ON 4-FIELDS AND 4-DISTRIBUTIONS
IN 8-DIMENSIONAL VECTOR BUNDLES
OVER 8-COMPLEXES

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Abstract. Let $\xi$ be an oriented 8-dimensional spin vector bundle over an 8-complex. Necessary and sufficient conditions for $\xi$ to have 4 linearly independent sections or to be a sum of two 4-dimensional spin vector bundles are given in terms of characteristic classes and higher order cohomology operations. On closed connected spin smooth 8-manifolds these operations can be computed.

1. Introduction. While the existence of 3-fields and 3-distributions in vector bundles over manifolds has been treated by many authors (see for instance [AD], [CS], [D], [K1], [K2], [N2], [R2], [T4]) and more or less completely solved, the results on the existence of 4-fields and 4-distributions are rare and not so complete (see [AR], [N1], [N2], [R1], [K1]). Especially, the case of $4k$-dimensional vector bundles over $4k$-manifolds seems to be difficult to deal with.

In this paper we solve the problem for 8-dimensional oriented spin vector bundles over 8-manifolds. The method of the Postnikov tower enables us to reveal that there is a generating class (see [T2]) in this case and that the obstructions can be computed using secondary and tertiary cohomology operations. The computation of these operations over closed connected smooth spin 8-manifolds has been carried out in our previous paper [CV3] which serves as an important preliminary material for this paper.

Our main results are Theorem 3.1 and Corollary 3.2 on the existence of 4-dimensional spin vector bundles over 8-manifolds in Section 3, Theorem 4.1 with Corollaries 4.2, 4.3 on the existence of 4-fields and Theorem 4.4 on the existence of 4-distributions in Section 4. Section 2 has auxiliary character and summarizes facts needed for the statements and proofs of the main results.

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2. Notation and auxiliary results. In this section we introduce notation and recall some facts about the singular cohomology of classifying spaces.

We will use $w_m(\xi)$ for the $m$-th Stiefel–Whitney class of the vector bundle $\xi$, $p_m(\xi)$ for the $m$-th Pontrjagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle $\xi$ the symbol $c_m(\xi)$ denotes the $m$-th Chern class. The letters $w_m, p_m, e$ and $c_m$ will stand for the characteristic classes of the universal vector bundles over the classifying spaces $BSO(n)$, and $BU(n)$, respectively. The pullbacks of the Stiefel–Whitney, Pontrjagin and Euler classes in $H^*(BSpin(n))$ will be denoted by the same letters.

The mappings $i_* : H^*(X, \mathbb{Z}_2) \to H^*(X, \mathbb{Z}_4)$ and $\rho_m : H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ are induced from the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and the reduction mod $m$, respectively. We will also use the Steenrod operations $Sq^i : H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2)$ and $P^n_i : H^n(X; \mathbb{Z}_3) \to H^{n+4i}(X; \mathbb{Z}_3)$.

We say that $x \in H^*(X; \mathbb{Z})$ is an element of order $n$ ($n = 2, 3, 4, \ldots$) if and only if $x \neq 0$ and $n$ is the least positive integer such that $nx = 0$ (if it exists).

The Eilenberg–MacLane space with the $n$-th homotopy group $G$ will be denoted $K(G, n)$, and $\iota_n$ will stand for the fundamental class in $H^n(K(G, n); G)$. Writing the fundamental class, it will be always clear which group $G$ we have in mind.

Now we summarize some facts about the groups $Spin(3)$ and $Spin(4)$ and the cohomologies of their classifying spaces. It is well known that $Spin(3)$ is isomorphic with the group $Sp(1)$ of unit quaternions. So, identifying $Spin(3) \times Spin(3)$ with $Sp(1) \times Sp(1)$, we can define a homomorphism $\vartheta : Spin(3) \times Spin(3) \to SO(4)$ using the representation

$$(\alpha, \beta) \cdot v = \alpha v \bar{\beta},$$

where $\alpha, \beta \in Sp(1), \ v \in \mathbb{H} \cong \mathbb{R}^4$ and $\bar{\cdot}$ stands for the conjugation. Since the kernel of this homomorphism is $\{(1, 1), (-1, -1)\} \cong \mathbb{Z}_2$, there is an isomorphism

$$\vartheta : Spin(3) \times Spin(3) \to Spin(4).$$

It induces a homeomorphism on the level of classifying spaces which will be denoted by the same letter.

**Lemma 2.1.** The cohomology ring of $BSpin(3)$ is

$$H^*(BSpin(3); \mathbb{Z}) \cong \mathbb{Z}[r],$$

where $p_1 = 4r$.

The cohomology ring of $BSpin(4)$ is

$$H^*(BSpin(4); \mathbb{Z}) \cong \mathbb{Z}[q, s],$$

where $q$ and $s$ are defined with the aid of the first Pontrjagin class and the Euler class by the relations

$$p_1 = 2q, \ e = 2s - q.$$

Moreover,

$$\vartheta^*(q) = r \otimes 1 + 1 \otimes r, \ \vartheta^*(s) = 1 \otimes r.$$
Proof. The cohomology of $B\text{Spin}(3)$ is well known. The existence of $q$ and $s \in \text{H}^4(B\text{Spin}(4); \mathbb{Z})$ follows from the relation

$$\rho_4 p_1 = \mathfrak{P} w_2 + i_* w_4,$$

where $\mathfrak{P}$ is the Pontrjagin square, which in $\text{H}^*(B\text{Spin}(4); \mathbb{Z})$ reads as

$$\rho_4(p_1 + 2e) = 0.$$

Since $\vartheta^*: \text{H}^*(B\text{Spin}(4); \mathbb{Z}) \to \text{H}^*(B\text{Spin}(4) \times B\text{Spin}(3); \mathbb{Z}) \cong \mathbb{Z}[r \otimes 1, 1 \otimes r]$, it is sufficient to prove the last part of our lemma.

