PROBLEMS IN PLANE AND SOLID GEOMETRY

v.1 Plane Geometry

Viktor Prasolov

translated and edited by Dimitry Leites

ABSTRACT. This book has no equal. The priceless treasures of elementary geometry are nowhere else exposed in so complete and at the same time transparent form. The short solutions take barely $1.5 - 2$ times more space than the formulations, while still remaining *complete*, with no gaps whatsoever, although many of the problems are quite difficult. Only this enabled the author to squeeze about 2000 problems on plane geometry in the book of volume of ca 600 pages thus embracing practically all the known problems and theorems of elementary geometry.

The book contains non-standard geometric problems of a level higher than that of the problems usually offered at high school. The collection consists of two parts. It is based on three Russian editions of Prasolov's books on plane geometry.

The text is considerably modified for the English edition. Many new problems are added and detailed structuring in accordance with the methods of solution is adopted.

The book is addressed to high school students, teachers of mathematics, mathematical clubs, and college students.

Contents

Editor's preface

The enormous number of problems and theorems of elementary geometry was considered too wide to grasp in full even in the last century. Even nowadays the stream of new problems is still wide. (The majority of these problems, however, are either well-forgotten old ones or those recently pirated from a neighbouring country.)

Any attempt to collect an encyclopedia of all the problems seems to be doomed to failure for many reasons.

First of all, this is an impossible task because of the huge number of the problems, an enormity too vast to grasp. Second, even if this might have been possible, the book would be terribly overloaded, and therefore of no interest to anybody.

However, in the book *Problems in plane geometry* followed by *Problems in solid geometry* this task is successfully perfomed.

In the process of writing the book the author used the books and magazines published in the last century as well as modern ones. The reader can judge the completeness of the book by, for instance, the fact that *American Mathematical Monthly* yearly[1] publishes, as "new", 1–2 problems already published in the Russian editions of this book.

The book turned out to be of interest to a vast audience: about 400 000 copies of the first edition of each of the Parts (Parts 1 and 2 — Plane and Part 3 — Solid) were sold; the second edition, published 5 years later, had an even larger circulation, the total over 1 000 000 copies. The 3rd edition of *Problems in Plane Geometry* was issued in 1996 and the latest one in 2001.

The readers' interest is partly occasioned by a well-thought classification system.

The collection consists of three parts.

Part 1 covers classical subjects of plane geometry. It contains nearly 1000 problems with complete solutions and over 100 problems to be solved on one's own. Still more will be added for the English version of the book.

Part 2 includes more recent topics, geometric transformations and problems more suitable for contests and for use in mathematical clubs. The problems cover cuttings, colorings, the pigeonhole (or Dirichlet's) principle, induction, and so on.

Part 3 is devoted to solid geometry.

A rather detailed table of contents serves as a guide in the sea of geometric problems. It helps the experts to easily find what they need while the uninitiated can quickly learn what exactly is that they are interested in in geometry. Splitting the book into small sections (5 to 10 problems in each) made the book of interest to the readers of various levels.

FOR THE ENGLISH VERSION of the book about 150 new problems are already added and several hundred more of elementary and intermideate level problems will be added to make the number of more elementary problems sufficient to use the book in the ordinary school: the Russian editions are best suited for coaching for a mathematical Olympiad than for a regular class work: the level of difficulty increases rather fast.

Problems in each section are ordered difficulty-wise. The first problems of the sections are simple; they are a match for many. Here are some examples:

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[1] Here are a few samples: v. 96, n. 5, 1989, p. 429–431 (here the main idea of the solution is the right illustration — precisely the picture from the back cover of the 1st Russian edition of *Problems in Solid Geometry*, Fig. to Problem 13.22); v. 96, n. 6, p. 527, Probl. E3192 corresponds to Problems 5.31 and 18.20 of *Problems in Plane Geometry* — with their two absolutely different solutions, the one to Problem 5.31, unknown to AMM, is even more interesting.

Plane 1.1. The bases of a trapezoid are a and b. Find the length of the segment that the diagonals of the trapezoid intersept on the trapezoid's midline.

Plane 1.52. Let AA_1 and BB_1 be the altitudes of $\triangle ABC$. Prove that $\triangle A_1B_1C$ is similar to $\triangle ABC$. What is the similarity coefficient?

Plane 2.1. A line segment connects vertex A of an acute $\triangle ABC$ with the center O of the circumscribed circle. The altitude AH is dropped from A. Prove that $\angle BAH = \angle OAC$.

Plane 6.1. Prove that if the center of the circle inscribed in a quadrilateral coincides with the intersection point of the quadrilateral's diagonals, then the quadrilateral is a rhombus.

Solid 1. Arrange 6 match sticks to get 4 equilateral triangles with side length equal to the length of a stick.

Solid 1.1. Consider the cube $ABCDA_1B_1C_1D_1$ with side length a. Find the angle and the distance between the lines A_1B and AC_1.

Solid 6.1. Is it true that in every tetrahedron the heights meet at one point?

The above problems are not difficult. The last problems in the sections are a challenge for the specialists in geometry. It is important that the passage from simple problems to complicated ones is not too long; there are no boring and dull long sequences of simple similar problems. (In the Russian edition these sequences are, perhaps, too short, so more problems are added.)

The final problems of the sections are usually borrowed from scientific journals. Here are some examples:

Plane 10.20. Prove that $l_a + l_b + m_c \leq \sqrt{3}p$, where l_a, l_b are the lengths of the bisectors of the angles $\angle A$ and $\angle B$ of the triangle $\triangle ABC$, m_c is the length of the median of the side AB, and p is the semiperimeter.

Plane 19.55. Let O be the center of the circle inscribed in $\triangle ABC$, K the Lemoine's point, P and Q Brocard's points. Prove that P and Q belong to the circle with diameter KO and that $OP = OQ$.

Plane 22.29. The numbers $\alpha_1, \dots, \alpha_n$, whose sum is equal to $(n-2)\pi$, satisfy inequalities $0 < \alpha_i < 2\pi$. Prove that there exists an n-gon $A_1 \dots A_n$ with the angles $\alpha_1, \dots, \alpha_n$ at the vertices $A_1, \dots, A_n$, respectively.

Plane 24.12. Prove that for any n there exists a circle on which there lie precisely n points with integer coordinates.

Solid 4.48. Consider several arcs of great circles on a sphere with the sum of their angle measures $< \pi$. Prove that there exists a plane that passes through the center of the sphere but does not intersect any of these arcs.

Solid 14.22. Prove that if the centers of the escribed spheres of a tetrahedron belong to the circumscribed sphere, then the tetrahedron's faces are equal.

Solid 15.34. In space, consider 4 points not in one plane. How many various parallelipipeds with vertices in these points are there?

From the Author's preface

The book underwent extensive revision. The solutions to many of the problems were rewritten and about 600 new problems were added, particularly those concerning the geometry of the triangle. I was greatly influenced in the process by the second edition of the book by I. F. Sharygin *Problems on Geometry. Plane geometry*, Nauka, Moscow,1986 and a wonderful and undeservedly forgotten book by D. Efremov *New Geometry of the Triangle*, Matezis, Odessa, 1902.

The present book can be used not only as a source of optional problems for students but also as a self-guide for those who wish (or have no other choice but) to study geometry independently. Detailed headings are provided for the reader's convenience. Problems in the two parts of Plane are spread over 29 Chapters, each Chapter comprising 6 to 14 sections. The classification is based on the methods used to solve geometric problems. The purpose of the division is basically to help the reader find his/her bearings in this large array of problems. Otherwise the huge number of problems might be somewhat depressingly overwhelming.

Advice and comments given by Academician A. V. Pogorelov, and Professors A. M. Abramov, A. Yu. Vaintrob, N. B. Vasiliev, N. P. Dolbilin, and S. Yu. Orevkov were a great help to me in preparing the first Soviet edition. I wish to express my sincere gratitude to all of them.

To save space, sections with background only contain the material directly pertinent to the respective chapter. It is collected just to remind the reader of notations. Therefore, the basic elements of a triangle are only defined in chapter 5, while in chapter 1 we assume that their definition is known. For the reader's convenience, cross references in this translation are facilitated by a very detailed index.

Chapter 1. SIMILAR TRIANGLES

Background

1) Triangle ABC is said to be *similar* to triangle $A_1B_1C_1$ (we write $\triangle ABC \sim \triangle A_1B_1C_1$) if and only if one of the following equivalent conditions is satisfied:

a) $AB : BC : CA = A_1B_1 : B_1C_1 : C_1A_1$;

b) $AB : BC = A_1B_1 : B_1C_1$ and $\angle ABC = \angle A_1B_1C_1$;

c) $\angle ABC = \angle A_1B_1C_1$ and $\angle BAC = \angle B_1A_1C_1$.

2) *Triangles AB_1C_1 and AB_2C_2 cut off from an angle with vertex A by parallel lines are similar and* $AB_1 : AB_2 = AC_1 : AC_2$ (here points B_1 and B_2 lie on one leg of the angle and C_1 and C_2 on the other leg).

3) A *midline* of a triangle is the line connecting the midpoints of two of the triangle's sides. *The midline is parallel to the third side and its length is equal to a half length of the third side.*

The *midline of a trapezoid* is the line connecting the midpoints of the trapezoid's sides. *This line is parallel to the bases of the trapezoid and its length is equal to the halfsum of their lengths.*

4) *The ratio of the areas of similar triangles is equal to the square of the similarity coefficient*, i.e., to the squared ratio of the lengths of respective sides. This follows, for example, from the formula $S_{ABC} = \frac{1}{2}AB \cdot AC \sin \angle A$.

5) Polygons $A_1A_2 \dots A_n$ and $B_1B_2 \dots B_n$ are called *similar* if $A_1A_2 : A_2A_3 : \dots : A_nA_1 = B_1B_2 : B_2B_3 : \dots : B_nB_1$ and the angles at the vertices $A_1, \dots, A_n$ are equal to the angles at the vertices $B_1, \dots, B_n$, respectively.

The ratio of the respective diagonals of similar polygons is equal to the similarity coefficient. For the circumscribed similar polygons, the ratio of the radii of the inscribed circles is also equal to the similarity coefficient.

Introductory problems

1. Consider heights AA_1 and BB_1 in acute triangle ABC. Prove that $A_1C \cdot BC = B_1C \cdot AC$.

2. Consider height CH in right triangle ABC with right angle $\angle C$. Prove that $AC^2 = AB \cdot AH$ and $CH^2 = AH \cdot BH$.

3. Prove that the medians of a triangle meet at one point and this point divides each median in the ratio of $2 : 1$ counting from the vertex.

4. On side BC of $\triangle ABC$ point A_1 is taken so that $BA_1 : A_1C = 2 : 1$. What is the ratio in which median CC_1 divides segment AA_1?

5. Square $PQRS$ is inscribed into $\triangle ABC$ so that vertices P and Q lie on sides AB and AC and vertices R and S lie on BC. Express the length of the square's side through a and h_a.

§1. Line segments intercepted by parallel lines

1.1. Let the lengths of bases AD and BC of trapezoid $ABCD$ be a and b ($a > b$).

a) Find the length of the segment that the diagonals intercept on the midline.

b) Find the length of segment MN whose endpoints divide AB and CD in the ratio of $AM : MB = DN : NC = p : q$.

1.2. Prove that the midpoints of the sides of an arbitrary quadrilateral are vertices of a parallelogram. For what quadrilaterals this parallelogram is a rectangle, a rhombus, a square?

1.3. Points A_1 and B_1 divide sides BC and AC of $\triangle ABC$ in the ratios $BA_1 : A_1C = 1 : p$ and $AB_1 : B_1C = 1 : q$, respectively. In what ratio is AA_1 divided by BB_1?

1.4. Straight lines AA_1 and BB_1 pass through point P of median CC_1 in $\triangle ABC$ (A_1 and B_1 lie on sides BC and CA, respectively). Prove that $A_1B_1 \parallel AB$.

1.5. The straight line which connects the intersection point P of the diagonals in quadrilateral $ABCD$ with the intersection point Q of the lines AB and CD bisects side AD. Prove that it also bisects BC.

1.6. A point P is taken on side AD of parallelogram $ABCD$ so that $AP : AD = 1 : n$; let Q be the intersection point of AC and BP. Prove that $AQ : AC = 1 : (n + 1)$.

1.7. The vertices of parallelogram $A_1B_1C_1D_1$ lie on the sides of parallelogram $ABCD$ (point A_1 lies on AB, B_1 on BC, etc.). Prove that the centers of the two parallelograms coincide.

1.8. Point K lies on diagonal BD of parallelogram $ABCD$. Straight line AK intersects lines BC and CD at points L and M, respectively. Prove that $AK^2 = LK \cdot KM$.

1.9. One of the diagonals of a quadrilateral inscribed in a circle is a diameter of the circle. Prove that (the lengths of) the projections of the opposite sides of the quadrilateral on the other diagonal are equal.

1.10. Point E on base AD of trapezoid $ABCD$ is such that $AE = BC$. Segments CA and CE intersect diagonal BD at O and P, respectively. Prove that if $BO = PD$, then $AD^2 = BC^2 + AD \cdot BC$.

1.11. On a circle centered at O, points A and B single out an arc of 60°. Point M belongs to this arc. Prove that the straight line passing through the midpoints of MA and OB is perpendicular to that passing through the midpoints of MB and OA.

1.12. a) Points A, B, and C lie on one straight line; points A_1, B_1, and C_1 lie on another straight line. Prove that if $AB_1 \parallel BA_1$ and $AC_1 \parallel CA_1$, then $BC_1 \parallel CB_1$.

b) Points A, B, and C lie on one straight line and A_1, B_1, and C_1 are such that $AB_1 \parallel BA_1$, $AC_1 \parallel CA_1$, and $BC_1 \parallel CB_1$. Prove that A_1, B_1 and C_1 lie on one line.

1.13. In $\triangle ABC$ bisectors AA_1 and BB_1 are drawn. Prove that the distance from any point M of A_1B_1 to line AB is equal to the sum of distances from M to AC and BC.

1.14. Let M and N be the midpoints of sides AD and BC in rectangle $ABCD$. Point P lies on the extension of DC beyond D; point Q is the intersection point of PM and AC. Prove that $\angle QNM = \angle MNP$.

1.15. Points K and L are taken on the extensions of the bases AD and BC of trapezoid $ABCD$ beyond A and C, respectively. Line segment KL intersects sides AB and CD at M and N, respectively; KL intersects diagonals AC and BD at O and P, respectively. Prove that if $KM = NL$, then $KO = PL$.

1.16. Points P, Q, R, and S on sides AB, BC, CD and DA, respectively, of convex quadrilateral $ABCD$ are such that $BP : AB = CR : CD = \alpha$ and $AS : AD = BQ : BC = \beta$. Prove that PR and QS are divided by their intersection point in the ratios $\beta : (1 - \beta)$ and $\alpha : (1 - \alpha)$, respectively.

§2. The ratio of sides of similar triangles

1.17. a) In $\triangle ABC$ bisector BD of the external or internal angle $\angle B$ is drawn. Prove that $AD : DC = AB : BC$.

b) Prove that the center O of the circle inscribed in $\triangle ABC$ divides the bisector AA_1 in the ratio of $AO : OA_1 = (b + c) : a$, where a, b and c are the lengths of the triangle's sides.

1.18. The lengths of two sides of a triangle are equal to a while the length of the third side is equal to b. Calculate the radius of the circumscribed circle.

1.19. A straight line passing through vertex A of square $ABCD$ intersects side CD at E and line BC at F. Prove that $\frac{1}{AE^2} + \frac{1}{AF^2} = \frac{1}{AB^2}$.

1.20. Given points B_2 and C_2 on heights BB_1 and CC_1 of $\triangle ABC$ such that $AB_2C = AC_2B = 90^\circ$, prove that $AB_2 = AC_2$.

1.21. A circle is inscribed in trapezoid $ABCD$ ($BC \parallel AD$). The circle is tangent to sides AB and CD at K and L, respectively, and to bases AD and BC at M and N, respectively.

a) Let Q be the intersection point of BM and AN. Prove that $KQ \parallel AD$.

b) Prove that $AK \cdot KB = CL \cdot LD$.

1.22. Perpendiculars AM and AN are dropped to sides BC and CD of parallelogram $ABCD$ (or to their extensions). Prove that $\triangle MAN \sim \triangle ABC$.

1.23. Straight line l intersects sides AB and AD of parallelogram $ABCD$ at E and F, respectively. Let G be the intersection point of l with diagonal AC. Prove that $\frac{AB}{AE} + \frac{AD}{AF} = \frac{AC}{AG}$.

1.24. Let AC be the longer of the diagonals in parallelogram $ABCD$. Perpendiculars CE and CF are dropped from C to the extensions of sides AB and AD, respectively. Prove that $AB \cdot AE + AD \cdot AF = AC^2$.

1.25. Angles α and β of $\triangle ABC$ are related as $3\alpha + 2\beta = 180^\circ$. Prove that $a^2 + bc = c^2$.

1.26. The endpoints of segments AB and CD are gliding along the sides of a given angle, so that straight lines AB and CD are moving parallelly (i.e., each line moves parallelly to itself) and segments AB and CD intersect at a point, M. Prove that the value of $\frac{AM \cdot BM}{CM \cdot DM}$ is a constant.

1.27. Through an arbitrary point P on side AC of $\triangle ABC$ straight lines are drawn parallelly to the triangle's medians AK and CL. The lines intersect BC and AB at E and F, respectively. Prove that AK and CL divide EF into three equal parts.

1.28. Point P lies on the bisector of an angle with vertex C. A line passing through P intercepts segments of lengths a and b on the angle's legs. Prove that the value of $\frac{1}{a} + \frac{1}{b}$ does not depend on the choice of the line.

1.29. A semicircle is constructed outwards on side BC of an equilateral triangle ABC as on the diameter. Given points K and L that divide the semicircle into three equal arcs, prove that lines AK and AL divide BC into three equal parts.

1.30. Point O is the center of the circle inscribed in $\triangle ABC$. On sides AC and BC points M and K, respectively, are selected so that $BK \cdot AB = BO^2$ and $AM \cdot AB = AO^2$. Prove that M, O and K lie on one straight line.

1.31. Equally oriented similar triangles AMN, NBM and MNC are constructed on segment MN (Fig. 1).

Prove that $\triangle ABC$ is similar to all these triangles and the center of its curcumscribed circle is equidistant from M and N.

1.32. Line segment BE divides $\triangle ABC$ into two similar triangles, their similarity ratio being equal to $\sqrt{3}$.

Find the angles of $\triangle ABC$.

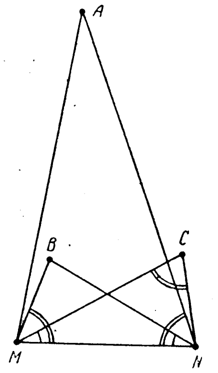

Figure 1 (1.31)

§3. The ratio of the areas of similar triangles

1.33. A point E is taken on side AC of $\triangle ABC$. Through E pass straight lines DE and EF parallel to sides BC and AB, respectively; D and E are points on AB and BC, respectively. Prove that $S_{BDEF} = 2\sqrt{S_{ADE} \cdot S_{EFG}}$.

1.34. Points M and N are taken on sides AB and CD, respectively, of trapezoid $ABCD$ so that segment MN is parallel to the bases and divides the area of the trapezoid in halves. Find the length of MN if $BC = a$ and $AD = b$.

1.35. Let Q be a point inside $\triangle ABC$. Three straight lines are pass through Q parallelly to the sides of the triangle. The lines divide the triangle into six parts, three of which are triangles of areas S_1, S_2 and S_3. Prove that the area of $\triangle ABC$ is equal to $\left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2$.

1.36. Prove that the area of a triangle whose sides are equal to the medians of a triangle of area S is equal to $\frac{3}{4}S$.

1.37. a) Prove that the area of the quadrilateral formed by the midpoints of the sides of convex quadrilateral $ABCD$ is half that of $ABCD$.

b) Prove that if the diagonals of a convex quadrilateral are equal, then its area is the product of the lengths of the segments which connect the midpoints of its opposite sides.

1.38. Point O lying inside a convex quadrilateral of area S is reflected symmetrically through the midpoints of its sides. Find the area of the quadrilateral with its vertices in the images of O under the reflections.

§4. Auxiliary equal triangles

1.39. In right triangle ABC with right angle $\angle C$, points D and E divide leg BC of into three equal parts. Prove that if $BC = 3AC$, then $\angle AEC + \angle ADC + \angle ABC = 90^\circ$.

1.40. Let K be the midpoint of side AB of square $ABCD$ and let point L divide diagonal AC in the ratio of $AL : LC = 3 : 1$. Prove that $\angle KLD$ is a right angle.

1.41. In square $ABCD$ straight lines l_1 and l_2 pass through vertex A. The lines intersect the square's sides. Perpendiculars BB_1, BB_2, DD_1, and DD_2 are dropped to these lines. Prove that segments B_1B_2 and D_1D_2 are equal and perpendicular to each other.

1.42. Consider an isosceles right triangle ABC with $CD = CE$ and points D and E on sides CA and CB, respectively. Extensions of perpendiculars dropped from D and C to AE intersect the hypotenuse AB at K and L. Prove that $KL = LB$.

1.43. Consider an inscribed quadrilateral $ABCD$. The lengths of sides AB, BC, CD, and DA are a, b, c, and d, respectively. Rectangles are constructed outwards on the sides of the quadrilateral; the sizes of the rectangles are $a \times c$, $b \times d$, $c \times a$ and $d \times b$, respectively. Prove that the centers of the rectangles are vertices of a rectangle.

1.44. Hexagon $ABCDEF$ is inscribed in a circle of radius R centered at O; let $AB = CD = EF = R$. Prove that the intersection points, other than O, of the pairs of circles circumscribed about $\triangle BOC$, $\triangle DOE$ and $\triangle FOA$ are the vertices of an equilateral triangle with side R.

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1.45. Equilateral triangles BCK and DCL are constructed outwards on sides BC and CD of parallelogram $ABCD$. Prove that AKL is an equilateral triangle.

1.46. Squares are constructed outwards on the sides of a parallelogram. Prove that their centers form a square.

1.47. Isosceles triangles with angles 2α, 2β and 2γ at vertices A', B' and C' are constructed outwards on the sides of triangle ABC; let $\alpha+\beta+\gamma = 180^\circ$. Prove that the angles of $\triangle A'B'C'$ are equal to α, β and γ.

1.48. On the sides of $\triangle ABC$ as on bases, isosceles similar triangles AB_1C and AC_1B are constructed outwards and an isosceles triangle BA_1C is constructed inwards. Prove that $AB_1A_1C_1$ is a parallelogram.

1.49. a) On sides AB and AC of $\triangle ABC$ equilateral triangles ABC_1 and AB_1C are constructed outwards; let $\angle C_1 = \angle B_1 = 90^\circ$, $\angle ABC_1 = \angle ACB_1 = \varphi$; let M be the midpoint of BC. Prove that $MB_1 = MC_1$ and $\angle B_1MC_1 = 2\varphi$.

b) Equilateral triangles are constructed outwards on the sides of $\triangle ABC$. Prove that the centers of the triangles constructed form an equilateral triangle whose center coincides with the intersection point of the medians of $\triangle ABC$.

1.50. Isosceles triangles AC_1B and AB_1C with an angle φ at the vertex are constructed outwards on the unequal sides AB and AC of a scalene triangle $\triangle ABC$.

a) Let M be a point on median AA_1 (or on its extension), let M be equidistant from B_1 and C_1. Prove that $\angle B_1MC_1 = \varphi$.

b) Let O be a point of the midperpendicular to segment BC, let O be equidistant from B_1 and C_1. Prove that $\angle B_1OC = 180^\circ - \varphi$.

1.51. Similar rhombuses are constructed outwards on the sides of a convex rectangle $ABCD$, so that their acute angles (equal to α) are adjacent to vertices A and C. Prove that the segments which connect the centers of opposite rhombuses are equal and the angle between them is equal to α.

§5. The triangle determined by the bases of the heights

1.52. Let AA_1 and BB_1 be heights of $\triangle ABC$. Prove that $\triangle A_1B_1C \sim \triangle ABC$. What is the similarity coefficient?

1.53. Height CH is dropped from vertex C of acute triangle ABC and perpendiculars HM and HN are dropped to sides BC and AC, respectively. Prove that $\triangle MNC \sim \triangle ABC$.

1.54. In $\triangle ABC$ heights BB_1 and CC_1 are drawn.

a) Prove that the tangent at A to the circumscribed circle is parallel to B_1C_1.

b) Prove that $B_1C_1 \perp OA$, where O is the center of the circumscribed circle.

1.55. Points A_1, B_1 and C_1 are taken on the sides of an acute triangle ABC so that segments AA_1, BB_1 and CC_1 meet at H. Prove that $AH \cdot A_1H = BH \cdot B_1H = CH \cdot C_1H$ if and only if H is the intersection point of the heights of $\triangle ABC$.

1.56. a) Prove that heights AA_1, BB_1 and CC_1 of acute triangle ABC bisect the angles of $\triangle A_1B_1C_1$.

b) Points C_1, A_1 and B_1 are taken on sides AB, BC and CA, respectively, of acute triangle ABC. Prove that if $\angle B_1A_1C = \angle BA_1C_1$, $\angle A_1B_1C = \angle AB_1C_1$ and $\angle A_1C_1B = \angle AC_1B_1$, then points A_1, B_1 and C_1 are the bases of the heights of $\triangle ABC$.

1.57. Heights AA_1, BB_1 and CC_1 are drawn in acute triangle ABC. Prove that the point symmetric to A_1 through AC lies on B_1C_1.

1.58. In acute triangle ABC, heights AA_1, BB_1 and CC_1 are drawn. Prove that if $A_1B_1 \parallel AB$ and $B_1C_1 \parallel BC$, then $A_1C_1 \parallel AC$.

1.59. Let p be the semiperimeter of acute triangle ABC and q the semiperimeter of the triangle formed by the bases of the heights of $\triangle ABC$. Prove that $p : q = R : r$, where R and r are the radii of the circumscribed and the inscribed circles, respectively, of $\triangle ABC$.

§6. Similar figures

1.60. A circle of radius r is inscribed in a triangle. The straight lines tangent to the circle and parallel to the sides of the triangle are drawn; the lines cut three small triangles off the triangle. Let r_1, r_2 and r_3 be the radii of the circles inscribed in the small triangles. Prove that $r_1 + r_2 + r_3 = r$.

1.61. Given $\triangle ABC$, draw two straight lines x and y such that the sum of lengths of the segments MX_M and MY_M drawn parallel to x and y from a point M on AC to their intersections with sides AB and BC is equal to 1 for any M.

1.62. In an isosceles triangle ABC perpendicular HE is dropped from the midpoint of base BC to side AC. Let O be the midpoint of HE. Prove that lines AO and BE are perpendicular to each other.

1.63. Prove that projections of the base of a triangle's height to the sides between which it lies and on the other two heights lie on the same straight line.

1.64. Point B lies on segment AC; semicircles S_1, S_2, and S_3 are constructed on one side of AC, as on diameter. Let D be a point on S_3 such that $BD \perp AC$. A common tangent line to S_1 and S_2 touches these semicircles at F and E, respectively.

a) Prove that EF is parallel to the tangent to S_3 passing through D.

b) Prove that $BFDE$ is a rectangle.

1.65. Perpendiculars MQ and MP are dropped from an arbitrary point M of the circle circumscribed about rectangle $ABCD$ to the rectangle's two opposite sides; the perpendiculars MR and MT are dropped to the extensions of the other two sides. Prove that lines $PR \perp QT$ and the intersection point of PR and QT belongs to a diagonal of $ABCD$.

1.66. Two circles enclose non-intersecting areas. Common tangent lines to the two circles, one external and one internal, are drawn. Consider two straight lines each of which passes through the tangent points on one of the circles. Prove that the intersection point of the lines lies on the straight line that connects the centers of the circles.

Problems for independent study

1.67. The (length of the) base of an isosceles triangle is a quarter of its perimeter. From an arbitrary point on the base straight lines are drawn parallel to the sides of the triangle. How many times is the perimeter of the triangle greater than that of the parallelogram?

1.68. The diagonals of a trapezoid are mutually perpendicular. The intersection point divides the diagonals into segments. Prove that the product of the lengths of the trapezoid's bases is equal to the sum of the products of the lengths of the segments of one diagonal and those of another diagonal.

1.69. A straight line is drawn through the center of a unit square. Calculate the sum of the squared distances between the four vertices of the square and the line.

1.70. Points A_1, B_1 and C_1 are symmetric to the center of the circumscribed circle of $\triangle ABC$ through the triangle's sides. Prove that $\triangle ABC = \triangle A_1B_1C_1$.

1.71. Prove that if $\angle BAC = 2\angle ABC$, then $BC^2 = (AC + AB)AC$.

1.72. Consider points A, B, C and D on a line l. Through A, B and through C, D parallel straight lines are drawn. Prove that the diagonals of the parallelograms thus formed (or their extensions) intersect l at two points that do not depend on parallel lines but depend on points A, B, C, D only.

1.73. In $\triangle ABC$ bisector AD and midline A_1C_1 are drawn. They intersect at K. Prove that $2A_1K = |b - c|$.

1.74. Points M and N are taken on sides AD and CD of parallelogram $ABCD$ such that $MN \parallel AC$. Prove that $S_{ABM} = S_{CBN}$.

1.75. On diagonal AC of parallelogram $ABCD$ points P and Q are taken so that $AP = CQ$. Let M be such that $PM \parallel AD$ and $QM \parallel AB$. Prove that M lies on diagonal BD.

1.76. Consider a trapezoid with bases AD and BC. Extensions of the sides of $ABCD$ meet at point O. Segment EF is parallel to the bases and passes through the intersection point of the diagonals. The endpoints of EF lie on AB and CD. Prove that $AE : CF = AO : CO$.

1.77. Three straight lines parallel to the sides of the given triangle cut three triangles off it leaving an equilateral hexagon. Find the length of the side of the hexagon if the lengths of the triangle's sides are a, b and c.

1.78. Three straight lines parallel to the sides of a triangle meet at one point, the sides of the triangle cutting off the line segments of length x each. Find x if the lengths of the triangle's sides are a, b and c.

1.79. Point P lies inside $\triangle ABC$ and $\angle ABP = \angle ACP$. On straight lines AB and AC, points C_1 and B_1 are taken so that $BC_1 : CB_1 = CP : BP$. Prove that one of the diagonals of the parallelogram whose two sides lie on lines BP and CP and two other sides (or their extensions) pass through B_1 and C_1 is parallel to BC.

Solutions

1.1. a) Let P and Q be the midpoints of AB and CD; let K and L be the intersection points of PQ with the diagonals AC and BD, respectively. Then $PL = \frac{a}{2}$ and $PK = \frac{1}{2}b$ and so $KL = PL - PK = \frac{1}{2}(a - b)$.

b) Take point F on AD such that $BF \parallel CD$. Let E be the intersection point of MN with BF. Then

$$MN = ME + EN = \frac{q \cdot AF}{p+q} + b = \frac{q(a-b) + (p+q)b}{p+q} = \frac{qa + pb}{p+q}.$$

1.2. Consider quadrilateral $ABCD$. Let K, L, M and N be the midpoints of sides AB, BC, CD and DA, respectively. Then $KL = MN = \frac{1}{2}AC$ and $KL \parallel MN$, that is $KLMN$ is a parallelogram. It becomes clear now that $KLMN$ is a *rectangle* if the diagonals AC and BD are perpendicular, a *rhombus* if $AC = BD$, and a *square* if AC and BD are of equal length and perpendicular to each other.

1.3. Denote the intersection point of AA_1 with BB_1 by O. In $\triangle B_1BC$ draw segment A_1A_2 so that $A_1A_2 \parallel BB_1$. Then $\frac{B_1C}{B_1A_2} = 1 + p$ and so $AO : OA_1 = AB_1 : B_1A_2 = B_1C : qB_1A_2 = (1 + p) : q$.

1.4. Let A_2 be the midpoint of A_1B. Then $CA_1 : A_1A_2 = CP : PC_1$ and $A_1A_2 : A_1B = 1 : 2$. So $CA_1 : A_1B = CP : 2PC_1$. Similarly, $CB_1 : B_1A = CP : 2PC_1 = CA_1 : A_1B$.

1.5. Point P lies on the median QM of $\triangle AQD$ (or on its extension). It is easy to verify that the solution of Problem 1.4 remains correct also for the case when P lies on the extension of the median. Consequently, $BC \parallel AD$.

1.6. We have $AQ : QC = AP : BC = 1 : n$ because $\triangle AQP \sim \triangle CQB$. So $AC = AQ + QC = (n+1)AQ$.

1.7. The center of $A_1B_1C_1D_1$ being the midpoint of B_1D_1 belongs to the line segment which connects the midpoints of AB and CD. Similarly, it belongs to the segment which connects the midpoints of BC and AD. The intersection point of the segments is the center of $ABCD$.

1.8. Clearly, $AK : KM = BK : KD = LK : AK$, that is $AK^2 = LK \cdot KM$.

1.9. Let AC be the diameter of the circle circumscribed about $ABCD$. Drop perpendiculars AA_1 and CC_1 to BD (Fig. 2).

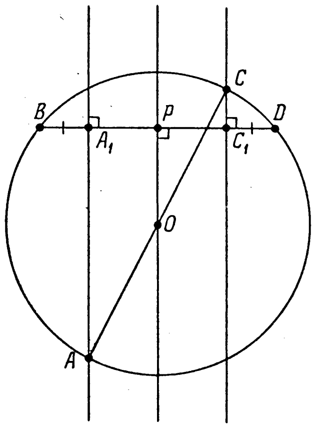


Figure 2 (Sol. 1.9)

We must prove that $BA_1 = DC_1$. Drop perpendicular OP from the center O of the circumscribed circle to BD. Clearly, P is the midpoint of BD. Lines AA_1, OP and CC_1 are parallel to each other and $AO = OC$. So $A_1P = PC_1$ and, since P is the midpoint of BD, it follows that $BA_1 = DC_1$.

1.10. We see that $BO : OD = DP : PB = k$, because $BO = PD$. Let $BC = 1$. Then $AD = k$ and $ED = \frac{1}{k}$. So $k = AD = AE + ED = 1 + \frac{1}{k}$, that is $k^2 = 1 + k$. Finally, observe that $k^2 = AD^2$ and $1 + k = BC^2 + BC \cdot AD$.

1.11. Let C, D, E and F be the midpoints of sides AO, OB, BM and MA, respectively, of quadrilateral $AOMB$. Since $AB = MO = R$, where R is the radius of the given circle, $CDEF$ is a rhombus by Problem 1.2. Hence, $CE \perp DF$.

1.12. a) If the lines containing the given points are parallel, then the assertion of the problem is obviously true. We assume that the lines meet at O. Then $OA : OB = OB_1 : OA_1$ and $OC : OA = OA_1 : OC_1$. Hence, $OC : OB = OB_1 : OC_1$ and so $BC_1 \parallel CB_1$ (the ratios of the segment should be assumed to be *oriented*).

b) Let AB_1 and CA_1 meet at D, let CB_1 and AC_1 meet at E. Then $CA_1 : A_1D = CB : BA = EC_1 : C_1A$. Since $\triangle CB_1D \sim \triangle EB_1A$, points A_1, B_1 and C_1 lie on the same line.

1.13. A point that lies on the bisector of an angle is equidistant from the angle's legs. Let a be the distance from point A_1 to lines AC and AB, let b be the distance from point B_1 to lines AB and BC. Further, let $A_1M : B_1M = p : q$, where $p + q = 1$. Then the distances from point M to lines AC and BC are equal to qa and pb, respectively. On the other hand, by Problem 1.1 b) the distance from point M to line AB is equal to $qa + pb$.

