orientation) with $c_1 = l$ if and only if $w_2(\eta) = \rho_2(l)$ and $p_1(\eta) + 2e(\eta) = l^2$.

3. The spin^c index

We turn next to the cohomological description of the fundamental splitting (1.2). The spin^c index is given by

$$y \in K^0(M) \mapsto \text{ind } y = (\mathrm{e}^{c/2}\hat{\mathcal{A}}(\tau M)\mathrm{ch}(y))[M] \in \mathbb{Z}. \tag{3.1}$$

Here the Chern character, $\mathrm{ch}(y)$, of y has the explicit expansion:

$$\begin{aligned} \mathrm{ch}(y) = \dim y + c_1 + (c_1^2 - 2c_2)/2 + (c_1^3 - 3c_1c_2 + 3c_3)/6 + \\ (c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4)/24 + \dots, \end{aligned} \tag{3.2}$$

where the c_i are the Chern classes of y, and

$$\hat{\mathcal{A}}(\tau M) = 1 - p_1(\tau M)/24 + (-4p_2(\tau M) + 7p_1^2(\tau M))/5760 + \cdots .$$

The Chern character of the complexification of a real class $x \in K\mathrm{O}^0(M)$ is written in terms of the Pontrjagin classes p_i of x as:

$$\dim x + p_1 + (p_1^2 - 2p_2)/12 + \dots; \tag{3.3}$$

for $p_1(x) = -c_2(y)$ and $p_2(x) = c_4(y)$, where y is the complexification of x.

Remark 3.1. From the Universal Coefficient theorem and Poincaré duality for the oriented 8-dimensional manifold M one obtains a short exact sequence

$$0 \to T(H^2(M;\, \mathbb{Z})) \otimes \mathbb{F}_2 \to H^2(M;\, \mathbb{F}_2) \to \mathrm{Hom}(H^6(M;\, \mathbb{Z}), \mathbb{F}_2) \to 0,$$

in which T denotes the torsion subgroup of an abelian group.

When $w_2(M)\cdot H^6(M;\, \mathbb{Z}) = 0$, or equivalently, by the Wu formula, when $\mathrm{Sq}^2\rho_2 H^6(M;\, \mathbb{Z}) = 0$, we may choose a spin^c-structure for M for which the characteristic class c is torsion and so disappears from the rational cohomology formulae.

On the other hand, if $w_2(M) \cdot H^6(M;\, \mathbb{Z}) \neq 0$, then there is, for any chosen class c, an element $t \in H^6(M;\, \mathbb{Z})$ such that $(c \cdot t)[M] \in \mathbb{Z}$ is odd.

We can use the splitting to extend the description (2.3) of the Chern classes of complex vector bundles over $M - B$ to bundles over M.

Proposition 3.2. *Consider cohomology classes $l \in H^2(M;\mathbb{Z})$, $u \in H^4(M;\mathbb{Z})$, $v \in H^6(M;\mathbb{Z})$, $w \in H^8(M;\mathbb{Z})$. According to (2.3), there is a 4-dimensional complex vector bundle ζ over $M - B$ with $c_1(\zeta) = l$, $c_2(\zeta) = u$ and $c_3(\zeta) = v$ if and only if $\mathrm{Sq}^2\rho_2 u = \rho_2(v + lu)$. In that case*

$$\frac{1}{2}(u(u + q_1(\tau M;\, c) - 2l^2 - 3cl - c^2) + (2l + 3c)v)[M] \in \mathbb{Q}$$

is an integer, I say, and $I\,(\mathrm{mod}\,6)$ is independent of the choice of c. Such a bundle extends over M to a complex vector bundle ζ with $c_4(\zeta) = w$ if and only if

$$w[M] \equiv I\,(\mathrm{mod}\,6).$$

Proof. We observe, first of all, that 4-dimensional complex vector bundles over an 8-dimensional space lie in the stable range, and so the problem is K-theoretic. By (1.2), the stable complex bundles $y \in K^0(M)$ over M extending a given bundle over $M - B$ are classified by the integer $\mathrm{ind}\, y \in \mathbb{Z}$. A computation using (3.1) and (3.2) shows that, for a bundle ζ with the given Chern classes and a complex line bundle λ with $c_1(\lambda) = l$, we have

$$\mathrm{ind}(\zeta - \mathbb{C}^4) - \mathrm{ind}(\lambda - \mathbb{C}) = \frac{1}{6}(I - w[M]).$$

The result follows. □

In the same way, we may use the splitting of $K\mathrm{O}^0(M)$ to distinguish oriented real vector bundles over M.

Proposition 3.3. *Let ξ and ξ' be oriented 8-dimensional real vector bundles over M with $w_2(\xi) = w_2(\xi') = \rho_2(l)$, where $l \in H^2(M;\mathbb{Z})$. Then ξ and ξ' are isomorphic if and only if*

a) $q_1(\xi;\, l) = q_1(\xi';\, l)$, b) $p_2(\xi) = p_2(\xi')$, c) $e(\xi) = e(\xi')$.

Proof. Condition a) is necessary and sufficient for the restrictions of ξ and ξ' to $M - B$ to be isomorphic. It implies, in particular, that $p_1(\xi) = p_1(\xi')$. And then $\mathrm{ind}(\mathbb{C} \otimes \xi) = \mathrm{ind}(\mathbb{C} \otimes \xi')$ if and only if $p_2(\xi) = p_2(\xi')$, from (3.1) and (3.3). Hence a) and b) are necessary and sufficient conditions for ξ and ξ' to be stably isomorphic, by (1.2). Finally, equality of the Euler classes, (1.1), gives isomorphism as oriented vector bundles. □

4. Reduction to U(4) : a complex structure

Consider an oriented real vector bundle ξ of dimension 8 over M such that $w_2(\xi)$ lifts to an integral class $l \in H^2(M;\mathbb{Z})$. We shall determine necessary and sufficient conditions for ξ to admit a complex structure with first Chern class $c_1 = l$. Notice that $p_2(\xi) - q_1^2(\xi;\, l)$ is divisible by 2 in $H^8(M;\mathbb{Z})$, since $\rho_2(p_2(\xi) - q_1^2(\xi;\, l)) = w_4^2(\xi) - w_4^2(\xi) = 0$.

Proposition 4.1. (Heaps, Čadek and Vanžura) *Suppose that ξ is an oriented 8-dimensional real vector bundle over M admitting a spin^c structure. Let $l \in H^2(M;\mathbb{Z})$*

be a class such that $w_2(\xi) = \rho_2(l)$*. Then the structure group* $\mathrm{SO}(8)$ *of* ξ *admits a reduction to* $\mathrm{U}(4)$ *with* $c_1 = l$ *if and only if there is a class* $v \in H^6(M;\,\mathbb{Z})$ *with* $\rho_2(v) = w_6(\xi)$ *such that*

$$\text{a)}\quad \frac{1}{2}(p_2(\xi) - q_1^2(\xi;\,l))[M] \equiv J \,(\mathrm{mod}\, 2) \quad \textit{and} \quad \text{b)}\quad p_2(\xi) - q_1^2(\xi;\,l) = 2(e(\xi) - lv),$$

where

$$J = \frac{1}{2}(q_1(\xi;\,l)(q_1(\xi;\,l) - q_1(\tau M;\,c) + 2l^2 + 3lc + c^2) + 3cv)[M]$$