Computing $\bar{\vartheta}^*: Sp(1) \times Sp(1) \to SO(4)$ on the standard tori and using the classical results of Borel and Hirzebruch (see [BH]), we get easily

$$\bar{\vartheta}^*(p_1) = 2(r \otimes 1 + 1 \otimes r), \quad \bar{\vartheta}^* e = 1 \otimes r - r \otimes 1.$$

Hence

$$\vartheta^* q = r \otimes 1 + 1 \otimes r, \quad \vartheta^* s = 1 \otimes r.$$

Further, we recall the definition of two higher order cohomology operations introduced in [CV3].

**Definition 2.2.** Let $\Sigma$ denote the secondary cohomology operation associated with the relation

$$\text{Sq}^2 \circ \text{Sq}^2 \rho_2 = 0$$

on integral cohomology classes of dimension 4.

Let $\Phi$ be the tertiary cohomology operation associated with the relation

$$i_* \text{Sq}^2 \circ \Sigma = 0$$

on integral cohomology classes of dimension 4 and uniquely determined by the properties

$$\Phi(r) = 0 \quad \Phi(2r) = -\rho_4 r^2$$

for $r \in \text{H}^4(B\text{Spin}(3); \mathbb{Z})$.

Let $\Omega$ be the secondary cohomology operation associated with the relation

$$i_* \text{Sq}^2 \circ \text{Sq}^2 = 0$$

in dimension 5.

Let $X$ be a CW-complex. The operations $\Sigma$, $\Omega$ and $\Phi$ are defined on the sets

$$\text{Def}(\Sigma, X) = \{x \in \text{H}^4(X; \mathbb{Z}); \text{Sq}^2 \rho_2 x = 0\}, \text{Def}(\Omega, X) = \{x \in \text{H}^5(X; \mathbb{Z}); \text{Sq}^2 \rho_2 x = 0, 0 \in \Sigma(x)\},$$

respectively. The values of $\Sigma(x)$ form a subset of $\text{H}^7(X; \mathbb{Z}_2)$, while $\Omega(x)$ and $\Phi(x)$ are the subsets of $\text{H}^8(X; \mathbb{Z}_2)$. The indeterminacies of $\Sigma$ and $\Omega$ are $\text{Indet}(\Sigma, X) = \text{Sq}^2 \text{H}^5(X; \mathbb{Z}_2)$ and $\text{Indet}(\Omega, X) = i_* \text{Sq}^2 \text{H}^6(X; \mathbb{Z}_2)$, respectively. The indeterminacy of the remaining operation is $\text{Indet}(\Phi, X) = \Omega \text{Def}(\Omega, X)$.

For further properties of $\Sigma$, $\Omega$ and $\Phi$ we refer to [CV3]. Particularly, the following formula holds

$$\Phi(x + y) = \Phi(x) + \Phi(y) - \rho_4(xy)$$

for all $x, y \in \text{Def}(\Phi, X)$ ([CV3, Lemma 3.9]).
Lemma 2.3. For \( q \) and \( s \in H^4(BSpin(4); \mathbb{Z}) \) there are
\[
\Sigma(q) = 0 \quad \Sigma(s) = 0 \\
\Phi(q) = \rho_4(s^2 - qs) \quad \Phi(s) = 0.
\]

Proof. Since \( H^5(BSpin(4); \mathbb{Z}_2) = H^7(BSpin(4); \mathbb{Z}_2) = 0 \), we have \( \Sigma(q) = \Sigma(s) = 0 \) and \( \text{Indet}(\Phi, BSpin(4)) = 0 \). Since \( \Phi(r) = 0 \) for \( r \in H^4(BSpin(3); \mathbb{Z}) \), using the formula for \( \Phi(x+y) \), we get
\[
\Phi(s) = \Phi(\vartheta^*)^{-1}(1 \otimes r) = -(\vartheta^*)^{-1}\Phi(1 \otimes r) = -\rho_4(q-s)s = \rho_4(s^2 - qs).
\]

Using the first Steenrod operation with \( \mathbb{Z}_3 \) coefficients we obtain other two relations in \( H^8(BSpin(4); \mathbb{Z}_3) \).

Lemma 2.4. For \( q \) and \( s \in H^4(BSpin(4); \mathbb{Z}) \) the formulas
\[
P^1_3 \rho_3 q + \rho_3 q^2 = \rho_3(s^2 - sq), \\
P^1_3 \rho_3 s + \rho_3 s^2 = 0
\]
hold.

Proof. According to the proof of Theorem 3.8 in [CV3] we know that
\[
P^1_3 \rho_3 r + \rho_3 r^2 = 0
\]
in \( H^8(BSpin(3); \mathbb{Z}_3) \). It yields immediately the latter relation. Further, according to [BS]
\[
P^1_3 \rho_3 p_1 = \rho_3(2p_2 - p_1^2)
\]
where \( p_2 = e^2 \) in \( H^8(BSpin(4); \mathbb{Z}) \). Substitute \( p_1 = 2q \) and \( e = 2s - q \) to get
\[
P^1_3 \rho_3 2q = \rho_3(8s^2 - 8sq - 2q^2).
\]
It yields the former relation in our lemma.

Finally, we remind the cohomology of \( BSpin(8) \) and the spin characteristic classes.