1.14. Let the line that passes through the center O of the given rectangle parallel to BC intersect line segment QN at point K (Fig. 3).

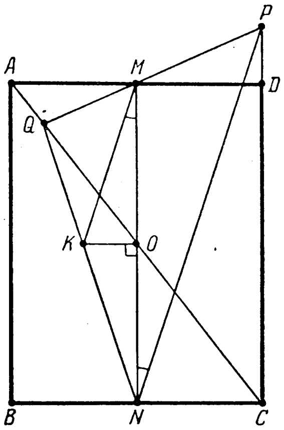


Figure 3 (Sol. 1.14)

Since $MO \parallel PC$, it follows that $QM : MP = QO : OC$ and, since $KO \parallel BC$, it follows that $QO : OC = QK : KN$. Therefore, $QM : MP = QK : KN$, i.e., $KM \parallel NP$. Hence, $\angle MNP = \angle KMO = \angle QNM$.

1.15. Let us draw through point M line EF so that $EF \parallel CD$ (points E and F lie on lines BC and AD). Then $PL : PK = BL : KD$ and $OK : OL = KA : CL = KA : KF = BL : EL$. Since $KD = EL$, we have $PL : PK = OK : OL$ and, therefore, $PL = OK$.

1.16. Consider parallelogram $ABCD_1$. We may assume that points D and D_1 do not coincide (otherwise the statement of the problem is obvious). On sides AD_1 and CD_1 take points S_1 and R_1, respectively, so that $SS_1 \parallel DD_1$ and $RR_1 \parallel DD_1$. Let segments PR_1 and QS_1 meet at N; let N_1 and N_2 be the intersection points of the line that passes through N parallel to DD_1 with segments PR and QS, respectively.

Then $\overrightarrow{N_1N} = \beta\overrightarrow{RR_1} = \alpha\beta\overrightarrow{DD_1}$ and $\overrightarrow{N_2N} = \alpha\overrightarrow{SS_1} = \alpha\beta\overrightarrow{DD_1}$. Hence, segments PR and QS meet at $N_1 = N_2$. Clearly, $PN_1 : PR = PN : PR_1 = \beta$ and $QN_2 : QS = \alpha$.

REMARK. If $\alpha = \beta$, there is a simpler solution. Since $BP : BA = BQ : BC = \alpha$, it follows that $PQ \parallel AC$ and $PQ : AC = \alpha$. Similarly, $RS \parallel AC$ and $RS : AC = 1 - \alpha$. Therefore, segments PR and QS are divided by their intersection point in the ratio of $\alpha : (1 - \alpha)$.

1.17. a) From vertices A and C drop perpendiculars AK and CL to line BD. Since $\angle CBL = \angle ABK$ and $\angle CDL = \angle KDA$, we see that $\triangle BLC \sim \triangle BKA$ and $\triangle CLD \sim \triangle AKD$. Therefore, $AD : DC = AK : CL = AB : BC$.

b) Taking into account that $BA_1 : A_1C = BA : AC$ and $BA_1 + A_1C = BC$ we get $BA_1 = \frac{ac}{b+c}$. Since BO is the bisector of triangle ABA_1, it follows that $AO : OA_1 = AB : BA_1 = (b + c) : a$.

1.18. Let O be the center of the circumscribed circle of isosceles triangle ABC, let B_1 be the midpoint of base AC and A_1 the midpoint of the lateral side BC. Since $\triangle BOA_1 \sim \triangle BCB_1$, it follows that $BO : BA_1 = BC : BB_1$ and, therefore, $R = BO = \frac{a^2}{\sqrt{4a^2-b^2}}$.

1.19. If $\angle EAD = \varphi$, then $AE = \frac{AD}{\cos\varphi} = \frac{AB}{\cos\varphi}$ and $AF = \frac{AB}{\sin\varphi}$. Therefore,

$$\frac{1}{AE^2} + \frac{1}{AF^2} = \frac{\cos^2\varphi + \sin^2\varphi}{AB^2} = \frac{1}{AB^2}.$$

1.20. It is easy to verify that $AB_2^2 = AB_1 \cdot AC = AC_1 \cdot AB = AC_2^2$.

1.21. a) Since $BQ : QM = BN : AM = BK : AK$, we have: $KQ \parallel AM$.

b) Let O be the center of the inscribed circle. Since $\angle CBA + \angle BAD = 180°$, it follows that $\angle ABO + \angle BAO = 90°$. Therefore, $\triangle AKO \sim \triangle OKB$, i.e., $AK : KO = OK : KB$. Consequently, $AK \cdot KB = KO^2 = R^2$, where R is the radius of the inscribed circle. Similarly, $CL \cdot LD = R^2$.

1.22. If angle $\angle ABC$ is obtuse (resp. acute), then angle $\angle MAN$ is also obtuse (resp. acute). Moreover, the legs of these angles are mutually perpendicular. Therefore, $\angle ABC = \angle MAN$. Right triangles ABM and ADN have equal angles $\angle ABM = \angle ADN$, therefore, $AM : AN = AB : AD = AB : CB$, i.e., $\triangle ABC \sim \triangle MAN$.

1.23. On diagonal AC, take points D' and B' such that $BB' \parallel l$ and $DD' \parallel l$. Then $AB : AE = AB' : AG$ and $AD : AF = AD' : AG$. Since the sides of triangles ABB' and CDD' are pairwise parallel and $AB = CD$, these triangles are equal and $AB' = CD'$. Therefore,

$$\frac{AB}{AE} + \frac{AD}{AF} = \frac{AB'}{AG} + \frac{AD'}{AG} = \frac{CD' + AD'}{AG} = \frac{AC}{AG}.$$

1.24. Let us drop from vertex B perpendicular BG to AC (Fig. 4).

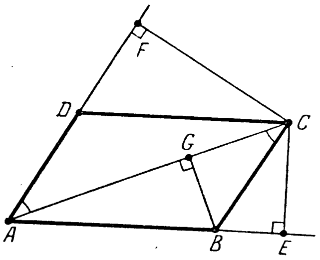


Figure 4 (Sol. 1.24)

Since triangles ABG and ACE are similar, $AC \cdot AG = AE \cdot AB$. Lines AF and CB are parallel, consequently, $\angle GCB = \angle CAF$. We also infer that right triangles CBG and ACF are similar and, therefore, $AC \cdot CG = AF \cdot BC$. Summing the equalities obtained we get

$$AC \cdot (AG + CG) = AE \cdot AB + AF \cdot BC.$$

Since $AG + CG = AC$, we get the equality desired.

1.25. Since $\alpha + \beta = 90° - \frac{1}{2}\alpha$, it follows that $\gamma = 180° - \alpha - \beta = 90° + \frac{1}{2}\alpha$. Therefore, it is possible to find point D on side AB so that $\angle ACD = 90° - \frac{1}{2}\alpha$, i.e., $AC = AD$. Then $\triangle ABC \sim \triangle CBD$ and, therefore, $BC : BD = AB : CB$, i.e., $a^2 = c(c - b)$.

1.26. As segments AB and CD move, triangle AMC is being replaced by another triangle similar to the initial one. Therefore, the quantity $\frac{AM}{CM}$ remains a constant. Analogously, $\frac{BM}{DM}$ remains a constant.

1.27. Let medians meet at O; denote the intersection points of median AK with lines FP and FE by Q and M, respectively; denote the intersection points of median CL with lines EP and FE by R and N, respectively (Fig. 5).

Clearly, $FM : FE = FQ : FP = LO : LC = 1 : 3$, i.e., $FM = \frac{1}{3}FE$. Similarly, $EN = \frac{1}{3}FE$.

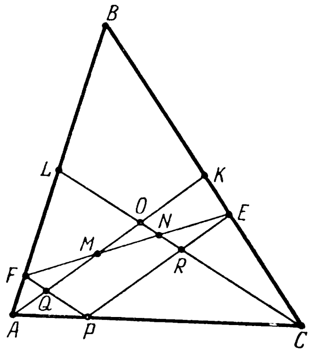


Figure 5 (Sol. 1.27)

1.28. Let A and B be the intersection points of the given line with the angle's legs. On segments AC and BC, take points K and L, respectively, so that $PK \parallel BC$ and $PL \parallel AC$. Since $\triangle AKP \sim \triangle PLB$, it follows that $AK : KP = PL : LB$ and, therefore, $(a-p)(b-p) = p^2$, where $p = PK = PL$. Hence, $\frac{1}{a} + \frac{1}{b} = \frac{1}{p}$.

1.29. Denote the midpoint of side BC by O and the intersection points of AK and AL with side BC by P and Q, respectively. We may assume that $BP < BQ$. Triangle LCO is an equilateral one and $LC \parallel AB$. Therefore, $\triangle ABQ \sim \triangle LCQ$, i.e., $BQ : QC = AB : LC = 2 : 1$. Hence, $BC = BQ + QC = 3QC$. Similarly, $BC = 3BP$.

1.30. Since $BK : BO = BO : AB$ and $\angle KBO = \angle ABO$, it follows that $\triangle KOB \sim \triangle OAB$. Hence, $\angle KOB = \angle OAB$. Similarly, $\angle AOM = \angle ABO$. Therefore,

$$\angle KOM = \angle KOB + \angle BOA + \angle AOM = \angle OAB + \angle BOA + \angle ABO = 180^\circ,$$

i.e., points K, O and M lie on one line.

1.31. Since $\angle AMN = \angle MNC$ and $\angle BMN = \angle MNA$, we see that $\angle AMB = \angle ANC$. Moreover, $AM : AN = NB : NM = BM : CN$. Hence, $\triangle AMB \sim \triangle ANC$ and, therefore, $\angle MAB = \angle NAC$. Consequently, $\angle BAC = \angle MAN$. For the other angles the proof is similar.

Let points B_1 and C_1 be symmetric to B and C, respectively, through the midperpendicular to segment MN. Since $AM : NB = MN : BM = MC : NC$, it follows that $MA \cdot MC_1 = AM \cdot NC = NB \cdot MC = MB_1 \cdot MC$. Therefore, point A lies on the circle circumscribed about trapezoid BB_1CC_1.

1.32. Since $\angle AEB + \angle BEC = 180^\circ$, angles $\angle AEB$ and $\angle BEC$ cannot be different angles of similar triangles ABE and BEC, i.e., the angles are equal and BE is a perpendicular.

Two cases are possible: either $\angle ABE = \angle CBE$ or $\angle ABE = \angle BCE$. The first case should be discarded because in this case $\triangle ABE = \triangle CBE$.

In the second case we have $\angle ABC = \angle ABE + \angle CBE = \angle ABE + \angle BAE = 90^\circ$. In right triangle ABC the ratio of the legs' lengths is equal to $1 : \sqrt{3}$; hence, the angles of triangle ABC are equal to 90°, 60°, 30°.

1.33. We have $\frac{S_{BDEF}}{2S_{ADE}} = \frac{S_{BDE}}{S_{ADE}} = \frac{DB}{AD} = \frac{EF}{AD} = \sqrt{\frac{S_{EFC}}{S_{ADE}}}$. Hence,

$$S_{BDEF} = 2\sqrt{S_{ADE} \cdot S_{EFC}}.$$

1.34. Let $MN = x$; let E be the intersection point of lines AB and CD. Triangles EBC, EMN and EAD are similar, hence, $S_{EBC} : S_{EMN} : S_{EAD} = a^2 : x^2 : b^2$. Since $S_{EMN} - S_{EBC} = S_{MBCN} = S_{MADN} = S_{EAD} - S_{EMN}$, it follows that $x^2 - a^2 = b^2 - x^2$, i.e., $x^2 = \frac{1}{2}(a^2 + b^2)$.

1.35. Through point Q inside triangle ABC draw lines DE, FG and HI parallel to BC, CA and AB, respectively, so that points F and H would lie on side BC, points E and I on side AC, points D and G on side AB (Fig. 6).

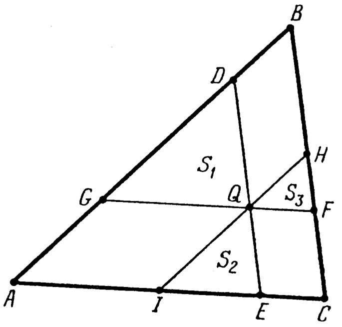


Figure 6 (Sol. 1.35)

Set $S = S_{ABC}$, $S_1 = S_{GDQ}$, $S_2 = S_{IEQ}$, $S_3 = S_{HFQ}$. Then

$$\sqrt{\frac{S_1}{S}} + \sqrt{\frac{S_2}{S}} + \sqrt{\frac{S_3}{S}} = \frac{GQ}{AC} + \frac{IE}{AC} + \frac{FQ}{AC} = \frac{AI + IE + EC}{AC} = 1,$$

i.e., $S = (\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3})^2$.

1.36. Let M be the intersection point of the medians of triangle ABC; let point A_1 be symmetric to M through the midpoint of segment BC. The ratio of the lengths of sides of triangle CMA_1 to the lengths of the corresponding medians of triangle ABC is to $2:3$. Therefore, the area to be found is equal to $\frac{9}{4}S_{CMA_1}$. Clearly, $S_{CMA_1} = \frac{1}{3}S$ (cf. the solution of Problem 4.1).

1.37. Let E, F, G and H be the midpoints of sides AB, BC, CD and DA, respectively.

a) Clearly, $S_{AEH} + S_{CFG} = \frac{1}{4}S_{ABD} + \frac{1}{4}S_{CBD} = \frac{1}{4}S_{ABCD}$. Analogously, $S_{BEF} + S_{DGH} = \frac{1}{4}S_{ABCD}$; hence, $S_{EFGH} = S_{ABCD} - \frac{1}{4}S_{ABCD} - \frac{1}{4}S_{ABCD} = \frac{1}{2}S_{ABCD}$.

b) Since $AC = BD$, it follows that $EFGH$ is a rhombus (Problem 1.2). By heading a) we have $S_{ABCD} = 2S_{EFGH} = EG \cdot FH$.

1.38. Let E, F, G and H be the midpoints of sides of quadrilateral $ABCD$; let points E_1, F_1, G_1 and H_1 be symmetric to point O through these points, respectively. Since EF is the midline of triangle E_1OF_1, we see that $S_{E_1OF_1} = 4S_{EOF}$. Similarly, $S_{F_1OG_1} = 4S_{FOG}$, $S_{G_1OH_1} = 4S_{GOH}$, $S_{H_1OE_1} = 4S_{HOE}$. Hence, $S_{E_1F_1G_1H_1} = 4S_{EFGH}$. By Problem 1.37 a) $S_{ABCD} = 2S_{EFGH}$. Hence, $S_{E_1F_1G_1H_1} = 2S_{ABCD} = 2S$.

1.39. First solution. Let us consider square $BCMN$ and divide its side MN by points P and Q into three equal parts (Fig. 7).

Then $\triangle ABC = \triangle PDQ$ and $\triangle ACD = \triangle PMA$. Hence, triangle $\triangle PAD$ is an isosceles right triangle and $\angle ABC + \angle ADC = \angle PDQ + \angle ADC = 45^\circ$.

Second solution. Since $DE = 1$, $EA = \sqrt{2}$, $EB = 2$, $AD = \sqrt{5}$ and $BA = \sqrt{10}$, it follows that $DE : AE = EA : EB = AD : BA$ and $\triangle DEA \sim \triangle AEB$. Therefore, $\angle ABC = \angle EAD$. Moreover, $\angle AEC = \angle CAE = 45^\circ$. Hence,

$$\angle ABC + \angle ADC + \angle AEC = (\angle EAD + \angle CAE) + \angle ADC = \angle CAD + \angle ADC = 90^\circ.$$

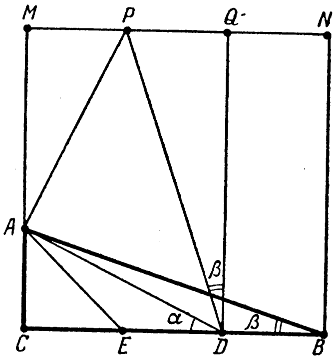


Figure 7 (Sol. 1.39)

1.40. From point L drop perpendiculars LM and LN on AB and AD, respectively. Then $KM = MB = ND$ and $KL = LB = DL$ and, therefore, right triangles KML and DNL are equal. Hence, $\angle DLK = \angle NLM = 90^\circ$.

1.41. Since $D_1A = B_1B, AD_2 = BB_2$ and $\angle D_1AD_2 = \angle B_1BB_2$, it follows that $\triangle D_1AD_2 = \triangle B_1BB_2$. Sides AD_1 and BB_1 (and also AD_2 and BB_2) of these triangles are perpendicular and, therefore, $B_1B_2 \perp D_1D_2$.

1.42. On the extension of segment AC beyond point C take point M so that $CM = CE$ (Fig. 8).

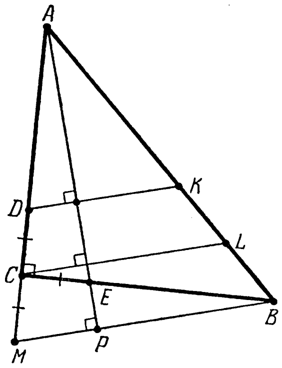


Figure 8 (Sol. 1.42)

Then under the rotation with center C through an angle of 90° triangle ACE turns into triangle BCM. Therefore, line MB is perpendicular to line AE; hence, it is parallel to line CL. Since $MC = CE = DC$ and lines DK, CL and MB are parallel, $KL = LB$.

1.43. Let rectangles ABC_1D_1 and A_2BCD_2 be constructed on sides AB and BC; let P, Q, R and S be the centers of rectangles constructed on sides AB, BC, CD and DA, respectively. Since $\angle ABC + \angle ADC = 180^\circ$, it follows that $\triangle ADC = \triangle A_2BC_1$ and, therefore, $\triangle RDS = \triangle PBQ$ and $RS = PQ$. Similarly, $QR = PS$. Therefore, $PQRS$ is a parallelogram such that one of triangles RDS and PBQ is constructed on its sides outwards and on the other side inwards; a similar statement holds for triangles QCR and SAP as well. Therefore, $\angle PQR + \angle RSP = \angle BQC + \angle DSA = 180^\circ$ because $\angle PQB = \angle RSD$ and $\angle RQC = \angle PSA$. It follows that $PQRS$ is a rectangle.

1.44. Let K, L and M be the intersection points of the circumscribed circles of triangles FOA and BOC, BOC and DOE, DOE and FOA, respectively; 2α, 2β and 2γ the angles at the vertices of isosceles triangles BOC, DOE and FOA, respectively (Fig. 9).

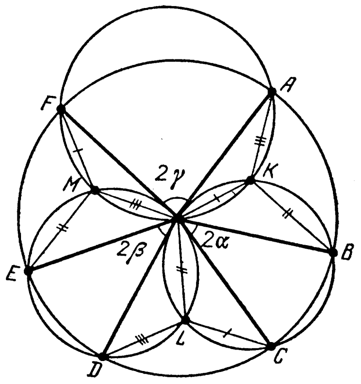


Figure 9 (Sol. 1.44)

Point K lies on arc $\smile OB$ of the circumscribed circle of the isosceles triangle BOC and, therefore, $\angle OKB = 90^\circ + \alpha$. Similarly, $\angle OKA = 90^\circ + \gamma$. Since $\alpha + \beta + \gamma = 90^\circ$, it follows that $\angle AKB = 90^\circ + \beta$. Inside equilateral triangle AOB there exists a unique point K that serves as the vertex of the angles that subtend its sides and are equal to the given angles.

Similar arguments for a point L inside triangle COD show that $\triangle OKB = \triangle CLO$.

Now, let us prove that $\triangle KOL = \triangle OKB$. Indeed, $\angle COL = \angle KBO$; hence, $\angle KOB + \angle COL = 180^\circ - \angle OKB = 90^\circ - \alpha$ and, therefore, $\angle KOL = 2\alpha + (90^\circ - \alpha) = 90^\circ + \alpha = \angle OKB$. It follows that $KL = OB = R$. Similarly, $LM = MK = R$.

1.45. Let $\angle A = \alpha$. It is easy to verify that both angles $\angle KCL$ and $\angle ADL$ are equal to $240^\circ - \alpha$ (or $120^\circ + \alpha$). Since $KC = BC = AD$ and $CL = DL$, it follows that $\triangle KCL = \triangle ADL$ and, therefore, $KL = AL$. Similarly, $KL = AK$.

1.46. Let P, Q and R be the centers of the squares constructed on sides DA, AB and BC, respectively, in parallelogram $ABCD$ with an acute angle of α at vertex A. It is easy to verify that $\angle PAQ = 90^\circ + \alpha = \angle RBQ$; hence, $\triangle PAQ = \triangle RBQ$. Sides AQ and BQ of these triangles are perpendicular, hence, $PQ \perp QR$.

1.47. First, observe that the sum of the angles at vertices A, B and C of hexagon $AB'CA'BC'$ is equal to 360° because by the hypothesis the sum of its angles at the other vertices is equal to 360°. On side AC', construct outwards triangle $\triangle AC'P$ equal to triangle $\triangle BC'A'$ (Fig. 10).

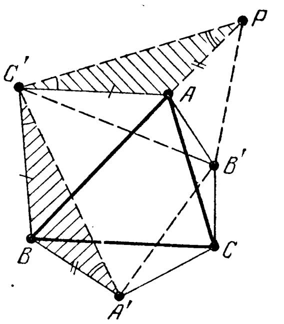


Figure 10 (Sol. 1.47)

Then $\triangle AB'P = \triangle CB'A'$ because $AB' = CB'$, $AP = CA'$ and

$$\angle PAB' = 360^\circ - \angle PAC' - \angle C'AB' = 360^\circ - \angle A'BC' - \angle C'AB' = \angle A'CB'.$$

Hence, $\triangle C'B'A' = \triangle C'B'P$ and, therefore, $2\angle A'B'C' = \angle PB'A' = \angle AB'C$ because $\angle PB'A = \angle A'B'C$.

1.48. Since $BA : BC = BC_1 : BA_1$ and $\angle ABC = \angle C_1BA_1$, it follows that $\triangle ABC \sim \triangle C_1BA_1$. Similarly, $\triangle ABC \sim \triangle B_1A_1C$. Since $BA_1 = A_1C$, it follows that $\triangle C_1BA_1 = \triangle B_1A_1C$. Therefore, $AC_1 = C_1B = B_1A_1$ and $AB_1 = B_1C = C_1A_1$. It is also clear that quadrilateral $AB_1A_1C_1$ is a convex one.

1.49. a) Let P and Q be the midpoints of sides AB and AC. Then $MP = \frac{1}{2}AC = QB_1$, $MQ = \frac{1}{2}AB = PC_1$ and $\angle C_1PM = \angle C_1PB + \angle BPM = \angle B_1QC + \angle CQM = \angle B_1QM$. Hence, $\triangle MQB_1 = \triangle C_1PM$ and, therefore, $MC_1 = MB_1$. Moreover,

$$\angle PMC_1 + \angle QMB_1 = \angle QB_1M + \angle QMB_1 = 180^\circ - \angle MQB_1$$

and

$$\angle MQB_1 = \angle A + \angle CQB_1 = \angle A + (180^\circ - 2\varphi).$$

Therefore, $\angle B_1MC_1 = \angle PMQ + 2\varphi - \angle A = 2\varphi$. (The case when $\angle C_1PB + \angle BPM > 180^\circ$ is analogously treated.)

b) On sides AB and AC, take points B' and C', respectively, such that $AB' : AB = AC' : AC = 2 : 3$. The midpoint M of segment $B'C'$ coincides with the intersection point of the medians of triangle ABC. On sides AB' and AC', construct outwards right triangles $AB'C_1$ and AB_1C' with angle $\varphi = 60^\circ$ as in heading a). Then B_1 and C_1 are the centers of right triangles constructed on sides AB and AC; on the other hand, by heading a), $MB_1 = MC_1$ and $\angle B_1MC_1 = 120^\circ$.

Remark. Statements of headings a) and b) remain true for triangles constructed inwards, as well.

1.50. a) Let B' be the intersection point of line AC and the perpendicular to line AB_1 erected from point B_1; define point C' similarly. Since $AB' : AC' = AC_1 : AB_1 = AB : AC$, it follows that $B'C' \parallel BC$. If N is the midpoint of segment $B'C'$, then, as follows from Problem 1.49, $NC_1 = NB_1$ (i.e., $N = M$) and $\angle B_1NC_1 = 2\angle AB'B_1 = 180^\circ - 2\angle CAB_1 = \varphi$.

b) On side BC construct outwards isosceles triangle BA_1C with angle $360^\circ - 2\varphi$ at vertex A_1 (if $\varphi < 90^\circ$ construct inwards a triangle with angle 2φ). Since the sum of the angles at the vertices of the three constructed isosceles triangles is equal to 360°, it follows that the angles of triangle $A_1B_1C_1$ are equal to $180^\circ - \varphi$, $\frac{1}{2}\varphi$ and $\frac{1}{2}\varphi$ (cf. Problem 1.47). In particular, this triangle is an isosceles one, hence, $A_1 = O$.

1.51. Let O_1, O_2, O_3 and O_4 be the centers of rhombuses constructed on sides AB, BC, CA and DA, respectively; let M be the midpoint of diagonal AC. Then $MO_1 = MO_2$ and $\angle O_1MO_2 = \alpha$ (cf. Problem 1.49). Similarly, $MO_3 = MO_4$ and $\angle O_3MO_4 = \alpha$. Therefore, under the rotation through an angle of α about point M triangle $\triangle O_1MO_3$ turns into $\triangle O_2MO_4$.

1.52. Since $A_1C = AC|\cos C|$, $B_1C = BC|\cos C|$ and angle $\angle C$ is the common angle of triangles ABC and A_1B_1C, these triangles are similar; the similarity coefficient is equal to $|\cos C|$.

1.53. Since points M and N lie on the circle with diameter CH, it follows that $\angle CMN = \angle CHN$ and since $AC \perp HN$, we see that $\angle CHN = \angle A$. Similarly, $\angle CNM = \angle B$.

1.54. a) Let l be the tangent to the circumscribed circle at point A. Then $\angle(l, AB) = \angle(AC, CB) = \angle(C_1B_1, AC_1)$ and, therefore, $l \parallel B_1C_1$.

b) Since $OA \perp l$ and $l \parallel B_1C_1$, it follows that $OA \perp B_1C_1$.

1.55. If AA_1, BB_1 and CC_1 are heights, then right triangles AA_1C and BB_1C have equal angles at vertex C and, therefore, are similar. It follows that $\triangle A_1BH \sim \triangle B_1AH$, consequently, $AH \cdot A_1H = BH \cdot B_1H$. Similarly, $BH \cdot B_1H = CH \cdot C_1H$.

If $AH \cdot A_1H = BH \cdot B_1H = CH \cdot C_1H$, then $\triangle A_1BH \sim \triangle B_1AH$; hence, $\angle BA_1H = \angle AB_1H = \varphi$. Thus, $\angle CA_1H = \angle CB_1H = 180^\circ - \varphi$.

Similarly, $\angle AC_1H = \angle CA_1H = 180^\circ - \varphi$ and $\angle AC_1H = \angle AB_1H = \varphi$. Hence, $\varphi = 90^\circ$, i.e., AA_1, BB_1 and CC_1 are heights.

1.56. a) By Problem 1.52 $\angle C_1A_1B = \angle CA_1B_1 = \angle A$. Since $AA_1 \perp BC$, it follows that $\angle C_1A_1A = \angle B_1A_1A$. The proof of the fact that rays B_1B and C_1C are the bisectors of angles $A_1B_1C_1$ and $A_1C_1B_1$ is similar.

b) Lines AB, BC and CA are the bisectors of the outer angles of triangle $A_1B_1C_1$, hence, A_1A is the bisector of angle $\angle B_1A_1C_1$ and, therefore, $AA_1 \perp BC$. For lines BB_1 and CC_1 the proof is similar.

1.57. From the result of Problem 1.56 a) it follows that the symmetry through line AC sends line B_1A_1 into line B_1C_1.

1.58. By Problem 1.52 $\angle B_1A_1C = \angle BAC$. Since $A_1B_1 \parallel AB$, it follows that $\angle B_1A_1C = \angle ABC$. Hence, $\angle BAC = \angle ABC$. Similarly, since $B_1C_1 \parallel BC$, it follows that $\angle ABC = \angle BCA$. Therefore, triangle ABC is an equilateral one and $A_1C_1 \parallel AC$.

1.59. Let O be the center of the circumscribed circle of triangle ABC. Since $OA \perp B_1C_1$ (cf. Problem 1.54 b), it follows that $S_{AOC_1} + S_{AOB_1} = \frac{1}{2}(R \cdot B_1C_1)$. Similar arguments for vertices B and C show that $S_{ABC} = qR$. On the other hand, $S_{ABC} = pr$.

1.60. The perimeter of the triangle cut off by the line parallel to side BC is equal to the sum of distances from point A to the tangent points of the inscribed circle with sides AB and AC; therefore, the sum of perimeters of small triangles is equal to the perimeter of triangle ABC, i.e., $P_1 + P_2 + P_3 = P$. The similarity of triangles implies that $\frac{r_i}{r} = \frac{P_i}{P}$. Summing these equalities for all the i we get the statement desired.

1.61. Let $M = A$. Then $X_A = A$; hence, $AY_A = 1$. Similarly, $CX_C = 1$. Let us prove that $y = AY_A$ and $x = CX_C$ are the desired lines. On side BC, take point D so that $AB \parallel MD$, see Fig. 11. Let E be the intersection point of lines CX_C and MD. Then, $X_MM + Y_MM = X_CE + Y_MM$. Since $\triangle ABC \sim \triangle MDC$, it follows that $CE = Y_MM$. Therefore, $CE = Y_MM$. Hence, $X_MM + Y_MM = X_CE + CE = X_CC = 1$.

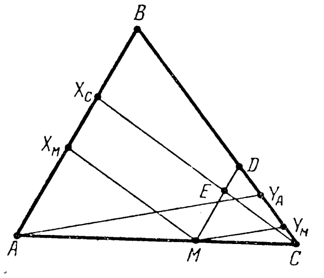


Figure 11 (Sol. 1.61)

1.62. Let D be the midpoint of segment BH. Since $\triangle BHA \sim \triangle HEA$, it follows that $AD : AO = AB : AH$ and $\angle DAH = \angle OAE$. Hence, $\angle DAO = \angle BAH$ and, therefore, $\triangle DAO \sim \triangle BAH$ and $\angle DOA = \angle BAH = 90^\circ$.

1.63. Let AA_1, BB_1 and CC_1 be heights of triangle ABC. Let us drop from point B_1 perpendiculars B_1K and B_1N to sides AB and BC, respectively, and perpendiculars B_1L and B_1M to heights AA_1 and CC_1, respectively. Since $KB_1 : C_1C = AB_1 : AC = LB_1 : A_1C$, it follows that $\triangle KLB_1 \sim \triangle C_1A_1C$ and, therefore, $KL \parallel C_1A_1$. Similarly, $MN \parallel C_1A_1$. Moreover, $KN \parallel C_1A_1$ (cf. Problem 1.53). It follows that points K, L, M and N lie on one line.

1.64. a) Let O be the midpoint of AC, let O_1 be the midpoint of AB and O_2 the midpoint of BC. Assume that $AB \leq BC$. Through point O_1 draw line O_1K parallel to EF (point K lies on segment EO_2). Let us prove that right triangles DBO and O_1KO_2 are equal. Indeed,

$O_1O_2 = DO = \frac{1}{2}AC$ and $BO = KO_2 = \frac{1}{2}(BC - AB)$. Since triangles DBO and O_1KO_2 are equal, we see that $\angle BOD = \angle O_1O_2E$, i.e., line DO is parallel to EO_2 and the tangent drawn through point D is parallel to line EF.

b) Since the angles between the diameter AC and the tangents to the circles at points F, D, E are equal, it follows that $\angle FAB = \angle DAC = \angle EBC$ and $\angle FBA + \angle DCA = \angle ECB$, i.e., F lies on line segment AD and E lies on line segment DC. Moreover, $\angle AFB = \angle BEC = \angle ADC = 90^\circ$ and, therefore, $FDEB$ is a rectangle.

1.65. Let MQ and MP be perpendiculars dropped on sides AD and BC, let MR and MT be perpendiculars dropped on the extensions of sides AB and CD (Fig. 12). Denote by M_1 and P_1 the other intersection points of lines RT and QP with the circle.

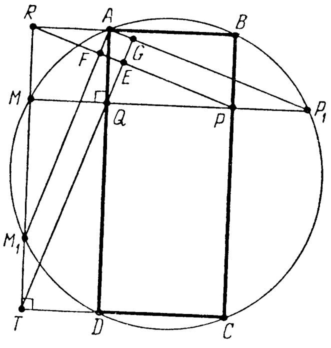


Figure 12 (Sol. 1.65)

Since $TM_1 = RM = AQ$ and $TM_1 \parallel AQ$, it follows that $AM_1 \parallel TQ$. Similarly, $AP_1 \parallel RP$. Since $\angle M_1AP_1 = 90^\circ$, it follows that $RP \perp TQ$.

Denote the intersection points of lines TQ and RP, M_1A and RP, P_1A and TQ by E, F, G, respectively. To prove that point E lies on line AC, it suffices to prove that rectangles $AFEG$ and AM_1CP_1 are similar. Since $\angle ARF = \angle AM_1R = \angle M_1TG = \angle M_1CT$, we may denote the values of these angles by the same letter α. We have: $AF = RA\sin\alpha = M_1A\sin^2\alpha$ and $AG = M_1T\sin\alpha = M_1C\sin^2\alpha$. Therefore, rectangles $AFEG$ and AM_1CP_1 are similar.

1.66. Denote the centers of the circles by O_1 and O_2. The outer tangent is tangent to the first circle at point K and to the other circle at point L; the inner tangent is tangent to the first circle at point M and to the other circle at point N (Fig. 13).

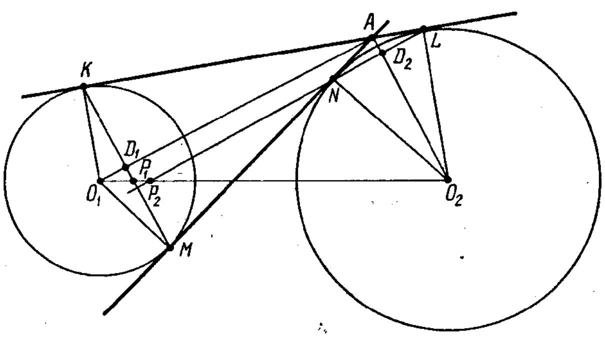


Figure 13 (Sol. 1.66)

Let lines KM and LN intersect line O_1O_2 at points P_1 and P_2, respectively. We have to prove that $P_1 = P_2$. Let us consider points A, D_1, D_2 — the intersection points of KL with MN, KM with O_1A, and LN with O_2A, respectively. Since $\angle O_1AM + \angle NAO_2 = 90^\circ$, right triangles O_1MA and ANO_2 are similar; we also see that $AO_2 \parallel KM$ and $AO_1 \parallel LN$. Since these lines are parallel, $AD_1 : D_1O_1 = O_2P_1 : P_1O_1$ and $D_2O_2 : AD_2 = O_2P_2 : P_2O_1$. The similarity of quadrilaterals AKO_1M and O_2NAL yields $AD_1 : D_1O_1 = D_2O_2 : AD_2$. Therefore, $O_2P_1 : P_1O_1 = O_2P_2 : P_2O_1$, i.e., $P_1 = P_2$.

CHAPTER 2. INSCRIBED ANGLES

Background

1. Angle $\angle ABC$ whose vertex lies on a circle and legs intersect this circle is called *inscribed* in the circle. Let O be the center of the circle. Then

$$\angle ABC = \begin{cases} \frac{1}{2}\angle AOC & \text{if points } B \text{ and } O \text{ lie on one side of } AC \\ 180^\circ - \frac{1}{2}\angle AOC & \text{otherwise.} \end{cases}$$

The most important and most often used corollary of this fact is that *equal chords subtend angles that either are equal or the sum of the angles is equal to* 180°.