(which is an integer if $\rho_2(v) = w_6(\xi)$*).*

Remark 4.2. The condition a), given that $\rho_2(v) = w_6(\xi)$, is necessary and sufficient for the existence of a stable complex structure on ξ, that is, a complex structure on $\xi \oplus \mathbb{R}^2$. (More precisely, if ξ has a stable complex structure outside a finite subset of M, then the sum of the local obstructions in $\pi_7(\mathrm{SO}(\infty)/\mathrm{U}(\infty)) = \mathbb{Z}/2$ to extending the stable structure to M is equal to $\frac{1}{2}(p_2(\xi) - q_1^2(\xi;\,l))[M] - J$ (mod 2).) If the class c can be chosen to be torsion, then the expression for J simplifies, and, in particular, J does not depend on v. If not, then, as was observed in (3.1), there is a class t such that $(ct)[M]$ is odd, and v can be modified by addition of a multiple of $2t$ to satisfy a). Thus, if $w_2(M)H^6(M;\,\mathbb{Z}) \neq 0$, the only condition required for the existence of a stable complex structure with $c_1 = l$ is that $w_6(\xi)$ should be the reduction of an integral class. (Compare [15].)

Proof. Suppose that there is a 4-dimensional complex vector bundle ζ, with Chern classes $c_1 = l$, $c_2 = u$, $c_3 = v$, $c_4 = w$, that is stably isomorphic to ξ. Then

$$\rho_2(v) = w_6(\xi), \quad u = -q_1(\xi;\,l) \quad \text{and} \quad 2w = p_2(\xi) - q_1^2(\xi;\,l) + 2lv,$$

because $p_2(\xi) = c_4(\zeta \oplus \overline{\zeta}) = 2c_4(\zeta) - 2c_1(\zeta)c_3(\zeta) + c_2^2(\zeta)$.

Necessary and sufficient conditions for the existence of a stable complex structure on ξ are, therefore, given by (3.2) and (3.3) with $J = I - (lv)[M]$. (Notice that $\mathrm{Sq}^2 w_4(\xi) = w_6(\xi) + w_2(\xi)w_4(\xi)$.) This reduces to the condition a), except that congruence is required (mod 6). But the congruence is automatically satisfied (mod 3). For the integrality of $\mathrm{ind}(\mathbb{C} \otimes_{\mathbb{R}} \xi - \mathbb{C}^8) - \mathrm{ind}(\mathbb{C} \otimes_{\mathbb{R}} \lambda - \mathbb{C}^2)$, where λ as before is a complex line bundle with $c_1 = l$, gives

$$(4.1)\quad (p_2(\xi) - q_1^2(\xi;\,l))[M] \equiv (q_1(\xi;\,l)(q_1(\xi;\,l) - q_1(\tau M;\,c) + 2l^2 + c^2))[M] \,(\mathrm{mod}\, 6).$$

(It is, of course, clear that there can be no 3-primary obstruction to the existence of a stable complex structure on ξ, because $2\xi = \xi \oplus \xi$ admits a complex structure as $\mathbb{C} \otimes_{\mathbb{R}} \xi$.)

Condition b) expresses the equality of the Euler classes required for isomorphism rather than stable isomorphism. □

5. Reduction to $\mathrm{Spin}^c(6)$: a 2-field

In this section we consider reduction to

$$\mathrm{Spin}^c(6) \to \mathrm{SO}(6) \to \mathrm{SO}(8),$$

where the first homomorphism is the canonical projection and the second is the standard inclusion.

There is an isomorphism

$$\{(z, g) \in \mathbb{T} \times \mathrm{U}(4) \mid z^2 = \det g\} \to \mathrm{Spin}^c(6).$$

Suppose that ζ is a 4-dimensional $\mathbb{C}$-vector bundle over M, and λ a complex line bundle, and that we are given an isomorphism $\Lambda^4\zeta \to \lambda \otimes \lambda$. Then $\lambda^* \otimes \Lambda^2\zeta$ has a real structure as $\mathbb{C} \otimes_{\mathbb{R}} \eta$, where η is a 6-dimensional real vector bundle over M. The trivialization of $\Lambda^6(\lambda^* \otimes \Lambda^2\zeta) = \lambda^{-6} \otimes (\Lambda^4\zeta)^3$ gives an orientation of η.

Lemma 5.1. *The characteristic classes of η are as follows:*

$$w_2(\eta) = \rho_2(l), \qquad q_1(\eta;\, l) = l^2 - c_2(\zeta),$$
$$e(\eta) = c_3(\zeta) + l q_1(\eta;\, l), \qquad p_2(\eta) = q_1^2(\eta;\, l) + 2le(\eta) - 4c_4(\zeta),$$

where $l = c_1(\lambda)$ and the orientation of η is chosen appropriately.

Proof. Over the 7-skeleton, the 4-dimensional complex vector bundle $(\lambda^*)^2 \otimes \zeta$ has a non-zero section, and hence we can write $\zeta = \zeta' \oplus \lambda^2$, with $\Lambda^3\zeta' = \mathbb{C}$. So $\lambda^* \otimes \Lambda^2\zeta = (\lambda^* \otimes \Lambda^2\zeta') \oplus (\lambda \otimes \zeta')$ and η is the underlying real bundle of $\lambda \otimes \zeta'$. This allows the computation of w_2, q_1 and e. The second Pontrjagin class is $c_4(\lambda^* \otimes \Lambda^2\zeta)$. □

Proposition 5.2. *Let ξ be an oriented 8-dimensional real vector bundle over M with $w_2(\xi) = \rho_2(l)$, $l \in H^2(M;\, \mathbb{Z})$. Let $v \in H^6(M;\, \mathbb{Z})$. Then ξ admits a reduction of structure group to* $\mathrm{Spin}^c(6)$ *with spin^c characteristic class l and Euler class v if and only if the following conditions are satisfied.*

$$e(\xi) = 0; \quad \rho_2(v) = w_6(\xi);$$
$$(\frac{1}{2}(p_2(\xi) - q_1^2(\xi;\, l)) + q_1(\xi;\, l)(q_1(\xi;\, l) - q_1(\tau M;\, c) + 2l^2 + 3lc + c^2) + 3(l+c)v)[M]$$
$$\equiv 0 \pmod 4.$$

Proof. We require a 4-dimensional complex vector bundle ζ over M with first Chern class $c_1(\zeta) = 2l$ such that the associated bundle η satisfies:

$$w_2(\xi) = w_2(\eta); \quad q_1(\xi;\, l) = q_1(\eta;\, l); \quad v = e(\eta); \quad p_2(\xi) = p_2(\eta).$$

This will guarantee that ξ and $\eta \oplus \mathbb{R}^2$ are stably isomorphic. The vanishing of the Euler class $e(\xi)$ will then be enough for the two vector bundles to be isomorphic.