Lemma 2.5. The cohomology rings of \( BSpin(8) \) are
\[
H^*(BSpin(8); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon]
\]
and
\[
H^*(BSpin(8); \mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e, \delta w_6]/(2\delta w_6)
\]
where $q_1$, $q_2$ and $\varepsilon$ are defined by the relations

$$p_1 = 2q_1, \quad p_2 = q_1^2 + 2\varepsilon + 4q_2, \quad \rho_2 q_2 = \varepsilon.$$ 

**Proof.** See [Q] and [CV2].

Let $\xi$ be an oriented 8-dimensional vector bundle over a CW-complex $X$ given by the homotopy class of some mapping $\xi : X \to BSO(8)$. $\xi$ has a spin structure iff $w_2(\xi) = 0$. If some lifting $\bar{\xi} : X \to BSpin(8)$ is fixed, we talk about a given spin structure. In this case we can define the spin characteristic classes

$$q_1(\xi) = \bar{\xi}^* q_1, \quad q_2(\xi) = \bar{\xi}^* q_2.$$

The first spin characteristic class is always independent of the choice of $\bar{\xi}$. Moreover, if $H^4(X; \mathbb{Z})$ has no element of order 4, then it is uniquely determined by the relations

$$2q_1(\xi) = p_1(\xi), \quad \rho_2 q_1(\xi) = w_4(\xi).$$

The second spin characteristic class is independent of the spin structure $\bar{\xi}$ if $X$ is simply connected or $H^8(X; \mathbb{Z}) \cong \mathbb{Z}$. In the case of 8-dimensional manifold $q_2(\xi)$ is uniquely determined by the relation

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8\varepsilon(\xi).$$

See [CV2].

### 3. Four-dimensional spin vector bundles over 8-complexes.

The previous section enables us to prove the following result on the existence of 4-dimensional spin vector bundles over CW-complexes of dimension 8.

**Theorem 3.1.** Let $X$ be a connected CW-complex of dimension $\leq 8$ and let $P, E \in H^4(X; \mathbb{Z})$. Then there exists an oriented 4-dimensional vector bundle $\eta$ over $X$ with

$$w_2(\eta) = 0, \quad p_1(\eta) = P, \quad e(\eta) = E$$

if and only if there are $Q, S \in H^4(X; \mathbb{Z})$ such that

1. $P = 2Q, \ E = 2S - Q$
2. $Sq^2 \rho_2 Q = Sq^2 \rho_2 S = 0$
3. $0 \in \Sigma(Q), \ 0 \in \Sigma(S)$
4. $\rho_4(S^2 - QS) \in \Phi(Q), \ 0 \in \Phi(S)$
5. $P_4^1 \rho_3 Q + \rho_3 Q^2 = \rho_3(S^2 - QS), \ P_4^1 \rho_3 S + \rho_3 S^2 = 0.$

**Proof.** Every oriented 4-dimensional spin vector bundle $\eta$ over a CW-complex $X$ is determined by a mapping $\eta : X \to BSpin(4)$. Let $\eta$ have the prescribed characteristic classes. Then $\eta^*(p_1) = P_1$ and $\eta^*(e) = E$. Put $Q = \eta^*(q)$ and $S = \eta^*(s)$. Now Lemmas 2.1, 2.3 and 2.4 imply that $Q$ and $S$ satisfy conditions (1) – (5).

Conversely, let there be $Q$ and $S$ such that (1) – (5) hold. Consider the fibration

$$F \to BSpin(4) \xrightarrow{\alpha} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)$$
where \( \alpha \) is determined by elements \( q, s \in H^4(B\text{Spin}(4);\mathbb{Z}) \). Next consider the mapping \( f : X \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \) determined by elements \( Q, S \in H^4(X;\mathbb{Z}) \). Then \( \eta \) with the prescribed properties exists if \( f \) can be lifted in the fibration \( \alpha \).

\[
\begin{array}{c}
\text{BSpin}(4) \\
\downarrow \alpha \\
K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)
\end{array}
\]

Therefore we will build the Postnikov tower for the fibration \( \alpha \). The fibre \( F \) is 4-connected and the next homotopy groups are

\[
\pi_5(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \pi_6(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \pi_7(F) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}.
\]

The first invariants can be easily obtained from the Serre exact sequence for the fibration \( F \to B\text{Spin}(4) \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \). They are \( Sq^2\rho_{24} \otimes 1 \) and \( 1 \otimes Sq^2\rho_{24} \).

The universal example for the secondary operation \( \Sigma \) is the fibration

\[
\begin{array}{c}
K(\mathbb{Z}_2, 5) \xrightarrow{j_1} Y_{1} \xrightarrow{\pi_1} K(\mathbb{Z}, 4)
\end{array}
\]

induced from the path fibration \( K(\mathbb{Z}_2, 5) \to PK(\mathbb{Z}_2, 6) \to K(\mathbb{Z}_2, 6) \) by the mapping \( Sq^2\rho_{24} : K(\mathbb{Z}, 4) \to K(\mathbb{Z}_2, 6) \). That is why the first stage of the Postnikov tower is the product \( Y_1 \times Y_1 \).

\[ F_1 \xrightarrow{\alpha_1} \mathbb{w}K(\mathbb{Z}_2, 5) \times K(\mathbb{Z}_2, 5) \]

\[ F_1 \xrightarrow{\alpha} Y_1 \times Y_1 \]

\[ K(\mathbb{Z}_2, 5) \times K(\mathbb{Z}_2, 5) \]

The next invariants are \( \sigma \otimes 1, 1 \otimes \sigma \in H^7(Y_1 \times Y_1;\mathbb{Z}_2) \) where \( \sigma \in H^7(Y_1;\mathbb{Z}_2) \) is the element defining the operation \( \Sigma \). The universal example for the tertiary cohomology operation \( \Phi \) is the fibration

\[
\begin{array}{c}
K(\mathbb{Z}_2, 6) \xrightarrow{j_2} Y_{2} \xrightarrow{\pi_2} Y_{1}
\end{array}
\]

induced from the path fibration \( K(\mathbb{Z}_2, 6) \to PK(\mathbb{Z}_2, 7) \to K(\mathbb{Z}_2, 7) \) by the mapping \( \sigma : Y_1 \to K(\mathbb{Z}_2, 7) \). Hence the second stage of the Postnikov tower is the product \( Y_2 \times Y_2 \).