2. The value of the angle between chord AB and the tangent to the circle that passes through point A is equal to half the angle value of arc $\smile AB$.

3. The angle values of arcs confined between parallel chords are equal.

4. As we have already said, if two angles subtend the same chord, either they are equal or the sum of their values is 180°. In order not to consider various variants of the positions of points on the circle let us introduce the notion of an *oriented angle* between lines. The *value of the oriented angle between lines* AB and CD (notation: $\angle(AB, CD)$) is the value of the angle by which we have to rotate line AB counterclockwise in order for it to become parallel to line CD. The angles that differ by $n \cdot 180^\circ$ are considered equal.

Notice that, generally, the oriented angle between lines CD and AB is not equal to the oriented angle between lines AB and CD (the sum of $\angle(AB, CD)$ and $\angle(CD, AB)$ is equal to 180° which, according to our convention, is the same as 0°).

It is easy to verify the following properties of the oriented angles:

a) $\angle(AB, BC) = -\angle(BC, AB)$;

b) $\angle(AB, CD) + \angle(CD, EF) = \angle(AB, EF)$;

c) *points A, B, C, D not on one line lie on one circle if and only if* $\angle(AB, BC) = \angle(AD, DC)$. (To prove this property we have to consider two cases: points B and D lie on one side of AC; points B and D lie on different sides of AC.)

Introductory problems

1. a) From point A lying outside a circle rays AB and AC come out and intersect the circle. Prove that the value of angle $\angle BAC$ is equal to half the difference of the angle measures of the arcs of the circle confined inside this angle.

b) The vertex of angle $\angle BAC$ lies inside a circle. Prove that the value of angle $\angle BAC$ is equal to half the sum of angle measures of the arcs of the circle confined inside angle $\angle BAC$ and inside the angle symmetric to it through vertex A.

2. From point P inside acute angle $\angle BAC$ perpendiculars PC_1 and PB_1 are dropped on lines AB and AC. Prove that $\angle C_1AP = \angle C_1B_1P$.

3. Prove that all the angles formed by the sides and diagonals of a regular n-gon are integer multiples of $\frac{180^\circ}{n}$.

4. The center of an inscribed circle of triangle ABC is symmetric through side AB to the center of the circumscribed circle. Find the angles of triangle ABC.

5. The bisector of the exterior angle at vertex C of triangle ABC intersects the circumscribed circle at point D. Prove that $AD = BD$.

§1. Angles that subtend equal arcs

2.1. Vertex A of an acute triangle ABC is connected by a segment with the center O of the circumscribed circle. From vertex A height AH is drawn. Prove that $\angle BAH = \angle OAC$.

2.2. Two circles intersect at points M and K. Lines AB and CD are drawn through M and K, respectively; they intersect the first circle at points A and C, the second circle at points B and D, respectively. Prove that $AC \parallel BD$.

2.3. From an arbitrary point M inside a given angle with vertex A perpendiculars MP and MQ are dropped to the sides of the angle. From point A perpendicular AK is dropped on segment PQ. Prove that $\angle PAK = \angle MAQ$.

2.4. a) The continuation of the bisector of angle $\angle B$ of triangle ABC intersects the circumscribed circle at point M; O is the center of the inscribed circle, O_b is the center of the escribed circle tangent to AC. Prove that points A, C, O and O_b lie on a circle centered at M.

b) Point O inside triangle ABC is such that lines AO, BO and CO pass through the centers of the circumscribed circles of triangles BCO, ACO and ABO, respectively. Prove that O is the center of the inscribed circle of triangle ABC.

2.5. Vertices A and B of right triangle ABC with right angle $\angle C$ slide along the sides of a right angle with vertex P. Prove that in doing so point C moves along a line segment.

2.6. Diagonal AC of square $ABCD$ coincides with the hypothenuse of right triangle ACK, so that points B and K lie on one side of line AC. Prove that

$$BK = \frac{|AK - CK|}{\sqrt{2}} \quad \text{and} \quad DK = \frac{AK + CK}{\sqrt{2}}.$$

2.7. In triangle ABC medians AA_1 and BB_1 are drawn. Prove that if $\angle CAA_1 = \angle CBB_1$, then $AC = BC$.

2.8. Each angle of triangle ABC is smaller than 120°. Prove that inside $\triangle ABC$ there exists a point that serves as the vertex for three angles each of value 120° and subtending the side of the triangle different from the sides subtended by the other angles.

2.9. A circle is divided into equal arcs by n diameters. Prove that the bases of the perpendiculars dropped from an arbitrary point M inside the circle to these diameters are vertices of a regular n-gon.

2.10. Points A, B, M and N on a circle are given. From point M chords MA_1 and MB_1 perpendicular to lines NB and NA, respectively, are drawn. Prove that $AA_1 \parallel BB_1$.

2.11. Polygon $ABCDEF$ is an inscribed one; $AB \parallel DE$ and $BC \parallel EF$. Prove that $CD \parallel AF$.

2.12. Polygon $A_1A_2 \dots A_{2n}$ as an inscribed one. We know that all the pairs of its opposite sides except one are parallel. Prove that for any odd n the remaining pair of sides is also parallel and for any even n the lengths of the exceptional sides are equal.

2.13. Consider triangle ABC. Prove that there exist two families of equilateral triangles whose sides (or extensions of the sides) pass through points A, B and C. Prove also that the centers of triangles from these families lie on two concentric circles.

§2. The value of an angle between two chords

The following fact helps to solve problems from this section. Let A, B, C, D be points on a circle situated in the order indicated. Then

$$\angle(AC,BD)=\frac{\smile AB+\smile CD}{2}\quad\text{and}\quad\angle(AB,CD)=\frac{|\smile AD-\smile CB|}{2}.$$

To prove this, we have to draw a chord parallel to another chord through the endpoint of one of the chords.

2.14. Points A, B, C, D in the indicated order are given on a circle. Let M be the midpoint of arc $\smile AB$. Denote the intersection points of chords MC and MD with chord AB by E and K. Prove that $KECD$ is an inscribed quadrilateral.

2.15. Concider an equilateral triangle. A circle with the radius equal to the triangle's height rolls along a side of the triangle. Prove that the angle measure of the arc cut off the circle by the sides of the triangle is always equal to 60°.

2.16. The diagonals of an isosceles trapezoid $ABCD$ with lateral side AB intersect at point P. Prove that the center O of the inscribed circle lies on the inscribed circle of triangle APB.

2.17. Points A, B, C, D in the indicated order are given on a circle; points A_1, B_1, C_1 and D_1 are the midpoints of arcs $\smile AB$, $\smile BC$, $\smile CD$ and $\smile DA$, respectively. Prove that $A_1C_1 \perp B_1D_1$.

2.18. Point P inside triangle ABC is taken so that $\angle BPC = \angle A + 60^\circ$, $\angle APC = \angle B + 60^\circ$ and $\angle APB = \angle C + 60^\circ$. Lines AP, BP and CP intersect the circumscribed circle of triangle ABC at points A', B' and C', respectively. Prove that triangle $A'B'C'$ is an equilateral one.

2.19. Points A, C_1, B, A_1, C, B_1 in the indicated order are taken on a circle.

a) Prove that if lines AA_1, BB_1 and CC_1 are the bisectors of the angles of triangle ABC, then they are the heights of triangle $A_1B_1C_1$.

b) Prove that if lines AA_1, BB_1 and CC_1 are the heights of triangle ABC, then they are the bisectors of the angles of triangle $A_1B_1C_1$.

2.20. Triangles T_1 and T_2 are inscribed in a circle so that the vertices of triangle T_2 are the midpoints of the arcs into which the circle is divided by the vertices of triangle T_1. Prove that in the hexagon which is the intersection of triangles T_1 and T_2 the diagonals that connect the opposite vertices are parallel to the sides of triangle T_1 and meet at one point.

§3. The angle between a tangent and a chord

2.21. Two circles intersect in points P and Q. Through point A on the first circle lines AP and AQ are drawn. The lines intersect the second circle in points B and C. Prove that the tangent at A to the first circle is parallel to line BC.

2.22. Circles S_1 and S_2 intersect at points A and P. Tangent AB to circle S_1 is drawn through point A, and line CD parallel to AB is drawn through point P (points B and C lie on S_2, point D on S_1). Prove that $ABCD$ is a parallelogram.

2.23. The tangent at point A to the inscribed circle of triangle ABC intersects line BC at point E; let AD be the bisector of triangle ABC. Prove that $AE = ED$.

2.24. Circles S_1 and S_2 intersect at point A. Through point A a line that intersects S_1 at point B and S_2 at point C is drawn. Through points C and B tangents to the circles are drawn; the tangents intersect at point D. Prove that angle $\angle BDC$ does not depend on the choice of the line that passes through A.

2.25. Two circles intersect at points A and B. Through point A tangents AM and AN, where M and N are points of the respective circles, are drawn. Prove that:

a) $\angle ABN + \angle MAN = 180°$;

b) $\frac{BM}{BN} = \left(\frac{AM}{AN}\right)^2$.

2.26. Inside square $ABCD$ a point P is taken so that triangle ABP is an equilateral one. Prove that $\angle PCD = 15°$.

2.27. Two circles are internally tangent at point M. Let AB be the chord of the greater circle which is tangent to the smaller circle at point T. Prove that MT is the bisector of angle AMB.

2.28. Through point M inside circle S chord AB is drawn; perpendiculars MP and MQ are dropped from point M to the tangents that pass through points A and B respectively. Prove that the value of $\frac{1}{PM} + \frac{1}{QM}$ does not depend on the choice of the chord that passes through point M.

2.29. Circle S_1 is tangent to sides of angle ABC at points A and C. Circle S_2 is tangent to line AC at point C and passes through point B, circle S_2 intersects circle S_1 at point M. Prove that line AM divides segment BC in halves.

2.30. Circle S is tangent to circles S_1 and S_2 at points A_1 and A_2; let B be a point of circle S, let K_1 and K_2 be the other intersection points of lines A_1B and A_2B with circles S_1 and S_2, respectively. Prove that if line K_1K_2 is tangent to circle S_1, then it is also tangent to circle S_2.

§4. Relations between the values of an angle and the lengths of the arc and chord associated with the angle

2.31. Isosceles trapezoids $ABCD$ and $A_1B_1C_1D_1$ with parallel respective sides are inscribed in a circle. Prove that $AC = A_1C_1$.

2.32. From point M that moves along a circle perpendiculars MP and MQ are dropped on diameters AB and CD, respectively. Prove that the length of segment PQ does not depend on the position of point M.

2.33. In triangle ABC, angle $\angle B$ is equal to $60°$; bisectors AD and CE intersect at point O. Prove that $OD = OE$.

2.34. In triangle ABC the angles at vertices B and C are equal to $40°$; let BD be the bisector of angle B. Prove that $BD + DA = BC$.

2.35. On chord AB of circle S centered at O a point C is taken. The circumscribed circle of triangle AOC intersects circle S at point D. Prove that $BC = CD$.

2.36. Vertices A and B of an equilateral triangle ABC lie on circle S, vertex C lies inside this circle. Point D lies on circle S and $BD = AB$. Line CD intersects S at point E. Prove that the length of segment EC is equal to the radius of circle S.

2.37. Along a fixed circle another circle whose radius is half that of the fixed one rolls on the inside without gliding. What is the trajectory of a fixed point K of the rolling circle?

§5. Four points on one circle

2.38. From an arbitrary point M on leg BC of right triangle ABC perpendicular MN is dropped on hypothenuse AP. Prove that $\angle MAN = \angle MCN$.

2.39. The diagonals of trapezoid $ABCD$ with bases AD and BC intersect at point O; points B' and C' are symmetric through the bisector of angle $\angle BOC$ to vertices B and C, respectively. Prove that $\angle C'AC = \angle B'DB$.

2.40. The extensions of sides AB and CD of the inscribed quadrilateral $ABCD$ meet at point P; the extensions of sides BC and AD meet at point Q. Prove that the intersection points of the bisectors of angles $\angle AQB$ and $\angle BPC$ with the sides of the quadrilateral are vertices of a rhombus.

2.41. The inscribed circle of triangle ABC is tangent to sides AB and AC at points M and N, respectively. Let P be the intersection point of line MN with the bisector (or its extension) of angle $\angle B$. Prove that:

a) $\angle BPC = 90^\circ$;

b) $S_{ABP} : S_{ABC} = 1 : 2$.

2.42. Inside quadrilateral $ABCD$ a point M is taken so that $ABMD$ is a parallelogram. Prove that if $\angle CBM = \angle CDM$, then $\angle ACD = \angle BCM$.

2.43. Lines AP, BP and CP intersect the circumscribed circle of triangle ABC at points A_1, B_1 and C_1, respectively. On lines BC, CA and AB points A_2, B_2 and C_2, respectively, are taken so that $\angle(PA_2, BC) = \angle(PB_2, CA) = \angle(PC_2, AB)$. Prove that $\triangle A_2B_2C_2 \sim \triangle A_1B_1C_1$.

2.44. About an equilateral triangle APQ a rectangular $ABCD$ is circumscribed so that points P and Q lie on sides BC and CD, respectively; P' and Q' are the midpoints of sides AP and AQ, respectively. Prove that triangles $BQ'C$ and $CP'D$ are equilateral ones.

2.45. Prove that if for inscribed quadrilateral $ABCD$ the equality $CD = AD + BC$ holds, then the intersection point of the bisectors of angles $\angle A$ and $\angle B$ lies on side CD.

2.46. Diagonals AC and CE of a regular hexagon $ABCDEF$ are divided by points M and N, respectively, so that $AM : AC = CN : CE = \lambda$. Find λ if it is known that points B, M and N lie on a line.

2.47. The corresponding sides of triangles ABC and $A_1B_1C_1$ are parallel and sides AB and A_1B_1 lie on one line. Prove that the line that connects the intersection points of the circumscribed circles of triangles A_1BC and AB_1C contains point C_1.

2.48. In triangle ABC heights AA_1, BB_1 and CC_1 are drawn. Line KL is parallel to CC_1; points K and L lie on lines BC and B_1C_1, respectively. Prove that the center of the circumscribed circle of triangle A_1KL lies on line AC.

2.49. Through the intersection point O of the bisectors of triangle ABC line MN is drawn perpendicularly to CO so that M and N lie on sides AC and BC, respectively. Lines AO and BO intersect the circumscribed circle of triangle ABC at points A' and B', respectively. Prove that the intersection point of lines $A'N$ and $B'M$ lies on the circumscribed circle.

§6. The inscribed angle and similar triangles

2.50. Points A, B, C and D on a circle are given. Lines AB and CD intersect at point M. Prove that

$$\frac{AC \cdot AD}{AM} = \frac{BC \cdot BD}{BM}.$$

2.51. Points A, B and C on a circle are given; the distance BC is greater than the distance from point B to line l tangent to the circle at point A. Line AC intersects the line drawn through point B parallelly to l at point D. Prove that $AB^2 = AC \cdot AD$.

2.52. Line l is tangent to the circle of diameter AB at point C; points M and N are the projections of points A and B on line l, respectively, and D is the projection of point C on AB. Prove that $CD^2 = AM \cdot BN$.

2.53. In triangle ABC, height AH is drawn and from vertices B and C perpendiculars BB_1 and CC_1 are dropped on the line that passes through point A. Prove that $\triangle ABC \sim \triangle HB_1C_1$.

2.54. On arc $\smile BC$ of the circle circumscribed about equilateral triangle ABC, point P is taken. Segments AP and BC intersect at point Q. Prove that

$$\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}.$$

2.55. On sides BC and CD of square $ABCD$ points E and F are taken so that $\angle EAF = 45^\circ$. Segments AE and AF intersect diagonal BD at points P and Q, respectively. Prove that $\frac{S_{AEF}}{S_{APQ}} = 2$.

2.56. A line that passes through vertex C of equilateral triangle ABC intersects base AB at point M and the circumscribed circle at point N. Prove that

$$CM \cdot CN = AC^2 \quad \text{and} \quad \frac{CM}{CN} = \frac{AM \cdot BM}{AN \cdot BN}.$$

2.57. Consider parallelogram $ABCD$ with an acute angle at vertex A. On rays AB and CB points H and K, respectively, are marked so that $CH = BC$ and $AK = AB$. Prove that:

a) $DH = DK$;

b) $\triangle DKH \sim \triangle ABK$.

2.58. a) The legs of an angle with vertex C are tangent to a circle at points A and B. From point P on the circle perpendiculars PA_1, PB_1 and PC_1 are dropped on lines BC, CA and AB, respectively. Prove that $PC_1^2 = PA_1 \cdot PB_1$.

b) From point O of the inscribed circle of triangle ABC perpendiculars OA', OB', OC' are dropped on the sides of triangle ABC opposite to vertices A, B and C, respectively, and perpendiculars OA'', OB'', OC'' are dropped to the sides of the triangle with vertices at the tangent points. Prove that

$$OA' \cdot OB' \cdot OC' = OA'' \cdot OB'' \cdot OC''.$$

2.59. Pentagon $ABCDE$ is inscribed in a circle. Distances from point E to lines AB, BC and CD are equal to a, b and c, respectively. Find the distance from point E to line AD.

2.60. In triangle ABC, heights AA_1, BB_1 and CC_1 are drawn; B_2 and C_2 are the midpoints of heights BB_1 and CC_1, respectively. Prove that $\triangle A_1B_2C_2 \sim \triangle ABC$.

2.61. On heights of triangle ABC points A_1, B_1 and C_1 that divide them in the ratio $2 : 1$ counting from the vertex are taken. Prove that $\triangle A_1B_1C_1 \sim \triangle ABC$.

2.62. Circle S_1 with diameter AB intersects circle S_2 centered at A at points C and D. Through point B a line is drawn; it intersects S_2 at point M that lies inside S_1 and it intersects S_1 at point N. Prove that $MN^2 = CN \cdot ND$.

2.63. Through the midpoint C of an arbitrary chord AB on a circle chords KL and MN are drawn so that points K and M lie on one side of AB. Segments KN and ML intersect AB at points Q and P, respectively. Prove that $PC = QC$.

2.64. a) A circle that passes through point C intersects sides BC and AC of triangle ABC at points A_1 and B_1, respectively, and it intersects the circumscribed circle of triangle ABC at point M. Prove that $\triangle AB_1M \sim \triangle BA_1M$.

b) On rays AC and BC segments AA_1 and BB_1 equal to the semiperimeter of triangle ABC are drawn. Let M be a point on the circumscribed circle such that $CM \parallel A_1B_1$. Prove that $\angle CMO = 90^\circ$, where O is the center of the inscribed circle.

§7. The bisector divides an arc in halves

2.65. In triangle ABC, sides AC and BC are not equal. Prove that the bisector of angle $\angle C$ divides the angle between the median and the height drawn from this vertex in halves if and only if $\angle C = 90^\circ$.

2.66. It is known that in a triangle the median, the bisector and the height drawn from vertex C divide the angle $\angle C$ into four equal parts. Find the angles of this triangle.

2.67. Prove that in triangle ABC bisector AE lies between median AM and height AH.

2.68. Given triangle ABC; on its side AB point P is chosen; lines PM and PN parallel to AC and BC, respectively, are drawn through P so that points M and N lie on sides BC and AC, respectively; let Q be the intersection point of the circumscribed circles of triangles APN and BPM. Prove that all lines PQ pass through a fixed point.

2.69. The continuation of bisector AD of acute triangle ABC inersects the circumscribed circle at point E. Perpendiculars DP and DQ are dropped on sides AB and AC from point D. Prove that $S_{ABC} = S_{APEQ}$.

§8. An inscribed quadrilateral with perpendicular diagonals

In this section $ABCD$ is an inscribed quadrilateral whose diagonals intersect at a right angle. We will also adopt the following notations: O is the center of the circumscribed circle of quadrilateral $ABCD$ and P is the intersection point of its diagonals.

2.70. Prove that the broken line AOC divides $ABCD$ into two parts whose areas are equal.

2.71. The radius of the circumscribed circle of quadrilateral $ABCD$ is equal to R.

a) Find $AP^2 + BP^2 + CP^2 + DP^2$.

b) Find the sum of squared lengths of the sides of $ABCD$.

2.72. Find the sum of squared lengths of the diagonals of $ABCD$ if the length of segment OP and the radius of the circumscribed circle R are known.

2.73. From vertices A and B perpendiculars to CD that intersect lines BD and AC at points K and L, respectively, are drawn. Prove that $AKLB$ is a rhombus.

2.74. Prove that the area of quadrilateral $ABCD$ is equal to $\frac{1}{2}(AB \cdot CD + BC \cdot AD)$.

2.75. Prove that the distance from point O to side AB is equal to half the length of side CD.

2.76. Prove that the line drawn through point P perpendicularly to BC divides side AD in halves.

2.77. Prove that the midpoints of the sides of quadrilateral $ABCD$ and the projections of point P on the sides lie on one circle.

2.78. a) Through vertices A, B, C and D tangents to the circumscribed circle are drawn. Prove that the quadrilateral formed by them is an inscribed one.

b) Quadrilateral $KLMN$ is simultaneously inscribed and circumscribed; A and B are the tangent points of the inscribed circle with sides KL and LM, respectively. Prove that $AK \cdot BM = r^2$, where r is the radius of the inscribed circle.

§9. Three circumscribed circles intersect at one point

2.79. On sides of triangle ABC triangles ABC', $AB'C$ and $A'BC$ are constructed outwards so that the sum of the angles at vertices A', B' and C' is a multiple of 180°. Prove that the circumscribed circles of the constructed triangles intersect at one point.

2.80. a) On sides (or their extensions) BC, CA and AB of triangle ABC points A_1, B_1 and C_1 distinct from the vertices of the triangle are taken (one point on one side). Prove that the circumscribed circles of triangles AB_1C_1, A_1BC_1 and A_1B_1C intersect at one point.

b) Points A_1, B_1 and C_1 move along lines BC, CA and AB, respectively, so that all triangles $A_1B_1C_1$ are similar and equally oriented. Prove that the intersection point of the circumscribed circles of triangles AB_1C_1, A_1BC_1 and A_1B_1C remains fixed in the process.

2.81. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1 are taken. Prove that if triangles $A_1B_1C_1$ and ABC are similar and have opposite orientations, then circumscribed circles of triangles AB_1C_1, ABC_1 and A_1B_1C pass through the center of the circumscribed circle of triangle ABC.

2.82. Points A', B' and C' are symmetric to a point P relative sides BC, CA and AB, respectively, of triangle ABC.

a) The circumscribed circles of triangles $AB'C'$, $A'BC'$, $A'B'C$ and ABC have a common point;

b) the circumscribed circles of triangles $A'BC$, $AB'C$, ABC' and $A'B'C'$ have a common point Q;

c) Let I, J, K and O be the centers of the circumscribed circles of triangles $A'BC$, $AB'C$, ABC' and $A'B'C'$, respectively. Prove that $QI : OI = QJ : OJ = QK : OK$.

§10. Michel's point

2.83. Four lines form four triangles. Prove that

a) The circumscribed circles of these triangles have a common point. (*Michel's point.*)

b) The centers of the circumscribed circles of these triangles lie on one circle that passes through Michel's point.

2.84. A line intersects sides (or their extensions) AB, BC and CA of triangle ABC at points C_1, B_1 and A_1, respectively; let O, O_a, O_b and O_c be the centers of the circumscribed circles of triangles ABC, AB_1C_1, A_1BC_1 and A_1B_1C, respectively; let H, H_a, H_b and H_c be the respective orthocenters of these triangles. Prove that

a) $\triangle O_aO_bO_c \sim \triangle ABC$.

b) the midperpendiculars to segments OH, O_aH_a, O_bH_b and O_cH_c meet at one point.

2.85. Quadrilateral $ABCD$ is an inscribed one. Prove that Michel's point of lines that contain its sides lies on the segment that connects the intersection points of the extensions of the sides.

2.86. Points A, B, C and D lie on a circle centered at O. Lines AB and CD intersect at point E and the circumscribed circles of triangles AEC and BED ilntersect at points E and P. Prove that

a) points A, D, P and O lie on one circle;

b) $\angle EPO = 90^\circ$.

2.87. Given four lines prove that the projections of Michel's point to these lines lie on one line.

See also Problem 19.45.

§11. Miscellaneous problems

2.88. In triangle ABC height AH is drawn; let O be the center of the circumscribed circle. Prove that $\angle OAH = |\angle B - \angle C|$.

2.89. Let H be the intersection point of the heights of triangle ABC; let AA' be a diameter of its circumscribed circle. Prove that segment $A'H$ divides side BC in halves.

2.90. Through vertices A and B of triangle ABC two parallel lines are drawn and lines m and n are symmetric to them through the bisectors of the corresponding angles. Prove that the intersection point of lines m and n lies on the circumscribed circle of triangle ABC.

2.91. a) Lines tangent to circle S at points B and C are drawn from point A. Prove that the center of the inscribed circle of triangle ABC and the center of its escribed circle tangent to side BC lie on circle S.

b) Prove that the circle that passes through vertices B and C of any triangle ABC and the center O of its inscribed circle intercepts on lines AB and AC chords of equal length.

2.92. On sides AC and BC of triangle ABC squares ACA_1A_2 and BCB_1B_2 are constructed outwards. Prove that lines A_1B, A_2B_2 and AB_1 meet at one point.

2.93. Circles S_1 and S_2 intersect at points A and B so that the tangents to S_1 at these points are radii of S_2. On the inner arc of S_1 a point C is taken; straight lines connect it with points A and B. Prove that the second intersection points of these lines with S_2 are the endpoints of a diameter.

2.94. From the center O of a circle the perpendicular OA is dropped to line l. On l, points B and C are taken so that $AB = AC$. Through points B and C two sections are drawn one of which intersects the circle at points P and Q and the other one at points M and N. Lines PM and QN intersect line l at points R and S, respectively. Prove that $AR = AS$.

Problems for independent study

2.95. In triangle ABC heights AA_1 and BB_1 are drawn; let M be the midpoint of side AB. Prove that $MA_1 = MB_1$.

2.96. In convex quadrilateral $ABCD$ angles $\angle A$ and $\angle C$ are right ones. Prove that $AC = BD \sin ABC$.

2.97. Diagonals AD, BE and CF of an inscribed hexagon $ABCDEF$ meet at one point. Prove that $AB \cdot CD \cdot EF = BC \cdot DE \cdot AF$.

2.98. In a convex quadrilateral $AB = BC = CD$, let M be the intersection point of diagonals, K is the intersection point of bisectors of angles $\angle A$ and $\angle D$. Prove that points A, M, K and D lie on one circle.

2.99. Circles centered at O_1 and O_2 intersect at points A and B. Line O_1A intersects the circle centered at O_2 at point N. Prove that points O_1, O_2, B and N lie on one circle.

2.100. Circles S_1 and S_2 intersect at points A and B. Line MN is tangent to circle S_1 at point M and to S_2 at point N. Let A be the intersection point of the circles, which is more distant from line MN. Prove that $\angle O_1AO_2 = 2\angle MAN$.

2.101. Given quadrilateral $ABCD$ inscribed in a circle and such that $AB = BC$, prove that $S_{ABCD} = \frac{1}{2}(DA+CD) \cdot h_b$, where h_b is the height of triangle ABD dropped from vertex B.

2.102. Quadrilateral $ABCD$ is an inscribed one and AC is the bisector of angle $\angle DAB$. Prove that $AC \cdot BD = AD \cdot DC + AB \cdot BC$.

2.103. In right triangle ABC, bisector CM and height CH are drawn from the vertex of the right angle $\angle C$. Let HD and HE be bisectors of triangles AHC and CHB. Prove that points C, D, H, E and M lie on one circle.

2.104. Two circles pass through the vertex of an angle and a point on its bisector. Prove that the segments cut by them on the sides of the angle are equal.

2.105. Triangle BHC, where H is the orthocenter of triangle ABC is complemented to the parallelogram $BHCD$. Prove that $\angle BAD = \angle CAH$.

2.106. Outside equilateral triangle ABC but inside angle $\angle BAC$, point M is taken so that $\angle CMA = 30^\circ$ and $\angle BMA = \alpha$. What is the value of angle $\angle ABM$?

2.107. Prove that if the inscribed quadrilateral with perpendicular diagonals is also a circumscribed one, then it is symmetric with respect to one of its diagonals.

Solutions

2.1. Let us draw diameter AD. Then $\angle CDA = \angle CBA$; hence, $\angle BAH = \angle DAC$ because $\angle BHA = \angle ACD = 90^\circ$.

2.2. Let us make use of the properties of oriented angles:

$$\angle(AC, CK) = \angle(AM, MK) = \angle(BM, MK) = \angle(BD, DK) = \angle(BD, CK),$$

i.e., $AC \parallel BD$.

2.3. Points P and Q lie on the circle with diameter AM. Therefore, $\angle QMA = \angle QPA$ as angles that intersect the same arc. Triangles PAK and MAQ are right ones, therefore, $\angle PAK = \angle MAQ$.

2.4. a) Since

$$\angle AOM = \angle BAO + \angle ABO = \frac{\angle A + \angle B}{2}$$

and

$$\angle OAM = \angle OAC + \angle CAM = \frac{\angle A}{2} + \angle CBM = \frac{\angle A + \angle B}{2},$$

we have $MA = MO$. Similarly, $MC = MO$.

Since triangle OAO_b is a right one and $\angle AOM = \angle MAO = \varphi$, it follows that $\angle MAO_b = \angle MO_bA = 90^\circ - \varphi$ and, therefore, $MA = MO_b$. Similarly, $MC = MO_b$.

b) Let P be the center of the circumscribed circle of triangle ACO. Then

$$\angle COP = \frac{180^\circ - \angle CPO}{2} = 90^\circ - \angle OAC.$$

Hence, $\angle BOC = 90^\circ + \angle OAC$. Similarly, $\angle BOC = 90^\circ + \angle OAB$ and, therefore, $\angle OAB = \angle OAC$. We similarly establish that point O lies on the bisectors of angles $\angle B$ and $\angle C$.

2.5. Points P and C lie on the circle with diameter AB, and, therefore, $\angle APC = \angle ABC$, i.e., the value of angle $\angle APC$ is a constant.

REMARK. A similar statement is true for any triangle ABC whose vertices are moving along the legs of angle $\angle MPN$ equal to $180^\circ - \angle C$.

2.6. Points B, D and K lie on the circle with diameter AC. Let, for definiteness sake, $\angle KCA = \varphi \leq 45^\circ$. Then

$$BK = AC \sin(45^\circ - \varphi) = \frac{AC(\cos\varphi - \sin\varphi)}{\sqrt{2}}$$

and

$$DK = AC \sin(45^\circ + \varphi) = \frac{AC(\cos\varphi + \sin\varphi)}{\sqrt{2}}.$$

Clearly, $AC\cos\varphi = CK$ and $AC\sin\varphi = AK$.

2.7. Since $\angle B_1AA_1 = \angle A_1BB_1$, it follows that points A, B, A_1 and B_1 lie on one circle. Parallel lines AB and A_1B_1 intercept on it equal chords AB_1 and BA_1. Hence, $AC = BC$.

2.8. On side BC of triangle ABC construct outwards an equilateral triangle A_1BC. Let P be the intersection point of line AA_1 with the circumscribed circle of triangle A_1BC. Then point P lies inside triangle ABC and

$$\angle APC = 180^\circ - \angle A_1PC = 180^\circ - \angle A_1BC = 120^\circ.$$

Similarly, $\angle APB = 120^\circ$.

2.9. The bases of perpendiculars dropped from point M on the diameters lie on the circle S with diameter OM (where O is the center of the initial circle). The intersection points of the given diameters with circle S distinct from O divide the circle into n arcs. Since the angles $\frac{180^\circ}{n}$ intersect all the circles that do not contain point O, the angle measure of each of these arcs is equal to $\frac{360^\circ}{n}$. Therefore, the angle measure of the arc on which point O lies is equal to $360^\circ - (n-1)\cdot\frac{360^\circ}{n} = \frac{360^\circ}{n}$. Thus, the bases of the perpendiculars divide the circle S into n equal arcs.

2.10. Clearly,

$$\angle(AA_1, BB_1) = \angle(AA_1, AB_1) + \angle(AB_1, BB_1) = \angle(MA_1, MB_1) + \angle(AN, BN).$$

Since $MA_1 \perp BN$ and $MB_1 \perp AN$, it follows that

$$\angle(MA_1, MB_1) = \angle(BN, AN) = -\angle(AN, BN).$$

Therefore, $\angle(AA_1, BB_1) = 0^\circ$, i.e., $AA_1 \parallel BB_1$.

2.11. Since $AB \parallel DE$, it follows that $\angle ACE = \angle BFD$ and since $BC \parallel EF$, it follows that $\angle CAE = \angle BDF$. Triangles ACE and BDF have two pairs of equal angles and, therefore, their third angles are also equal. The equality of these angles implies the equality of arcs $\smile AC$ and $\smile DF$, i.e., chords CD and AF are parallel.

2.12. Let us carry out the proof by induction on n. For the quadrilateral the statement is obvious; for the hexagon it had been proved in the preceding problem. Assume that the statement is proved for the $2(n-1)$-gon; let us prove the statement for the $2n$-gon. Let $A_1 \dots A_{2n}$ be a $2n$-gon in which $A_1A_2 \parallel A_{n+1}A_{n+2}, \dots, A_{n-1}A_n \parallel A_{2n-1}A_{2n}$. Let us consider $2(n-1)$-gon $A_1A_2 \dots A_{n-1}A_{n+1} \dots A_{2n-1}$. By the inductive hypothesis for n odd we have $A_{n-1}A_{n+1} = A_{2n-1}A_1$, and for n even we have $A_{n-1}A_{n+1} \parallel A_{2n-1}A_1$.

Let us consider triangles $A_{n-1}A_nA_{n+1}$ and $A_{2n-1}A_{2n}A_1$. Let n be even. Then vectors $\{A_{n-1}A_n\}$ and $\{A_{2n-1}A_{2n}\}$, as well as $\{A_{n-1}A_{n+1}\}$ and $\{A_{2n-1}A_1\}$ are parallel and directed towards each other; hence, $\angle A_nA_{n-1}A_{n+1} = \angle A_1A_{2n-1}A_{2n}$ and $A_nA_{n+1} = A_{2n}A_1$ as chords that cut equal arcs, as required.

Let n be odd. Then $A_{n-1}A_{n+1} = A_{2n-1}A_1$, i.e., $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$. In hexagon $A_{n-1}A_nA_{n+1}A_{2n-1}A_{2n}A_1$ we have $A_1A_{n-1} \parallel A_{n+1}A_{2n-1}$ and $A_{n-1}A_n \parallel A_{2n-1}A_{2n}$; hence, thanks to the preceding problem $A_nA_{n+1} \parallel A_{2n}A_1$, as required.

2.13. Let lines FG, GE and EF pass through points A, B and C, respectively, so that triangle EFG is an equilateral one, i.e.,

$$\angle(GE, EF) = \angle(EF, FG) = \angle FG, GE) = \pm 60^\circ.$$

Then

$$\angle(BE, EC) = \angle(CF, FA) = \angle(AG, GB) = \pm 60^\circ.$$

Selecting one of the signs we get three circles S_E, S_F and S_G on which points E, F and G should lie. Any point E of circle S_E uniquely determines triangle EFG.

Let O be the center of triangle EFG; let P, R and Q be the intersection points of lines OE, OF and OG with the corresponding circles S_E, S_F and S_G. Let us prove that P, Q and R are the centers of equilateral triangles constructed on sides of triangle ABC (outwards for one family and inwards for the other one), and point O lies on the circumscribed circle of triangle PQR.