The conditions for the existence of ζ are given in Proposition 3.2. One has first

$$\mathrm{Sq}^2(\rho_2(l^2 - q_1(\xi;\, l))) = \rho_2(v - lq_1(\xi;\, l)),$$

that is, $\mathrm{Sq}^2(w_2^2(\xi) + w_4(\xi)) = \rho_2(v) + w_2(\xi)w_4(\xi)$, which reduces to the obvious condition that $\rho_2(v) = w_6(\xi)$. In the notation of Proposition 3.2 we find that

$$2I - 2w[M] = (q_1(\xi;\, l)(q_1(\xi;\, l) - q_1(\tau M;\, c) + 2l^2 + 3lc + c^2) + l^2 q_1(\tau M;\, c)$$
$$- l^2(7l^2 + 6cl + c^2) + (4l + 3c)v)[M] + (\frac{1}{2}(p_2(\xi) - q_1^2(\xi;\, l)) - lv)[M] \in 12\mathbb{Z}.$$

By considering the special case in which $\xi = \lambda \oplus \mathbb{R}^6$, with $q_1(\xi;\, l) = 0$ and $p_2(\xi) = 0$, when we may take $v = 0$, we see that

$$(l^2(q_1(\tau M;\, c) - (7l^2 + 6cl + c^2)))[M] \in 12\mathbb{Z}.$$

This leads to the stated congruence, but (mod 12) instead of (mod 4). However the congruence automatically holds (mod 3), by (4.1). □

Remark 5.3. Given a complex line bundle μ over M, there is a similar necessary and sufficient condition for ξ to split as a sum $\eta \oplus \mu$. There are other methods when $w_2(\xi) = w_2(M)$ or $w_2(\mu) = w_2(\xi) + w_2(M)$; see [12] and [3]. The problems of reduction to U(4) and $\mathrm{Spin}^c(6)$ are related, or rather the SU(4) and Spin(6) reductions - the cases in which $l = 0$. The correspondence is explained in [4].

6. Reduction to U(3) : the adjoint representation

For a finite dimensional complex Hilbert space V, let us write $\mathfrak{su}(V)$ for the Lie algebra of the special unitary group. Thus $\dim_{\mathbb{R}} \mathfrak{su}(V) = (\dim_{\mathbb{C}} V)^2 - 1$. In particular, taking $V = \mathbb{C}^3$, we have an 8-dimensional real representation $\mathfrak{su}(\mathbb{C}^3)$ of U(3), and so, up to conjugation, a homomorphism $\rho : \mathrm{U}(3) \to \mathrm{SO}(8)$. (In fact, since ρ factors through the projective unitary group PU(3), which is a quotient of SU(3) by the central subgroup of order 3, the homomorphism must lift to $\mathrm{U}(3) \to \mathrm{Spin}(8)$.) In this section we investigate the reduction of the structure group of ξ over M to U(3).

So suppose that ζ is a 3-dimensional complex (unitary) vector bundle over M. We form the associated Lie algebra bundle $\mathfrak{su}(\zeta)$ of real dimension 8.

Lemma 6.1. *The vector bundle $\mathfrak{su}(\zeta)$ is spin and its characteristic classes are as follows:*

$$\begin{aligned} w_2(\mathfrak{su}(\zeta)) &= 0, & e(\mathfrak{su}(\zeta)) &= 0,\\ q_1(\mathfrak{su}(\zeta)) &= c_1^2(\zeta) - 3c_2(\zeta), & p_2(\mathfrak{su}(\zeta)) &= c_1^4(\zeta) - 6c_1^2(\zeta)c_2(\zeta) + 9c_2^2(\zeta),\\ w_4(\mathfrak{su}(\zeta)) &= \rho_2(c_1^2(\zeta) + c_2(\zeta)), & w_6(\mathfrak{su}(\zeta)) &= \rho_2(c_1(\zeta)c_2(\zeta) + c_3(\zeta)). \end{aligned}$$

Proof. We may calculate in the torsion-free cohomology of $B\mathrm{U}(3)$ and so reduce to the case that $\zeta = \mu_1 \oplus \mu_2 \oplus \mu_3$ is a sum of 3 complex line bundles. In that case we have the explicit description (using a bar for the complex conjugate):

$$\mathfrak{su}(\zeta) = \mathrm{i}\mathbb{R}^2 \oplus ((\overline{\mu_1} \otimes \mu_2) \oplus (\overline{\mu_1} \otimes \mu_3) \oplus (\overline{\mu_2} \otimes \mu_3))$$

(as a real bundle). The computations are then straightforward. □

The standard method leads to necessary and sufficient conditions for reduction of the structure group.

Proposition 6.2. *Let ξ be an oriented 8-dimensional real vector bundle over M with $w_2(\xi) = 0$. Consider cohomology classes $l \in H^2(M;\mathbb{Z})$, $u \in H^4(M;\mathbb{Z})$, $v \in H^6(M;\mathbb{Z})$. Then there is a 3-dimensional complex vector bundle ζ over M such that $\mathfrak{su}(\zeta) \cong \xi$, with $c_1(\zeta) = l$, $c_2(\zeta) = u$ and $c_3(\zeta) = v$, if and only if*

$$q_1(\xi) = -3u + l^2; \quad \rho_2(v - lu) = w_6(\xi); \quad p_2(\xi) = q_1^2(\xi); \quad e(\xi) = 0;$$

and

$$\frac{1}{2}(u(u + q_1(\tau M;\, c) - c^2) + (2l + 3c)(v - lu))[M] \equiv 0 \pmod 6.$$

(It is to be understood in the statement that the expression in the last line is integral if $\rho_2(v - lu) = w_6(\xi)$.)

Proof. We require first the existence of a complex vector bundle ζ as in Proposition 3.2 with $w = 0$, so that ζ admits a nowhere zero cross-section. Then the conditions that $q_1(\xi) = l^2 - 3u$ and $p_2(\xi) = l^4 - 6l^2u + 9u^2$ ensure that $\mathfrak{su}(\zeta)$ and ξ are stably isomorphic. Recall that $\rho_2(q_1(\xi)) = w_4(\xi)$, so that $\mathrm{Sq}^2\rho_2(u) = \mathrm{Sq}^2 w_4(\xi) = w_6(\xi)$, because $w_2(\xi) = 0$.

Finally, equality of the Euler classes $e(\mathfrak{su}(\zeta)) = 0 = e(\xi)$ is necessary and sufficient for the two bundles to be isomorphic. □

Remark 6.3. If μ is a complex line bundle, then $\mathfrak{su}(\mu \otimes \zeta) = \mathfrak{su}(\zeta)$. Hence, the conditions above must be invariant under the transformation $\zeta \mapsto \mu \otimes \zeta$, that is: $l \mapsto l + 3m$, $u \mapsto u + 2lm + 3m^2$, $v \mapsto v + um + lm^2 + m^3$, where $c_1(\mu) = m$.

The conditions simplify if the manifold M is spin ($c = 0$) and we look for an SU(3)-structure ($l = 0$) on ξ.

Corollary 6.4. *Suppose that M is a spin manifold. Then the bundle ξ admits an* SU(3)*-structure if and only if $w_6(\xi)$ is the reduction of an integral class, $e(\xi) = 0$, and there is a class $u \in H^4(M;\,\mathbb{Z})$ such that*

$$q_1(\xi) = -3u, \quad p_2(\xi) = 9u^2 \text{ and } \frac{1}{2}(u(u + q_1(\tau M)))[M] \equiv 0 \pmod 6.$$

For the special case in which $\xi = \tau M$, see [18] (although the argument there purporting to show that $w_6(M) = 0$ seems to be incorrect).

7. Quaternionic bundles

Modules over a quaternion algebra are understood to be *right* modules unless explicit reference is made to left multiplication.