\[ F_2 \xrightarrow{\alpha_2} \mathbb{w}K(\mathbb{Z}_2, 6) \times K(\mathbb{Z}_2, 6) \]

\[ F_2 \xrightarrow{\alpha} \mathbb{w}Y_2 \times Y_2 \]

\[ Y_1 \times Y_1 \]

\[ Y_1 \times Y_1 \]

\[ \mathbb{w}K(\mathbb{Z}_2, 7) \times K(\mathbb{Z}_2, 7) \]
In the stage \( Y_2 \times Y_2 \) there are two \( \mathbb{Z}_4 \)-invariants and two \( \mathbb{Z}_3 \)-invariants in dimension 8. \( (F_2 \text{ is 6-connected and } \pi_7(F_2) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4) \)

First, consider \( \mathbb{Z}_4 \)-coefficients. According to \([CV3, \text{Section 3}]\), \( H^8(Y_2; \mathbb{Z}_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \) with the generators \( \rho_4 \pi_2^* \pi_1^* t_4 \) and \( \phi \) which is the element defining the tertiary cohomology operation \( \Phi \). Using the Künneth formula for \( \mathbb{Z}_2 \) coefficients, the exact sequence associated with the short exact sequence \( 0 \rightarrow \mathbb{Z}_2 \rightarrow \mathbb{Z}_4 \rightarrow \mathbb{Z}_2 \rightarrow 0 \) and the knowledge of \( H^*(Y_2; \mathbb{Z}_2) \) and \( H^*(Y_2; \mathbb{Z}_4) \) from \([CV3]\) (see Section 3), we get that \( H^8(Y_2 \times Y_2; \mathbb{Z}_4) \cong (\mathbb{Z}_4)^5 \) with the generators \( \rho_4 \pi_2^* \pi_1^* t_4 \otimes 1, 1 \otimes \rho_4 \pi_2^* \pi_1^* t_4^2, \rho_4(\pi_2^* \pi_1^* t_4 \otimes \pi_2^* \pi_1^* t_4) \), \( \varphi \otimes 1 \) and \( 1 \otimes \varphi \). Moreover,

\[
\alpha_2^*(\rho_4 \pi_2^* \pi_1^* t_4^2 \otimes 1) = \rho_4 q^2, \quad \alpha_2^*(1 \otimes \rho_4 \pi_2^* \pi_1^* t_4^2) = \rho_4 s^2, \\
\alpha_2^*(\rho_4(\pi_2^* \pi_1^* t_4 \otimes \pi_2^* \pi_1^* t_4)) = \rho_4(qs),
\]

and using Lemma 2.3

\[
\alpha_2^*(\varphi \otimes 1) = \Phi(q) = \rho_4(s^2 - qs) \\
\alpha_2^*(1 \otimes \varphi) = \Phi(s) = 0.
\]

Hence the invariants are \( \varphi \otimes 1 - 1 \otimes \rho_4 \pi_2^* \pi_1^* t_4^2 + \rho_4(\pi_2^* \pi_1^* t_4 \otimes \pi_2^* \pi_1^* t_4) \) and \( 1 \otimes \varphi \).

Analogously, using Lemma 2.4 we find that the \( \mathbb{Z}_3 \)-invariants are \( P_3^1(\rho_3 \pi_2^* \pi_1^* t_4 \otimes 1) + \rho_3(\pi_2^* \pi_1^* t_4 \otimes 1 - \rho_3(1 \otimes \pi_2^* \pi_1^* t_4 - \pi_2^* \pi_1^* t_4 \otimes \pi_2^* \pi_1^* t_4) \) and \( P_3^1(1 \otimes \rho_3 \pi_2^* \pi_1^* t_4) \).

This shows that \( f : X \rightarrow K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4) \) given by the cohomology classes \( Q \) and \( S \) can be lifted to the third stage of the Postnikov tower if and only if the conditions \( (2) - (5) \) are satisfied. But because \( \dim X = 8 \), we can see that these conditions are necessary and sufficient for the existence of a lift of \( f \) to \( B\text{Spin}(4) \) in the fibration \( \alpha \).

Now we apply Theorem 3.1 to a closed connected smooth spin manifold of dimension 8.

**Corollary 3.2.** Let \( M \) be a closed connected smooth spin manifold of dimension 8 and let \( P, E \in H^4(M; \mathbb{Z}) \). Then there exists an oriented 4-dimensional vector bundle \( \eta \) over \( M \) with

\[
w_2(\eta) = 0, \quad p_1(\eta) = P, \quad e(\eta) = E
\]

if and only if there are \( Q, S \in H^4(M; \mathbb{Z}) \) such that

1. \( P = 2Q, \ E = 2S - Q \)
2. \( S q^2 \rho_2 Q = S q^2 \rho_2 S = 0 \)
3. \( \{4E^2 + P^2 - 2Pp_1(M)\}[M] \equiv 0 \mod 64 \)
4. \( \{2(2E + P)p_1(M) - (2E + P)^2\}[M] \equiv 0 \mod 128 \)
5. \( P_3^1 \rho_3 E = \rho_3 EP \).