Clearly,

$$\angle(CB, BP) = \angle(CE, EP) = \angle(EF, EO) = \mp 30^\circ$$

and

$$\angle(BP, CP) = \angle(BE, EC) = \angle(GE, EF) = \pm 60^\circ.$$

Hence,

$$\angle(CB, CP) = \angle(CB, BP) + \angle(BP, CP) = \pm 30^\circ.$$

Therefore, P is the center of an equilateral triangle with side AB.

For points Q and R the proof is similar. Triangle PQR is an equilateral one and its center coincides with the intersection point of medians of triangle ABC (cf. Problem 1.49 b)). As is not difficult to verify,

$$\angle(PR, RQ) = \mp 60^\circ = \angle(OE, OG) = \angle(OP, OQ),$$

i.e., point O lies on the circumscribed circle of triangle PQR.

2.14. Clearly,

$$2(\angle KEC + \angle KDC) = (\smile MB + \smile AC) + (\smile MB + \smile BC) = 360^\circ,$$

since $\smile MB = \smile AM$.

2.15. Denote the angle measure of the arc intercepted on the circle by the sides of triangle ABC by α. Denote the angle measure of the arc intercepted by the extensions of the sides of the triangle on the circle by α'. Then $\frac{1}{2}(\alpha + \alpha') = \angle BAC = 60^\circ$. But $\alpha = \alpha'$ because these arcs are symmetric through the line that passes through the center of the circle parallel to side BC. Hence, $\alpha = \alpha' = 60^\circ$.

2.16. Since $\angle APB = \frac{1}{2}(\smile AB + \smile CD) = \angle AOB$, point O lies on the circumscribed circle of triangle APB.

2.17. Let O be the point where lines A_1C_1 and B_1D_1 meet; let α, β, γ and δ be angle measures of arcs AB, BC, CD and DA. Then

$$\angle A_1OB_1 = \frac{\smile A_1B + \smile BB_1 + \smile C_1D + \smile DD_1}{2} = \frac{\alpha + \beta + \gamma + \delta}{4} = 90^\circ.$$

2.18. By summing up the equalities we get

$$\smile C'A + \smile CA' = 2(180^\circ - \angle APC) = 240^\circ - 2\angle B \quad \text{and} \quad \smile AB' + \smile BA' = 240^\circ - 2\angle C.$$

Then by subtracting from their sum the equality $\smile BA' + \smile CA' = 2\angle A$ we get

$$\smile C'B' = \smile C'A + \smile AB' = 480^\circ - 2(\angle A + \angle B + \angle C) = 120^\circ.$$

Similarly, $\smile B'A' = \smile C'A' = 120^\circ$.

2.19. a) Let us prove, for example, that $AA_1 \perp C_1B_1$. Let M be the intersection point of these segments. Then

$$\angle AMB_1 = \frac{\smile AB_1 + \smile A_1B + \smile BC_1}{2} = \angle ABB_1 + \angle A_1AB + \angle BCC_1 = \frac{\angle B + \angle A + \angle C}{2} = 90^\circ.$$

b) Let M_1 and M_2 be the intersection points of segments AA_1 with BC and BB_1 with AC. Right triangles AM_1C and BM_2C have a common angle $\angle C$; hence, $\angle B_1BC = \angle A_1AC$. Consequently, $\smile B_1C = \smile A_1C$ and $\angle B_1C_1C = \angle A_1C_1C$, i.e., CC_1 is the bisector of angle $\angle A_1C_1B_1$.

2.20. Denote the vertices of triangle T_1 by A, B and C; denote the midpoints of arcs $\smile BC$, $\smile CA$, $\smile AB$ by A_1, B_1, C_1, respectively. Then $T_2 = A_1B_1C_1$. Lines AA_1, BB_1, CC_1 are the bisectors of triangle T_1; hence, they meet at one point, O. Let lines AB and C_1B_1 intersect at point K. It suffices to verify that $KO \parallel AC$. In triangle AB_1O, line B_1C_1 is a bisector and height, hence, this triangle is an isosceles one. Therefore, triangle AKO is also an isosceles one. Lines KO and AC are parallel, since $\angle KOA = \angle KAO = \angle OAC$.

2.21. Let l be tangent to the first circle at point A. Then $\angle(l, AP) = \angle(AQ, PQ) = \angle(BC, PB)$, hence, $l \parallel BC$.

2.22. Since

$$\angle(AB, AD) = \angle(AP, PD) = \angle(AB, BC),$$

we have $BC \parallel AD$.

2.23. Let, for definiteness, point E lie on ray BC. Then $\angle ABC = \angle EAC$ and

$$\angle ADE = \angle ABC + \angle BAD = \angle EAC + \angle CAD = \angle DAE.$$

2.24. Let P be the other intersection point of the circles. Then $\angle(AB, DB) = \angle(PA, PB)$ and $\angle(DC, AC) = \angle(PC, PA)$. By summing these equalities we get

$$\angle(DC, DB) = \angle(PC, PB) = \angle(PC, CA) + \angle(BA, PB).$$

The latter two angles subtend constant arcs.

2.25. a) Since $\angle MAB = \angle BNA$, the sum of angles $\angle ABN$ and $\angle MAN$ is equal to the sum of the angles of triangle ABN.

b) Since $\angle BAM = \angle BNA$ and $\angle BAN = \angle BMA$, it follows that $\triangle AMB \sim \triangle NAB$ and, therefore, $AM : NA = MB : AB$ and $AM : NA = AB : NB$. By multiplying these equalities we get the desired statement.

2.26. Point P lies on the circle of radius BC with center B and line DC is tangent to this circle at point C. Hence, $\angle PCD = \frac{1}{2}\angle PBC = 15^\circ$.

2.27. Let A_1 and B_1 be intersection points of lines MA and MB, respectively, with the smaller circle. Since M is the center of homothety of the circles, $A_1B_1 \parallel AB$. Hence, $\angle A_1MT = \angle A_1TA = \angle B_1A_1T = \angle B_1MT$.

2.28. Let φ be the angle between chord AB and the tangent that passes through one of the chord's endpoints. Then $AB = 2R\sin\varphi$, where R is the radius of circle S. Moreover, $PM = AM\sin\varphi$ and $QM = BM\sin\varphi$. Hence,

$$\frac{1}{PM} + \frac{1}{QM} = \left(\frac{AM + BM}{\sin\varphi}\right) AM \cdot BM = \frac{2R}{AM \cdot BM}.$$

The value $AM \cdot BM$ does not depend on the choice of chord AB.

2.29. Let line AM intersect circle S_2 at point D. Then $\angle MDC = \angle MCA = \angle MAB$; hence, $CD \parallel AB$. Further, $\angle CAM = \angle MCB = \angle MDB$; hence, $AC \parallel BD$. Therefore, $ABCD$ is a parallelogram and its diagonal AD divides diagonal BC in halves.

2.30. Let us draw line l_1 tangent to S_1 at point A_1. Line K_1K_2 is tangent to S_1 if and only if $\angle(K_1K_2, K_1A_1) = \angle(K_1A_1, l_1)$. It is also clear that

$$\angle(K_1A_1, l_1) = \angle(A_1B, l_1) = \angle(A_2B, A_1A_2).$$

Similarly, line K_1K_2 is tangent to S_2 if and only if $\angle(K_1K_2, K_2A_2) = \angle(A_1B, A_1A_2)$. It remains to observe that if $\angle(K_1K_2, K_1A_1) = \angle(A_2B, A_1A_2)$, then

$$\angle(K_1K_2, K_2A_2) = \angle(K_1K_2, A_2B) =$$
$$\angle(K_1K_2, A_1B) + \angle(A_1B, A_1A_2) + \angle(A_1A_2, A_2B) = \angle(A_1B, A_1A_2).$$

2.31. Equal angles ABC and $A_1B_1C_1$ intersect chords AC and A_1C_1, hence, $AC = A_1C_1$.

2.32. Let us denote the center of the circle by O. Points P and Q lie on the circle with diameter OM, i.e., points O, P, Q and M lie on a circle of radius $\frac{1}{2}R$. Moreover, either $\angle POQ = \angle AOD$ or $\angle POQ = \angle BOD = 180^\circ - \angle AOD$, i.e., the length of chord PQ is a constant.

2.33. Since $\angle AOC = 90^\circ + \frac{1}{2}\angle B$ (cf. Problem 5.3), it follows that

$$\angle EBD + \angle EOD = 90^\circ + \frac{3}{2}\angle B = 180^\circ$$

and, therefore, quadrilateral $BEOD$ is an inscribed one. Equal angles $\angle EBO$ and $\angle OBD$ subtend chords EO and OD, hence, $EO = OD$.

2.34. On the extension of segment BD beyond point D take a point Q such that $\angle ACQ = 40^\circ$. Let P be the intersection point of lines AB and QC. Then $\angle BPC = 60^\circ$ and D is the intersection point of the bisectors of angles of triangle BCP. By Problem 2.33 $AD = DQ$. Moreover, $\angle BQC = \angle BCQ = 80^\circ$. Therefore, $BC = BD + DQ = BD + DA$.

2.35. It suffices to verify that the exterior angle ACD of triangle BCD is twice greater than the angle at vertex B. Clearly, $\angle ACD = \angle AOD = 2\angle ABD$.

2.36. Let O be the center of circle S. Point B is the center of the circumscribed circle of triangle ACD, hence, $\angle CDA = \frac{1}{2}\angle ABC = 30^\circ$ and, therefore, $\angle EOA = 2\angle EDA = 60^\circ$, i.e., triangle EOA is an equilateral one. Moreover, $\angle AEC = \angle AED = \angle AOB = 2\angle AOC$; hence, point E is the center of the circumscribed circle of triangle AOC. Therefore, $EC = EO$.

2.37. Let us consider two positions of the moving circle: at the first moment, when point K just gets to the fixed circle (the tangent point of the circles at this moment will be denoted by K_1) and at some other (second) moment.

Let O be the center of the fixed circle, O_1 and O_2 be the positions of the center of the moving circle at the first and the second moments, respectively, K_2 be the position of point K at the second moment. Let A be the tangent point of the circles at the second moment. Since the moving circle rolls without gliding, the length of arc $\smile K_1A$ is equal to the length of arc $\smile K_2A$. Since the radius of the moving circle is one half of the radius of the fixed circle, $\angle K_2O_2A = 2\angle K_1OA$. Point O lies on the moving circle, hence, $\angle K_2OA = \frac{1}{2}\angle K_2O_2A = \angle K_1OA$, i.e., points K_2, K_1 and O lie on one line.

The trajectory of point K is the diameter of the fixed circle.

2.38. Points N and C lie on the circle with diameter AM. Angles $\angle MAN$ and $\angle MCN$ subtend the same arc and therefore, are equal.

2.39. The symmetry through the bisector of angle $\angle BOC$ sends lines AC and DB into each other and, therefore, we have to prove that $\angle C'AB' = \angle B'DC'$. Since $BO = B'O$, $CO = C'O$ and $AO : DO = CO : BO$, it follows that $AO \cdot B'O = DO \cdot C'O$, i.e., the quadrilateral $AC'B'D$ is an inscribed one and $\angle C'AB' = \angle B'DC'$.

2.40. Denote the intersection points and angles as indicated on Fig. 14.

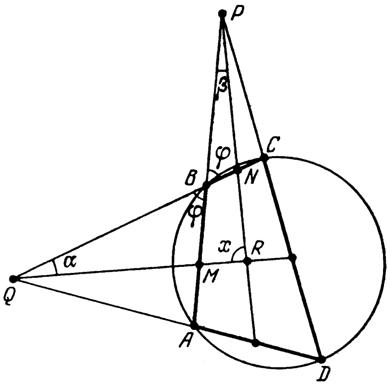


Figure 14 (Sol. 2.40)

It suffices to verify that $x = 90^\circ$. The angles of quadrilateral $BMRN$ are equal to $180^\circ - \varphi$, $\alpha + \varphi$, $\beta + \varphi$ and x, hence, the equality $x = 90^\circ$ is equivalent to the equality $(2\alpha+\varphi)+(2\beta+\varphi) = 180^\circ$. It remains to notice that $2\alpha+\varphi = \angle BAD$ and $2\beta+\varphi = \angle BCD$.

2.41. a) It suffices to prove that if P_1 is the point on the bisector (or its extension) of angle $\angle B$ that serves as the vertex of an angle of 90° that subtends segment BC, then P_1 lies on line MN. Points P_1 and N lie on the circle with diameter CO, where O is the intersection point of bisectors, hence,

$$\angle(P_1N, NC) = \angle(P_1O, OC) = \frac{1}{2}(180^\circ - \angle A) = \angle(MN, NC).$$

b) Since $\angle BPC = 90^\circ$, it follows that $BP = BC \cdot \cos\frac{\angle B}{2}$; hence,

$$S_{ABP} : S_{ABC} = \left(BP \cdot \sin\frac{\angle B}{2}\right) : (BC \sin B) = 1 : 2.$$

2.42. Take point N so that $BN \parallel MC$ and $NC \parallel BM$. Then $NA \parallel CD$, $\angle NCB = \angle CBM = \angle CDM = \angle NAB$, i.e., points A, B, N and C lie on one circle. Hence, $\angle ACD = \angle NAC = \angle NBC = \angle BCM$.

2.43. Points A_2, B_2, C and P lie on one circle, hence,

$$\angle(A_2B_2, B_2P) = \angle(A_2C, CP) = \angle(BC, CP).$$

Similarly, $\angle(B_2P, B_2C_2) = \angle(AP, AB)$. Therefore,

$$\begin{aligned}\angle(A_2B_2, B_2C_2) &= \angle(BC, CP) + \angle(AP, AB) = \angle(B_1B, B_1C_1) + \angle(A_1B_1, B_1B)\\ &= \angle(A_1B_1, B_1C_1).\end{aligned}$$

We similarly verify that all the other angles of triangles $A_1B_1C_1$ and $A_2B_2C_2$ are either equal or their sum is equal to 180°; therefore, these triangles are similar (cf. Problem 5.42).

2.44. Points Q' and C lie on the circle with diameter PQ, hence, $\angle Q'CQ = \angle Q'PQ = 30^\circ$. Therefore, $\angle BCQ' = 60^\circ$. Similarly, $\angle CBQ' = 60^\circ$ and, therefore, triangle $BQ'C$ is equilateral one. By similar reasons triangle $CP'D$ is an equilateral one.

2.45. Let $\angle BAD = 2\alpha$ and $\angle CBA = 2\beta$; for definiteness we will assume that $\alpha \geq \beta$. On side CD take point E so that $DE = DA$. Then $CE = CD - AD = CB$. The angle at vertex C of an isosceles triangle BCE is equal to $180^\circ - 2\alpha$; hence, $\angle CBE = \alpha$. Similarly, $\angle DAE = \beta$. The bisector of angle B intersects CD at a point F. Since $\angle FBA = \beta = \angle AED$, quadrilateral $ABFE$ is an inscribed one and, therefore, $\angle FAE = \angle FBE = \alpha - \beta$. It follows that $\angle FAD = \beta + (\alpha - \beta) = \alpha$, i.e., AF is the bisector of angle $\angle A$.

2.46. Since $ED = CB$, $EN = CM$ and $\angle DEC = \angle BCA = 30^\circ$ (Fig. 15), it follows that $\triangle EDN = \triangle CBM$. Let $\angle MBC = \angle NDE = \alpha$, $\angle BMC = \angle END = \beta$.

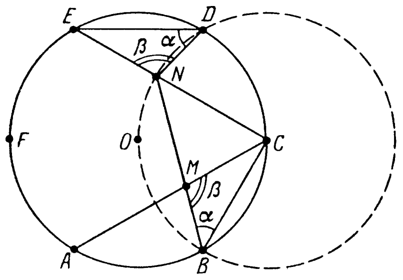


Figure 15 (Sol. 2.46)

It is clear that $\angle DNC = 180^\circ - \beta$. Considering triangle BNC we get $\angle BNC = 90^\circ - \alpha$. Since $\alpha + \beta = 180^\circ - 30^\circ = 150^\circ$, it follows that

$$\angle DNB = \angle DNC + \angle CNB = (180^\circ - \beta) + (90^\circ - \alpha) = 270^\circ - (\alpha + \beta) = 120^\circ.$$

Therefore, points B, O, N and D, where O is the center of the hexagon, lie on one circle. Moreover, $CO = CB = CD$, i.e., C is the center of this circle, hence, $\lambda = CN : CE = CB : CA = 1 : \sqrt{3}$.

2.47. Let D be the other intersection point of the circumscribed circles of triangles A_1BC and AB_1C. Then $\angle(AC, CD) = \angle(AB_1, B_1D)$ and $\angle(DC, CB) = \angle(DA_1, A_1B)$.

Hence,

$$\begin{aligned}\angle(A_1C_1, C_1B_1) &= \angle(AC, CB) = \angle(AC, CD) + \angle(DC, CB)\\ &= \angle(AB_1, B_1D) + \angle(DA_1, A_1B) = \angle(A_1D, DB_1),\end{aligned}$$

i.e., points A_1, B_1, C_1 and D lie on one circle. Therefore, $\angle(A_1C_1, C_1B) = \angle(A_1B_1, B_1D) = \angle(AC, CD)$. Taking into account that $A_1C_1 \parallel AC$, we get the desired statement.

2.48. Let point M be symmetric to point A_1 through line AC. By Problem 1.57 point M lies on line B_1C_1. Therefore,

$$\angle(LM, MA_1) = \angle(C_1B_1) = \angle(C_1C, CB) = \angle(LK, KA_1),$$

i.e., point M lies on the circumscribed circle of triangle A_1KL. It follows that the center of this circle lies on line AC — the midperpendicular to segment A_1M.

2.49. Let PQ be the diameter perpendicular to AB and such that Q and C lie on one side of AB; let L be the intersection point of line QO with the circumscribed circle; let M' and N' be the intersection points of lines LB' and LA' with sides AC and BC, respectively. It suffices to verify that $M' = M$ and $N' = N$.

Since $\smile PA + \smile AB' + \smile B'Q = 180^\circ$, it follows that $\smile B'Q = \angle A$ and, therefore, $\angle B'LQ = \angle M'AO$. Hence, quadrilateral $AM'OL$ is an inscribed one and $\angle M'OA = \angle M'LA = \frac{1}{2}\angle B$. Therefore, $\angle CMO = \frac{1}{2}(\angle A + \angle B)$, i.e., $M' = M$. Similarly, $N' = N$.

2.50. Since $\triangle ADM \sim \triangle CBM$ and $\triangle ACM \sim \triangle DBM$, it follows that $AD : CB = DM : BM$ and $AC : DB = AM : DM$. It remains to multiply these equalities.

2.51. Let D_1 be the intersection point of line BD with the circle distinct from point B. Then $\smile AB = \smile AD_1$; hence, $\angle ACB = \angle AD_1B = \angle ABD_1$. Triangles ACB and ABD have a common angle, $\angle A$, and, moreover, $\angle ACB = \angle ABD$; hence, $\triangle ACB \sim \triangle ABD$. Therefore, $AB : AC = AD : AB$.

2.52. Let O be the center of the circle. Since $\angle MAC = \angle ACO = \angle CAO$, it follows that $\triangle AMC = \triangle ADC$. Similarly, $\triangle CDB = \triangle CNB$. Since $\triangle ACD \sim \triangle CDB$, it follows that $CD^2 = AD \cdot DB = AM \cdot NB$.

2.53. Points B_1 and H lie on the circle with diameter AB, hence,

$$\angle(AB, BC) = \angle(AB, BH) = \angle(AB_1, B_1H) = \angle(B_1C_1, B_1H).$$

Similarly, $\angle(AC, BC) = \angle(B_1C_1, C_1H)$.

2.54. On an extension of segment BP beyond point P take point D such that $PD = CP$. Then triangle CDP is an equilateral one and $CD \parallel QP$. Therefore, $BP : PQ = BD : DC = (BP + CP) : CP$, i.e., $\frac{1}{PQ} = \frac{1}{CP} + \frac{1}{BP}$.

2.55. Segment QE subtends angles of 45° with vertices at points A and B, hence, quadrilateral $ABEQ$ is an inscribed one. Since $\angle ABE = 90^\circ$, it follows that $\angle AQE = 90^\circ$. Therefore, triangle AQE is an isosceles right triangle and $\frac{AE}{AQ} = \sqrt{2}$. Similarly, $\frac{AF}{AP} = \sqrt{2}$.

2.56. Since $\angle ANC = \angle ABC = \angle CAB$, it follows that $\triangle CAM \sim \triangle CNA$ and, therefore, $CA : CM = CN : CA$, i.e., $CM \cdot CN = AC^2$ and $AM : NA = CM : CA$. Similarly, $BM : NB = CM : CB$. Therefore,

$$\frac{AM \cdot BM}{AN \cdot BN} = \frac{CM^2}{CA^2} = \frac{CM^2}{CM \cdot CN} = \frac{CM}{CN}.$$

2.57. Since $AK = AB = CD$, $AD = BC = CH$ and $\angle KAD = \angle DCH$, it follows that $\triangle ADK = \triangle CHD$ and $DK = DH$. Let us show that points A, K, H, C and D lie on one circle. Let us circumscribe the circle about triangle ADC. Draw chord CK_1 in this circle parallel to AD and chord AH_1 parallel to DC. Then $K_1A = DC$ and $H_1C = AD$. Hence, $K_1 = K$ and $H_1 = H$, i.e., the constructed circle passes through points K and H and angles

$\angle KAH$ and $\angle KDH$ are equal because they subtend the same arc. Moreover, as we have already proved, KDH is an isosceles triangle.

2.58. a) $\angle PBA_1 = \angle PAC_1$ and $\angle PBC_1 = \angle PAB_1$ and, therefore, right triangles PBA_1 and PAC_1, PAB_1 and PBC_1 are similar, i.e., $PA_1 : PB = PC_1 : PA$ and $PB_1 : PA = PC_1 : PB$. By multiplying these equalities we get $PA_1 \cdot PB_1 = PC_1^2$.

b) According to heading a)

$$OA'' = \sqrt{OB' \cdot OC'}, \quad OB'' = \sqrt{OA' \cdot OC'}, \quad OC'' = \sqrt{OA' \cdot OB'}.$$

By multiplying these equalities we get the desired statement.

2.59. Let K, L, M and N be the bases of perpendiculars dropped from point E to lines AB, BC, CD and DA, respectively. Points K and N lie on the circle with diameter AE, hence, $\angle(EK, KN) = \angle(EA, AN)$. Similarly, $\angle(EL, LM) = \angle(EC, CM) = \angle(EA, AN)$ and, therefore, $\angle(EK, KN) = \angle(EL, LM)$. Similarly, $\angle(EN, NK) = \angle(EM, ML)$ and $\angle(KE, EN) = \angle(LE, EM)$. It follows that $\triangle EKN \sim \triangle ELM$ and, therefore, $EK : EN = EL : EM$, i.e., $EN = \frac{EK \cdot EM}{EL} = \frac{ac}{b}$.

2.60. Let H be the intersection point of heights, M the midpoint of side BC. Points A_1, B_2 and C_2 lie on the circle with diameter MH, hence, $\angle(B_2A_1, A_1C_2) = \angle(B_2M, MC_2) = \angle(AC, AB)$. Moreover, $\angle(A_1B_2, B_2C_2) = \angle(A_1H, HC_2) = \angle(BC, AB)$ and $\angle(A_1C_2, C_2B_2) = \angle(BC, AC)$.

2.61. Let M be the intersection point of medians, H the intersection point of heights of triangle ABC. Points A_1, B_1 and C_1 are the projections of point M on the heights and, therefore, these points lie on the circle with diameter MH. Hence, $\angle(A_1B_1, B_1C_1) = \angle(AH, HC) = \angle(BC, AB)$. By writing similar equalities for the other angles we get the desired statement.

2.62. Let lines BM and DN meet S_2 at points L and C_1, respectively. Let us prove that lines DC_1 and CN are symmetric through line AN. Since $BN \perp NA$, it suffices to verify that $\angle CNB = \angle BND$. But arcs $\smile CB$ and $\smile BD$ are equal. Arcs $\smile C_1M$ and $\smile CL$ are symmetric through line AN, hence, they are equal and, therefore, $\angle MDC_1 = \angle CML$. Besides, $\angle CNM = \angle MND$. Thus, $\triangle MCN \sim \triangle DMN$, i.e., $CN : MN = MN : DN$.

2.63. Let us drop from point Q perpendiculars QK_1 and QN_1 to KL and NM, respectively, and from point P perpendiculars PM_1 and PL_1 to NM and KL, respectively. Clearly, $\frac{QC}{PC} = \frac{QK_1}{PL_1} = \frac{QN_1}{PM_1}$, i.e., $\frac{QC^2}{PC^2} = \frac{QK_1 \cdot QN_1}{PL_1 \cdot PM_1}$. Since $\angle KNC = \angle MLC$ and $\angle NKC = \angle LMC$, it follows that $QN_1 : PL_1 = QN : PL$ and $QK_1 : PM_1 = QK : PM$. Therefore,

$$\frac{QC^2}{PC^2} = \frac{QK \cdot QN}{PL \cdot PM} = \frac{AQ \cdot QB}{PB \cdot AP} = \frac{(AC - QC) \cdot (AC + QC)}{(AC - PC) \cdot (AC + PC)} = \frac{AC^2 - QC^2}{AC^2 - PC^2}.$$

This implies that $QC = PC$.

2.64. a) Since $\angle CAM = \angle CBM$ and $\angle CB_1M = \angle CA_1M$, it follows that $\angle B_1AM = \angle A_1BM$ and $\angle AB_1M = \angle BA_1M$.

b) Let M_1 be a point of the circle S with diameter CO such that $CM_1 \parallel A_1B_1$; let M_2 be an intersection point of circle S with the circumscribed circle of triangle ABC; let A_2 and B_2 be the tangent points of of the inscribed circle with sides BC and AC, respectively. It suffices to verify that $M_1 = M_2$. By Problem a) $\triangle AB_2M_2 \sim \triangle BA_2M_2$, hence, $B_2M_2 : A_2M_2 = AB_2 : BA_2$. Since $CA_1 = p - b - BA_2$ and $CB_1 = AB_2$, it follows that

$$\frac{B_2M_1}{A_2M_1} = \frac{\sin B_2CM_1}{\sin A_2CM_1} = \frac{\sin CA_1B_1}{\sin CB_1A_1} = \frac{CB_1}{CA_1} = \frac{AB_2}{BA_2}.$$

On arc $\smile A_2CB_2$ of circle S, there exists a unique point X for which $B_2X : A_2X = k$ (Problem 7.14), hence, $M_1 = M_2$.

2.65. Let O be the center of the circumscribed circle of the triangle, M the midpoint of side AB, H the base of height CH, D the midpoint of the arc on which point C does not lie and with endpoints A and B. Since $OD \parallel CH$, it follows that $\angle DCH = \angle MDC$. The bisector divides the angle between the median and the height in halves if and only if $\angle MCD = \angle DCH = \angle MDC = \angle ODC = \angle OCD$, i.e., $M = O$ and AB is the diameter of the circle.

2.66. Let $\alpha = \angle A < \angle B$. By the preceding problem $\angle C = 90^\circ$. Median CM divides triangle ABC into two isosceles triangles. Since $\angle ACM = \angle A = \alpha$, $\angle MCB = 3\alpha$, it follows that $\alpha + 3\alpha = 90^\circ$, i.e., $\alpha = 22.5^\circ$. Therefore, $\angle A = 22.5^\circ$, $\angle B = 67.5^\circ$, $\angle C = 90^\circ$.

2.67. Let D be a point at which line AE intersects the circumscribed circle. Point D is the midpoint of arc $\smile BC$. Therefore, $MD \parallel AH$, moreover, points A and D lie on different sides of line MH. It follows that point E lies on segment MH.

2.68. Clearly,

$$\angle(AQ, QP) = \angle(AN, NP) = \angle(PM, MB) = \angle(QP, QB).$$

Therefore, point Q lies on the circle such that segment AB subtends an angle of $2\angle(AC, CB)$ with vertex at Q and line QP divides arc $\smile AB$ of this circle in halves.

2.69. Points P and Q lie on the circle with diameter AD; this circle intersects side BC at point F. (Observe that F does not coincide with D if $AB \neq AC$.) Clearly,

$$\angle(FC, CE) = \angle(BA, AE) = \angle(DA, AQ) = \angle(DF, FQ), \text{ i.e., } EC \parallel FQ.$$

Similarly, $BE \parallel FP$. To complete the proof it suffices to notice that the areas of triangles adjacent to the lateral sides of the trapezoid are equal.

2.70. Let $\angle AOB = \alpha$ and $\angle COD = \beta$. Then $\frac{\alpha}{2} + \frac{\beta}{2} = \angle ADP + \angle PAD = 90^\circ$. Since $2S_{AOB} = R^2 \sin\alpha$ and $2S_{COD} = R^2 \sin\beta$, where R is the radius of the circumscribed circle, it follows that $S_{AOB} = S_{COD}$. Similarly, $S_{BOC} = S_{AOD}$.

2.71. Let $\angle AOB = 2\alpha$ and $\angle COD = 2\beta$. Then $\alpha + \beta = \angle ADP + \angle PAD = 90^\circ$. Hence,

$$(AP^2 + BP^2) + (CP^2 + DP^2) = AB^2 + CD^2 = 4R^2(\sin^2\alpha + \cos^2\alpha) = 4R^2.$$

Similarly, $BC^2 + AD^2 = 4R^2$.

2.72. Let M be the midpoint of AC, N the midpoint of BD. We have $AM^2 = AO^2 - OM^2$ and $BN^2 = BO^2 - ON^2$; hence,

$$AC^2 + BD^2 = 4(R^2 - OM^2) + 4(R^2 - ON^2) = 8R^2 - 4(OM^2 + ON^2) = 8R^2 - 4OP^2$$

since $OM^2 + ON^2 = OP^2$.

2.73. The correspondiong legs of acute angles $\angle BLP$ and $\angle BDC$ are perpendicular, hence, the angles are equal.

Therefore, $\angle BLP = \angle BDC = \angle BAP$. Moreover, $AK \parallel BL$ and $AL \perp BK$. It follows that $AKLB$ is a rhombus.

2.74. In the circumscribed circle take a point D' so that $DD' \parallel AC$. Since $DD' \perp BD$, it follows that BD' is a diameter and, therefore, $\angle D'AB = \angle D'CB = 90^\circ$. Hence,

$$S_{ABCD} = S_{ABCD} = \frac{1}{2}(AD' \cdot AB + BC \cdot CD') = \frac{1}{2}(AB \cdot CD + BC \cdot AD).$$

2.75. Let us draw diameter AE. Since $\angle BEA = \angle BCP$ and $\angle ABE = \angle BPC = 90^\circ$, it follows that $\angle EAB = \angle CBP$. The angles that intersect chords EB and CD are equal, hence, $EB = CD$. Since $\angle EBA = 90^\circ$, the distance from point O to AB is equal to $\frac{1}{2}EB$.

2.76. Let the perpendicular dropped from point P to BC intersect BC at point H and AD at point M (Fig. 16).

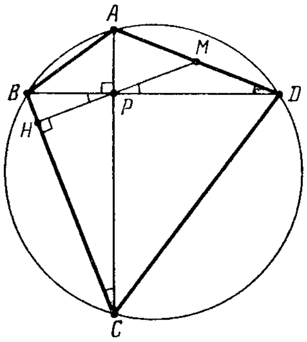


Figure 16 (Sol. 2.76)

Therefore, $\angle BDA = \angle BCA = \angle BPH = \angle MPD$. Since angles MDP and MPD are equal, MP is a median of right triangle APD. Indeed,

$$\angle APM = 90^\circ - \angle MPD = 90^\circ - \angle MDP = \angle PAM,$$

i.e., $AM = PM = MD$.

2.77. The midpoints of the sides of quadrilateral $ABCD$ are vertices of a rectangle (cf. Problem 1.2), hence, they lie on one circle. Let K and L be the midpoints of sides AB and CD, let M be the intersection point of lines KP and CD. By Problem 2.76 $PM \perp CD$; hence, M is the projection of point P on side CD and point M lies on the circle with diameter KL.

For the other projections the proof is similar.

2.78. a) It is worth to observe that since points A, B, C and D divide the circle into arcs smaller than 180° each, then the quadrilateral constructed contains this circle. The angle φ between the tangents drawn through points A and B is equal to $180^\circ - \angle AOB$ and the angle ψ between the tangents drawn through points C and D is equal to $180^\circ - \angle COD$. Since $\angle AOB + \angle COD = 180^\circ$, it follows that $\varphi + \psi = 180^\circ$.

Remark. Conversely, the equality $\varphi + \psi = 180^\circ$ implies that $\angle AOB + \angle COD = 180^\circ$, i.e., $AC \perp BD$.

b) Let O be the center of the inscribed circle. Since $\angle AKO + \angle BMO = 90^\circ$, it follows that $\angle AKO = \angle BOM$ and $\triangle AKO \sim \triangle BOM$. Therefore, $AK \cdot BM = BO \cdot AO = r^2$.

2.79. First, let us suppose that the circumscribed circles of triangles $A'BC$ and $AB'C$ are not tangent to each other and P is their common point distinct from C. Then

$$\angle(PA, PB) = \angle(PA, PC) + \angle(PC, PB) = \angle(B'A, B'C) + \angle(A'C, A'B) = \angle(C'A, C'B),$$

i.e., point P lies on the circumscribed circle of triangle ABC'.

If the the circumscribed circles of triangles $A'BC$ and $AB'C$ are tangent to each other, i.e., $P = C$, then our arguments require an insignificant modifications: instead of line PC we have to take the common tangent.

2.80. a) By applying the statement of Problem 2.79 to triangles AB_1C_1, A_1BC_1 and A_1B_1C constructed on the sides of triangle $A_1B_1C_1$ we get the desired statement.

b) Let P be the intersection point of the indicated circles. Let us prove that the value of the angle $\angle(AP, PC)$ is a constant. Since

$$\angle(AP, PC) = \angle(AP, AB) + \angle(AB, BC) + \angle(BC, PC)$$

and angle $\angle(AB, BC)$ is a constant, it remains to verify that the sum $\angle(AP, AB)+\angle(BC, PC)$ is a constant. Clearly,

$$\begin{array}{rcl}\angle(AP, AB) + \angle(BC, CP) & = & \angle(AP, AC_1) + \angle(CA_1, CP) = \\ \angle(B_1P, B_1C_1) + \angle(B_1A_1, B_1P) & = & \angle(B_1A_1, B_1C_1)\end{array}$$

and the value of the latter angle is constant by hypothesis.

We similarly prove that the values of angles $\angle(AP, PB)$ and $\angle(BP, PC)$ are constants. Hence, point P remains fixed.

2.81. As follows from Problem 2.80 b) it suffices to carry out the proof for one such triangle $A_1B_1C_1$ only; for instance, for the triangle with vertices in the midpoints of sides of triangle ABC. Let H be the intersection point of heights of triangle $A_1B_1C_1$, i.e., the center of the circumscribed circle of triangle ABC. Since $A_1H \perp B_1C_1$ and $B_1H \perp A_1C_1$, it follows that $\angle(A_1H, HB_1) = \angle(B_1C_1, A_1C_1) = \angle(A_1C, CB_1)$, i.e., point H lies on the circumscribed circle of triangle A_1B_1C.

A similar argument shows that point H lies on the circumscribed circles of triangles A_1BC_1 and AB_1C_1.