Consider a complex line bundle λ over M, with $c_1(\lambda) = l$. In order to use a spinor index in KO-theory, we require that $\rho_2(l) = w_2(M)$.

Let $\mathbb{H}_\lambda$ be the bundle of quaternion algebras associated with λ: $\mathbb{H}_\lambda = \mathbb{C} \oplus \lambda$, where $jz = \bar{z}j$ for $j \in S(\lambda)$, $z \in \mathbb{C}$, and $j^2 = -1$. An $\mathbb{H}_\lambda$-vector bundle is a complex vector bundle by restriction of scalars. We will say that a complex vector bundle ζ admits an $\mathbb{H}_\lambda$-structure if it has a compatible $\mathbb{H}_\lambda$-multiplication. One can show that ζ has an $\mathbb{H}_\lambda$-structure if and only if it has a non-singular skew-symmetric form $\zeta \otimes \zeta \to \lambda$. The next result, which goes back to an idea of Dupont [13], is fundamental.

Lemma 7.1. *Let β be a 3-dimensional complex vector bundle over $M - B$ such that $c_1(\beta) = l$ and $c_3(\beta) = 0$. Then β has a nowhere-zero section over $M - B$ and so splits as a direct sum $\beta = \mathbb{C} \oplus \mu$, where μ is an $\mathbb{H}_\lambda$-line bundle.*

Proof. We have to show that the sphere bundle $S(\beta)$ has a section over $M-B$. The top obstruction is a class $\mathfrak{o}$ in $H^7(M - B;\,\mathbb{Z}/2)$ $(= H^7(M;\,\mathbb{Z}/2))$, because $\pi_6(S^5) = \mathbb{Z}/2$. Consider an element x of $H^1(M;\,\mathbb{Z}/2)$, represented by a real line bundle ν. The zero-set of a generic cross-section of ν is a 7-dimensional submanifold N of $M-B$. It suffices to prove that $\mathfrak{o}$ maps to zero in $H^7(N;\,\mathbb{Z}/2)$. For evaluation on the fundamental class $[N]$ gives $(\mathfrak{o} \cdot x)[M] \in \mathbb{Z}/2$. This is established by a KO-theory Hopf theorem. It will be convenient, as in [10] and [11], to use a local coefficient notation $KO^*(M;\,\alpha)$ for the reduced KO-theory $\widetilde{K}\mathrm{O}^*(M^\alpha)$ of the Thom space of a (virtual) real vector bundle α over M.

The composition

$$K\mathrm{O}^0(N;\,-\beta) \xrightarrow{\eta(\nu)} K\mathrm{O}^0(N;\,\nu-\beta) \xrightarrow{\pi_!} K\mathrm{O}^{-2}(*) = \mathbb{Z}/2,$$

where $\eta(\nu) \in K\mathrm{O}^0(N;\,\nu)$ is a twisted Hopf element (lifting the generator η of $K\mathrm{O}^{-1}(*) = \mathbb{Z}/2$) and $\pi_!$ is the index determined by the spin structure on $\tau M - \beta$, maps the KO-Euler class $\gamma(\beta)$ to $(\mathfrak{o}\cdot x)[M] \in \mathbb{Z}/2$. Now there is a $\mathbb{Z}/2$-equivariant lifting:

$$K\mathrm{O}^0_{\mathbb{Z}/2}(N;\,-L\otimes\beta) \xrightarrow{\eta(\nu)} K\mathrm{O}^0_{\mathbb{Z}/2}(N;\,\nu-L\otimes\beta) \xrightarrow{\pi_!} K\mathrm{O}^{-8}_{\mathbb{Z}/2}(*;\,-6L) = \mathbb{Z},$$

where L is the non-trivial 1-dimensional real representation of $\mathbb{Z}/2$. (For more details, see [10].) But $\eta(\nu)$ is a torsion class. So the image of $\gamma(L\otimes\beta)$ in $\mathbb{Z}$ must be zero.

Of course, a 2-dimensional complex bundle μ with $c_1 = l$ admits an $\mathbb{H}_\lambda$-structure given by the exterior product to the determinant bundle regarded as a skew-symmetric form $\mu\otimes\mu \to \Lambda^2\mu \cong \lambda$. □

8. Reduction from U(4) to Sp(2) $\times_{\{\pm 1\}}$ U(1)

Let ζ be a 4-dimensional complex vector bundle over M. The structure group U(4) of bundle ζ reduces to the subgroup Sp(2) $\times_{\{\pm1\}}$ U(1) if and only if there is a complex line bundle λ over M such that ζ has an $\mathbb{H}_\lambda$-structure. For more details see [2].

So consider a complex line bundle λ with $c_1(\lambda) = l$ and $\rho_2(l) = w_2(M)$. We ask when the structure group of ζ can be reduced to an $\mathbb{H}_\lambda$-structure. Simultaneously, we want to characterize the Chern classes of such vector bundles.

If an $\mathbb{H}_\lambda$-vector bundle has a nowhere zero section, we can split off a trivial $\mathbb{H}_\lambda$-line bundle. As a first consequence, an m-dimensional $\mathbb{H}_\lambda$-bundle over a 3-dimensional space must be trivial, that is, isomorphic to $\mathbb{H}_\lambda^m$, and so has $c_1 = ml$, where $l = c_1(\lambda)$.

Again we can treat the problem in three stages, starting with the stable question over $M-B$. An $\mathbb{H}_\lambda$-bundle over $M-B$ can be reduced to dimension 1 over $\mathbb{H}_\lambda$ because a higher dimensional bundle has a nowhere zero section. So a necessary condition for lifting ζ is that $c_3(\zeta - \mathbb{H}_\lambda) = 0$, that is,

$$c_3(\zeta) - lc_2(\zeta) + l^3 = 0.$$

In that case, we show that $(\zeta - \mathbb{H}_\lambda)\,|\,M - B$ can be reduced to complex dimension 2 and so it has an $\mathbb{H}_\lambda$-structure. Let β be a 3-dimensional complex vector bundle over $M-B$ such that $\beta\oplus\lambda$ is stably isomorphic to ζ. (Such a bundle always exists.) Note that $c_1(\beta) = l$. We are assuming that $c_3(\beta) = 0$, so that β has a nowhere-zero cross-section over $M-B$ by Lemma 7.1. This establishes:

Lemma 8.1. *A 4-dimensional complex bundle ζ has a stable $\mathbb{H}_\lambda$-structure over $M-B$ if and only if $c_1(\zeta) = 2l$ and $c_3(\xi) - lc_2(\xi) + l^3 = 0$.*

The rest of the argument follows the familiar pattern. The stable $\mathbb{H}_\lambda$-bundles are classified by a twisted K-group $K\mathrm{Sp}^0_\lambda(M)$, which can be identified with the KO-group $\widetilde{K}\mathrm{O}^{-2}(M^\lambda)$ of the Thom space of λ. (See [1] pp. 135–136 and [9] Chapter 9 for the twisted K-theory.) To describe the isomorphism, we think of $\widetilde{K}\mathrm{O}^{-2}(M^\lambda)$ as the KO-theory with compact supports $K\mathrm{O}^0(\mathbb{C}\oplus\lambda)$ of the total space of the bundle $\mathbb{C}\oplus\lambda = \mathbb{H}_\lambda$.