**Proof.** Let \( M \) be a closed connected smooth spin manifold of dimension 8. Theorems 5.3 and 5.5 in \([CV3]\) assert that

\[
\Phi(z) = \rho_4 \frac{1}{2} \{z q_1(M) - z^2\}
\]

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for all \( z \in \text{Def}(\Phi, M) \) and \( 0 \in \Sigma(z) \)
for all \( z \in \text{Def}(\Sigma, M) \). Using this and the fact that \( H^3(M; \mathbb{Z}) \cong \mathbb{Z} \) we will show that (iii) is equivalent to (4) of Theorem 3.1 on \( M \).

\[
4E^2 + P^2 - 2Pp_1(M) = 4(2S - Q)^2 + (2Q)^2 - 8Qq_1(M) = 16S^2 + 8Q^2 - 16QS - 8Qq_1(M) = 8\{2(S^2 - QS) - (Qq_1(M) - Q^2)\}.
\]

Next
\[
2(2E + P)p_1(M) = (2E + P)^2 = 16(Sq_1(M) - S^2).
\]

Similarly, substituting for \( P \) and \( E \) in (iv) we get (5) of Theorem 3.1.

\[
P_3^1\rho_4P - 2\rho_3E^2 + \rho_3P^2 = 2P_3^1\rho_3Q + 2\rho_3Q^2 - 8\rho_3S^2 + 8\rho_3SQ = 2\{P_3^1\rho_3Q + \rho_3Q^2 - \rho_3(S^2 - SQ)\}
\]

Further, using the fact that \( P_3^1\rho_3P = 2\rho_3E^2 - \rho_3P^2 \), we have
\[
P_3^1\rho_3E - \rho_3EP = -2P_3^1\rho_3E - \rho_3EP - P_3^1\rho_3P + 2\rho_3E^2 - \rho_3P^2 = -4P_3^1\rho_3S - \rho_3(2S - Q)^2 - 2\rho_3(2S - Q)Q - 4\rho_3Q^2 = -P_3^1\rho_3S - \rho_3S^2.
\]

4. Four linearly independent sections and 4-distributions. In this section we will find necessary and sufficient conditions for an oriented 8-dimensional spin vector bundle over an 8-complex to have 4 linearly independent sections or to be a sum of two 4-dimensional spin vector bundles.

**Theorem 4.1.** Let \( \xi \) be an oriented 8-dimensional vector bundle over a connected CW-complex \( X \) of dimension \( \leq 8 \) with \( w_2(\xi) = 0 \). Then \( \xi \) has 4 linearly independent sections if and only if for some spin structure on \( \xi \) there is \( S \in H^4(X; \mathbb{Z}) \) such that the following conditions are satisfied:

1. \( w_6(\xi) = 0 \), \( Sq_2^2 = 0 \)
2. \( 0 \in \Sigma(q_1(\xi)) \), \( 0 \in \Sigma(S) \)
3. \( e(\xi) = 0 \)
4. \( q_2(\xi) = S^2 - q_1(\xi)S \)
5. \( \rho_3q_2(\xi) \in \Phi(q_1(\xi)) \)
6. \( 0 \in \Phi(S) \)
7. \( P_3^1\rho_3S + \rho_3S^2 = 0 \).

**Proof.** The vector bundle \( \xi \) over \( X \) has 4 linearly independent sections if and only if the mapping \( \xi : X \to B\text{Spin}(8) \) which is determined up to homotopy by the spin structure of the vector bundle can be lifted in the standard fibration

\[
V_{8,4} \longrightarrow B\text{Spin}(4) \xrightarrow{\kappa} B\text{Spin}(8).
\]

So, we will build the Postnikov tower for this fibration.
The Stiefel manifold $V_{8,4}$ is 3-connected and the next homotopy groups are

$$\pi_4(V_{8,4}) \cong \mathbb{Z}, \quad \pi_5(V_{8,4}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \pi_6(V_{8,4}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \pi_7(V_{8,4}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$ 

Moreover, we have

$$\kappa^*(q_1) = q, \quad \kappa^*(e) = 0, \quad \kappa^*(p_2) = (2s - q)^2,$$

and hence

$$\kappa^*(q_2) = s^2 - sq.$$ 

The first invariant lies in $H^5(BSpin(8); \mathbb{Z}) \cong 0$ and that is why it is zero. So the first stage is $BSpin(8) \times K(Z, 4)$ and the mapping $\mu = (\kappa, \beta) : BSpin(4) \to BSpin(8) \times K(Z, 4)$ can be chosen to be a fibration in such a way that

$$\beta^*(t_4) = s.$$ 

The next invariants can be easily obtained from the Serre exact sequence for the fibration $\mu$. They are $u_6 \otimes 1$ and $1 \otimes Sq^2 \rho_2 t_4$. So the first stage of the Postnikov tower for the fibration $\mu$ is $E_1 \times Y_1$, where $Y_1$ is the universal example for the operation $\Sigma$ and $E_1 \xrightarrow{\kappa_1} BSpin(8)$ is the fibration induced from the path fibration $K(Z_2, 5) \rightarrow PK(Z_2, 6) \rightarrow K(Z_2, 6)$ by the mapping $u_6 = Sq^2 \rho_2 q_1$. We have the following commutative diagram

$$\begin{array}{ccc}
K(Z_2, 5) \times K(Z_2, 5) & \longrightarrow & \mathfrak{v}K(Z_2, 5)^2 \\
\downarrow & & \downarrow \\
BSpin(4) & \xrightarrow{\gamma_1} & \mathfrak{v}E_1 \times Y_1 \\
\downarrow \mu \downarrow & & \downarrow \kappa_1 \times \pi_1 \\
BSpin(8) \times K(Z, 4) & = & BSpin(8) \times K(Z, 4) \xrightarrow{q_1 \times \id} \mathfrak{v}K(Z, 4)^2 \xrightarrow{\eta_2 \circ \rho_2 \rho_1} \mathfrak{v}K(Z_2, 6)^2
\end{array}$$

where $f_1$ exists due to the fact that $Sq^2 \rho_2 \kappa_1^* q_1 = 0$. The next invariants are generators of $H^7(E_1 \times Y_1; \mathbb{Z}_2)$ and these are $f_1^*(\sigma) \otimes 1 = \Sigma(\kappa_1^* q_1 \otimes 1)$ and $1 \otimes \sigma = 1 \otimes \Sigma(\pi_1^* t_4)$. Consequently, the second stage of the Postnikov tower has the form $E_2 \times Y_2$ where $Y_2$ is the universal example for the operation $\Phi$ and $E_2 \xrightarrow{\kappa_2} E_1$ is the fibration induced from the path fibration $K(Z_2, 6) \rightarrow PK(Z_2, 7) \rightarrow K(Z_2, 7)$ by the mapping $f_1^*(\sigma) : E_1 \rightarrow K(Z_2, 7)$, which is the same as the fibration induced from the $K(Z_2, 6) \rightarrow Y_2 \rightarrow Y_1$ by the mapping $f_1$. 