2.82. a) Let X be the intersection point of the circumscribed circles of triangles ABC and $AB'C'$. Then

$$\angle(XB', XC) = \angle(XB', XA) + \angle(XA, XC) = \angle(C'B', C'A) + \angle(BA, BC).$$

Since $AC' = AP = AB'$, triangle $C'AB'$ is an isosceles one and $\angle C'AB' = 2\angle A$; hence, $\angle(C'B', C'A) = \angle A - 90^\circ$. Therefore,

$$\angle(XB', XC) = \angle A - 90^\circ + \angle B = 90^\circ - \angle C = \angle(A'B', A'C),$$

i.e., point X lies on the circumscribed circle of triangle $A'B'C$. For the circumscribed circle of triangle $A'BC'$ the proof is similar.

b) Let X be the intersection point of the circumscribed circles of triangles $A'B'C'$ and $A'BC$. Let us prove that X lies on the circumscribed circle of triangle ABC'. Clearly,

$$\angle(XB, XC') = \angle(XB, XA') + \angle(XA', XC') = \angle(CB, CA') + \angle(B'A', B'C').$$

Let A_1, B_1 and C_1 be the midpoints of segments PA', PB' and PC'. Then

$$\angle(CB, CA') = \angle(CP, CA_1) = \angle(B_1P, B_1A_1), \angle(B'A', B'C') = \angle(B_1A_1, B_1C_1)$$

and

$$\angle(AB, AC') = \angle(AP, AC_1) = \angle(B_1P, B_1C_1).$$

It follows that $\angle(XB, XC') = \angle(AB, AC')$.

We similarly prove that point X lies on the circumscribed circle of triangle $AB'C$.

c) Since QA' is the common chord of circles centered at O and I, it follows that $QA' \perp OI$. Similarly, $QB' \perp OJ$ and $QC \perp IJ$. Therefore, sides of angles OJI and $B'QC$, as well as sides of angles OIJ and $A'QC$, are mutually perpendicular, hence, $\sin OJI = \sin B'QC$ and $\sin OIJ = \sin A'QC$. Therefore, $OI : OJ = \sin OJI : \sin OIJ = \sin B'QC : \sin A'QC$. It is also clear that

$$\frac{QI}{QJ} = \frac{\sin QJI}{\sin QIJ} = \frac{\sin(\frac{1}{2}QJC)}{\sin(\frac{1}{2}QIC)} = \frac{\sin QB'C}{\sin QA'C}.$$

Taking into account that $\sin B'QC : \sin QB'C = B'C : QC$ and $\sin A'QC : \sin QA'C = A'C : QC$ we get

$$\frac{OI}{OJ} : \frac{QI}{QJ} = \frac{B'C}{QC} : \frac{A'C}{QC} = 1.$$

2.83. a) The conditions of the problem imply that no three lines meet at one point. Let lines AB, AC and BC intersect the fourth line at points D, E, and F, respectively (Fig. 17).

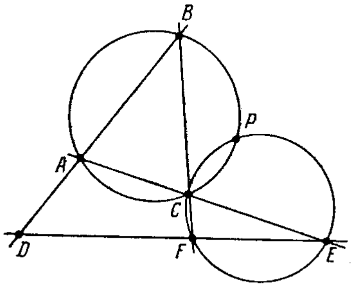


Figure 17 (Sol. 2.83)

Denote by P the intersection point of circumscribed circles of triangles ABC and CEF distinct from point C. Let us prove that point P belongs to the circumscribed circle of triangle BDF. For this it suffices to verify that $\angle(BP, PF) = \angle(BD, DF)$. Clearly,

$$\begin{aligned}\angle(BP, PF) &= \angle(BP, PC) + \angle(PC, PF) = \angle(BA, AC) + \angle(EC, EF)\\ &= \angle(BD, AC) + \angle(AC, DF) = \angle(BD, DF).\end{aligned}$$

We similarly prove that point P belongs to the circumscribed circle of triangle ADE.

b) Let us make use of notations of Fig. 17. Thanks to heading a), the circumscribed circles of triangles ABC, ADE and BDF pass through point P and, therefore, we can consider them as the circumscribed circles of triangles ABP, ADP and BDP respectively. Therefore, their centers lie on a circle that passes through point P (cf. Problem 5.86).

We similarly prove that the centers of any of the three of given circles lie on a circle that passes through point P. It follows that all the four centers lie on a circle that passes through point P.

2.84. a) Let P be Michel's point for lines AB, BC, CA and A_1B_1. The angles between rays PA, PB, PC and the tangents to circles S_a, S_b, S_c are equal to $\angle(PB_1, B_1A) = \angle(PC_1, C_1A)$, $\angle(PC_1, C_1B) = \angle(PA_1, A_1B)$, $\angle(PA_1, A_1C) = \angle(PB_1, B_1C)$, respectively. Since $\angle(PC_1, C_1A) = \angle(PC_1, C_1B) = \angle(PA_1, A_1C) = \varphi$, it follows that after a rotation through an angle of φ about point P lines PA, PB and PC turn into the tangents to circles S_a, S_b and S_c, respectively, and, therefore, after a rotation through an angle of $90^\circ - \varphi$ these lines turn into lines PO_a, PO_b and PO_c respectively. Moreover,

$$\frac{PO_a}{PA} = \frac{PO_b}{PB} = \frac{PO_c}{PC} = \frac{1}{2}\sin\varphi.$$

Therefore, the composition of the rotation through an angle of $90^\circ - \varphi$ and the homothety (see ???) with center P and coefficient $\frac{1}{2}\sin\varphi$ sends triangle ABC to $O_aO_bO_c$.

b) The transformation considered in the solution of heading a) sends the center O of the circumscribed circle of triangle ABC into the center O' of the circumscribed circle of triangle $O_aO_bO_c$ and the orthocenter H of triangle ABC to orthocenter H' of triangle $O_aO_bO_c$. Let us complement triangle $OO'H'$ to parallelogram $OO'H'M$. Since $\frac{OH}{OM} = \frac{OH}{O'H'} = 2\sin\varphi$ and $\angle HOM = \angle(HO, O'H') = 90^\circ - \varphi$, it follows that $MH = MO$, i.e., point M lies on the midperpendicular of segment OH. It remains to notice that for the inscribed quadrilateral $OO_aO_bO_c$ point M is uniquely determined: taking instead of point O any of the points O_a, O_b or O_c we get the same point M (cf. Problem 13.33).

2.85. We may assume that rays AB and DC meet at point E and rays BC and AD meet at point F. Let P be the intersection point of circumscribed circles of triangles BCE

and CDF. Then $\angle CPE = \angle ABC$ and $\angle CPF = \angle ADC$. Hence, $\angle CPE + \angle CPF = 180^\circ$, i.e., point P lies on segment EF.

2.86. a) Since

$$\angle(AP, PD) = \angle(AP, PE) + \angle(PE, PD) = \angle(AC, CD) + \angle(AB, BD) + \angle(AO, OD),$$

points A, P, D and O lie on one circle.

b) Clearly,

$$\angle(EP, PO) = \angle(EP, PA) + \angle(PA, PO) = \angle(DC, CA) + \angle(DA, DO) = 90^\circ,$$

because the arcs intersected by these angles constitute a half of the circle.

2.87. Let us make use of notations of Fig. 17. The projections of point P on lines CA and CB coincide with its projection to CE and CF, respectively. Therefore, Simson's lines of point P relative triangles ABC and CEF coincide (cf. Problem 5.85 a).

2.88. Let point A' be symmetric to point A through the midperpendicular to segment BC. Then $\angle OAH = \frac{1}{2}\angle AOA' = \angle ABA' = |\angle B - \angle C|$.

2.89. Since AA' is a diameter, $A'C \perp AC$; hence, $BH \parallel A'C$. Similarly, $CH \parallel A'B$. Therefore, $BA'CH$ is a parallelogram.

2.90. Let l be a line parallel to the two given lines, D the intersection point of lines m and n. Then

$$\angle(AD, DB) = \angle(m, AB) + \angle(AB, n) = \angle(AC, l) + \angle(l, CB) = \angle(AC, CB)$$

and, therefore, point D lies on the circumscribed circle of triangle ABC.

2.91. a) Let O be the midpoint of the arc of circle S that lies inside triangle ABC. Then $\angle CBO = \angle BCO$ and due to a property of the angle between a tangent and a chord, $\angle BCO = \angle ABO$. Therefore, BO is the bisector of angle ABC, i.e., O is the center of the inscribed circle of triangle ABC. We similarly prove that the midpoint of the arc of circle S that lies outside triangle ABC is the center of its escribed circle.

b) We have to prove that the center of the considered circle S lies on the bisector of angle BAC. Let D be the intersection point of the bisector of the angle with the circumscribed circle of triangle ABC. Then $DB = DO = DC$ (cf. Problem 2.4 a), i.e., D is the center of circle S.

2.92. If angle $\angle C$ is a right one, then the solution of the problem is obvious: C is the intersection point of lines A_1B, A_2B_2, AB_1. If $\angle C \neq 90^\circ$, then the circumscribed circles of squares ACA_1A_2 and BCB_1B_2 have in addition to C one more common point, C_1. Then

$$\begin{array}{rcl} \angle(AC_1, A_2C_1) & = & \angle(A_2C_1, A_1C_1) = \angle(A_1C_1, C_1C) = \angle(C_1C, C_1B_1) \\ = \angle(C_1B_1, C_1B_2) & = & \angle(C_1B_2, C_1B) = 45^\circ \end{array}$$

(or -45°; it is only important that all the angles are of the same sign). Hence, $\angle(AC, C_1B_1) = 4 \cdot 45^\circ = 180^\circ$, i.e., line AB_1 passes through point C_1.

Similarly, A_2B_2 and A_1B pass through point C_1.

2.93. Let P and O be the centers of circles S_1 and S_2, respectively; let $\alpha = \angle APC$, $\beta = \angle BPC$; lines AC and BC intersect S_2 at points K and L, respectively. Since $\angle OAP = \angle OBP = 90^\circ$, it follows that $\angle AOB = 180^\circ - \alpha - \beta$. Furthermore,

$$\angle LOB = 180^\circ - 2\angle LBO = 2\angle CBP = 180^\circ - \beta.$$

Similarly, $\angle KOA = 180^\circ - \alpha$. Therefore,

$$\angle LOK = \angle LOB + \angle KOA - \angle AOB = 180^\circ,$$

i.e., KL is a diameter.

2.94. Let us consider points M', P', Q' and R' symmetric to points M, P, Q and R, respectively, through line OA. Since point C is symmetric to point B through OA, it follows that line $P'Q'$ passes through point C. The following equalities are easy to verify:

$$\angle(CS, NS) = \angle(Q'Q, NQ) = \angle(Q'P, NP') = \angle(CP', NP');$$
$$\angle(CR', P'R') = \angle(MM', P'M') = \angle(MN, P'N) = \angle(CN, P'N).$$

From these equalities we deduce that points C, N, P', S and R' lie on one circle. But points S, R' and C lie on one line, therefore, $S = R'$.

CHAPTER 3. CIRCLES

Background

1. A line that has exactly one common point with a circle is called a line *tangent* to the circle. Through any point A outside the circle exactly two tangents to the circle can be drawn.

Let B and C be the tangent points and O the center of the circle. Then:

a) $AB = AC$;

b) $\angle BAO = \angle CAO$;

c) $OB \perp AB$.

(Sometimes the word "tangent" is applied not to the whole line AB but to the segment AB. Then property a), for example, is formulated as: *the tangents to one circle drawn from one point are equal.*)

2. Let lines l_1 and l_2 that pass through point A intersect a circle at points B_1, C_1 and B_2, C_2, respectively. Then $AB_1 \cdot AC_1 = AB_2 \cdot AC_2$. Indeed, $\triangle AB_1C_2 \sim \triangle AB_2C_1$ in three angles. (We advise the reader to prove this making use of the properties of the inscribed angles and considering two cases: A lies outside the circle and A lies inside the circle.)

If line l_2 is tangent to the circle, i.e., $B_2 = C_2$, then $AB_1 \cdot AC_1 = AB_2^2$. The proof runs along the same lines as in the preceding case except that now we have to make use of *the properties of the angle between a tangent and a chord.*

3. The line that connects the centers of tangent circles passes through their tangent point.

4. The *value of the angle between two intersecting circles* is the value of the angle between the tangents to these circles drawn through the intersection point. It does not matter which of the two of intersection points we choose: the corresponding angles are equal.

The angle between tangent circles is equal to 0°.

5. In solutions of problems from §6 a property that has no direct relation to circles is used: *the heights of a triangle meet at one point.* The reader can find the proof of this fact in solutions of Problems 5.45 and 7.41 or can take it for granted for the time being.

6. It was already in the middle of the V century A.D. that **Hyppocratus** from island Chios (do not confuse him with the famous doctor Hyppocratus from island Kos who lived somewhat later) and Pythagoreans began to solve the *quadrature of the circle* problem. It is formulated as follows: *with the help of a ruler and compass construct a square of the same area as the given circle.*

In 1882 the German mathematician **Lindemann** proved that number π is transcendental, i.e., is not a root of a polynomial with integer coefficients. This implies, in particular, that the problem on the quadrature of the circle is impossible to solve as stated (using other tools one can certainly solve it).

It seems that it was the problem on *Hyppocratus' crescents* (Problem 3.38) that induced in many a person great expectations to the possibility of squaring the circle: *the area of the figure formed by arcs of circles is equal to the area of a triangle.* Prove this statement and try to understand why such expectations were not grounded in this case.

Introductory problems

1. Prove that from a point A outside a circle it is possible to draw exactly two tangents to the circle and the lengths of these tangents (more exactly, the lengths from A to the tangent points) are equal.

2. Two circles intersect at points A and B. Point X lies on line AB but not on segment AB. Prove that the lengths of all the tangents drawn from point X to the circles are equal.

3. Two circles whose radii are R and r are tangent from the outside (i.e., none of them lies inside the other one). Find the length of the common tangent to these circles.

4. Let a and b be the lengths of the legs of a right triangle, c the length of its hypothenuse. Prove that:

a) the radius of the inscribed circle of this triangle is equal to $\frac{1}{2}(a+b-c)$;

b) the radius of the circle tangent to the hypothenuse and the extensions of the legs is equal to $\frac{1}{2}(a+b+c)$.

§1. The tangents to circles

3.1. Lines PA and PB are tangent to a circle centered at O; let A and B be the tangent points. A third tangent to the circle is drawn; it intersects with segments PA and PB at points X and Y, respectively. Prove that the value of angle XOY does not depend on the choice of the third tangent.

3.2. The inscribed circle of triangle ABC is tangent to side BC at point K and an escribed circle is tangent at point L. Prove that $CK = BL = \frac{1}{2}(a+b-c)$, where a, b, c are the lengths of the triangle's sides.

3.3. On the base AB of an isosceles triangle ABC a point E is taken and circles tangent to segment CE at points M and N are inscribed into triangles ACE and ECB, respectively. Find the length of segment MN if the lengths of segments AE and BE are known.

3.4. Quadrilateral $ABCD$ is such that there exists a circle inscribed into angle $\angle BAD$ and tangent to the extensions of sides BC and CD. Prove that $AB + BC = AD + DC$.

3.5. The common inner tangent to circles whose radii are R and r intersects their common outer tangents at points A and B and is tangent to one of the circles at point C. Prove that $AC \cdot CB = Rr$.

3.6. Common outer tangents AB and CD are drawn to two circles of distinct radii. Prove that quadrilateral $ABCD$ is a circumscribed one if and only if the circles are tangent to each other.

3.7. Consider parallelogram $ABCD$ such that the escribed circle of triangle ABD is tangent to the extensions of sides AD and AB at points M and N, respectively. Prove that the intersection points of segment MN with BC and CD lie on the inscribed circle of triangle BCD.

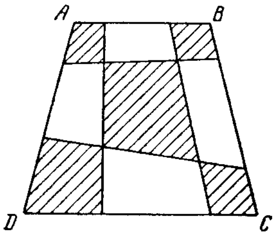


Figure 18 (3.7)

3.8. On each side of quadrilateral $ABCD$ two points are taken; these points are connected as shown on Fig. 18. Prove that if all the five dashed quadrilaterals are circumscribed ones, then the quadrilateral $ABCD$ is also a circumscribed one.

§2. The product of the lengths of a chord's segments

3.9. Through a point P lying on the common chord AB of two intersecting circles chord KM of the first circle and chord LN of the second circle are drawn. Prove that quadrilateral $KLMN$ is an inscribed one.

3.10. Two circles intersect at points A and B; let MN be their common tangent. Prove that line AB divides MN in halves.

3.11. Line OA is tangent to a circle at point A and chord BC is parallel to OA. Lines OB and OC intersect the circle for the second time at points K and L, respectively. Prove that line KL divides segment OA in halves.

3.12. In parallelogram $ABCD$, diagonal AC is longer than diagonal BD; let M be a point on diagonal AC such that quadrilateral $BCDM$ is an inscribed one. Prove that line BD is a common tangent to the circumscribed circles of triangles ABM and ADM.

3.13. Given circle S and points A and B outside it. For each line l that passes through point A and intersects circle S at points M and N consider the circumscribed circle of triangle BMN. Prove that all these circles have a common point distinct from point B.

3.14. Given circle S, points A and B on it and point C on chord AB. For every circle S' tangent to chord AB at point C and intersecting circle S at points P and Q consider the intersection point M of lines AB and PQ. Prove that the position of point M does not depend on the choice of circle S'.

§3. Tangent circles

3.15. Two circles are tangent at point A. A common (outer) tangent line is drawn to them; it is tangent to the circles at points C and D, respectively. Prove that $\angle CAD = 90^\circ$.

3.16. Two circles S_1 and S_2 centered at O_1 and O_2 are tangent to each other at point A. A line that intersects S_1 at point A_1 and S_2 at point A_2 is drawn through point A. Prove that $O_1A_1 \parallel O_2A_2$.

3.17. Three circles S_1, S_2 and S_3 are pairwise tangent to each other at three distinct points. Prove that the lines that connect the tangent point of circles S_1 and S_2 with the other two tangent points intersect circle S_3 at points that are the endpoints of its diameter.

3.18. Two tangent circles centered at O_1 and O_2, respectively, are tangent from the inside to the circle of radius R centered at O. Find the perimeter of triangle OO_1O_2.

3.19. Circles S_1 and S_2 are tangent to circle S from the inside at points A and B so that one of the intersection points of circles S_1 and S_2 lies on segment AB. Prove that the sum of the radii of circles S_1 and S_2 is equal to the radius of circle S.

3.20. The radii of circles S_1 and S_2 tangent at point A are equal to R and r ($R > r$). Find the length of the tangent drawn to circle S_2 from point B on circle S_1 if $AB = a$ (consider the cases of the inner and outer tangent).

3.21. A point C is taken on segment AB. A line that passes through point C intersects circles with diameters AC and BC at points K and L and the circle with diameter AB at points M and N, respectively. Prove that $KM = LN$.

3.22. Given four circles S_1, S_2, S_3 and S_4 such that S_i and S_{i+1} are tangent from the outside for $i = 1, 2, 3, 4$ ($S_5 = S_1$). Prove that the tangent points are the vertices of an inscribed quadrilateral.

3.23. a) Three circles centered at A, B and C are tangent to each other and line l; they are placed as shown on Fig. 19. Let a, b and c be radii of circles centered at A, B and C, respectively. Prove that $\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$.

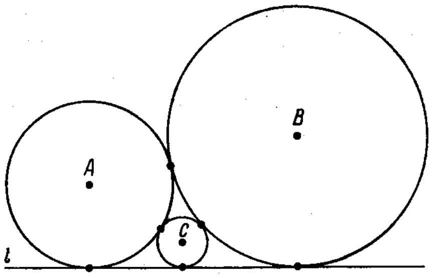


Figure 19 (3.23)

b) Four circles are pairwise tangent from the outside (at 6 distinct points). Let a, b, c and d be their radii; $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$ and $\delta = \frac{1}{d}$. Prove that

$$2(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (\alpha + \beta + \gamma + \delta)^2.$$

§4. Three circles of the same radius

3.24. Three circles of radius R pass through point H; let A, B and C be points of their pairwise intersection distinct from H. Prove that

a) H is the intersection point of heights of triangle ABC;

b) the radius of the circumscribed circle of the triangle ABC is also equal to R.

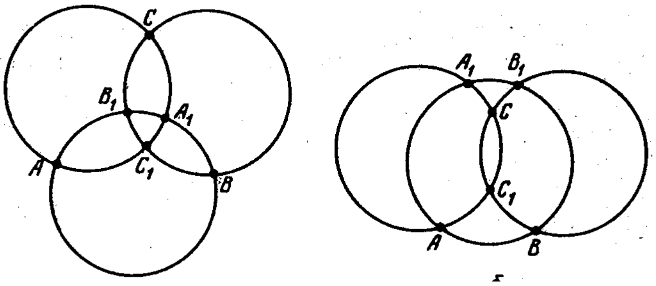


Figure 20 (3.24)

3.25. Three equal circles intersect as shown on Fig. 20 a) or b). Prove that $\smile AB_1 + \smile BC_1 \pm \smile CA_1 = 180^\circ$, where the minus sign is taken in case b) and plus in case a).

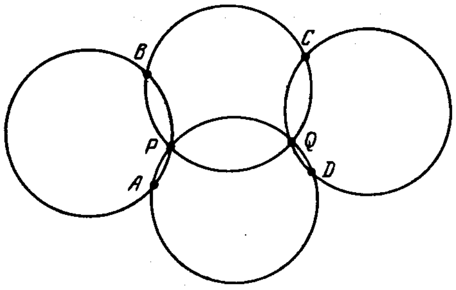


Figure 21 (3.26)

3.26. Three circles of the same radius pass through point P; let A, B and Q be points of their pairwise intersections. A fourth circle of the same radius passes through point Q

and intersects the other two circles at points C and D. The triangles ABQ and CDP thus obtained are acute ones and quadrilateral $ABCD$ is a convex one (Fig. 21). Prove that $ABCD$ is a parallelogram.

§5. Two tangents drawn from one point

3.27. Tangents AB and AC are drawn from point A to a circle centered at O. Prove that if segment AO subtends a right angle with vertex at point M, then segments OB and OC subtend equal angles with vertices at M.

3.28. Tangents AB and AC are drawn from point A to a circle centered at O. Through point X on segment BC line KL perpendicular to XO is drawn so that points K and L lie on lines AB and AC, respectively. Prove that $KX = XL$.

3.29. On the extension of chord KL of a circle centered at O a point A is taken and tangents AP and AQ to the circle are drawn from it; let M be the midpoint of segment PQ. Prove that $\angle MKO = \angle MLO$.

3.30. From point A tangents AB and AC to a circle and a line that intersects the circle at points D and E are drawn; let M be the midpoint of segment BC. Prove that $BM^2 = DM \cdot ME$ and either $\angle DME = 2\angle DBE$ or $\angle DME = 2\angle DCE$; moreover, $\angle BEM = \angle DEC$.

3.31. Quadrilateral $ABCD$ is inscribed in a circle so that tangents to this circle at points B and D intersect at a point K that lies on line AC.

a) Prove that $AB \cdot CD = BC \cdot AD$.

b) A line parallel to KB intersects lines BA, BD and BC at points P, Q and R, respectively. Prove that $PQ = QR$.

* * *

3.32. A circle S and a line l that has no common points with S are given. From point P that moves along line l tangents PA and PB to circle S are drawn. Prove that all chords AB have a common point.

Let point P lie outside circle S; let PA and PB be tangents to the circle. Then line AB is called the *polar line* of point P relative circle S.

3.33. Circles S_1 and S_2 intersect at points A and B so that the center O of circle S_1 lies on S_2. A line that passes through point O intersects segment AB at point P and circle S_2 at point C. Prove that point P lies on the polar line of point C relative circle S_1.

§6. Application of the theorem on triangle's heights

3.34. Points C and D lie on the circle with diameter AB. Lines AC and BD, AD and BC meet at points P and Q, respectively. Prove that $AB \perp PQ$.

3.35. Lines PC and PD are tangent to the circle with diameter AB so that C and D are tangent points. Prove that the line that connects P with the intersection point of lines AC and BD is perpendicular to AB.

3.36. Given diameter AB of a circle and point C outside AB. With the help of the ruler alone (no compasses) drop the perpendicular from C to AB if:

a) point C does not lie on the circle;

b) point C lies on the circle.

3.37. Let O_a, O_b and O_c be the centers of circumscribed circles of triangles PBC, PCA and PAB. Prove that if points O_a and O_b lie on lines PA and PB, then point O_c lies on line PC.

§7. Areas of curvilinear figures

3.38. On the hypothenuse and legs of a rectangular triangle semicircles are constructed as shown on Fig. 22. Prove that the sum of the areas of the crescents obtained (shaded) is equal to the area of the given triangle.

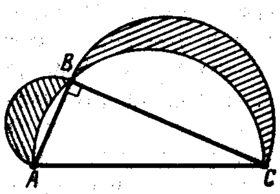


Figure 22 (3.38)

3.39. In a disc two perpendicular diameters, i.e., four radii, are constructed. Then there are constructed four disks whose diameters are these radii. Prove that the total area of the pairwise common parts of these four disks is equal to the area of the initial (larger) disk that lies outside the considered four disks (Fig. 23).

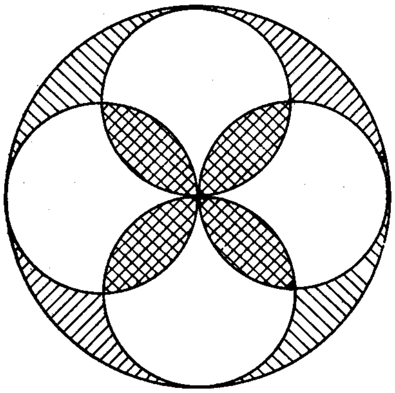


Figure 23 (3.39)

3.40. On three segments OA, OB and OC of the same length (point B lies outside angle AOC) circles are constructed as on diameters. Prove that the area of the curvilinear triangle bounded by the arcs of these circles and not containing point O is equal to a half area of the (common) triangle ABC.

3.41. On sides of an arbitrary acute triangle ABC as on diameters circles are constructed. They form three "outer" curvilinear triangles and one "inner" triangle (Fig. 24). Prove that if we subtract the area of the "inner" triangle from the sum of the areas of "outer" triangles we get the doubled area of triangle ABC.

§8. Circles inscribed in a disc segment

In this section a *segment* is always a disc segment.

3.42. Chord AB divides circle S into two arcs. Circle S_1 is tangent to chord AB at point M and one of the arcs at point N. Prove that:

a) line MN passes through the midpoint P of the second arc;

b) the length of tangent PQ to circle S_1 is equal to that of PA.

3.43. From point D of circle S the perpendicular DC is dropped to diameter AB. Circle S_1 is tangent to segment CA at point E and also to segment CD and to circle S. Prove that DE is a bisector of triangle ADC.

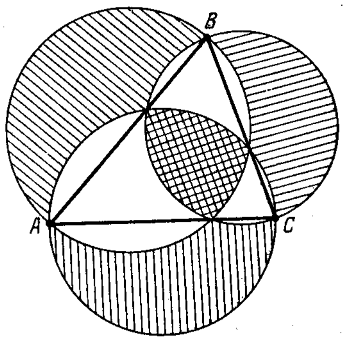


Figure 24 (3.41)

3.44. Two circles inscribed in segment AB of the given circle intersect at points M and N. Prove that line MN passes through the midpoint C of arc AB complementary for the given segment.

3.45. A circle tangent to sides AC and BC of triangle ABC at points M and N, respectively is also tangent to its circumscribed circle (from the inside). Prove that the midpoint of segment MN coincides with the center of the inscribed circle of triangle ABC.

3.46. Triangles ABC_1 and ABC_2 are inscribed in circle S so that chords AC_2 and BC_1 intersect. Circle S_1 is tangent to chord AC_2 at point M_2, to chord BC_1 at point N_1 and to circle S (???where?). Prove that the centers of the inscribed circles of triangles ABC_1 and ABC_2 lie on segment M_2N_1.

§9. Miscellaneous problems

3.47. The radii of two circles are equal to R_1 and R_2 and the distance between the centers of the circles is equal to d. Prove that these circles are orthogonal if and only if $d^2 = R_1^2 + R_2^2$.

3.48. Three circles are pairwise tangent from the outside at points A, B and C. Prove that the circumscribed circle of triangle ABC is perpendicular to all the three circles.

3.49. Two circles centered at O_1 and O_2 intersect at points A and B. A line is drawn through point A; the line intersects the first circle at point M_1 and the second circle at point M_2. Prove that $\angle BO_1M_1 = \angle BO_2M_2$.

§10. The radical axis

3.50. Circle S and point P are given on the plane. A line drawn through point P intersects the circle at points A and B. Prove that the product $PA \cdot PB$ does not depend on the choice of a line.

This product taken with the plus sign if point P is outside the circle and with minus sign if P is inside of the circle is called the *degree of point P with respect to circle S.*

3.51. Prove that for a point P outside circle S its degree with respect to S is equal to the square of the length of the tangent drawn to the circle from point P.

3.52. Prove that the degree of point P with respect to circle S is equal to $d^2 - R^2$, where R is the radius of S and d is the distance from P to the center of S.

3.53. Two nonconcentric circles S_1 and S_2 are given in plane. Prove that the locus of points whose degree with respect to S_1 is equal to the degree with respect to S_2 is a line.

This line is called *the radical axis* of circles S_1 and S_2.

3.54. Prove that the radical axis of two intersecting circles passes through the intersection points.

3.55. Given three circles in plane whose centers do not lie on one line. Let us draw radical axes for each pair of these circles. Prove that all the three radical axes meet at one point.

This point is called the *radical center* of the three circles.

3.56. Consider three pairwise intersecting circles in plane. Through the intersection points of any two of them a line is drawn. Prove that either these three lines meet at one point or are parallel.

3.57. Two nonconcentric circles S_1 and S_2 are given. Prove that the set of centers of circles that intersect both these circles at a right angle is their radical axis (without their common chord if the given circles intersect).

3.58. a) Prove that the midpoints of the four common tangents to two nonintersecting circles lie on one line.

b) Through two of the tangent points of common exterior tangents with two circles a line is drawn, see Fig. . Prove that the circles cut on this line equal chords.

3.59. On sides BC and AC of triangle ABC, points A_1 and B_1, respectively, are taken; let l be the line that passes through the common points of circles with diameters AA_1 and BB_1. Prove that:

a) Line l passes through the intersection point H of heights of triangle ABC;

b) line l passes through point C if and only if $AB_1 : AC = BA_1 : BC$.

3.60. The extensions of sides AB and CD of quadrilateral $ABCD$ meet at point F and the extensions of sides BC and AD meet at point E. Prove that the circles with diameters AC, BD and EF have a common radical axis and the orthocenters of triangles ABE, CDE, ADF and BCF lie on it.

3.6l. Three circles intersect pairwise at points A_1 and A_2, B_1 and B_2, C_1 and C_2. Prove that $A_1B_2 \cdot B_1C_2 \cdot C_1A_2 = A_2B_1 \cdot B_2C_1 \cdot C_2A_1$.

3.62. On side BC of triangle ABC point A' is taken. The midperpendicular to segment $A'B$ intersects side AB at point M and the midperpendicular to segment $A'C$ intersects side AC at point N. Prove that point symmetric to point A' through line MN lies on the circumscribed circle of triangle ABC.

3.63. Solve Problem 1.66 making use of the properties of the radical axis.

3.64. Inside a convex polygon several pairwise nonintersecting disks of distinct radii are placed. Prove that it is possible to cut the polygon into smaller polygons so that all these small polygons are convex and each of them contains exactly one of the given disks.

3.65. a) In triangle ABC, heights AA_1, BB_1 and CC_1 are drawn. Lines AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 intersect at points C', A' and B', respectively. Prove that points A', B' and C' lie on the radical axis of the circle of nine points (cf. Problem 5.106) and on that of the circumscribed circle.

b) The bisectors of the outer angles of triangle ABC intersect the extensions of the opposite sides at points A', B' and C'. Prove that points A', B' and C' lie on one line and this line is perpendicular to the line that connects the centers of the inscribed and circumscribed circles of triangle ABC.

3.66. Prove that diagonals AD, BE and CF of the circumscribed hexagon $ABCDEF$ meet at one point. (*Brianchon's theorem.*)

3.67. Given four circles S_1, S_2, S_3 and S_4 such that the circles S_i and S_{i+1} are tangent from the outside for $i = 1, 2, 3, 4$, where $S_5 = S_1$. Prove that the radical axis of circles S_1 and S_3 passes through the intersection point of common outer tangents to S_2 and S_4.

3.68. a) Circles S_1 and S_2 intersect at points A and B. The degree of point P of circle S_1 with respect to circle S_2 is equal to p, the distance from point P to line AB is equal to h and the distance between the centers of circles is equal to d. Prove that $|p| = 2dh$.

b) The degrees of points A and B with respect to the circumscribed circles of triangles BCD and ACD are equal to p_a and p_b, respectively. Prove that $|p_a|S_{BCD} = |p_b|S_{ACD}$.

Problems for independent study

3.69. An easy chair of the form of a disc sector of radius R is swinging on a horizontal table. What is the trajectory of the vertex of the sector?

3.70. From a point A outside a circle of radius R two tangents AB and AC are drawn, B and C are tangent points. Let $BC = a$. Prove that $4R^2 = r^2 + r_a^2 + \frac{1}{2}a^2$, where r and r_a are the radii of the inscribed and escribed circles of triangle ABC.

3.71. Two circles have an inner tangent. The line that passes through the center of a smaller circle intersects the greater one at points A and D and the smaller one at points B and C. Find the ratio of the radii of the circles if $AB : BC : CD = 2 : 3 : 4$.

3.72. The centers of three circles each of radius R, where $1 < R < 2$, form an equilateral triangle with side 2. What is the distance between the intersection points of these circles that lie outside the triangle?

3.73. A point C is taken on segment AB and semicircles with diameters AB, AC and BC are constructed (on one side of line AB). Find the ratio of the area of the curvilinear triangle bounded by these semicircles to the area of the triangle formed by the midpoints of the arcs of these semicircles.

3.74. A circle intersects side BC of triangle ABC at points A_1 and A_2, side AC at points B_1 and B_2, side AB at points C_1 and C_2. Prove that

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \left(\frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A}\right)^{-1}.$$

3.75. From point A tangents AB and AC to a circle are drawn (B and C are tangent points); PQ is a diameter of the circle; line l is tangent to the circle at point Q. Lines PA, PB and PC intersect line l at points A_1, B_1 and C_1. Prove that $A_1B_1 = A_1C_1$.

Solutions

3.1. Let line XY be tangent to the given circle at point Z. The corresponding sides of triangles XOA and XOZ are equal and, therefore, $\angle XOA = \angle XOZ$. Similarly, $\angle ZOY = \angle BOY$. Therefore,

$$\angle XOY = \angle XOZ + \angle ZOY = \frac{\angle AOZ + \angle ZOB}{2} = \frac{\angle AOB}{2}.$$

3.2. Let M and N be the tangent points of the inscribed circle with sides AB and BC. Then $BK + AN = BM + AM = AB$, hence, $CK + CN = a + b - c$.

Let P and Q be the tangent points of the escribed circle with the extensions of sides AB and BC. Then $AP = AB + BP = AB + BL$ and $AQ = AC + CQ = AC + CL$. Hence, $AP + AQ = a + b + c$. Therefore, $BL = BP = AP - AB = \frac{1}{2}(a + b - c)$.

3.3. By Problem 3.2 $CM = \frac{1}{2}(AC + CE - AE)$ and $CN = \frac{1}{2}(BC + CE - BE)$. Taking into account that $AC = BC$ we get $MN = |CM - CN| = \frac{1}{2}|AE - BE|$.

3.4. Let lines AB, BC, CD and DA be tangent to the circle at points P, Q, R and S, respectively. Then $CQ = CR = x$, hence, $BP = BC + CQ = BC + x$ and $DS = DC + CR = DC + x$. Therefore, $AP = AB + BP = AB + BC + x$ and $AS = AD + DS = AD + DC + x$. Taking into account that $AP = AS$, we get the statement desired.