The isomorphism maps the class $[\mu] \in K\mathrm{Sp}_\lambda(M)$ of an $\mathbb{H}_\lambda$-module μ over M to the class in $K\mathrm{O}(\mathbb{C} \oplus \lambda)$ given by the $\mathbb{R}$-linear map

$$\mu \to \mu : \quad x \mapsto xv$$

over $v \in \mathbb{H}_\lambda = \mathbb{C} \oplus \lambda$, which is an isomorphism outside the (compact) zero-section.

A spin structure on $\tau M + \lambda$ gives an index, or Gysin map $K\mathrm{Sp}^0_\lambda(M) = \widetilde{K}\mathrm{O}^{-2}(M^\lambda) \to K\mathrm{O}^{-4}(*) = \mathbb{Z}$. The corresponding splitting of the complex K-groups is that defined by taking the spin^c structure on M with characteristic class $c = -l$.

Now we have a map from the $K\mathrm{Sp}_\lambda$-sequence to the complex K-theory giving a commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z} = K\mathrm{Sp}^0_\lambda(M, M-B) & \underset{\longleftarrow - -}{\longrightarrow} & K\mathrm{Sp}^0_\lambda(M) & \to & K\mathrm{Sp}^0_\lambda(M-B) & \to & 0 \\
\downarrow\ 2\times & & \downarrow & & \downarrow & & \\
\mathbb{Z} = K^0(M, M-B) & \underset{\longleftarrow - -}{\longrightarrow} & K^0(M) & \to & K^0(M-B) & \to & 0
\end{array}$$

in which the restriction from the twisted quaternionic to the complex theory is multiplication by 2 on the top cell $\mathbb{Z}$.

Taking the spin^c structure on M with characteristic class $c = -l$, for ζ to admit an $\mathbb{H}_\lambda$-structure we, thus, require that

$$\operatorname{ind} \zeta = (\mathrm{e}^{-l/2} \hat{\mathcal{A}}(\tau M) \mathrm{ch}(\zeta))[M] \in 2\mathbb{Z},$$

since two stably isomorphic $\mathbb{H}_\lambda$-bundles of dimension 2 over M are isomorphic. This leads to the following characterization.

Proposition 8.2. *Let λ be a complex line bundle over M with $c_1(\lambda) = l$ and $\rho_2(l) = w_2(M)$. Consider cohomology classes $u \in H^4(M\ ;\mathbb{Z})$ and $w \in H^8(M\ ;\mathbb{Z})$. Then there is a 2-dimensional $\mathbb{H}_\lambda$-vector bundle ζ over $M - B$ with $c_2(\zeta) = u$ if and only if $\mathrm{Sq}^2 \rho_2 u = \rho_2(lu - l^3)$. In that case $c_1(\zeta) = 2l$, $c_3(\zeta) = lu - l^3$ and the expression*

$$K = \frac{1}{4}(p_1(\tau M)u + 2u^2 + l^2(3l^2 - p_1(\tau M) - 5u))[M] \in \mathbb{Q}$$

is an integer. Such a bundle extends to an $\mathbb{H}_\lambda$-bundle over M with $c_4(\zeta) = w$ if and only if

$$w[M] \equiv K \pmod{12}.$$

Proof. A complex vector bundle over M of dimension 4 lies in the stable range, and so the problem is K-theoretic. Over $M - B$ the solution is given by Corollary 2.3 and Lemma 8.1. Now elements $y \in K^0(M)$ having $\mathbb{H}_\lambda$-structure and extending a given element in $K^0(M - B)$ are classified by even integers $\operatorname{ind} y$. One can compute that

$$\operatorname{ind} \zeta - \operatorname{ind} \mathbb{H}^2_\lambda = \frac{1}{6}(K - w[M]),$$

and this completes the proof. □

As immediate corrollaries we get

Corollary 8.3. *Let ξ be a 4-dimensional complex bundle over M, and let λ be a complex line bundle, with $c_1(\lambda) = l$, such that $\rho_2(l) = w_2(M)$. Then ξ admits an $\mathbb{H}_\lambda$-structure if and only if*

$$c_1(\xi) = 2l; \qquad c_3(\xi) - lc_2(\xi) + l^3 = 0$$

and the expression

$$K = \frac{1}{4}(p_1(\tau M)c_2(\xi) + 2c_2^2(\xi) + l^2(3l^2 - p_1(\tau M) - 5c_2(\xi)))[M] \in \mathbb{Q},$$

which is always an integer (if the conditions above hold), satisfies

$$K \equiv c_4(\xi)[M] \pmod 4.$$

Corollary 8.4. *Let λ be a complex line bundle over M with $c_1(\lambda) = l$ and $\rho_2(l) = w_2(M)$. Let $u \in H^4(M; \mathbb{Z})$. Then there is an $\mathbb{H}_\lambda$-line bundle η over M with ($c_1(\eta) = l$ and) $c_2(\eta) = u$ if and only if $\mathrm{Sq}^2\rho_2(u) = \rho_2(lu)$ and the integer*

$$\frac{1}{4}(p_1(\tau M)u + 2u^2 - l^2u)[M] \equiv 0 \pmod{12}.$$

Proof. If η is such an $\mathbb{H}_\lambda$-line bundle, then $\zeta = \eta \oplus \mathbb{H}_\lambda$ is a 4-dimensional complex vector bundle with $c_1(\xi) = 2l$, $c_2(\xi) = u + l^2$, $c_3(\xi) = lu$, $c_4(\xi) = 0$. Conversely, if a 4-dimensional complex vector bundle ζ has these Chern classes and admits an $\mathbb{H}_\lambda$-structure, then, since its Euler class is zero, it splits as a direct sum of an $\mathbb{H}_\lambda$-line and a trivial $\mathbb{H}_\lambda$ summand. □

9. Reduction to $\mathrm{Spin}^c(4)$ and $\mathrm{Spin}^c(3)$

Again let λ be a complex line bundle over M with $c_1(\lambda) = l$ and $w_2(M) = \rho_2(l)$. We shall use the results of the last section on $\mathbb{H}_\lambda$-line bundles to investigate the existence of $\mathrm{Spin}^c(4)$ and $\mathrm{Spin}^c(3)$-structures on an 8-dimensional vector bundle ξ.

Since $\mathrm{Spin}^c(4) = (\mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{U}(1))/\{\pm(1,1,1)\}$ any 4-dimensional real vector bundle η over M having a $\mathrm{Spin}^c(4)$-structure with characteristic class l can be expressed as

$$\eta = \mathrm{Hom}_{\mathbb{H}_\lambda}(\zeta_-, \zeta_+) = \zeta_+ \otimes_{\mathbb{H}_\lambda} \zeta_-^*,$$

where ζ_+ and ζ_- are right $\mathbb{H}_\lambda$-line bundles over M and $\zeta_-^* = \mathrm{Hom}_{\mathbb{H}_\lambda}(\zeta_-, \mathbb{H}_\lambda)$ is considered as a left $\mathbb{H}_\lambda$-line bundle.

We may also think of η as the real form of the complex bundle $\zeta_+ \otimes_{\mathbb{C}} \zeta_-^*$ with the real structure given by the symmetric $\mathbb{C}$-valued bilinear form arizing as the tensor product of a λ-valued skew-symmetric form on ζ_+ and a λ^*-valued skew-symmetric form on ζ_-^*.