$$\begin{array}{ccc}
K(Z_2, 6) \times K(Z_2, 6) & \xrightarrow{\sigma \otimes 1} & \mathfrak{v}K(Z_2, 6)^2 \\
\downarrow & & \downarrow \\
BSpin(4) & \xrightarrow{\gamma_2} & \mathfrak{v}E_2 \times Y_2 \\
\downarrow \gamma_1 \downarrow & & \downarrow \kappa_2 \times \pi_2 \\
E_1 \times Y_1 & \xrightarrow{f_1 \times \id} & \mathfrak{v}Y_1 \times Y_1 \xrightarrow{\sigma \otimes 1} \mathfrak{v}K(Z_2, 7)^2
\end{array}$$
The mapping $f_2$ exists since $\Sigma(k_2^1 q_1) = 0$.

Further invariants lie in $H^8(E_2 \times Y_2; \mathbb{Z})$, $H^8(E_2 \times Y_2; \mathbb{Z}_4)$ and $H^8(E_2 \times Y_2; \mathbb{Z}_3)$.

The cohomologies of $E_2$ were computed in the proof of Theorem 4.1 in [CV3]. Hence, we have

$$H^8(E_2 \times Y_2; \mathbb{Z}) \cong \mathbb{Z}$$

with generators $k_2^1 q_1 \otimes 1, 1 \otimes \pi_2^1 \pi_1^2 \xi_2, k_2^1 q_1 \otimes \pi_2^1 \pi_1^4, k_2^1 q_2 \otimes 1, k_2^1 e \otimes 1$.

Since $H^8(BSpin(4); \mathbb{Z}) \cong \mathbb{Z}^3$ with generators $q_2^2, s^2$ and $qs$, the integral invariants are the generators of $\text{ker} \gamma_1$

$$A = k_2^1 q_2 \otimes 1 - 1 \otimes \pi_2^1 \pi_1^2 + k_2^1 q_1 \otimes \pi_2^1 \pi_1^4.$$ 

Next,

$$H^8(E_2 \times Y_2; \mathbb{Z}_4) \cong (\mathbb{Z}_4)^7$$

with generators $k_2^1 \rho q_1 \otimes 1, k_2^1 \rho q_2 \otimes 1, k_2^1 \rho e \otimes 1, f_2^2(\varphi) \otimes 1, 1 \otimes \pi_2^1 \pi_1^2 \xi_2$, $1 \otimes \varphi, k_2^1 \rho q_1 \otimes \pi_2^1 \pi_1^4, H^8(BSpin(4); \mathbb{Z}_4) \cong (\mathbb{Z}_4)^3$ with generators $\rho q_2^2, \rho s^2, \rho q s$.

So using Lemma 2.3, we get that $\text{ker} \gamma_2$ is generated by

$$\rho_4 A, \rho_4 B, 1 \otimes \varphi, f_2^2(\varphi) \otimes 1 - 1 \otimes \pi_2^1 \pi_1^2 \xi_2 + k_2^1 q_1 \otimes \pi_2^1 \pi_1^4.$$

It remains to compute the $\mathbb{Z}_3$-invariant.

$$H^8(E_2 \times Y_2; \mathbb{Z}_3) \cong (\mathbb{Z}_3)^6$$

with generators $k_2^1 \rho q_2 \otimes 1, k_2^1 \rho e \otimes 1, k_2^1 \rho q_1 \otimes \pi_2^1 \pi_1^4, 1 \otimes \pi_2^1 \pi_1^4 \xi_3, 1 \otimes \pi_2^1 \pi_1^4 P_3^1 \rho s_4$. So using Lemma 2.4, we obtain that $\text{ker} \gamma_2$ is generated by

$$\rho_3 A, \rho_3 B, 1 \otimes \pi_2^1 \pi_1^4 P_3^1 \rho s_4 + 1 \otimes \pi_2^1 \pi_1^4 \xi_3.$$ 

Now, because $\dim X \leq 8$, we can immediately see that the vector bundle $\xi : X \to BSpin(8)$ has 4 linearly independent sections if and only if all the conditions (1) – (7) of Theorem 4.1 are satisfied.

**Corollary 4.2.** Let $\xi$ be an oriented 8-dimensional vector bundle over a closed connected smooth spin 8-manifold $M$ with $w_2(\xi) = 0$. Then $\xi$ has 4 linearly independent sections if and only if there is $S \in H^4(M; \mathbb{Z})$ and the following conditions are satisfied:

1. $w_6(\xi) = 0, S q^2 \rho S = 0$
2. $e(\xi) = 0$
3. $4 p_2(\xi) - p_1^2(\xi) = 16 S^2 - 8 p_1(\xi) S$
4. $\{2 p_1(M) p_1(\xi) - p_1^2(\xi) - 4 p_2(\xi)\}[M] \equiv 0 \mod 64$
5. $\{p_1(M) S - 2 S^2\}[M] \equiv 0 \mod 16$
6. $P_3^1 \rho s + \rho S^2 = 0$.