3.5. Let line AB be tangent to the circles centered at O_1 and O_2 at points C and D, respectively. Since $\angle O_1AO_2 = 90^\circ$, the right triangles AO_1C and O_2AD are similar. Therefore, $O_1C : AC = AD : DO_2$. Moreover, $AD = CB$ (cf. Problem 3.2). Therefore, $AC \cdot CB = Rr$.

3.6. Let lines AB and CD intersect at point O. Let us assume for definiteness that points A and D lie on the first circle while points B and C lie on the second one. Suppose also that $OB < OA$ (Fig. 25).

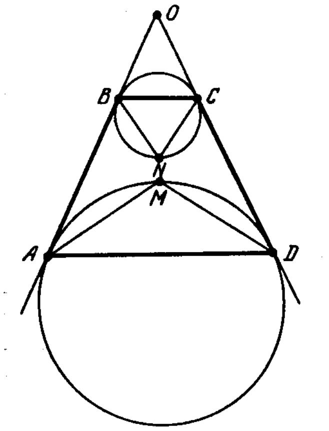


Figure 25 (Sol. 3.6)

The intersection point M of bisectors of angles $\angle A$ and $\angle D$ of quadrilateral $ABCD$ is the midpoint of the arc of the first circle that lies inside triangle AOD and the intersection point N of bisectors of angles $\angle B$ and $\angle C$ is the midpoint of the arc of the second circle that lies outside triangle BOC, cf. Problem 2.91 a). Quadrilateral $ABCD$ is a circumscribed one if and only if points M and N coincide.

3.7. Let R be the tangent point of the escribed circle with side BD, let P and Q be the intersection points of segment MN with BC and CD, respectively (Fig. 26).

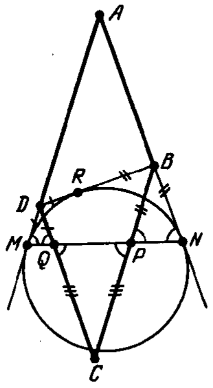


Figure 26 (Sol. 3.7)

Since $\angle DMQ = \angle BPN$, $\angle DQM = \angle BNP$ and $\angle DMQ = \angle BNP$, it follows that triangles MDQ, PBN and PCQ are isosceles ones. Therefore, $CP = CQ$, $DQ = DM = DR$ and $BP = BN = BR$. Therefore, P, Q and R are the tangent points of the inscribed circle of triangle BCD with its sides (cf. Problem 5.1).

3.8. Denote some of the tangent points as shown on Fig. 27. The sum of the lengths of one pair of the opposite sides of the inner quadrilateral is equal to the sum of the lengths of the pair of its other sides. Let us extend the sides of this quadrilateral to tangent points with inscribed circles of the other quadrilaterals (ST is one of the obtained segments).

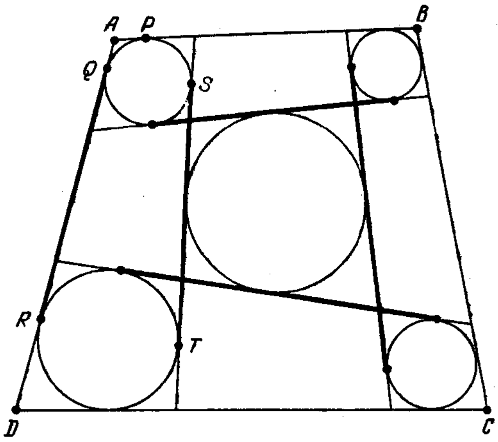


Figure 27 (Sol. 3.8)

Then both sums of lengths of pairs of opposite segments increase by the same number. Each of the obtained segments is the common tangent to a pair of “corner” circles; each segment can be replaced with another common outer tangent of equal length (i.e., replace ST with QR). To prove the equality $AB + CD = BC + AD$, it remains to make use of equalities of the form $AP = AQ$.

3.9. Let P be the intersection point of diagonals of convex quadrilateral $ABCD$. Quadrilateral $ABCD$ is an inscribed one if and only if $\triangle APB \sim \triangle DPC$, i.e., $PA \cdot PC = PB \cdot PD$. Since quadrilaterals $ALBN$ and $AMBK$ are inscribed ones, $PL \cdot PN = PA \cdot PB = PM \cdot PK$. Hence, quadrilateral $KLMN$ is an inscribed one.

3.10. Let O be the intersection point of line AB and segment MN. Then $OM^2 = OA \cdot OB = ON^2$, i.e., $OM = ON$.

3.11. Let, for definiteness, rays OA and BC be codirected, M the intersection point of lines KL and OA. Then $\angle LOM = \angle LCB = \angle OKM$ and, therefore, $\triangle KOM \sim \triangle OLM$. Hence, $OM : KM = LM : OM$, i.e., $OM^2 = KM \cdot LM$. Moreover, $MA^2 = MK \cdot ML$. Therefore, $MA = OM$.

3.12. Let O be the intersection point of diagonals AC and BD. Then $MO \cdot OC = BO \cdot OD$. Since $OC = OA$ and $BO = OD$, we have $MO \cdot OA = BO^2$ and $MO \cdot OA = DO^2$. These equalities mean that OB is tangent to the circumscribed circle of triangle ABM and OD is tangent to the circumscribed circle of triangle ADM.

3.13. Let C be the intersection point of line AB with the circumscribed circle of triangle BMN distinct from point B; let AP be the tangent to circle S. Then $AB \cdot AC = AM \cdot AN = AP^2$ and, therefore, $AC = \frac{AP^2}{AB}$, i.e., point C is the same for all lines l.

Remark. We have to exclude the case when the length of the tangent drawn to S from A is equal to AB.

3.14. Clearly, $MC^2 = MP \cdot MQ = MA \cdot MB$ and point M lies on ray AB if $AC > BC$ and on ray BA if $AC < BC$. Let, for definiteness sake, point M lie on ray AB. Then $(MB + BC)^2 = (MB + BA) \cdot MB$. Therefore, $MB = \frac{BC^2}{AB-2BC}$ and we deduce that the position of point M does not depend on the choice of circle S'.

3.15. Let M be the intersection point of line CD and the tangent to circles at point A. Then $MC = MA = MD$. Therefore, point A lies on the circle with diameter CD.

3.16. Points O_1, A and O_2 lie on one line, hence, $\angle A_2AO_2 = \angle A_1AO_1$. Triangles AO_2A_2 and AO_1A_1 are isosceles ones, hence, $\angle A_2AO_2 = \angle AA_2O_2$ and $\angle A_1AO_1 = \angle AA_1O_1$. Therefore, $\angle AA_2O_2 = \angle AA_1O_1$, i.e., $O_1A_1 \parallel O_2A_2$.

3.17. Let O_1, O_2 and O_3 be the centers of circles S_1, S_2 and S_3; let A, B, C be the tangent points of circles S_2 and S_3, S_3 and S_1, S_1 and S_2, respectively; A_1 and B_1 the intersection points of lines CA and CB, respectively, with circle S_3. By the previous problem $B_1O_3 \parallel CO_1$ and $A_1O_3 \parallel CO_2$. Points O_1, C and O_2 lie on one line and, therefore, points A_1, O_3 and B_1 also lie on one line, i.e., A_1B_1 is a diameter of cicle S_3.

3.18. Let A_1, A_2 and B be the tangent points of the circles centered at O and O_1, O and O_2, O_1 and O_2, respectively. Then $O_1O_2 = O_1B + BO_2 = O_1A_1 + O_2A_2$. Therefore,

$$OO_1 + OO_2 + O_1O_2 = (OO_1 + O_1A_1) + (OO_2 + O_2A_2) = OA_1 + OA_2 = 2R.$$

3.19. Let O, O_1 and O_2 be centers of circles S, S_1 and S_2; let C be the common point of circles S_1 and S_2 that lies on segment AB. Triangles AOB, AO_1C and CO_2B are isosceles ones; consequently, OO_1CO_2 is a parallelogram and $OO_1 = O_2C = O_2B$; hence, $AO = AO_1 + O_1O = AO_1 + O_2B$.

3.20. Let O_1 and O_2 be the centers of circles S_1 and S_2; let X be the other intersection point of line AB with circle S_2. The square of the length of the tangent in question is equal to $BA \cdot BX$. Since $AB : BX = O_1A : O_1O_2$, it follows that $AB \cdot BX = \frac{AB^2 \cdot O_1O_2}{R} = \frac{a^2(R \pm r)}{R}$, where the minus sign is taken for the inner tangent and the plus sign for the outer tangent.

3.21. Let O, O_1 and O_2 be the centers of the circles with diameters AB, AC and BC, respectively. It suffices to verify that $KO = OL$. Let us prove that $\triangle O_1KO = \triangle O_2OL$. Indeed, $O_1K = \frac{1}{2}AC = O_2O$, $O_1O = \frac{1}{2}BC = O_2L$ and $\angle KO_1O = \angle OO_2L = 180^\circ - 2\alpha$, where α is the value of the angle between lines KL and AB.

3.22. Let O_i be the center of circle S_i and A_i the tangent point of circles S_i and S_{i+1}. Quadrilateral $O_1O_2O_3O_4$ is a convex one; let α_1, α_2, α_3 and α_4 be the values of its angles. It is easy to verify that $\angle A_{i-1}A_iA_{i+1} = \frac{1}{2}(\alpha_i + \alpha_{i+1})$ and, therefore,

$$\angle A_1 + \angle A_3 = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = \angle A_2 + \angle A_4.$$

3.23. a) Let A_1, B_1 and C_1 be the projections of points A, B and C, respectively, to line l; let C_2 be the projection of point C to line AA_1. By Pythagorus theorem $CC_2^2 = AC^2 - AC_2^2$, i.e., $A_1C_1^2 = (a+c)^2 - (a-c)^2 = 4ac$. Similarly, $B_1C_1^2 = 4bc$ and $A_1B_1^2 = 4ab$. Since $A_1C_1 + C_1B_1 = A_1B_1$, it follows that $\sqrt{ac} + \sqrt{bc} = \sqrt{ab}$, i.e., $\frac{1}{\sqrt{b}} + \frac{1}{\sqrt{a}} = \frac{1}{\sqrt{c}}$.

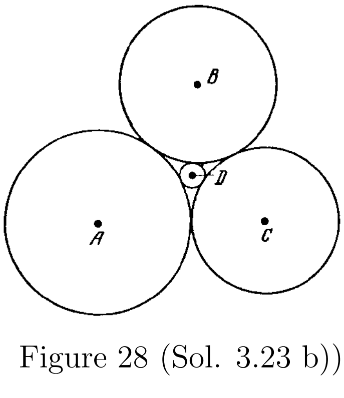


Figure 28 (Sol. 3.23 b))

b) Let A, B, C be the centers of "outer" circles, D the center of the "inner" circle (Fig. 28). The semiperimeter of triangle BDC is equal to $b+c+d$, and, therefore,

$$\cos^2\left(\frac{\angle BDC}{2}\right)=\frac{d(b+c+d)}{(b+d)(c+d)},\quad \sin^2\left(\frac{\angle BDC}{2}\right)=\frac{bc}{(b+d)(c+d)}$$

(cf. Problem 12.13). As is easy to see the law of cosines is equivalent to the statement:

$$\alpha'+\beta'+\gamma'=180^\circ \Longrightarrow \sin^2\alpha'-\sin^2\beta'-\sin^2\gamma'+2\sin\beta'\sin\gamma'\cos\alpha'=0. \qquad (*)$$

Substituting the values $\alpha'=\frac{1}{2}\angle BDC$, $\beta'=\frac{1}{2}\angle ADC$ and $\gamma'=\frac{1}{2}\angle ADB$ into formula $(*)$, we get

$$\frac{bc}{(b+d)(c+d)}-\frac{ac}{(a+d)(c+d)}-\frac{ab}{(a+d)(b+d)}+2\frac{a\sqrt{bcd(b+c+d)}}{(a+d)(b+d)(c+d)}=0,$$

i.e.,

$$\frac{a+d}{a}-\frac{b+d}{b}-\frac{c+d}{c}+2\sqrt{\frac{d(b+c+d)}{bc}}=0.$$

Dividing this by d we get

$$\alpha-\beta-\gamma-\delta+2\sqrt{\beta\gamma+\gamma\delta+\delta\beta}=0.$$

Therefore,

$$\begin{gathered}(\alpha+\beta+\gamma+\delta)^2=(\alpha-\beta-\gamma-\delta)^2+4(\alpha\beta+\alpha\gamma+\alpha\delta)+\\ 4(\beta\gamma+\gamma\delta+\delta\beta)+4(\alpha\beta+\alpha\gamma+\alpha\delta)=\\ 2(\alpha+\beta+\gamma+\delta)^2-2(\alpha^2+\beta^2+\gamma^2+\delta^2),\end{gathered}$$

i.e.,

$$2(\alpha^2+\beta^2+\gamma^2+\delta^2)=(\alpha+\beta+\gamma+\delta)^2.$$

3.24. Let A_1, B_1 and C_1 be the centers of the given circles (Fig. 29). Then A_1BC_1H is a rhombus and, therefore, $BA_1 \parallel HC_1$. Similarly, $B_1A \parallel HC_1$; hence, $B_1A \parallel BA_1$ and B_1ABA_1 is a parallelogram.

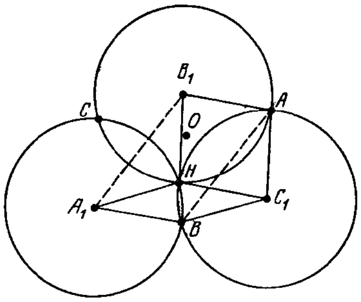


Figure 29 (Sol. 3.24)

a) Since $A_1B_1 \perp CH$ and $A_1B_1 \parallel AB$, it follows that $AB \perp CH$. We similarly prove that $BC \perp AH$ and $CA \perp BH$.

b) In the same way as we have proved that $B_1A \parallel BA_1$, we can prove that $B_1C \parallel BC_1$ and $A_1C \parallel AC_1$; moreover, the lengths of all these six segments are equal to R. Let us complement the triangle BA_1C to a rhombus BA_1CO. Then AB_1CO is also a rhombus. Therefore, $AO=BO=CO=R$, i.e., O is the center of the circumscribed circle of triangle ABC and its radius is equal to R.

3.25. It is easy to verify that

$$\begin{array}{rcl} \smile AB_1 \pm \smile B_1A_1 & = & \smile AC_1 + \smile C_1A_1, \\ \smile BC_1 + \smile C_1B_1 & = & \smile BA_1 \pm \smile B_1A_1 \\ \smile C_1A_1 \pm \smile CA_1 & = & \smile C_1B_1 \pm \smile B_1C, \end{array}$$

where the minus sign is only taken in case b). Adding up these equalities we get

$$\smile AB_1 + \smile BC_1 \pm \smile CA_1 = \smile AC_1 + \smile BA \pm \smile CB_1.$$

On the other hand, the doubled values of the angles of triangle ABC are equal to $\smile BA_1 \pm \smile CA_1$, $\smile AB_1 \pm \smile CB_1$ and $\smile BC_1 + \smile AC_1$, and their sum is equal to 360°.

3.26. Since $\smile AP + \smile BP + \smile PQ = 180^\circ$ (cf. Problem 3.25), it follows that $\smile AB = 180^\circ - \smile PQ$. Similarly, $\smile CD = 180^\circ - \smile PQ$, i.e., $\smile AB = \smile CD$ and, therefore, $AB = CD$. Moreover, $PQ \perp AB$ and $PQ \perp CD$ (cf. Problem 3.24) and, therefore, $AB \parallel CD$.

3.27. Points M, B and C lie on the circle with diameter AO. Moreover, chords OB and OC of the circle are equal.

3.28. Points B and X lie on the circle with diameter KO, and, therefore, $\angle XKO = \angle XBO$. Similarly, $\angle XLO = \angle XCO$. Since $\angle XBO = \angle XCO$, triangle KOL is an isosceles one and OX is its height.

3.29. It suffices to verify that $AK \cdot AL = AM \cdot AO$. Indeed, if such is the case, then points K, L, M and O lie on one circle and, therefore, $\angle MKO = \angle MLO$. Since $\triangle AOP \sim \triangle APM$, it follows that $AM \cdot AO = AP^2$; it is also clear that $AK \cdot AL = AP^2$.

3.30. Let O be the center of the circle; let points D' and E' be symmetric to points D and E through line AO. By Problem 28.7 the lines ED' and $E'D$ meet at point M. Hence, $\angle BDM = \angle EBM$ and $\angle BEM = \angle DBM$ and, therefore, $\triangle BDM \sim \triangle EBM$. It follows that $BM : DM = EM : BM$. Moreover, if line ED separates points B and M, then $\angle DME = \smile DE = 2\angle DCE$.

The equality $\angle BEM = \angle DBM$ implies that $\angle BEM = \angle DBC = \angle DEC$.

3.31. a) Since $\triangle KAB \sim \triangle KBC$, we have $AB : BC = KB : KC$. Similarly, $AD : DC = KD : KC$. Taking into account that $KB = KD$ we get the desired statement.

b) The problem of this heading reduces to that of the previous one, since

$$\frac{PQ}{BQ} = \frac{\sin\angle PBQ}{\sin\angle BPQ} = \frac{\sin\angle ABD}{\sin\angle KBA} = \frac{\sin\angle ABD}{\sin\angle ADB} = \frac{AD}{AB}, \quad \frac{QR}{BQ} = \frac{CD}{CB}.$$

3.32. Let us drop perpendicular OM to line l from center O of circle S. Let us prove that point X at which AB and OM intersect remains fixed. Points A, B and M lie on the circle with diameter PO. Hence, $\angle AMO = \angle ABO = \angle BAO$ and, therefore, $\triangle AMO \sim \triangle XAO$, because these triangles have a common angle at vertex O. It follows that $AO : MO = XO : AO$, i.e., $OX = \frac{OA^2}{MO}$ is a constant.

3.33. Since $\angle OBP = \angle OAB = \angle OCB$, we deduce that $\triangle OBP \sim \triangle OCB$ and, therefore, $OB^2 = OP \cdot OC$. Let us draw tangent CD to circle S_1 from point C. Then $OD^2 = OB^2 = OP \cdot OC$. Therefore, $\triangle ODC \sim \triangle OPD$ and $\angle OPD = \angle ODC = 90^\circ$.

3.34. Lines BC and AD are heights of triangle APB and, therefore, line PQ that passes through their intersection point Q is perpendicular to line AB.

3.35. Denote the intersection points of lines AC and BD, BC and AD by K and K_1, respectively. Thanks to the above problem, $KK_1 \perp AB$ and, therefore, it suffices to show that the intersection point of tangents at points C and D lies on line KK_1.

Let us prove that the tangent at point C passes through the midpoint of segment KK_1. Let M be the intersection point of the tangent at point C and segment KK_1. The respective sides of acute angles $\angle ABC$ and $\angle CKK_1$ are perpendicular and, therefore, the angles are equal. Similarly, $\angle CAB = \angle CK_1K$. It is also clear that $\angle KCM = \angle ABC$ and, therefore, triangle CMK is an isosceles one. Similarly, triangle CMK_1 is an isosceles one and $KM = CM = K_1M$, i.e., M is the midpoint of segment KK_1.

We similarly prove that the tangent at point D passes through the midpoint of segment KK_1.

3.36. a) Line AC intersects the circle at points A and A_1, line BC does same at points B and B_1. If $A = A_1$ (or $B = B_1$), then line AC (or BC) is the perpendicular to be constructed. If this is not the case, then AB_1 and BA_1 are heights of triangle ABC and the line to be constructed is the line that passes through point C and the intersection point of lines AB_1 and BA_1.

b) Let us take point C_1 that does not lie on the circle and drop from it perpendicular to AB. Let the perpendicular intersect the circle at points D and E. Let us construct the intersection point P of lines DC and AB and then the intersection point F of line PE with the circle. The symmetry through AB sends point C to point F. Therefore, CF is the perpendicular to be constructed.

3.37. Since $PA \perp O_bO_c$, line PA passes through point O_a if and only if line PO_a passes through the intersection point of heights of triangle $O_aO_bO_c$. Similar statements are true for points B and C as well.

The hypothesis of the problem implies that P is the intersection point of heights of triangle $O_aO_bO_c$ and, therefore, $PO_c \perp O_aO_b$.

3.38. Let $2a$ and $2b$ be the lengths of the legs, $2c$ the length of the hypothenuse. The sum of the areas of the "crescents" is equal to $\pi a^2 + \pi b^2 + S_{ABC} - \pi c^2$. But $\pi(a^2 + b^2 - c^2) = 0$.

3.39. It suffices to carry out the proof for each of the four parts into which the diameters divide the initial disc (Fig. 30).

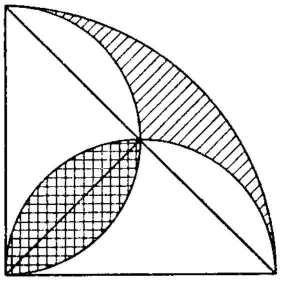


Figure 30 (Sol. 3.39)

In the disc, consider the segment cut off by the chord intercepted by the central angle of 90°; let S and s be the areas of such segments for the initial disc and any of the four constructed disks, respectively. Clearly, $S = 4s$. It remains to observe that the area of the part shaded once is equal to $S - 2s = 2s$ and the area of the part shaded twice is equal to $2s$.

3.40. Denote the intersection points of circles constructed on segments OB and OC, OA and OC, OA and OB as on diameters by A_1, B_1, C_1, respectively (Fig. 31). Since $\angle OA_1B = \angle OA_1C = 90°$, it follows that points B, A_1 and C lie on one line and since all the circles have equal radii, $BA_1 = A_1C$.

Points A_1, B_1, C_1 are the midpoints of sides of triangle ABC, therefore, $BA_1 = C_1B_1$ and $BC = A_1B_1$. Since the disks are of the same radius, the equal chords BA_1 and C_1B_1 cut off the disks parts of equal area and equal chords C_1B and B_1A_1 also cut off the disc's parts of equal area. Therefore, the area of curvilinear triangle $A_1B_1C_1$ is equal to the area of parallelogram $A_1B_1C_1B$, i.e., is equal to half the area of triangle ABC.

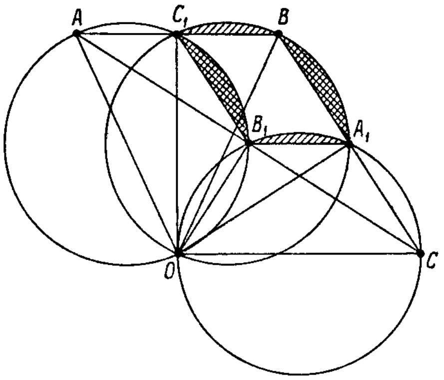


Figure 31 (Sol. 3.40)

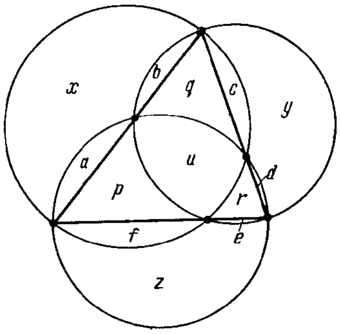


Figure 32 (Sol. 3.41)

3.41. The considered circles pass through the bases of the triangle's heights and, therefore, their intersection points lie on the triangle's sides. Let x, y, z and u be the areas of the considered curvilinear triangles; let a, b, c, d, e and f be the areas of the segments cut off the circles by the sides of the triangle; let p, q and r be the areas of the parts of the triangle that lie outside the inner curvilinear triangle (see Fig. 32). Then

$$\begin{aligned} x+(a+b) &= u+p+q+(c+f), \\ y+(c+d) &= u+q+r+(e+b), \\ z+(e+f) &= u+r+p+(a+d) \end{aligned}$$

By adding up these equalities we get

$$x+y+z=2(p+q+r+u)+u.$$

3.42. a) Let O and O_1 be the centers of circles S and S_1. The triangles MO_1N and PON are isosceles ones and $\angle MO_1N=\angle PON$. Therefore, points P, M and N lie on one line.

b) It is clear that $PQ^2=PM\cdot PN=PM\cdot(PM+MN)$. Let K be the midpoint of chord AB. Then

$$PM^2=PK^2+MK^2 \quad \text{and } PM\cdot MN=AM\cdot MB=AK^2-MK^2.$$

Therefore, $PQ^2=PK^2+AK^2=PA^2$.

3.43. By Problem 3.42 b) $BE=BD$. Hence,

$$\angle DAE+\angle ADE=\angle DEB=\angle BDE=\angle BDC+\angle CDE.$$

Since $\angle DAB=\angle BDC$, it follows that $\angle ADE=\angle CDE$.

3.44. Let O_1 and O_2 be the centers of the inscribed circles, CP and CQ the tangents to them. Then $CO_1^2=CP^2+PO_1^2=CP^2+O_1M^2$ and since $CQ=CA=CP$ (by Problem 3.42

b), we have $CO_2^2 = CQ^2 + QO_2^2 = CP^2 + O_2M^2$. It follows that $CO_1^2 - CO_2^2 = MO_1^2 - MO_2^2$ and, therefore, line CM is perpendicular to O_1O_2 (see Problem 7.6). Therefore, line MN passes through point C.

Remark. If the circles do not intersect but are tangent to each other the statement is still true; in this case, however, one should replace line MN with the tangent to the circles at their common point.

3.45. Let A_1 and B_1 be the midpoints of arcs $\smile BC$ and $\smile AC$;let O the center of the inscribed circle. Then $A_1B_1 \perp CO$ (cf. Problem 2.19 a) and $MN \perp CO$, consequently, $MN \parallel A_1B_1$. Let us move points M' and N' along rays CA and CB, respectively, so that $M'N' \parallel A_1B_1$. Only for one position of points M' and N' does point L at which lines B_1M' and A_1N' intersect lie on the circumscribed circle of triangle ABC.

On the other hand, if segment MN passes through point O, then point L lies on this circle (cf. Problem 2.49).

3.46. The solution of this problem generalizes the solution of the preceding problem. It suffices to prove that the center O_1 of the inscribed circle of triangle ABC_1 lies on segment M_2N_1. Let A_1 and A_2 be the midpoints of arcs $\smile BC_1$ and $\smile BC_2$; let B_1 and B_2 be the midpoints of arcs $\smile AC_1$ and $\smile AC_2$; let PQ be the diameter of circle S perpendicular to chord AB and let points Q and C_1 lie on one side of line AB. Point O_1 is the intersection point of chords AA_1 and BB_1 and point L of tangent of circles S and S_1 is the intersection point of lines A_1N_1 and B_2M_2 (Fig. 33).

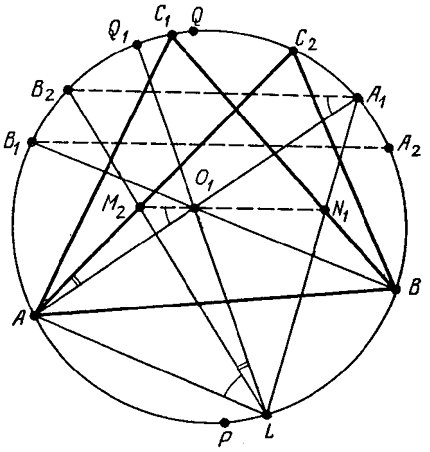


Figure 33 (Sol. 3.46)

Let $\angle C_1AB = 2\alpha$, $\angle C_1BA = 2\beta$, $\angle C_1AC_2 = 2\varphi$. Then $\smile A_1A_2 = 2\varphi = \smile B_1B_2$, i.e., $A_1B_2 \parallel B_1A_2$. For the angles between chords we have:

$$\angle(A_1B_2, BC_1) = \tfrac{1}{2}(\smile B_2C_1 + \smile A_1B) = \beta - \varphi + \alpha,$$
$$\angle(BC_1, AC_2) = \tfrac{1}{2}(\smile C_1C_2 + \smile AB) = 2\varphi + 180^\circ - 2\alpha - 2\beta.$$

Consequently, chord M_2N_1 constitutes equal angles with tangents BC_1 and AC_2, each angle equal to $\alpha + \beta - \varphi$. Therefore, $M_2N_1 \parallel A_1B_2$.

Now, suppose that points M_2' and N_1' are moved along chords AC_2 and BC_1 so that $M_2'N_1' \parallel A_1B_2$. Let lines A_1N_1' and B_2M_2' meet at point L'. Point L' lies on circle S for one position of points M_2' and N_1' only. Therefore, it suffices to indicate on arc $\smile AB$ a point L_1 such that if M_2'', N_1'' are the intersection points of chords AC_2 and L_1B_2, BC_1 and L_1A_1, respectively, then $M_2''N_1'' \parallel A_1B_2$ and point O_1 lies on segment $M_2''N_1''$. Let Q_1 be a point on circle S such that $2\angle(PQ, PQ_1) = \angle(PC_2, PC_1)$ and L_1 the intersection point of line Q_1O_1 with S.

Let us prove that L_1 is the desired point. Since $\smile B_1Q = 2\alpha$, it follows that $\smile B_2Q_1 = 2(\alpha - 2\varphi) = \smile C_2A_1$. Hence, quadrilateral $AM_2''O_1L_1$ is an inscribed one and, therefore, $\angle M_2''O_1A = \angle M_2''L_1A = \angle B_2A_1A$, i.e., $M_2''O_1 \parallel B_2A_1$.

Similarly, $N_1''O_1 \parallel B_2A_1$.

3.47. Let circles centered at O_1 and O_2 pass through point A. The radii O_1A and O_2A are perpendicular to the tangents to circles at point A and, therefore, the circles are orthogonal if and only if $\angle O_1AO_2 = 90^\circ$, i.e., $\angle O_1O_2^2 = O_1A^2 + O_2A^2$.

3.48. Let A_1, B_1 and C_1 be the centers of the given circles so that points A, B and C lie on segments B_1C_1, C_1A_1 and A_1B_1, respectively. Since $A_1B = A_1C$, $B_1A = B_1C$ and $CA = C_1B$, it follows that A, B and C are the tangent points of the inscribed circle of triangle $A_1B_1C_1$ with its sides (cf. Problem 5.1). Therefore, the radii A_1B, BC and C_1A of the given circles are tangent to the circumscribed circle of triangle ABC.

3.49. It is easy to verify that the angle of rotation from vector $\overrightarrow{O_iB}$ to vector $\overrightarrow{O_iM_i}$ (counterclockwise) is equal to $2\angle(AB, AM_i)$. It is also clear that $\angle(AB, AM_1) = \angle(AB, AM_2)$.

3.50. Let us draw through point P another line that intersects the circle at points A_1 and B_1. Then $\triangle PAA_1 \sim \triangle PB_1B$ and, therefore, $PA : PA_1 = PB_1 : PB$.

3.51. Let us draw through point P tangent PC. Since $\triangle PAC \sim \triangle PCB$, it follows that $PA : PC = PC : PB$.

3.52. Let the line that passes through point P and the center of the circle intersect the circle at points A and B. Then $PA = d+R$ and $PB = |d-R|$. Therefore, $PA \cdot PB = |d^2-R^2|$. It is also clear that the signs of the expression $d^2 - R^2$ and of the degree of point P with respect to to S are the same.

3.53. Let R_1 and R_2 be the radii of the circles. Let us consider the coordinate system in which the coordinates of the centers of the circles are $(-a, 0)$ and $(a, 0)$. By Problem 3.52 the degrees of the point with coordinates (x, y) with respect to the given circles are equal to $(x + a)^2 + y^2 - R_1^2$ and $(x - a)^2 + y^2 - R_2^2$, respectively. By equating these expressions we get $x = \frac{R_1^2 - R_2^2}{4a}$. This equation determines the perpendicular to the segment that connects the centers of the circles.

3.54. The degrees of the intersection point of the circles with respect to each one of the cicles are equal to zero and, therefore, the point belongs to the radical axis. If there are two intersection points, then they uniquely determine the radical axis.

3.55. Since the centers of the circles do not lie on one line, the radical axis of the first and the second circles intersects with the radical axis of the second and third circles. The degrees of the intersection point with respect to all three circles are equal and, therefore, this intersection point lies on the radical axis of the first and third circles.

3.56. By Problem 3.54 the lines that contain chords are radical axes. By Problem 3.55 the radical axes meet at one point if the centers of the circles do not lie on one line. Otherwise they are perpendicular to this line.

3.57. Let O_1 and O_2 be the centers of given circles, r_1 and r_2 their radii. The circle S of radius r centered at O is orthogonal to circle S_i if and only if $r^2 = OO_i^2 - r_i^2$, i.e., the squared radius of S is equal to the degree of point O with respect to circle S_i. Therefore, the locus of the centers of the circles to be found is the set of the points of the radical axis whose degrees with respect to the given circles are positive.

3.58. a) The indicated points lie on the radical axis.

b) The tangent points of the outer tangents with the circles are vertices of trapezoid $ABCD$ with base AB. The midpoints of lateral sides AD and BC belong to the radical axis and, therefore, the midpoint O of diagonal AC also belongs to the radical axis. If line AC intersects the circles at points A_1 and C_1, then $OA_1 \cdot OA = OC_1 \cdot OC$; consequently, $OA_1 = OC_1$ and $AA_1 = CC_1$.

3.59. a) Let S_A and S_B be circles with diameters AA_1 and BB_1; let S be the circle with diameter AB. The common chords of circles S and S_A, S and S_B are heights AH_a and BH_b and, therefore, these heights (or their extensions) intersect at point H. By Problem 3.56 the common chord of circles S_A and S_B passes through the intersection point of chords AH_a and BH_b.

b) The common chord of circles S_A and S_B passes through the intersection point of lines A_1H_a and B_1H_b (i.e., through point C) if and only if $CB_1 \cdot CH_b = CA_1 \cdot CH_a$ (here we should consider the lengths of segments as oriented). Since

$$CH_b = \frac{a^2+b^2-c^2}{2b} \quad \text{and} \quad CH_a = \frac{a^2+b^2-c^2}{2a},$$

we deduce that $\frac{CB_1}{b} = \frac{CA_1}{a}$.

3.60. In triangle CDE, draw heights CC_1 and DD_1; let H be their intersection point. The circles with diameters AC and BD pass through points C_1 and D_1, respectively, therefore, the degree of point H with respect to each of these circles is equal to its degree with respect to the circle with diameter CD (this circle passes through points C_1 and D_1). We similarly prove that the degrees of point H with respect to to circles with diameters AC, BD and EF are equal, i.e., the radical axes of these circles pass through point H.

For the intersection points of heights of the other three triangles the proof is carried out in a similar way.

Remark. The centers of the considered circles lie on the Gauss' line (cf. Problem 4.55) and, therefore, their common radical axis is perpendicular to the Gauss line.

3.61. Lines A_1A_2, B_1B_2 and C_1C_2 meet at a point O (cf. Problem 3.56). Since $\triangle A_1OB_2 \sim \triangle B_1OA_2$, it follows that $A_1B_2 : A_2B_1 = OA_1 : OB_1$. Similarly, $B_1C_2 : B_2C_1 = OB_1 : OC_1$ and $C_1A_2 : C_2A_1 = OC_1 : OA_1$. By multiplying these equalities we get the statement desired.

3.62. Denote by B' and C' the intersection points of lines $A'M$ and $A'N$, respectively, with the line drawn through point A parallel to BC (Fig. 34).

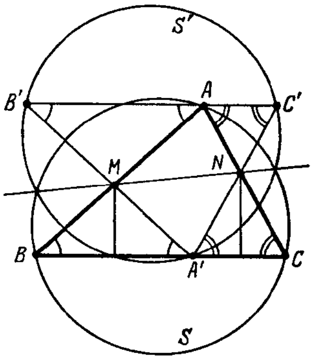


Figure 34 (Sol. 3.62)

Since triangles $A'BM$ and $A'NC$ are isosceles ones, $\triangle ABC = \triangle A'B'C'$. Since $AM \cdot BM = A'M \cdot B'M$, the degrees of point M with respect to circles S and S' circumscribed about triangles ABC and $A'B'C'$, respectively, are equal. This is true for point N as well and, therefore, line MN is the radical axis of circles S and S'. Circles S and S' have equal radii and, therefore, their radical axis is their axis of symmetry. The symmetry through line MN sends a point A' that lies on circle S' into a point that lies on circle S.