Lemma 9.1. *The characteristic classes of η are:*

$$w_2(\eta) = \rho_2(l); \qquad w_4(\eta) = w_4(\zeta_+) + w_4(\zeta_-);$$
$$q_1(\eta; l) = -(c_2(\zeta_+) + c_2(\zeta_-)); \qquad e(\eta) = c_2(\zeta_+) - c_2(\zeta_-),$$

with the appropriate choice of orientation.

Proof. Since $c_1(\zeta_+) = c_1(\zeta_-) = l$, by a splitting principle we may suppose that $\zeta_+ = \mu_+ \oplus \lambda \otimes \mu_+^*$ and $\zeta_- = \mu_- \oplus \lambda \otimes \mu_-^*$, where μ_+ and μ_- are complex line bundles. Using the description of $\mathbb{C} \otimes_{\mathbb{R}} \eta$ as $\zeta_+ \otimes_{\mathbb{C}} \zeta_-^*$, we may identify η with the complex bundle $(\mu_+^* \oplus \lambda^* \otimes_{\mathbb{C}} \mu_+) \otimes_{\mathbb{C}} \mu_-$. □

Proposition 9.2. (Čadek and Vanžura). *Let ξ satisfy $w_2(\xi) = w_2(M) = \rho_2(l)$. Then the structure group of ξ can be reduced to* $\mathrm{Spin}^c(4)$ *with characteristic class l if and only if there exist classes u_+, $u_- \in H^4(M;\,\mathbb{Z})$ such that the following conditions are satisfied.*

$$q_1(\xi;\,l) = -(u_+ + u_-); \qquad w_6(\xi) = 0; \qquad \mathrm{Sq}^2\rho_2 u_+ = \rho_2(lu_+);$$
$$e(\xi) = 0; \qquad p_2(\xi) = (u_+ - u_-)^2;$$
$$\frac{1}{4}(u_+(p_1(\tau M) + 2u_+ - l^2))[M] \equiv 0 \pmod{12};$$
$$\frac{1}{4}(u_-(p_1(\tau M) + 2u_- - l^2))[M] \equiv 0 \pmod{4}.$$

Remark 9.3. The first three conditions imply that $\mathrm{Sq}^2\rho_2(u_-) = \rho_2(lu_-)$. If the conditions on u_+ and u_- are satisfied, the last congruence holds also (mod 12). The integrality of $\mathrm{ind}(\xi - \mathbb{R}^8) \otimes \mathbb{C} - \mathrm{ind}(\lambda + \lambda^* - \mathbb{C}^2)$ gives

$$(q_1(\xi;\,l)^2 + q_1(\xi;\,l)l^2 + p_2(\xi) - q_1(\xi;\,l)p_1(\tau M))[M] \equiv 0 \pmod{3},$$

which is the sum of the (mod 3) congruences for u_+ and u_-.

Proof. Given $\mathbb{H}_\lambda$-line bundles ζ_+ and ζ_- with c_2 equal to u_+ and u_- respectively, we construct η as above. Then $\eta \oplus \mathbb{R}^4$ is isomorphic to ξ if and only if $q_1(\eta;\,l) = q_1(\xi;\,l)$, $p_2(\xi) = p_2(\eta) = e(\eta)^2$ and $e(\xi) = e(\eta \oplus \mathbb{R}^4) = 0$. The existence of ζ_+ and ζ_- is guaranteed by Corollary 8.4. □

The group $\mathrm{Spin}^c(3)$ is isomorphic to $\mathrm{Sp}(1) \times_{\{\pm 1\}} \mathrm{U}(1)$. By a similar argument, for any 3-dimensional spin^c vector bundle α over M with $w_2(\alpha) = \rho_2(l)$ there is just one $\mathbb{H}_\lambda$-line vector bundle ζ such that

$$\mathbb{R} \oplus \alpha = \mathrm{End}_{\mathbb{H}_\lambda}(\zeta) = \zeta \otimes_{\mathbb{H}_\lambda} \zeta^*.$$

Since $w_4(\alpha) = 0$, $q_1(\alpha; l) = 2u$ for an element $u \in H^4(M;\,\mathbb{Z})$. From Lemma 9.1 and Corollary 8.4 we get

Proposition 9.4. *Let $w_2(M) = \rho_2(l)$ and let $u \in H^4(M;\,\mathbb{Z})$. Then there is a 3-dimensional vector bundle α over M with $w_2(\alpha) = \rho_2(l)$ and $q_1(\alpha, l) = 2u$ if and only if $\mathrm{Sq}^2\rho_2(u) = \rho_2(lu)$ and the integer*

$$\frac{1}{4}u(p_1(\tau M) - 2u - l^2)[M] \equiv 0 \pmod{12}.$$

Proof. The conditions are necessary and sufficient for the existence of an $\mathbb{H}_\lambda$-line bundle ζ with $c_1(\zeta) = l$ and $c_2(\zeta) = -u$. Then $\alpha \oplus \mathbb{R} = \zeta \otimes_{\mathbb{H}_\lambda} \zeta^*$ has the prescribed characteristic classes by Lemma 9.1. □

Corollary 9.5. *Let ξ be an 8-dimensional real vector bundle over M with $w_2(\xi) = w_2(M) = \rho_2(l)$. Then the structure group of ξ can be reduced to* $\mathrm{Spin}^c(3)$ *with characteristic class l if and only if there exists a class $u \in H^4(M;\,\mathbb{Z})$ such that the following* *conditions are satisfied.*

$$q_1(\xi;\, l) = 2u; \qquad w_6(\xi) = 0; \qquad \mathrm{Sq}^2\rho_2 u = \rho_2(lu);$$
$$e(\xi) = 0; \qquad p_2(\xi) = 0;$$
$$\frac{1}{4}(u(p_1(\tau M) - 2u - l^2))[M] \equiv 0 \pmod 4.$$

10. Reduction to $\mathrm{Sp}(2)\times_{\{\pm1\}}\mathrm{U}(1)$ and $\mathrm{Sp}(2)\times_{\{\pm1\}}\mathrm{Sp}(1)$: a quaternionic structure

It is straightforward now to give conditions for an 8-dimensional real vector bundle ξ over M to admit an $\mathbb{H}_\lambda$-structure.

Proposition 10.1. (Čadek and Vanžura). *Let λ be a complex line bundle over M with $c_1(\lambda) = l$ and $w_2(M) = \rho_2(l)$. An oriented 8-dimensional real vector bundle ξ admits an $\mathbb{H}_\lambda$-structure if and only if:*

a) $w_2(\xi) = 0$ *and* $w_6(\xi) = w_2(M)(w_4(\xi) + w_2^2(M))$;

b) $\frac{1}{2}(p_2(\xi) - q_1^2(\xi))[M] \equiv \frac{1}{4}(2q_1^2(\xi) - p_1(\tau M)q_1(\xi) + l^2(p_1(\tau M) - 3q_1(\xi) + l^2))[M] \pmod 4$;

c) $2e(\xi) = p_2(\xi) - q_1^2(\xi)$.