**Proof.** It is easy to show that in the case $X = M$ the conditions (1) – (7) of Theorem 4.1 are equivalent to the conditions (1) – (6) of Corollary. It suffices to use the relations $0 \in \Sigma(z)$ for $z \in \text{Def}(\Sigma, M)$ (see [CV3], Theorem 5.5), $p_1(\xi) = 2q_1(\xi)$, $p_2(\xi) = q_1^2(\xi) + 2 e(\xi) + 4 q_2(\xi)$, $\Phi(\xi) = \rho_4 2 \{z q_1(M) - z^2\}$ for $z \in \text{Def}(\Phi, M)$ and the fact that $H^8(M; \mathbb{Z}) \cong \mathbb{Z}$.

Now we shall formulate nontrivial sufficient conditions for the existence of 4 linearly independent vector fields in tangent bundles without any reference to the element $S$. 

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Corollary 4.3. Let $M$ be a closed connected smooth spin manifold of dimension 8. If

(i) $w_6(M) = 0$
(ii) $e(M) = 0$
(iii) $\{4p_2(M) - p_1^2(M)\}[M] \equiv 0 \mod 128$

and there is $k \in \mathbb{Z}$ such that

(iv) $4p_2(M) = (2k - 1)^2 p_1^2(M)$
(v) $k(k + 2)p_2(M)[M] \equiv 0 \mod 3$,

then $M$ has 4 linearly independent vector fields.

Proof. We will prove that for $S = kq_1(M)$ the conditions (1) – (6) of Corollary 4.2 hold. Obviously, (1) and (2) are satisfied. As for (3)

$$4p_2(M) - p_1^2(M) = (2k - 1)^2 p_1^2(M) - p_1^2(M) = (4k^2 - 4k)p_1^2 = 16S^2 - 8p_1(M)S.$$  

(4) follows from the fact that

$$4p_2(M) - p_1^2(M) = 16k(k - 1)q_1^2(M)$$

and

$$p_2q_1^2(M) = w_4^2(M) = w_8(M) = 0.$$  

Because of

$$8(p_1(M)S - 2S^2) = -(2k - 1)^2 p_1^2(M) + p_1^2(M) = p_1^2(M) - 4p_2(M),$$

(iii) implies (5). Finally, the condition (6) follows from (v). On spin manifolds

$$\hat{A}[M] = \frac{1}{2^7 \cdot 45}\{7p_1^2(M) - 4p_2(M)\}[M]$$

is an integer (see [H], Theorem 26.3.1) and that is why $p_1^2(M)[M] \equiv p_2(M)[M] \mod 3$. So we have

$$4(P_1^3p_3S + \rho_3S^2) = 2kP_1^3\rho_3p_1(M) + k^2\rho_3p_1^2(M)$$

$$= 4kp_3p_2(M) - 2k\rho_3p_1^2(M) + k^2\rho_3p_1^2(M)$$

and

$$\{4kp_2(M) - 2kp_1^2(M) + k^2p_1^2(M)\}[M] \equiv (4k - 2k + k^2)p_2(M)[M]$$

$$\equiv k(k + 2)p_2(M)[M] \mod 3.$$  

Now, we will state and prove a result on the existence of 4-dimensional subbundles (4-distributions) in an 8-dimensional spin vector bundle. Since we want to avoid technical difficulties with the use of the Postnikov tower, our assumptions are a little more restrictive than in the case of Theorem 4.1 or Corollary 4.2.
**Theorem 4.4.** Let $M$ be a closed connected smooth spin manifold of dimension 8 such that $H^4(M; \mathbb{Z})$ has no element of order 4. Let $\xi$ be an oriented 8-dimensional vector bundle over $M$ with $w_2(\xi) = 0$. Then $\xi$ is the sum of two 4-dimensional spin vector bundles if and only if there are $S_1, S_2, Q_1, Q_2 \in H^4(M; \mathbb{Z})$ and the following conditions are satisfied for $n = 1, 2$.

1. $p_1(\xi) = 2(Q_1 + Q_2)$
2. $e(\xi) = (2S_1 - Q_1)(2S_2 - Q_2)$
3. $p_2(\xi) = (2S_1 - Q_1)^2 + (2S_2 - Q_2)^2 + 4Q_1Q_2$
4. $Sq^2\rho_2Q_n = Sq^2\rho_2S_n = 0$
5. $\{S_n p_1(M) - 2S^2_n\}[M] = 0 \mod 16$
6. $\{4S^2_n - 4Q_nS_n\}[M] = \{Q_n p_1(M) - 2Q^2_n\}[M] \mod 16$
7. $P^1_3\rho_3Q_n + \rho_3Q^2_n = P_3(S_n^2 - Q_nS_n)$
8. $P^1_3\rho_3S_n + \rho_3S^2_n = 0.$

**Proof.** First, we show that all the conditions are necessary. Let $\xi = \xi_1 \oplus \xi_2$ where $\xi_1$ and $\xi_2$ are 4-dimensional spin vector bundles with the Euler and the first Pontrjagin classes $2S_1 - Q_1, 2Q_1$ and $2S_2 - Q_2, 2Q_2$, respectively. According to Theorem 3.1 the classes $Q_1, Q_2, S_1, S_2 \in H^4(M; \mathbb{Z})$ satisfy the conditions (4) – (8). (Realize that $\Phi(\xi) = \rho_1\frac{1}{2}(zq_1(M) - z^2)$ on $\text{Def}(\Phi, M)$.) Moreover,

\[ p_1(\xi) = p_1(\xi_1) + p_1(\xi_2) \]
\[ e(\xi) = e(\xi_1) \cdot e(\xi_2) \]
\[ p_2(\xi) = p_2(\xi_1) + p_2(\xi_2) + p_1(\xi_1)p_1(\xi_2). \]

These conditions read as (1), (2) and (3).