3.63. Let AC and BD be the tangents; E and K the intersection points of lines AC and BD, AB and CD, respectively; O_1 and O_2 the centers of the circles (Fig. 35).

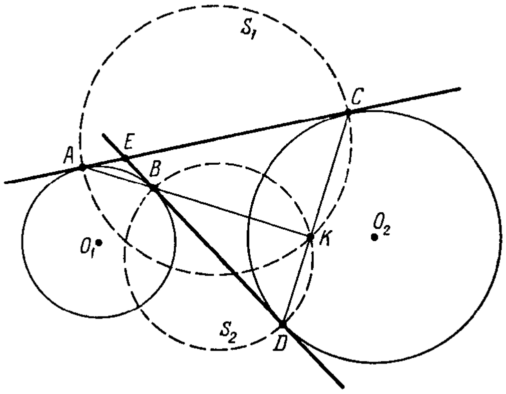


Figure 35 (Sol. 3.63)

Since $AB \perp O_1E$, $O_1E \perp O_2E$ and $O_2E \perp CD$, it follows that $AB \perp CD$ and, therefore, K is the intersection point of circles S_1 and S_2 with diameters AC and BD, respectively. Point K lies on the radical axis of circles S_1 and S_2; it remains to verify that line O_1O_2 is this radical axis. The radii O_1A and O_1B are tangent to S_1 and S_2, respectively, and, therefore, point O_1 lies on the radical axis. Similarly, point O_2 also lies on the radical axis.

3.64. Denote the given circles by $S_1, \dots, S_n$. For each circle S_i consider the set M_i that consists of all the points X whose degree with respect to S_i does not exceed their degrees with respect to the other circles $S_1, \dots, S_{i-1}, S_{i+1}, \dots, S_n$.

The set M_i is a convex one. Indeed, let M_{ij} be the set of points X whose degree with respect to S_i does not exceed the degree with respect to S_j. The set M_{ij} is a half plane that consists of the points that lie on the same side of the radical axis of circles S_i and S_j as S_i does. The set M_i is the intersection of the convex sets M_{ij} for all j and, therefore, is a convex set itself. Moreover, since each of the sets M_{ij} contains circle S_i, then M_i also contains S_i. Since for each point of the plane at least one of the degrees with respect to $S_1, \dots, S_n$ is the least one, the sets M_i cover the whole plane.

Now, by considering the parts of the sets M_i that lie inside the initial polygon we get the partition statement desired.

3.65. a) Points B_1 and C_1 lie on the circle with diameter BC and, therefore, the degrees of point A' with respect to the circumscribed circles of triangles $A_1B_1C_1$ and ABC are equal to the degrees of point A' with respect to this circle. This means that point A' lies on the radical axis of the Euler circle and the circumscribed circle of triangle ABC. For points B' and C' the proof is similar.

b) Let us consider triangle $A_1B_1C_1$ formed by the outer bisectors of triangle ABC (triangle $A_1B_1C_1$ is an acute one). Thanks to heading a) points A', B' and C' lie on the radical axis of the circumscribed circles of triangles ABC and $A_1B_1C_1$. The radical axis of these circles is perpendicular to the line that connects their centers, i.e., the Euler line of triangle $A_1B_1C_1$. It remains to notice that the intersection point of the heights of triangle $A_1B_1C_1$ is the intersection point of the bisectors of triangle ABC, cf. Problem 1.56 a).

3.66. Let a convex hexagon $ABCDEF$ be tangent to the circle at points R, Q, T, S, P, U (point R lies on AB, point Q lies on BC, etc.).

Take a number $a > 0$ and construct points Q' and P' on lines BC and EF so that $QQ' = PP' = a$ and vectors $\overrightarrow{QQ'}$ and $\overrightarrow{PP'}$ are codirected with vectors $\overrightarrow{CB}$ and $\overrightarrow{EF}$.

Let us similarly construct points R', S', T', U' (see Fig. 36, where $RR' = SS' = TT' = UU' = a$). Let us construct circle S_1 tangent to lines BC and EF at points Q' and P'. Let us similarly construct circles S_2 and S_3.

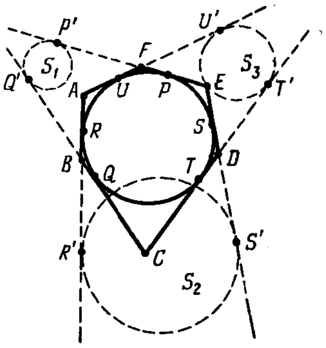


Figure 36 (Sol. 3.66)

Let us prove that points B and E lie on the radical axis of circles S_1 and S_2. We have

$$BQ' = QQ' - BQ = RR' - BR = BR'$$

(if $QQ' < BQ$, then $BQ' = BQ - QQ' = BR - RR' = BR'$) and

$$EP' = EP + PP' = ES + SS' = ES'.$$

We similarly prove that lines FC and AD are the radical axes of circles S_1 and S_3, S_2 and S_3, respectively. Since the radical axes of three circles meet at one point, lines AD, BE and CF meet at one point.

3.67. Let A_i be the tangent point of circles S_i and S_{i+1} and X be the intersection point of lines A_1A_4 and A_2A_3. Then X is the intersection point of the common outer tangents to circles S_2 and S_4 (cf. Problem 5.60). Since quadrilateral $A_1A_2A_3A_4$ is an inscribed one (by Problem 3.22), $XA_1 \cdot XA_4 = XA_2 \cdot XA_3$; consequently, point X lies on the radical axis of circles S_1 and S_3.

3.68. a) Let us consider the coordinate system whose origin O is at the center of the segment that connects the centers of the circles and the Ox-axis is directed along this segment. Let (x, y) be the coordinates of point P; let R and r be the radii of circles S_1 and S_2, respectively; $a = \frac{1}{2}d$. Then $(x+a)^2 + y^2 = R^2$ and

$$p = (x-a)^2 + y^2 - r^2 = ((x+a)^2 + y^2 - R^2) - 4ax - r^2 + R^2 = R^2 - r^2 - 4ax.$$

Let (x_0, y_0) be the coordinates of point A. Then

$$(x_0 + a)^2 + y_0^2 - R^2 = (x_0 - a)^2 + y_0^2 - r^2, \quad \text{i.e., } x_0 = \frac{R^2 - r^2}{4a}.$$

Therefore,

$$2dh = 4a|x_0 - x| = |R^2 - r^2 - 4ax| = |p|.$$

b) Let d be the distance between the centers of the circumscribed circles of triangles ACD and BCD; let h_a and h_b be the distances from points A and B to line CD. By heading a) $|p_a| = 2dh_a$ and $|p_b| = 2dh_b$. Taking into account that $S_{BCD} = \frac{1}{2}h_bCD$ and $S_{ACD} = \frac{1}{2}h_aCD$ we get the statement desired.

CHAPTER 4. AREA

Background

1. One can calculate the area S of triangle ABC with the help of the following formulas:

a) $S = \frac{1}{2}ah_a$, where $a = BC$ and h_a is the length of the height dropped to BC;

b) $S = \frac{1}{2}bc\sin\angle A$, where b, c are sides of the triangle, $\angle A$ the angle between these sides;

c) $S = pr$, where p is a semiperimeter, r the radius of the inscribed circle. Indeed, if O is the center of the inscribed circle, then

$$S = S_{ABO} + S_{AOC} + S_{OBC} = \frac{1}{2}(c+b+a)r = pr.$$

2. If a polygon is cut into several polygons, then the sum of their areas is equal to the area of the initial polygon.

Introductory problems

1. Prove that the area of a convex quadrilateral is equal to $\frac{1}{2}d_1d_2\sin\varphi$, where d_1 and d_2 are the lengths of the diagonals and φ is the angle between them.

2. Let E and F be the midpoints of sides BC and AD of parallelogram $ABCD$. Find the area of the quadrilateral formed by lines AE, ED, BF and FC if it is known that the area of $ABCD$ is equal to S.

3. A polygon is circumscribed about a circle of radius r. Prove that the area of the polygon is equal to pr, where p is the semiperimeter of the polygon.

4. Point X is inside parallelogram $ABCD$. Prove that $S_{ABX} + S_{CDX} = S_{BCX} + S_{ADX}$.

5. Let A_1, B_1, C_1 and D_1 be the midpoints of sides CD, DA, AB, BC, respectively, of square $ABCD$ whose area is equal to S. Find the area of the quadrilateral formed by lines AA_1, BB_1, CC_1 and DD_1.

§1. A median divides the triangle into triangles of equal areas

4.1. Prove that the medians divide any triangle into six triangles of equal area.

4.2. Given triangle ABC, find all points P such that the areas of triangles ABP, BCP and ACP are equal.

4.3. Inside given triangle ABC find a point O such that the areas of triangles BOL, COM and AON are equal (points L, M and N lie on sides AB, BC and CA so that $OL \parallel BC$, $OM \parallel AC$ and $ON \parallel AB$; see Fig. 37).

4.4. On the extensions of the sides of triangle ABC points A_1, B_1 and C_1 are taken so that $\overrightarrow{AB_1} = 2\overrightarrow{AB}$, $\overrightarrow{BC_1} = 2\overrightarrow{BC}$ and $\overrightarrow{CA_1} = 2\overrightarrow{AC}$. Find the area of triangle $A_1B_1C_1$ if it is known that the area of triangle ABC is equal to S.

4.5. On the extensions of sides DA, AB, BC, CD of convex quadrilateral $ABCD$ points A_1, B_1, C_1, D_1 are taken so that $\overrightarrow{DA_1} = 2\overrightarrow{DA}$, $\overrightarrow{AB_1} = 2\overrightarrow{AB}$, $\overrightarrow{BC_1} = 2\overrightarrow{BC}$ and $\overrightarrow{CD_1} = 2\overrightarrow{CD}$. Find the area of the obtained quadrilateral $A_1B_1C_1D_1$ if it is known that the area of quadrilateral $ABCD$ is equal to S.

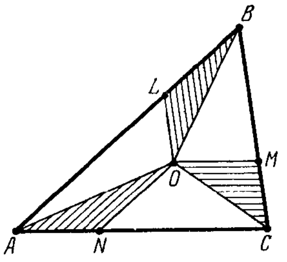


Figure 37 (4.3)

4.6. Hexagon $ABCDEF$ is inscribed in a circle. Diagonals AD, BE and CF are diameters of this circle. Prove that $S_{ABCDEF} = 2S_{ACE}$.

4.7. Inside a convex quadrilateral $ABCD$ there exists a point O such that the areas of triangles OAB, OBC, OCD and ODA are equal. Prove that one of the diagonals of the quadrilateral divides the other diagonal in halves.

§2. Calculation of areas

4.8. The height of a trapezoid whose diagonals are mutually perpendicular is equal to 4. Find the area of the trapezoid if it is known that the length of one of its diagonals is equal to 5.

4.9. Each diagonal of convex pentagon $ABCDE$ cuts off it a triangle of unit area. Calculate the area of pentagon $ABCDE$.

4.10. In a rectangle $ABCD$ there are inscribed two distinct rectangles with a common vertex K lying on side AB. Prove that the sum of their areas is equal to the area of rectangle $ABCD$.

4.11. In triangle ABC, point E is the midpoint of side BC, point D lies on side AC; let $AC = 1$, $\angle BAC = 60°$, $\angle ABC = 100°$, $\angle ACB = 20°$ and $\angle DEC = 80°$ (Fig. 38). Find $S_{\triangle ABC} + 2S_{\triangle CDE}$.

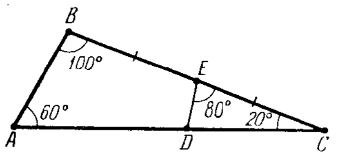


Figure 38 (4.11)

4.12. Triangle $T_a = \triangle A_1A_2A_3$ is inscribed in triangle $T_b = \triangle B_1B_2B_3$ and triangle T_b is inscribed in triangle $T_c = \triangle C_1C_2C_3$ so that the sides of triangles T_a and T_c are pairwise parallel. Express the area of triangle T_b in terms of the areas of triangles T_a and T_c.

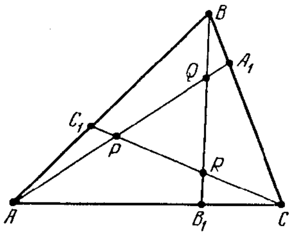


Figure 39 (4.12)

4.13. On sides of triangle ABC, points A_1, B_1 and C_1 that divide its sides in ratios $BA_1 : A_1C = p$, $CB_1 : B_1A = q$ and $AC_1 : C_1B = r$, respectively, are taken. The intersection points of segments AA_1, BB_1 and CC_1 are situated as depicted on Fig. 39. Find the ratio of areas of triangles PQR and ABC.

§3. The areas of the triangles into which a quadrilateral is divided

4.14. The diagonals of quadrilateral $ABCD$ meet at point O. Prove that $S_{AOB} = S_{COD}$ if and only if $BC \parallel AD$.

4.15. a) The diagonals of convex quadrilateral $ABCD$ meet at point P. The areas of triangles ABP, BCP, CDP are known. Find the area of triangle ADP.

b) A convex quadrilateral is divided by its diagonals into four triangles whose areas are expressed in integers. Prove that the product of these integers is a perfect square.

4.16. The diagonals of quadrilateral $ABCD$ meet at point P and $S^2_{ABP} + S^2_{CDP} = S^2_{BCP} + S^2_{ADP}$. Prove that P is the midpoint of one of the diagonals.

4.17. In a convex quadrilateral $ABCD$ there are three inner points P_1, P_2, P_3 not on one line and with the property that

$$S_{ABP_i} + S_{CDP_i} = S_{BCP_i} + S_{ADP_i}$$

for $i = 1$, 2, 3. Prove that $ABCD$ is a parallelogram.

§4. The areas of the parts into which a quadrilateral is divided

4.18. Let K, L, M and N be the midpoints of sides AB, BC, CD and DA, respectively, of convex quadrilateral $ABCD$; segments KM and LN intersect at point O. Prove that

$$S_{AKON} + S_{CLOM} = S_{BKOL} + S_{DNOM}.$$

4.19. Points K, L, M and N lie on sides AB, BC, CD and DA, respectively, of parallelogram $ABCD$ so that segments KM and LN are parallel to the sides of the parallelogram. These segments meet at point O. Prove that the areas of parallelograms $KBLO$ and $MDNO$ are equal if and only if point O lies on diagonal AC.

4.20. On sides AB and CD of quadrilateral $ABCD$, points M and N are taken so that $AM : MB = CN : ND$. Segments AN and DM meet at point K, and segments BN and CM meet at point L. Prove that $S_{KMLN} = S_{ADK} + S_{BCL}$.

4.21. On side AB of quadrilateral $ABCD$, points A_1 and B_1 are taken, on side CD points C_1 and D_1 are taken so that $AA_1 = BB_1 = pAB$ and $CC_1 = DD_1 = pCD$, where $p < 0.5$. Prove that $\frac{S_{A_1B_1C_1D_1}}{S_{ABCD}} = 1 - 2p$.

4.22. Each of the sides of a convex quadrilateral is divided into five equal parts and the corresponding points of the opposite sides are connected as on Fig. 40. Prove that the area of the middle (shaded) quadrilateral is 25 times smaller than the area of the initial quadrilateral.

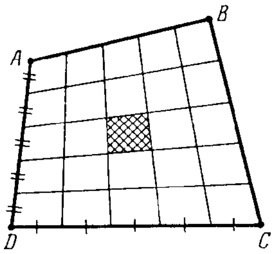


Figure 40 (4.22)

4.23. On each side of a parallelogram a point is taken. The area of the quadrilateral with vertices at these points is equal to half the area of the parallelogram. Prove that at least one of the diagonals of the quadrilateral is parallel to a side of the parallelogram.

4.24. Points K and M are the midpoints of sides AB and CD, respectively, of convex quadrilateral $ABCD$, points L and N lie on sides BC and AD so that $KLMN$ is a rectangle. Prove that $S_{ABCD} = S_{KLMN}$.

4.25. A square is divided into four parts by two perpendicular lines whose intersection point lies inside the square. Prove that if the areas of three of these parts are equal, then the area of all four parts are equal.

§5. Miscellaneous problems

4.26. Given parallelogram $ABCD$ and a point M, prove that

$$S_{ACM} = |S_{ABM} \pm S_{ADM}|.$$

4.27. On sides AB and BC of triangle ABC, parallelograms are constructed outwards; let P be the intersection point of the extensions of the sides of these parallelograms parallel to AB and BC. On side AC, a parallelogram is constructed whose other side is equal and parallel to BP. Prove that the area of this parallelogram is equal to the sum of areas of the first two parallelograms.

4.28. Point O inside a regular hexagon is connected with the vertices. The six triangles obtained in this way are alternately painted red and blue. Prove that the sum of areas of red triangles is equal to the sum of areas of blue ones.

4.29. The extensions of sides AD and BC of convex quadrilateral $ABCD$ meet at point O; let M and N be the midpoints of sides AB and CD; let P and Q be the midpoints of diagonals AC and BD. Prove that:

a) $S_{PMQN} = \frac{1}{2}|S_{ABD} - S_{ACD}|$;

b) $S_{OPQ} = \frac{1}{2}S_{ABCD}$.

4.30. On sides AB and CD of a convex quadrilateral $ABCD$ points E and F are taken. Let K, L, M and N be the midpoints of segments DE, BF, CE and AF, respectively. Prove that quadrilateral $KLMN$ is a convex one and its area does not depend on the choice of points E and F.

4.31. The midpoints of diagonals AC, BD, $CE, \dots$ of convex hexagon $ABCDEF$ are vertices of a convex hexagon. Prove that the area of the new hexagon is $\frac{1}{4}$ of that of the initial one.

4.32. The diameter PQ and the chord RS perpendicular to it intersect in point A. Point C lies on the circle, point B lies inside the circle and we know that $BC \parallel PQ$ and $BC = RA$. From points A and B perpendiculars AK and BL are dropped to line CQ. Prove that $S_{ACK} = S_{BCL}$.

* * *

4.33. Through point O inside triangle ABC segments are drawn parallel to its sides (Fig. 41). Segments AA_1, BB_1 and CC_1 divide triangle ABC into four triangles and three quadrilaterals. Prove that the sum of areas of the triangles adjacent to vertices A, B and C is equal to the area of the fourth triangle.

4.34. On the bisector of angle $\angle A$ of triangle ABC a point A_1 is taken so that $AA_1 = p - a = \frac{1}{2}(b + c - a)$ and through point A_1 line l_a perpendicular to the bisector is drawn. If we similarly construct lines l_b and l_c, then triangle ABC will be divided into parts among

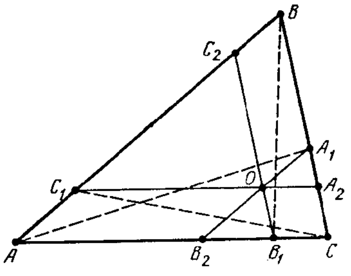


Figure 41 (4.33)

which there are four triangles. Prove that the area of one of these triangles is equal to the sum of areas of the three other triangles.

See also problems 3.38–3.41, 13.52–13.56, 16.5, 24.5.

§6. Lines and curves that divide figures into parts of equal area

4.35. Segment MN parallel to side CD of quadrilateral $ABCD$ divides its area in halves (points M and N lie on sides BC and AD). The lengths of segments drawn from points A and B parallel to CD till they intersect with lines BC and AD are equal to a and b, respectively. Prove that $MN^2 = \frac{1}{2}(ab + c^2)$, where $c = CD$.

4.36. Each of certain three lines divides the area of a figure in halves. Prove that the area of the part of the figure confined inside the triangle formed by these lines does not exceed $\frac{1}{4}$ of the area of the whole figure.

4.37. Line l divides the area of a convex polygon in halves. Prove that this line divides the projection of the given polygon onto a line perpendicular to l in the ratio that does not exceed $1 + \sqrt{2}$.

4.38. Prove that any convex polygon can be cut by two mutually perpendicular lines in four figures of equal area.

4.39. a) Prove that any line that divides the area and the perimeter of the triangle in halves passes through the center of the inscribed circle.

b) Prove a similar statement for any circumscribed polygon.

4.40. Points A and B of circle S_1 are connected by an arc of circle S_2 that divides the area of the disk bounded by S_1 into equal parts. Prove that the length of the arc of S_2 that connects A and B is greater than that of the diameter of S_1.

4.41. Curve Γ divides a square into two parts of equal area. Prove that on Γ we can select two points A and B so that line AB passes through the center O of the square.

See also problems 6.51, 6.52, 16.8, 18.29.

§7. Formulas for the area of a quadrilateral

4.42. The diagonals of quadrilateral $ABCD$ meet at point P. The distances from points A, B and P to line CD are equal to a, b and p, respectively. Prove that the area of quadrilateral $ABCD$ is equal to $\frac{ab\cdot CD}{2p}$.

4.43. Quadrilateral $ABCD$ is inscribed into a circle of radius R; let φ be the angle between the diagonals of $ABCD$. Prove that the area S of $ABCD$ is equal to $2R^2 \cdot \sin\angle A \cdot \sin\angle B \cdot \sin\varphi$.

4.44. Prove that the area of a quadrilateral whose diagonals are not perpendicular is equal to $\frac{1}{4}\tan\varphi \cdot |a^2+c^2-b^2-d^2|$, where a, b, c and d are the lengths of the consecutive sides and φ is the angle between the diagonals.

4.45. a) Prove that the area of a convex quadrilatral $ABCD$ can be computed with the help of the formula

$$S^2 = (p-a)(p-b)(p-c)(p-d) - abcd\cos^2\left(\frac{\angle B + \angle D}{2}\right),$$

where p is the semiperimeter, a, b, c, d are the lengths of the quadrilateral's sides.

b) Prove that if quadrilateral $ABCD$ is an inscribed one, then

$$S^2 = (p-a)(p-b)(p-c)(p-d).$$

c) Prove that if quadrilateral $ABCD$ is a circumscribed one, then

$$S^2 = abcd\sin^2\left(\frac{\angle B + \angle D}{2}\right).$$

See also Problem 11.34.

§8. An auxiliary area

4.46. Prove that the sum of distances from an arbitrary point within an equilateral triangle to the triangle's sides is constant (equal to the length of the triangle's height).

4.47. Prove that the length of the bisector AD of triangle ABC is equal to $\frac{2bc}{b+c}\cos\frac{1}{2}\alpha$.

4.48. Inside triangle ABC, point O is taken; lines AO, BO and CO meet the sides of the triangle at points A_1, B_1 and C_1, respectively. Prove that:

a) $\frac{OA_1}{AA_1} + \frac{OB_1}{BB_1} + \frac{OC_1}{CC_1} = 1$;

b) $\frac{AC}{CB} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1$.

4.49. A $(2n-1)$-gon $A_1 \dots A_{2n-1}$ and a point O are given. Lines A_kO and $A_{n+k-1}A_{n+k}$ meet at point B_k. Prove that the product of ratios $\frac{A_{n+k-1}B_k}{A_{n+k}B_k}$ for $k = 1, \dots, n$ is equal to 1.

4.50. A convex polygon $A_1A_2\dots A_n$ is given. On side A_1A_2 points B_1 and D_2 are taken, on side A_2A_3 points B_2 and D_3, etc. so that if we construct parallelograms $A_1B_1C_1D_1, \dots, A_nB_nC_nD_n$, then lines $A_1C_1, \dots, A_nC_n$ would meet at one point O. Prove that

$$A_1B_1 \cdot A_2B_2 \cdot \dots \cdot A_nB_n = A_1D_1 \cdot A_2D_2 \cdot \dots \cdot A_nD_n.$$

4.51. The lengths of the sides of a triangle form an arithmetic progression. Prove that the (length of the) radius of the inscribed circle is equal to one third of the length of one of the triangle's heights.

4.52. The distances from point X on side BC of triangle ABC to lines AB and AC are equal to d_b and d_c, respectively. Prove that $\frac{d_b}{d_c} = \frac{BX \cdot AC}{CX \cdot AB}$.

4.53. A polygon circumscribed about a circle of radius r is divided into triangles (in an arbitrary way). Prove that the sum of radii of the inscribed circles of these triangles is greater than r.

4.54. Through point M inside parallelogram $ABCD$ lines PR and QS parallel to sides BC and AB are drawn (points P, Q, R and S lie on sides AB, BC, CD and DA, respectively). Prove that lines BS, PD and MC meet at one point.

4.55. Prove that if no side of a quadrilateral is parallel to any other side, then the midpoint of the segment that connects the intersection points of the opposite sides lies on the line that connects the midpoints of the diagonals. (The *Gauss line*.)

4.56. In an acute triangle ABC heights BB_1 and CC_1 are drawn and points K and L are taken on sides AB and AC so that $AK = BC_1$ and $AL = CB_1$. Prove that line AO, where O is the center of the circumscribed circle of triangle ABC, divides segment KL in halves.

4.57. Medians AA_1 and CC_1 of triangle ABC meet at point M. Prove that if quadrilateral A_1BC_1M is a circumscribed one, then $AB = BC$.

4.58. Inside triangle ABC a point O is taken. Denote the distances from O to sides BC, CA, AB of the triangle by d_a, d_b, d_c, respectively, and the distances from point O to vertices A, B, C by R_a, R_b, R_c, respectively. Prove that:

a) $aR_a \geq cd_c + bd_b$;

b) $d_aR_a + d_bR_b + d_cR_c \geq 2(d_ad_b + d_bd_c + d_cd_a)$;

c) $R_a + R_b + R_c \geq 2(d_a + d_b + d_c)$;

d) $R_aR_bR_c \geq \frac{R}{2r}(d_a + d_b)(d_b + d_c)(d_c + d_a)$.

See also problems 5.5, 10.6.

§9. Regrouping areas

4.59. Prove that the area of a regular octagon is equal to the product of the lengths of its greatest and smallest diagonals.

4.60. From the midpoint of each side of an acute triangle perpendiculars are dropped to two other sides. Prove that the area of the hexagon bounded by these perpendiculars is equal to a half area of the initial triangle.

4.61. Sides AB and CD of parallelogram $ABCD$ of unit area are divided into n equal parts; sides AD and BC are divided into m equal parts. The division points are connected as indicated on a) Fig. 42 a); b) Fig. 42 b).

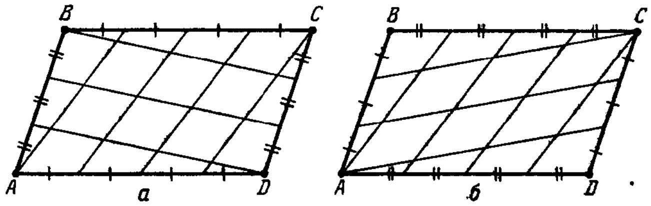


Figure 42 (4.61)

What are the areas of small parallelograms obtained in this way?

4.62. a) Four vertices of a regular 12-gon lie in the midpoints of a square (Fig. 43). Prove that the area of the shaded part is equal to $\frac{1}{12}$ that of the 12-gon.

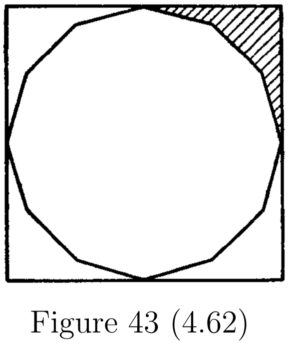


Figure 43 (4.62)

b) Prove that the area of a 12-gon inscribed in the unit circle is equal to 3.

Problems for independent study

4.63. The sides of an inscribed quadrilateral $ABCD$ satisfy the relation $AB \cdot BC = AD \cdot DC$. Prove that the areas of triangles ABC and ADC are equal.

4.64. Is it possible to use two straight cuts passing through two vertices of a triangle to divide the triangle into four parts so that three triangles (of these parts) were of equal area?

4.65. Prove that all the convex quadrilaterals with common midpoints of sides are of equal area.

4.66. Prove that if two triangles obtained by extention of sides of a convex quadrilateral to their intersection are of equal area, then one of the diagonals divides the other one in halves.

4.67. The area of a triangle is equal to S, its perimeter is equal to P. Each of the lines on which the sides of the triangle lie are moved (outwards) by a distance of h. Find the area and the perimeter of the triangle formed by the three obtained lines.

4.68. On side AB of triangle ABC, points D and E are taken so that $\angle ACD = \angle DCE = \angle ECB = \varphi$. Find the ratio $CD : CE$ if the lengths of sides AC and BC and angle φ are known.

4.69. Let AA_1, BB_1 and CC_1 be the bisectors of triangle ABC. Prove that

$$\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{2abc}{(a+b)\cdot(b+c)\cdot(c+a)}.$$

4.70. Points M and N are the midpoints of lateral sides AB and CD of trapezoid $ABCD$. Prove that if the doubled area of the trapezoid is equal to $AN \cdot NB + CM \cdot MD$, then $AB = CD = BC + AD$.

4.71. If a quadrilateral with sides od distinct lengths is inscribed into a circle of radius R, then there exist two more quadrilaterals not equal to it with the same lengths of sides inscribed in the same circle. These quadrilaterals have not more than three distinct lengths of diagonals: d_1, d_2 and d_3. Prove that the area of the quadrilateral is equal to $\frac{d_1d_2d_3}{4R}$.

4.72. On sides AB, BC and CA of triangle ABC points C_1, A_1 and B_1 are taken; points C_2, A_2 and B_2 are symmetric to these points through the midpoints of the corresponding sides. Prove that $S_{A_1B_1C_1} = S_{A_2B_2C_2}$.

4.73. Inside triangle ABC, point P is taken. The lines that pass through P and vertices of the triangle intersect the sides at points A_1, B_1 and C_1. Prove that the area of the triangle determined by the midpoints of segments AA_1, BB_1 and CC_1 is equal to $\frac{1}{4}$ of the area of triangle $A_1B_1C_1$.

Solutions

4.1. The triangles adjacent to one side have equal bases and a common height and, therefore, are of equal area. Let M be the intersection point of the medians of triangle ABC. Line BM divides each of the triangles ABC and AMC into two triangles of equal area; consequently, $S_{ABM} = S_{BCM}$. Similarly, $S_{BCM} = S_{CAM}$.

4.2. The equality of areas of triangles ABP and BCP implies that the distances from points A and C to line BP are equal. Therefore, either line BP passes through the midpoint of segment AC or it is parallel to it. The points to be found are depicted on Fig. 44.

4.3. Denote the intersection point of line LO with side AC by L_1. Since $S_{LOB} = S_{MOC}$ and $\triangle MOC = \triangle L_1OC$, it follows that $S_{LOB} = S_{L_1CO}$. The heights of triangles LOB and L_1OC are equal and, therefore, $LO = L_1O$, i.e., point O lies on the median drawn from vertex A.

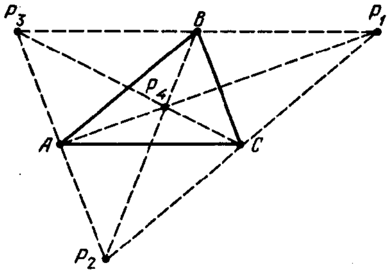


Figure 44 (Sol. 4.2)

We similarly prove that point O lies on the medians drawn from vertices B and C, i.e., O is the intersection point of the medians of the triangle. These arguments also demonstrate that the intersection point of the medians of the triangle possesses the necessary property.

4.4. Since $S_{A_1BB_1} = S_{A_1AB} = S_{ABC}$, it follows that $S_{AA_1B_1} = 2S$. Similarly, $S_{BB_1C_1} = S_{CC_1A_1} = 2S$. Therefore, $S_{ABC} = 7S$.

4.5. Since $AB = BB_1$, it follows that $S_{BB_1C} = S_{BAC}$. Since $BC = CC_1$, we have $S_{B_1C_1C} = S_{BB_1C} = S_{BAC}$ and $S_{BB_1C_1} = 2S_{BAC}$. Similarly, $S_{DD_1A_1} = 2S_{ACD}$ and, consequently,

$$S_{BB_1C_1} + S_{DD_1A_1} = 2S_{ABC} + 2S_{ACD} = 2S_{ABCD}.$$

Similarly, $S_{AA_1B_1} + S_{CC_1D_1} = 2S_{ABCD}$, consequently,

$$S_{A_1B_1C_1D_1} = S_{ABCD} + S_{AA_1B_1} + S_{BB_1C_1} + S_{CC_1D_1} + S_{DD_1A_1} = 5S_{ABCD}.$$

4.6. Let O be the center of the circumscribed circle. Since AD, BE and CF are diameters,

$$S_{ABO} = S_{DEO} = S_{AEO},\quad S_{BCO} = S_{EFO} = S_{CEO},\quad S_{CDO} = S_{AFO} = S_{ACO}.$$

It is also clear that $S_{ABCDEF} = 2(S_{ABO} + S_{BCO} + S_{CDO})$ and $S_{ACE} = S_{AEO} + S_{CEO} + S_{ACO}$. Therefore, $S_{ABCDEF} = 2S_{ACE}$.

4.7. Let E and F be the midpoints of diagonals AC and BD, respectively. Since $S_{AOB} = S_{AOD}$, point O lies on line AF. Similarly, point O lies on line CF. Suppose that the intersection point of the diagonals is not the midpoint of either of them. Then the lines AF and CF have a unique common point, F; hence, $O = F$. We similarly prove that $O = E$. Contradiction.

4.8. Let the length of diagonal AC of trapezoid $ABCD$ with base AD be equal to 5. Let us complement triangle ACB to parallelogram $ACBE$. The area of trapezoid $ABCD$ is equal to the area of the right triangle DBE. Let BH be a height of triangle DBE. Then $EH^2 = BE^2 - BH^2 = 5^2 - 4^2 = 3^2$ and $ED = \frac{BE^2}{EH} = \frac{25}{3}$. Therefore, $S_{DBE} = \frac{1}{2}ED \cdot BH = \frac{50}{3}$.

4.9. Since $S_{ABE} = S_{ABC}$, it follows that $EC \parallel AB$. The remaining diagonals are also parallel to the corresponding sides. Let P be the intersection point of BD and EC. If $S_{BPC} = x$, then

$$S_{ABCDE} = S_{ABE} + S_{EPB} + S_{EDC} + S_{BPC} = 3 + x.$$

(we have $S_{EPB} = S_{ABE} = 1$ because $ABPE$ is a parallelogram). Since $S_{BPC} : S_{DPC} = BP : DP = S_{EPB} : S_{EPD}$, it follows that $x : (1 - x) = 1 : x$ and, therefore, $x = \frac{\sqrt{5}-1}{2}$ and $S_{ABCDE} = \frac{\sqrt{5}+5}{2}$.