Proof. Suppose that there is a 4-dimensional complex $\mathbb{H}_\lambda$-vector bundle ζ with $c_2(\zeta) = u$ and $c_4(\zeta) = w$ stably isomorphic to ξ. Then

$$w_2(\xi) = \rho_2(c_1(\zeta)) = \rho_2(2l) = 0,$$
$$u = -q_1(\xi; 2l) = 2l^2 - q_1(\xi),$$
$$w_6(\xi) = \rho_2(c_3(\zeta)) = \rho_2(lu - l^3) = w_2(M)(w_4(\xi) + w_2^2(M)),$$
$$2w = p_2(\xi) - c_2^2(\zeta) + 2c_1(\zeta)c_3(\zeta) = p_2(\xi) - q_1^2(\xi).$$

According to Proposition 8.2 necessary and sufficient conditions for the existence of a stable $\mathbb{H}_\lambda$-structure on ξ are then given by conditions a) and b) except that the congruence in (8.2) is required $\pmod{12}$. But the congruence is always satisfied $\pmod 3$ since the integrality of $\mathrm{ind}(\xi\otimes\mathbb{C} - \mathbb{C}^8) - \mathrm{ind}(2(\lambda\oplus\overline{\lambda}) - \mathbb{C}^4)$ gives

$$(4q_1^2(\xi) - 2p_2(\xi) - q_1(\xi)p_1(\tau M) + l^2 p_1(\tau M) - 2l^4)[M] \equiv 0 \pmod 3.$$

Condition c) expresses the equality of the Euler classes. □

We next consider the extension to other twisted quaternionic structures. Let α be a 3-dimensional oriented Euclidean vector bundle over M. Recall that an oriented 3-dimensional real inner product space V has a vector product $\times : V\otimes V \to V$, which can be used to give $\mathbb{R}1\oplus V$ a quaternionic multiplication with $(0,x)\cdot(0,y) = (-\langle x,y\rangle, x\times y)$ for x, $y \in V$. Applied to the fibres of α, this gives the 4-dimensional real vector bundle $\mathbb{R}1\oplus\alpha$ the structure of a bundle of quaternion algebras.

When $\alpha = \mathbb{R}\mathrm{i}\oplus\lambda$, we may identify $\mathbb{R}1\oplus\alpha$ with the bundle $\mathbb{H}_\lambda$ already considered. We now investigate the more general problem of the existence of a right $\mathbb{R}1\oplus\alpha$-module structure on an 8-dimensional real vector bundle ξ. This is equivalent to reduction of the structure group to $\mathrm{Sp}(2)\times_{\{\pm1\}}\mathrm{Sp}(1)$. To apply the methods of this paper we must require that $w_2(\alpha) = w_2(M) = \rho_2(l)$. In that case, the bundle of algebras $\mathbb{R}1 \oplus \alpha$ is the endomorphism algebra of an $\mathbb{H}_\lambda$-line bundle ζ:

$$\mathbb{R}1 \oplus \alpha = \mathrm{End}_{\mathbb{H}_\lambda}(\zeta).$$

Notice that

$$\mathrm{End}_{\mathbb{H}_\lambda}(\mathbb{H}_\lambda) = \mathbb{H}_\lambda.$$

Lemma 10.2. *Let ξ be an 8-dimensional real vector bundle and let ζ be an $\mathbb{H}_\lambda$-line bundle. Then ξ has an $\mathrm{End}_{\mathbb{H}_\lambda}(\zeta)$-structure if and only if there is a 2-dimensional $\mathbb{H}_\lambda$-vector bundle η such that*

$$\xi \cong \mathrm{Hom}_{\mathbb{H}_\lambda}(\zeta, \eta) = \eta \otimes_{\mathbb{H}_\lambda} \zeta^*.$$

Proof. This is standard module theory: ζ is an invertible $(\mathrm{End}_{\mathbb{H}_\lambda}(\zeta), \mathbb{H}_\lambda)$-bimodule. □

Lemma 10.3. *The characteristic classes of $\xi = \eta \otimes_{\mathbb{H}_\lambda} \zeta^*$ where $c_1(\zeta^*) = -l$, $c_2(\zeta) = -u$ and $c_1(\eta) = 2l$ are*

$$w_2(\xi) = 0; \qquad q_1(\xi) = 2u - c_2(\eta) + 2l^2;$$
$$p_2(\xi) = 2c_4(\eta) + c_2(\eta)(c_2(\eta) - 2u - 4l^2) + 6u^2 + 6ul^2 + 4l^2;$$
$$e(\xi) = c_4(\eta) + c_2(\eta)u - l^2u + u^2.$$

Proof. To compute the characteristic classes we can suppose that η is a sum of two $\mathbb{H}_\lambda$-line bundles $\eta_1 \oplus \eta_2$ and use Lemma 9.1. For instance, the computation of the Euler class is as follows:

$$\begin{aligned} e(\xi) &= e((\eta_1 \oplus \eta_2) \otimes_{\mathbb{H}_\lambda} \zeta^*) = e(\eta_1 \otimes_{\mathbb{H}_\lambda} \zeta^*) e(\eta_2 \otimes_{\mathbb{H}_\lambda} \zeta^*) \\ &= (c_2(\eta_1) - c_2(\zeta^*))(c_2(\eta_2) - c_2(\zeta^*)) = c_4(\eta) + c_2(\eta)u - l^2u + u^2 \end{aligned}$$

□

Proposition 10.4. (See [5], Theorem 8.1). *Let $w_2(M) = \rho_2(l)$. Then an 8-dimensional oriented vector bundle ξ has an $\mathbb{H}_\alpha$-structure with $w_2(\alpha) = \rho_2(l)$ and $q_1(\alpha; l) = 2u$ if and only if $w_2(\xi) = 0$ and the following conditions are satisfied.*

$$p_2(\xi) - q_1^2(\xi) - 2e(\xi) = 0; \qquad w_6(\xi) + w_4(\xi)\rho_2(l) + \rho_2(l^3) = 0; \qquad \mathrm{Sq}^2\rho_2 u = \rho_2(lu);$$
$$\frac{1}{4}(u(p_1(\tau M) - 2u - l^2))[M] \equiv 0 \pmod{12};$$
$$\begin{aligned} \frac{1}{4}(4q_1^2(\xi) - 2p_2(\xi) - q_1(\xi)p_1(\tau M) - 3l^2q_1(\xi) + l^2p_1(\tau M) + l^4 \\ + u(20u + 10l^2 + 2p_1(\tau M)) - 12q_1(\xi)u)[M] \equiv 0 \pmod 4. \end{aligned}$$

Substituting $u = 0$ we get the conditions from Proposition 10.1.

Example 10.5. The quaternionic projective space $\mathbb{H}P^2$, the Grassmannian $G_{4,2}(\mathbb{C})$ and the homogeneous space $\mathrm{G}_2/\mathrm{SO}(4)$ are known to be quaternionic Kaehler manifolds and therefore the structure groups of their tangent bundles can be reduced to $\mathrm{Sp}(2) \times_{\{\pm 1\}} \mathrm{Sp}(1)$. Using our results we can say a little bit more.

The tangent bundle $\tau(\mathbb{H}P^2)$ has no complex structure (and hence no $\mathbb{H}_\lambda$-structure), but it has an $\mathbb{H}_\alpha$-structure if and only if $q_1(\alpha, 0) = 2ka$, where $k \equiv 1, 9 \pmod{24}$ and a is the generator of $H^4(\mathbb{H}P^2; \mathbb{Z})$.