Conversely, let the conditions (1) – (8) be satisfied. Then according to Theorem 3.1 there are two 4-dimensional vector bundles $\xi_1$ and $\xi_2$ over $M$. The conditions (1) – (3) say that the bundles $\xi$ and $\xi_1 \oplus \xi_2$ have the same characteristic classes. Hence, using Theorem 2 from [CV1] (and just here we need that $H^4(M; \mathbb{Z})$ has no element of order 4), we get that the both oriented vector bundles are isomorphic.

**Example 4.5.** The quaternionic projective space $\mathbb{H}P^2$ does not admit any 4-distribution in its tangent bundle. The characteristic classes are

\[ p_1(\mathbb{H}P^2) = 2u, \quad p_2(\mathbb{H}P^2) = 7u^2, \quad e(\mathbb{H}P^2) = 3u^2 \]

where $u \in H^4(\mathbb{H}P^2; \mathbb{Z})$ and $H^*(\mathbb{H}P^2; \mathbb{Z}) = \mathbb{Z}[u]/(u^3)$. There are no $Q_1, Q_2, S_1, S_2 \in H^4(\mathbb{H}P^2; \mathbb{Z})$ satisfying (2) and (3) of Theorem 4.4. From (2) follows that $2S_1 - Q_1 = \pm 3$ and $2S_2 - Q_2 = \pm 1$ or vice versa. Then from (3) we get

\[ 4Q_1Q_2 = 7u^2 - (2S_1 - Q_1)^2 - (2S_2 - Q_2)^2 = 3u^2, \]

which is a contradiction.

**Example 4.6.** The Grassmannian $G^+_6(\mathbb{R})$ (the space of oriented 2-planes in $\mathbb{R}^6$) does not admit any spin 4-distribution. We have $H^4(G^+_6(\mathbb{R}); \mathbb{Z}) = \mathbb{Z}[u]/(u^5)$ where $u \in H^2(G^+_6(\mathbb{R}); \mathbb{Z})$ and

\[ p_1(G^+_6(\mathbb{R})) = 2u^2, \quad p_2(1(G^+_6(\mathbb{R}))) = 7u^4, \quad e(1(G^+_6(\mathbb{R}))) = 5u^4. \]

the same considerations as above show the nonexistence of $Q_1, Q_2, S_1, S_2$ satisfying (2) and (3) of Theorem 4.4. It implies that neither $G^+_6(\mathbb{R})$ (the space of nonoriented 2-planes in $\mathbb{R}^6$) has any spin 4-distribution.
Example 4.7. Now consider the tangent bundle of the complex Grassmannian $G_{4,2}(\mathbb{C})$. In this case the situation is more complicated since

$$H^*(G_{4,2}(\mathbb{C}); \mathbb{Z}) = \mathbb{Z}[u, v]/(u^3 - 2uv, v^2 - u^2v)$$

where $u \in H^2(G_{4,2}(\mathbb{C}); \mathbb{Z})$ and $v \in H^4(G_{4,2}(\mathbb{C}); \mathbb{Z})$ and

$$p_1(G_{4,2}(\mathbb{C})) = 2u^2, \quad p_2(G_{4,2}(\mathbb{C})) = 14u^2v, \quad e(G_{4,2}(\mathbb{C})) = 6u^2v.$$ 

Nevertheless, neither in this case there is any spin 4-distribution in the tangent bundle. To prove it we have to explore nontrivial conditions (6) and (7) of Theorem 4.4 since for instance

$$Q_1 = -2u^2 - 2v \quad S_1 = -5u^2$$

$$Q_2 = 3u^2 + 2v \quad S_2 = 4v$$

satisfy (1), (2), (3), (4), (5), (8) and (9) of Theorem 4.4. The proof that the system of equations (1) – (7) has no solution can be carried out considering equations (1) – (3) "modulo 8" and using a computer to go through all the possible values.

Example 4.8. Using Theorem 3.1 and the characterization of 6-dimensional vector bundles over 6-complexes from [W], it can be shown that the tangent bundle of the complex projective space $\mathbb{C}P^3$ has only spin 4-distributions $\alpha$ with the characteristic classes

$$q(\alpha) = 0, \quad s(\alpha) = \pm y^2, \quad e(\alpha) = \pm 2y^2$$

where $y \in H^2(\mathbb{C}P^3; \mathbb{Z})$ and $H^*(\mathbb{C}P^3; \mathbb{Z}) = \mathbb{Z}[y]/(y^4)$. So

$$\tau(\mathbb{C}P^3) = \alpha \oplus \beta,$$

where $\beta$ is a 2-distribution with $e(\beta) = \pm 2y$. Hence the tangent bundle of $S^2 \times \mathbb{C}P^3$ is a sum of two spin 4-distributions

$$\tau(S^2 \times \mathbb{C}P^3) = \alpha \oplus (\beta \oplus \tau(S^2))$$

with

$$q(\beta \oplus \tau(S^2)) = 2y^2, \quad s(\beta \oplus \tau(S^2)) = \pm 2xy + y^2, \quad e(\beta \oplus \tau(S^2)) = \pm 4xy$$

where $x$ is a generator of $H^2(S^2; \mathbb{Z})$.

However, the tangent bundle of $S^2 \times \mathbb{C}P^3$ can be also written as a sum $\gamma \oplus \delta$ of spin 4-distributions with different characteristic classes, for instance

$$q(\gamma) = Q_1 = -4xy - 6y^2, \quad s(\gamma) = S_1 = -4xy + y^2, \quad e(\gamma) = -4xy + 8y^2$$

$$q(\delta) = Q_2 = 4xy + 8y^2, \quad s(\delta) = S_2 = -2xy - 5y^2, \quad e(\delta) = -8xy - 18y^2.$$ 

It is easy to show that these $Q_1, S_1, Q_2, S_2$ satisfy assumptions of Theorem 4.4.

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