There are infinitely many complex structures on $\tau(G_{4,2}(\mathbb{C}))$, but none of them extends to an $\mathbb{H}_\lambda$-structure. The tangent bundle admits $\mathbb{H}_\alpha$-structures with $q_1(\alpha, 0) = 2ka^2$, where $k \equiv 1, 9 \pmod{12}$ and a generates $H^2(G_{4,2}; \mathbb{Z})$.

In the case of $\tau(\mathrm{G}_2/\mathrm{SO}(4))$ our Proposition 10.4 can be used only to decide if there is an $\mathbb{H}_\alpha$-structure with $w_2(\alpha) = w_2(\mathrm{G}_2/\mathrm{SO}(4)) = 0$. And such a structure does not exist.

Consider next the complex manifolds $V_d = \{(z_0, \cdots, z_5) \in \mathbb{C}P^5;\ z_0^d + z_1^d + \cdots + z_5^d = 0\}$. Among them only $V_2 = G_{4,2}(\mathbb{C})$ and V_6 have almost quaternionic structures (see also [6]). The tangent bundle $\tau(V_6)$, like $\tau(G_{4,2})$, has infinitely many complex structures, none being a restriction of an $\mathbb{H}_\lambda$-structure. But $\tau(V_6)$ is an $\mathbb{H}_\alpha$-module for the bundles α with $q_1(\alpha, 0) = 2ka^2$, where $k \equiv 1 \pmod 4$ and a is a generator of $H^2(V_6; \mathbb{Z})$.

11. Reduction to $\mathrm{Spin}^c(5)$: a 3-field

Consider the composition

$$\mathrm{Sp}(2) \times_{\{\pm 1\}} U(1) = \mathrm{Spin}^c(5) \to \mathrm{SO}(5) \to \mathrm{SO}(8)$$

of the canonical map and the standard inclusion. Suppose that λ is a complex line bundle over M and that ζ is a 2-dimensional $\mathbb{H}_\lambda$-bundle. Then the 6-dimensional complex bundle $\lambda^* \otimes \Lambda^2\zeta$ has a trivial 1-dimensional summand given by the (dual of the) skew form $\zeta \otimes \zeta \to \lambda$ and a real structure given by a symmetric $\mathbb{C}$-valued form on $\lambda^* \otimes \Lambda^2\zeta$. So we may write $\lambda^* \otimes_{\mathbb{C}} \Lambda^2\zeta = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} \oplus \eta)$ for a 5-dimensional real vector bundle η.

Lemma 11.1. *The characteristic classes of η are:*

$$\begin{aligned} w_2(\eta) &= \rho_2(l); & w_4(\eta) &= \rho_2(l^2) + w_4(\zeta); \\ q_1(\eta; l) &= l^2 - c_2(\zeta) : & p_2(\eta) &= q_1^2(\eta; l) - 4c_4(\zeta). \end{aligned}$$

Proof. Apply Lemma 5.1 to $\eta \oplus \mathbb{R}$. □

Proposition 11.2. (Crabb and Steer). *Suppose that $w_2(M) = w_2(\xi) = \rho_2(l)$. Then ξ admits a* $\mathrm{Spin}^c(5)$*-structure with characteristic class l if and only if the following conditions hold.*

$$w_6(\xi) = 0; \qquad e(\xi) = 0;$$

$$(\frac{1}{2}(p_2(\xi) - q_1^2(\xi; l)) + \frac{1}{2}(2q_1^2(\xi;\, l) - q_1(\xi;\, l)p_1(\tau M) + q_1(\xi;\, l)l^2))[M] \equiv 0 \pmod 8.$$

Proof. The proof follows the same pattern as that of Proposition 5.2, only that instead of Proposition 3.2 we use Proposition 8.2. (If $e(\xi) = 0$, the last congruence is always satisfied $(\mathrm{mod}\, 3)$ by (4.1).) □

Next we describe when a given 8-dimensional vector bundle has a 3-dimensional subbundle. To do so, we will need the following statement:

Proposition 11.3. *Let $w_2(M) = \rho_2(l)$. Consider cohomology classes $u \in H^4(M; \mathbb{Z})$ and $z \in H^8(M; \mathbb{Z})$. Then there is a 5-dimensional vector bundle β over M with*

$w_2(\beta) = \rho_2(l)$, $q_1(\beta; l) = u$ and $p_2(\beta) = z$ if and only if $\mathrm{Sq}^2\rho_2(u) = \rho_2(lu)$ and there is a class $w \in H^8(M;\ \mathbb{Z})$ such that $4w = u^2 - z$ and the integer

$$w[M] \equiv \frac{1}{4}u(2u - p_1(\tau M) + l^2)[M] \pmod{12}.$$

Proof. This follows from Lemma 11.1 and Proposition 8.2. □

Corollary 11.4. (Čadek, Vanžura). *Let $w_2(M) = \rho_2(l)$ and let ξ satisfy $w_2(\xi) = 0$. Then ξ has a 3-dimensional subbundle α with characteristic classes $w_2(\alpha) = \rho_2(l)$ and $q_1(\alpha; l) = 2u$ if and only if the following conditions are satisfied.*

$$e(\xi) = 0; \qquad w_6(\xi) + w_4(\xi)\rho_2(l) + \rho_2(l^3) = 0; \qquad \mathrm{Sq}^2\rho_2 u = \rho_2(lu);$$

$$\frac{1}{4}(u(p_1(\tau M) - 2u - l^2))[M] \equiv 0 \pmod{12};$$

$$\frac{1}{4}(p_2(\xi) + q_1^2(\xi) - q_1(\xi)p_1(\tau M) - 3l^2q_1(\xi) + l^2p_1(\tau M) + l^4$$
$$+ u(20u + 10l^2 + 2p_1(\tau M) - 12q_1(\xi))[M] \equiv 0 \pmod{4}.$$

Proof. The conditions on u ensure the existence of a 3-dimensional vector bundle α with $q_1(\alpha; l) = 2u$. It remains to show that the other conditions ensure the existence of a 5-dimensional vector bundle β with $w_2(\beta) = \rho_2(l)$, $q_1(\beta; -l) = q_1(\xi) - q_1(\alpha; l) - l^2 = q_1(\xi) - 2u - l^2$ and $p_2(\beta) = p_2(\xi) - p_1(\beta)p_1(\alpha) = p_2(\xi) - (4u + l^2)(2q_1(\xi) - 4u - l^2)$. This can be done using the previous proposition. □

Remark 11.5. In [4] a certain automorphism φ of the group Spin(8) was found such that an 8-dimensional vector bundle ξ with the spin structure $\overline{\xi}$ admits reduction to $\mathrm{Sp}(2) \times_{\{\pm1\}} \mathrm{Sp}(1)$ if and only if a vector bundle η with spin structure $\overline{\eta} = \varphi^*(\overline{\xi})$ has a 3-dimensional subbundle ([4], Theorem 3.2). Moreover, φ^* in the integer cohomology of BSpin(8) was computed as

$$\varphi^*(q_1) = q_1; \qquad \varphi^*(e) = -q_2; \qquad \varphi^*(q_2) = -e.$$

Here q_2 is uniquely defined by the equation $p_2 = q_1^2 + 2e + 4q_2$. This relates reductions to $\mathrm{Sp}(2) \times_{\{\pm1\}} \mathrm{Sp}(1)$ with reductions to $\mathrm{Spin}(5) \times_{\{\pm1\}} \mathrm{Spin}(3)$ and allows us to deduce Proposition 10.4 from Corrolary 11.4.

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