4.10. The centers of all the three rectangles coincide (see Problem 1.7) and, therefore, two smaller rectangles have a common diagonal, KL. Let M and N be the vertices of these rectangles that lie on side BC. Points M and N lie on the circle with diameter KL. Let O be the center of the circle, O_1 the projection of O to BC. Then $BO_1 = CO_1$ and $MO_1 = NO_1$ and, therefore, $BM = NC$. To prove that $S_{KLM} + S_{KLN} = S_{KBCL}$ it suffices to verify that

$$(S_{KBM} + S_{LCM}) + (S_{KBN} + S_{LCN}) = S_{KBCL} = \frac{1}{2}BC(KB + CL) = \frac{1}{2}BC \cdot AB.$$

It remains to observe that

$$\begin{gathered} KB \cdot BM + KB \cdot BN = KB \cdot BC, \\ LC \cdot CM + LC \cdot CN = LC \cdot BC, \\ KB \cdot BC + LC \cdot BC = AB \cdot BC. \end{gathered}$$

4.11. Let us drop perpendicular l from point C to line AB. Let points A', B' and E' be symmetric to points A, B and E, respectively, through line l. Then triangle $AA'C$ is an equilateral one and $\angle ACB = \angle BCB' = \angle B'CA' = 20^\circ$. Triangles $EE'C$ and DEC are isosceles ones with the angle of 20° at the vertex and a common lateral side EC. Therefore, $S_{ABC} + 2S_{EDC} = S_{ABC} + 2S_{EE'C}$. Since E is the midpoint of BC, it follows that $2S_{EE'C} = S_{BE'C} = \frac{1}{2}S_{BB'C}$. Hence,

$$S_{ABC} + 2S_{EDC} = \frac{S_{AA'C}}{2} = \frac{\sqrt{3}}{8}.$$

4.12. Let the areas of triangles T_a, T_b and T_c be equal to a, b and c, respectively. Triangles T_a and T_c are homothetic and, therefore, the lines that connect their respective vertices meett at one point, O. The similarity coefficient k of these triangles is equal to $\sqrt{\frac{a}{c}}$. Clearly, $S_{A_1B_3O} : S_{C_1B_3O} = A_1O : C_1O = k$. Writing similar equations for $??? and adding them, we get $a : b = k$ and, therefore, $b = \sqrt{ac}$.

4.13. Making use of the result of Problem 1.3 it is easy to verify that

$$\begin{aligned} \frac{BQ}{BB_1} &= \frac{p + pq}{1 + p + pq}, \qquad & \frac{B_1R}{BB_1} &= qr1 + q + qr, \\ \frac{CR}{CC_1} &= \frac{q + qr}{1 + q + qr}, \qquad & \frac{CP}{CC_1} &= \frac{pr}{1 + r + pr}. \end{aligned} \tag{1}$$

It is also clear that $\frac{S_{PQR}}{S_{RB_1C}} = \frac{QR}{RB_1} \cdot \frac{PR}{RC}$ and $\frac{S_{RB_1C}}{S_{ABC}} = \frac{B_1C}{AC} \cdot \frac{B_1R}{BB_1}$. Hence,

$$\frac{S_{PQR}}{S_{ABC}} = \frac{QR}{BB_1} \cdot \frac{PR}{RC} \cdot \frac{B_1C}{AC} = \frac{QR}{BB_1} \cdot \frac{PR}{CC_1} \cdot \frac{CC_1}{CR} \cdot \frac{B_1C}{AC}.$$

Taking into account that

$$\begin{aligned} \tfrac{QR}{BB_1} &= 1 - \tfrac{p+pq}{1+p+pq} - \tfrac{qr}{1+q+rq} = \tfrac{1}{1+p+pq} - \tfrac{rq}{1+q+rq} \\ &= \tfrac{(1+q)(1-pqr)}{(1+p+pq)(1+q+qr)} \end{aligned}$$

and

$$\frac{PR}{CC_1} = \frac{(1 + r)(1 - pqr)}{(1 + q + qr)(1 + r + pr)}$$

we get

$$\frac{S_{PQR}}{S_{ABC}} = \frac{(1 - pqr)^2}{(1 + p + pq)(1 + q + qr)(1 + r + pr)}.$$

4.14. If $S_{AOB} = S_{COD}$, then $AO \cdot BO = CO \cdot DO$. Hence, $\triangle AOD \sim \triangle COB$ and $AD \parallel BC$. These arguments are invertible.

4.15. a) Since $S_{ADP} : S_{ABP} = DP : BP = S_{CDP} : S_{BCP}$, we have

$$S_{ADP} = \frac{S_{ABP} \cdot S_{CDP}}{S_{BCP}}.$$

b) Thanks to heading a) $S_{ADP}\cdot S_{CBP}=S_{ABP}\cdot S_{CDP}$. Therefore,

$$S_{ABP}\cdot S_{CBP}\cdot S_{CDP}\cdot S_{ADP}=(S_{ADP}\cdot S_{CBP})^2.$$

4.16. After division by $\frac{1}{4}\sin^2\varphi$, where φ is the angle between the diagonals, we rewrite the given equality of the areas in the form

$$(AP\cdot BP)^2+(CP\cdot DP)^2=(BP\cdot CP)^2+(AP\cdot DP)^2,$$

i.e.,

$$(AP^2-CP^2)(BP^2-DP^2)=0.$$

4.17. Suppose that quadrilateral $ABCD$ is not a parallelogram; for instance, let lines AB and CD intersect. By Problem 7.2 the set of points P that lie inside quadrilateral $ABCD$ for which

$$S_{ABP}+S_{CDP}=S_{BCP}+S_{ADP}=\frac{1}{2}S_{ABCD}$$

is a segment. Therefore, points P_1, P_2 and P_3 lie on one line. Contradiction.

4.18. Clearly,

$$S_{AKON}=S_{AKO}+S_{ANO}=\frac{1}{2}(S_{AOB}+S_{AOD}).$$

Similarly, $S_{CLOM}=\frac{1}{2}(S_{BCO}+S_{COD})$. Hence, $S_{AKON}+S_{CLOM}=\frac{1}{2}S_{ABCD}$.

4.19. If the areas of the parallelograms $KBLO$ and $MDNO$ are equal, then $OK\cdot OL=OM\cdot ON$. Taking into account that $ON=KA$ and $OM=LC$, we get $KO:KA=LC:LO$. Therefore, $\triangle KOA\sim\triangle LCO$ which means that point O lies on diagonal AC. These arguments are invertible.

4.20. Let h_1, h and h_2 be the distances from points A, M and B to line CD, respectively. By Problem 1.1 b) we have $h=ph_2+(1-p)h_1$, where $p=\frac{AM}{AB}$. Therefore,

$$S_{DMC}=\frac{h\cdot DC}{2}=\frac{h_2p\cdot DC+h_1(1-p)\cdot DC}{2}=S_{BCN}+S_{ADN}.$$

Subtracting $S_{DKN}+S_{CLN}$ from both sides of this equality we get the desired statement.

4.21. Thanks to Problem 4.20,

$$S_{ABD_1}+S_{CDB_1}=S_{ABCD}.$$

Hence,

$$\begin{aligned}S_{A_1B_1C_1D_1}&=S_{A_1B_1D_1}+S_{C_1D_1B_1}\\&=(1-2p)S_{ABD_1}+(1-2p)S_{CDB_1}=(1-2p)S_{ABCD}.\end{aligned}$$

4.22. By Problem 4.21 the area of the middle quadrilateral of those deteremined by segments that connect points of sides AB and CD is $\frac{1}{5}$ of the area of the initial quadrilateral. Since each of the considered segments is divided by segments that connect the corresponding points of the other pair of opposite sides into 5 equal parts (see Problem 1.16). By making use once again of the result of Problem 4.21, we get the desired statement.

4.23. On sides AB, BC, CD and AD points K, L, M and N, respectively, are taken. Suppose that diagonal KM is not parallel to side AD. Fix points K, M and N and let us move point L along side BC. In accordance with this movement the area of triangle KLM varies strictly monotonously. Moreover, if $LN\parallel AB$, then the equality $S_{AKN}+S_{BKL}+S_{CLM}+S_{DMN}=\frac{1}{2}S_{ABCD}$ holds, i.e., $S_{KLMN}=\frac{1}{2}S_{ABCD}$.

4.24. Let L_1 and N_1 be the midpoints of sides BC and AD, respectively. Then KL_1MN_1 is a parallelogram and its area is equal to a half area of quadrilateral $ABCD$, cf. Problem 1.37 a). Therefore, it suffices to prove that the areas of parallelograms $KLMN$ and KL_1MN_1 are equal. If these parallelograms coincide, then there is nothing more to prove and if they do not coincide, then $LL_1\parallel NN_1$ and $BC\parallel AD$ because the midpoint of segment KM is their center of symmetry. In this case the midline KM of trapezoid $ABCD$ is parallel to bases BC and AD and therefore, heights of triangles KLM and KL_1M dropped to side KM are equal, i.e., the areas of parallelograms $KLMN$ and KL_1MN_1 are equal.

4.25. Let the given lines l_1 and l_2 divide the square into four parts whose areas are equal to S_1, S_2, S_3 and S_4 so that for the first line the areas of the parts into which it divides the square are equal to $S_1 + S_2$ and $S_3 + S_4$ and for the second line they are equal to $S_2 + S_3$ and $S_1 + S_4$. Since by assumption $S_1 = S_2 = S_3$, it follows that $S_1 + S_2 = S_2 + S_3$. This means that the image of line l_1 under the rotation about the center of the square through an angle of $+90^\circ$ or -90° is not just parallel to line l_2 but coincides with it.

It remains to prove that line l_1 (hence, line l_2) passes through the center of the square. Suppose that this is not true. Let us consider the images of lines l_1 and l_2 under rotations through an angle of $\pm 90^\circ$ and denote the areas of the parts into which they divide the square as plotted on Fig. 45 (on this figure both distinct variants of the disposition of the lines are plotted).

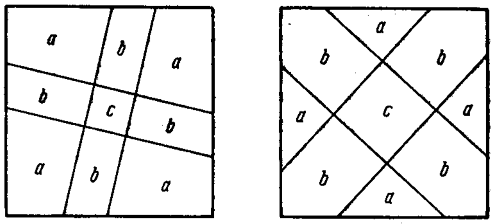


Figure 45 (Sol. 4.25)

Lines l_1 and l_2 divide the square into four parts whose areas are equal to a, $a+b$, $a+2b+c$ and $a+b$, where numbers a, b and c are nonzero. It is clear that three of the four numbers indicated cannot be equal. Contradiction.

4.26. All the three triangles considered have a common base AM. Let h_b, h_c and h_d be the distances from points B, C and D, respectively, to line AM. Since $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$, it follows that $h_c = |h_b \pm h_d|$.

4.27. We may assume that P is a common point of parallelograms constructed on sides AB and BC, i.e., these parallelograms are of the form $ABPQ$ and $CBPR$. It is clear that $S_{ACRQ} = S_{ABPQ} + S_{CBPR}$.

4.28. Let the length of the hexagon's side be equal to a. The extensions of red sides of the hexagon form an equilateral triangle with side $3a$ and the sum of areas of red triangles is equal to a half product of a by the sum of distances from point O to a side of this triangle and, therefore, it is equal to $\frac{3\sqrt{3}}{4}a^2$, cf. Problem 4.46.

The sum of areas of blue triangles is similarly calculated.

4.29. a) The area of parallelogram $PMQN$ is equal to $\frac{1}{4}BC \cdot AD \sin\alpha$, where α is the angle between lines AD and BC. The heights of triangles ABD and ACD dropped from vertices B and C are equal to $OB\sin\alpha$ and $OC\sin\alpha$, respectively; hence,

$$|S_{ABD} - S_{ACD}| = \frac{|OB - OC| \cdot AD\sin\alpha}{2} = \frac{BC \cdot AD\sin\alpha}{2}.$$

b) Let, for definiteness, rays AD and BC intersect. Since $PN \parallel AO$ and $QN \parallel CO$, point N lies inside triangle OPQ. Therefore,

$$\begin{aligned} S_{OPQ} &= S_{PQN} + S_{PON} + S_{QON} = \tfrac{1}{2}S_{PMQN} + \tfrac{1}{4}S_{ACD} + \tfrac{1}{4}S_{BCD} \\ &= \tfrac{1}{4}(S_{ABD} - S_{ACD} + S_{ACD} + S_{BCD} = \tfrac{1}{4}S_{ABCD}. \end{aligned}$$

4.30. Segments KM and LN are the midlines of triangles CED and AFB and, therefore, they have a common point — the midpoint of segment EF. Moreover, $KM = \frac{1}{2}CD$, $LN = \frac{1}{2}AB$ and the angle between lines KM and LN is equal to the angle α between lines AB and CD. Therefore, the area of quadrilateral $KLMN$ is equal to $\frac{1}{8}AB \cdot CD \sin\alpha$.

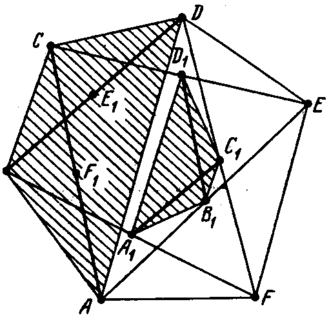


Figure 46 (Sol. 4.31)

4.31. Denote the midpoints of the diagonals of hexagon $ABCDEF$ as shown on Fig. 46. Let us prove that the area of quadrilateral $A_1B_1C_1D_1$ is $\frac{1}{4}$ of the area of quadrilateral $ABCD$. To this end let us make use of the fact that the area of the quadrilateral is equal to a half product of the lengths of the diagonals by the sine of the angle between them. Since A_1C_1 and B_1D_1 are the midlines of triangles BDF and ACE, we get the desired statement.

We similarly prove that the area of quadrilateral $D_1E_1F_1A_1$ is $\frac{1}{4}$ of the area of quadrilateral $DEFA$.

4.32. Let $\alpha = \angle PQC$. Then

$$\begin{aligned}2S_{ACK} &= CK \cdot AK = (AP\cos\alpha)\cdot(AQ\sin\alpha) = AR^2\sin\alpha\cdot\cos\alpha\\ &= BC^2\sin\alpha\cdot\cos\alpha = BL\cdot CL = 2S_{BCL}.\end{aligned}$$

4.33. Let S_a, S_b and S_c be the areas of the triangles adjacent to vertices A, B and C; let S be the area of the fourth of the triangles considered. Clearly,

$$S_{ACC_1} + S_{BAA_1} + S_{CBB_1} = S_{ABC} - S + S_a + S_b + S_c.$$

Moreover,

$$S_{ABC} = S_{AOC} + S_{AOB} + S_{BOC} = S_{ACC_1} + S_{BAA_1} + S_{CBB_1}.$$

4.34. Let O be the center of the inscribed circle of triangle ABC, let B_1 be the tangent point of the inscribed circle and side AC. Let us cut off triangle ABC triangle AOB_1 and reflect AOB_1 symmetrically through the bisector of angle OAB_1. Under this reflection line OB_1 turns into line l_a. Let us perform a similar operation for the remaining triangles. The common parts of the triangles obtained in this way are three triangles of the considered partition and the uncovered part of triangle ABC is the fourth triangle. It is also clear that the area of the uncovered part is equal to the sum of areas of the parts covered twice.

4.35. Let, for definiteness, rays AD and BC meet at point O. Then $S_{CDO} : S_{MNO} = c^2 : x^2$, where $x = MN$ and $S_{ABO} : S_{MNO} = ab : x^2$ because $OA : ON = a : x$ and $OB : OM = b : x$. It follows that $x^2 - c^2 = ab - x^2$, i.e., $2x^2 = ab + c^2$.

4.36. Denote the areas of the parts of the figure into which it is divided by lines as shown on Fig. 47. Let us denote by S the area of the whole figure. Since

$$S_3 + (S_2 + S_7) = \frac{1}{2}S = S_1 + S_6 + (S_2 + S_7),$$

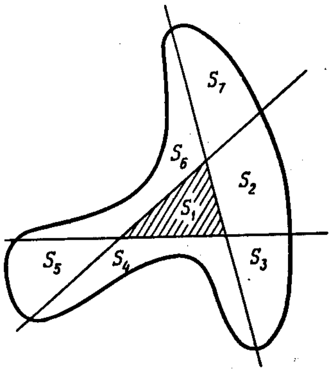


Figure 47 (Sol. 4.36)

it follows that $S_3 = S_1 + S_6$. Adding this equality to the equality $\frac{1}{2}S = S_1 + S_2 + S_3 + S_4$ we get

$$\frac{1}{2}S = 2S_1 + S_2 + S_4 + S_6 \geq 2S_1, \quad \text{i.e.,} \quad S_1 \leq \frac{1}{4}S.$$

4.37. Let us denote the projection of line l by B and the endpoints of the projection of the polygon by A and C. Let C_1 be a point of the polygon whose projection is C. Then line l intersects the polygon at points K and L; let points K_1 and L_1 be points on lines C_1K and C_1L that have point A as their projection (Fig. 48).

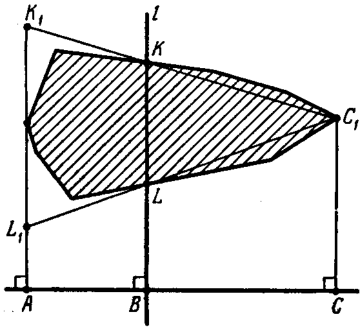


Figure 48 (Sol. 4.37)

One of the parts into which line l divides the polygon is contained in trapezoid K_1KLL_1, the other part contains triangle C_1KL. Therefore, $S_{K_1KLL_1} \geq S_{C_1KL}$, i.e., $AB \cdot (KL + K_1L_1) \geq BC \cdot KL$. Since $K_1L_1 = KL \cdot \frac{AB+BC}{BC}$, we have $AB \cdot \left(2 + \frac{AB}{BC}\right) \geq BC$. Solving this inequality we get $\frac{BC}{AB} \leq 1 + \sqrt{2}$.

Similarly, $\frac{AB}{BC} \leq 1 + \sqrt{2}$. (We have to perform the same arguments but interchange A and C.)

4.38. Let S denote the area of the polygon, l an arbitrary line. Let us introduce a coordinate system in which line l is Ox-axis. Let $S(a)$ be the area of the part of the polygon below the line $y = a$. Clearly, $S(a)$ varies continuously from 0 to S as a varies from $-\infty$ to $+\infty$ and, therefore (by Calculus, see, e.g., ??), $S(a) = \frac{1}{2}S$ for some a, i.e., the line $y = a$ divides the area of the polygon in halves.

Similarly, there exists a line perpendicular to l and this perpendicular also divides the area of the polygon in halves. These two lines divide the polygon into parts whose areas are equal to S_1, S_2, S_3 and S_4 (see Fig. 49). Since $S_1 + S_2 = S_3 + S_4$ and $S_1 + S_4 = S_2 + S_3$, we have $S_1 = S_3 = A$ and $S_2 = S_4 = B$. The rotation of line l by 90° interchanges points A

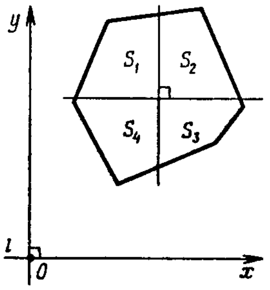


Figure 49 (Sol. 4.38)

and B. Since A and B vary continuously under the rotation of l, it follows that $A = B$ for a certain position of l, i.e., the areas of all the four figures are equal at this moment.

4.39. a) Let the line that divides the area and the perimeter of triangle ABC in halves intersect sides AC and BC at points P and Q, respectively. Denote the center of the inscribed circle of triangle ABC by O and the radius of the inscribed circle by r. Then $S_{ABQOP} = \frac{1}{2}r(AP + AB + BQ)$ and $S_{OQCP} = \frac{1}{2}r(QC + CP)$. Since line PQ divides the perimeter in halves, $AP{+}AB{+}BQ = QC{+}CP$ and, therefore, $S_{ABQOP} = S_{OQCP}$. Moreover, $S_{ABQP} = S_{QCP}$ by the hypothesis. Therefore, $S_{OQP} = 0$, i.e., line QP passes through point O.

b) Proof is carried out similarly to that of heading a).

4.40. By considering the image of circle S_2 under the symmetry through the center of circle S_1 and taking into account the equality of areas, it is possible to prove that diameter AA_1 of circle S_1 intersects S_2 at a point K distinct from A and so that $AK > A_1K$. The circle of radius KA_1 centered at K is tangent to S_1 at point A_1 and, therefore, $BK > A_1K$, i.e., $BK + KA > A_1A$. It is also clear that the sum of the lengths of segments BK and KA is smaller than the length of the arc of S_2 that connects points A and B.

4.41. The case when point O belongs to Γ is obvious; therefore, let us assume that O does not belong to Γ. Let Γ' be the image of the curve Γ under the symmetry through point O. If curves Γ and Γ' do not intersect, then the parts into which Γ divides the square cannot be of equal area. Let X be the intersection point of Γ and Γ'; let X' be symmetric to X through point O. Since under the symmetry through point O curve Γ' turns into Γ, it follows that X' belongs to Γ. Hence, line XX' is the desired one.

4.42. Let the areas of triangles APB, BPC, CPD and DPA be equal to S_1, S_2, S_3 and S_4, respectively. Then $\frac{a}{p} = \frac{S_3+S_4}{S_3}$ and $\frac{b\cdot CD}{2} = S_3 + S_2$; consequently,

$$\frac{ab \cdot CD}{2p} = \frac{(S_3 + S_4)(S_3 + S_2)}{S_3}.$$

Taking into account that $S_2S_4 = S_1S_3$ we get the desired statement.

4.43. By applying the law of sines to triangles ABC and ABD we get $AC = 2R\sin\angle B$ and $BD = 2R \cdot \sin\angle A$. Therefore,

$$S = \frac{1}{2}AC \cdot BD\sin\varphi = 2R^2\sin\angle A \cdot \sin\angle B \cdot \sin\varphi.$$

4.44. Since the area of the quadrilateral is equal to $\frac{1}{2}d_1d_2\sin\varphi$, where d_1 and d_2 are the lengths of the diagonals, it remains to verify that $2d_1d_2\cos\varphi = |a^2 + c^2 - b^2 - d^2|$. Let O be the intersection point of the diagonals of quadrilateral $ABCD$ and $\varphi = \angle AOB$. Then

$$AB^2 = AO^2 + BO^2 - 2AO \cdot OB\cos\varphi; \quad BC^2 = BO^2 + CO^2 + 2BO \cdot CO\cos\varphi.$$

Hence,

$$AB^2 - BC^2 = AO^2 - CO^2 - 2BO \cdot AC \cos\varphi.$$

Similarly,

$$CD^2 - AD^2 = CO^2 - AO^2 - 2DO \cdot AC \cos\varphi.$$

By adding these equalities we get the desired statement.

Remark. Since

$$16S^2 = 4d_1^2d_2^2 \sin^2\varphi = 4d_1^2d_2^2 - (2d_1d_2\cos\varphi)^2,$$

it follows that $16S^2 = 4d_1^2d_2^2 - (a^2 + c^2 - b^2 - d^2)^2$.

4.45. a) Let $AB = a$, $BC = b$, $CD = c$ and $AD = d$. Clearly,

$$S = S_{ABC} + S_{ADC} = \tfrac{1}{2}ab\sin\angle B + cd\sin\angle D;$$
$$a^2 + b^2 - 2ab\cos\angle B = AC^2 = c^2 + d^2 - 2cd\cos\angle D.$$

Therefore,

$$16S^2 = 4a^2b^2 - 4a^2b^2\cos^2\angle B + 8abcd\sin\angle B\sin\angle D + 4c^2d^2 - 4c^2d^2\cos^2\angle D,$$
$$(a^2 + b^2 - c^2 - d^2)^2 + 8abcd\cos\angle B\cos\angle D = 4a^2b^2 \cdot \cos^2\angle B + 4c^2d^2\cos^2\angle D.$$

By inserting the second equality into the first one we get

$$16S^2 = 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 8abcd(1 + \cos\angle B\cos\angle D - \sin\angle B\sin\angle D).$$

Clearly,

$$4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 = 16(p - a)(p - b)(p - c)(p - d);$$
$$1 + \cos\angle B\cos\angle D - \sin\angle B\sin\angle D = 2\cos^2\tfrac{\angle B + \angle D}{2}.$$

b) If $ABCD$ is an inscribed quadrilateral, then $\angle B + \angle D = 180^\circ$ and, therefore, $\cos^2\frac{\angle B+\angle D}{2} = 0$.

c) If $ABCD$ is a circumscribed quadrilateral, then $a + c = b + d$ and, therefore, $p = a + c = b + d$ and $p - a = c$, $p - b = d$, $p - c = a$, $p - d = b$. Hence,

$$S^2 = abcd\left(1 - \cos^2\frac{\angle B + \angle D}{2}\right) = abcd\sin^2\frac{\angle B + \angle D}{2}.$$

If $ABCD$ is simultaneously an inscribed and circumscribed quadrilateral, then $S^2 = abcd$.

4.46. Let us drop perpendiculars OA_1, OB_1 and OC_1 to sides BC, AC and AB, respectively, of an equilateral triangle ABC from a point O inside it. In triangle ABC, let a be the length of the side, h the length of the height. Clearly, $S_{ABC} = S_{BCO} + S_{ACO} + S_{ABO}$. Therefore, $ah = a \cdot OA_1 + a \cdot OB_1 + a \cdot OC_1$, i.e., $h = OA_1 + OB_1 + OC_1$.

4.47. Let $AD = l$. Then

$$2S_{ABD} = cl\sin\frac{\alpha}{2},\quad 2S_{ACD} = bl\sin\frac{\alpha}{2},\quad 2S_{ABD} = bc\sin\alpha.$$

Hence,

$$cl\sin\frac{\alpha}{2} + bl\sin\frac{\alpha}{2} = bc\sin\alpha = 2bc\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}.$$

4.48. a) Let the distances from points A and O to line BC be equal to h and h_1, respectively. Then $S_{OBC} : S_{ABC} = h_1 : H = OA_1 : AA_1$. Similarly, $S_{OAC} : S_{ABC} = OB_1 : BB_1$ and $S_{OAB} : S_{ABC} = OC_1 : CC_1$. By adding these equalities and taking into account that $S_{OBC} + S_{OAC} + S_{OAB} = S_{ABC}$ we get the desired statement.

b) Let the distances from points B and C to line AA_1 be equal to d_b and d_c, respectively. Then $S_{ABO} : S_{ACO} = d_b : d_c = BA_1 : A_1C$. Similarly, $S_{ACO} : S_{BCO} = AC_1 : C_1B$ and $S_{BCO} : S_{ABO} = CB_1 : B_1A$. It remains to multiply these equalities.

4.49. It is easy to verify that the ratio of the lengths of segments $A_{n+k-1}B_k$ and $A_{n+k}B_k$ is equal to the ratio of areas of triangles $A_{n+k-1}OA_k$ and A_kOA_{n+k}. By multiplying these equalities we get the desired statement.

4.50. Since $A_iB_iC_iD_i$ is a parallelogram and point O lies on the extension of its diagonal A_iC_i, it follows that $S_{A_iB_iO} = S_{A_iD_iO}$ and, therefore, $A_iB_i : A_iD_i = h_i : h_{i-1}$, where h_i is the distance from point O to side A_iA_{i+1}. It remains to multiply these equalities for $i = 1, \dots, n$.

4.51. Let the lengths of sides of triangle ABC be equal to a, b and c, where $a \leq b \leq c$. Then $2b = a + c$ and $2S_{ABC} = r(a + b + c) = 3rb$, where r is the radius of the inscribed circle. On the other hand, $2S_{ABC} = h_b b$. Therefore, $r = \frac{1}{3}h_b$.

4.52. It suffices to observe that

$$d_b \cdot AB = 2S_{AXB} = BX \cdot AX \sin\varphi,$$

where $\varphi = \angle AXB$ and $d_c \cdot AC = 2S_{AXC} = CX \cdot AX \sin\varphi$.

4.53. Let $r_1, \dots, r_n$ be the radii of the inscribed circles of the obtained triangles, let $P_1, \dots, P_n$ their perimeters and $S_1, \dots, S_n$ their areas. Let us denote the area and the perimeter of the initial polygon by S and P, respectively.

It is clear that $P_i < P$ (cf. Problem 9.27, b). Hence,

$$r_1 + \dots + r_n = 2\frac{S_1}{P_1} + \dots + 2\frac{S_n}{P_n} > 2\frac{S_1}{P} + \dots + 2\frac{S_n}{P} = 2\frac{S}{P} = r.$$

4.54. Let us draw lines Q_1S_1 and P_1R_1 parallel to lines QS and PR through the intersection point N of lines BS and CM (points P_1, Q_1, R_1 and S_1 lie on sides AB, BC, CD and DA, respectively). Let F and G be the intersection points of lines PR and Q_1S_1, P_1R_1 and QS, respectively. Since point M lies on diagonal NC of parallelogram NQ_1CR_1, it follows that $S_{FQ_1QM} = S_{MRR_1G}$ (by Problem 4.19) and, therefore, $S_{NQ_1QG} = S_{NFRR_1}$. Point N lies on diagonal BS of parallelogram $ABQS$ and, therefore, $S_{AP_1NS_1} = S_{NQ_1QG} = S_{NFRR_1}$. It follows that point N lies on diagonal PD of parallelogram $APRD$.

4.55. Let E and F be the intersection points of the extensions of sides of the given quadrilateral. Denote the vertices of the quadrilateral so that E is the intersection point of the extensions of sides AB and CD beyond points B and C and F is the intersection point of rays BC and AD. Let us complement triangles AEF and ABD to parallelograms $AERF$ and $ABLD$, respectively.

The homothety with center A and coefficient 2 sends the midpoint of the diagonal BD, the midpoint of the diagonal AC and the midpoint of segment EF to points L, C and R, respectively. Therefore, it suffices to prove that points L, C and R lie on one line. This is precisely the fact proved in the preceding problem.

4.56. It suffices to verify that $S_{AKO} = S_{ALO}$, i.e., $AO{\cdot}AL \sin\angle OAL = AO{\cdot}AK \sin\angle OAK$. Clearly,

$$AL = CB_1 = BC\cos\angle C, \quad \sin\angle OAL = \cos\angle B,$$
$$AK = BC_1 = BC\cos\angle B, \quad \sin\angle OAK = \cos\angle C.$$

4.57. Since quadrilateral A_1BC_1M is a circumscribed one, then, first, the sums of the lengths of its opposite sides are equal:

$$\frac{a}{2} + \frac{m_c}{3} = \frac{c}{2} + \frac{m_a}{3}$$

and, second, its inscribed circle is simultaneously the inscribed circle of triangles AA_1B and CC_1B. Since these triangles have equal areas, their perimeters are equal:

$$c + m_a + \frac{a}{2} = a + m_c + \frac{c}{2}.$$

By multiplying the first equality by 3 and adding to the second one we get the desired statement.

4.58. First, let us prove a general inequality that will be used in the proof of headings a)–d):

$$(*)\qquad BC_1 \cdot R_a \geq B_1K \cdot R_a + C_1L \cdot R_a = 2S_{AOB_1} + 2S_{AOC_1} = AB_1 \cdot d_c + AC_1 \cdot d_b.$$

On rays AB and AC take arbitrary points B_1 and C_1 and drop from them perpendiculars B_1K and C_1L to line AO. Since $B_1C_1 \geq B_1K + C_1L$, inequality $(*)$ follows.

a) Setting $B_1 = B$ and $C_1 = C$ we get the desired statement.

b) By multiplying both sides of the inequality $aR_a \geq cd_c + bd_b$ by $\frac{d_a}{a}$ we get

$$d_aR_a \geq \frac{c}{a}d_ad_c + \frac{b}{a}d_ad_b.$$

Taking the sum of this inequality with the similar inequalities for d_bR_b and d_cR_c and taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

c) Take points B_1 and C_1 such that $AB_1 = AC$ and $AC_1 = AB$. Then $aR_a \geq bd_c + cd_b$, i.e., $R_a \geq \frac{b}{a}d_c + \frac{c}{a}d_b$. Taking the sum of this inequality with similar inequalities for R_b and R_c and taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

d) Take points B_1 and C_1 such that $AB_1 = AC_1 = 1$; then $B_1C_1 = 2\sin\frac{1}{2}\angle A$ and, therefore, $2\sin\frac{1}{2}R_a \geq d_c + d_b$. By multiplying this inequality by similar inequalities for R_b and R_c and taking into account that $\sin\frac{1}{2}\angle A\sin\frac{1}{2}\angle B\sin\frac{1}{2}\angle C = \frac{r}{4R}$ (by Problem 12.36 a)) we get the desired statement.

4.59. Let us cut triangles off a regular octagon and replace the triangles as shown on Fig. 50. As a result we get a rectangle whose sides are equal to the longest and shortest diagonals of the octagon.

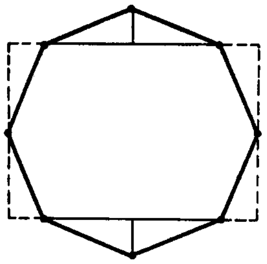


Figure 50 (Sol. 4.59)

4.60. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively, of triangle ABC. The drawn segments are heights of triangles AB_1C_1, A_1BC_1 and A_1B_1C, respectively. Let P, Q and R be the respective intersection points of the heights of these triangles and O the intersection point of the heights of triangle $A_1B_1C_1$ (Fig. 51).

The considered hexagon consists of triangle $A_1B_1C_1$ and triangles B_1C_1P, C_1A_1Q and A_1BR. Clearly, $\triangle B_1C_1P = \triangle C_1B_1O$, $\triangle C_1A_1Q = \triangle A_1C_1O$ and $\triangle A_1B_1R = \triangle B_1A_1O$. Therefore, the area of the considered hexagon is equal to the doubled area of triangle $A_1B_1C_1$. It remains to observe that $S_{ABC} = 4S_{A_1B_1C_1}$.

4.61. a) Let us cut two parts off the parallelogram (Fig. 52 a)) and replace these parts as shown on Fig. 52 b). We get a figure composed of $mn + 1$ small parallelograms. Therefore, the area of a small parallelogram is equal to $\frac{1}{mn+1}$.

b) Let us cut off the parallelogram three parts (Fig. 53 a)) and replace these parts as indicated on Fig. 53 b). We get a figure that consists of $mn - 1$ small parallelograms. Therefore, the area of a small parallelogram is equal to $\frac{1}{mn-1}$.

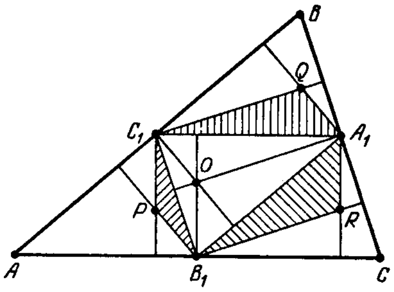


Figure 51 (Sol. 4.60)

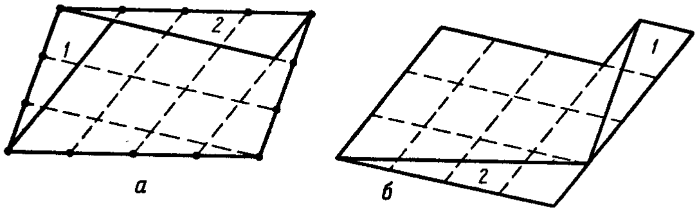


Figure 52 (Sol. 4.61 a))

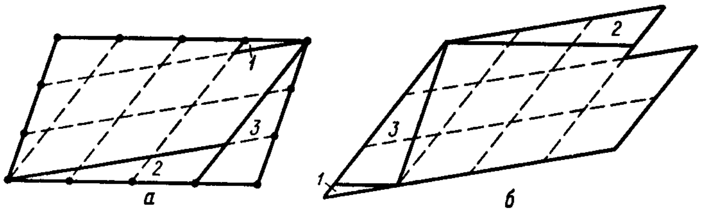


Figure 53 (Sol. 4.61 b))

4.62. a) Let us cut the initial square into four squares and consider one of them (Fig. 54). Let point B' be symmetric to point B through line PQ. Let us prove that $\triangle APB = \triangle OB'P$. Indeed, triangle APB is an isosceles one and angle at its base is equal to 15° (Problem 2.26), hence, triangle BPQ is an isosceles one. Therefore,

$$\angle OPB' = \angle OPQ - \angle B'PQ = 75^\circ - 60^\circ = 15^\circ$$

and $\angle POB' = \frac{1}{2}\angle OPQ = 15^\circ$. Moreover, $AB = OP$. We similarly prove that $\triangle BQC = \triangle OB'Q$. It follows that the area of the shaded part on Fig. 43 is equal to the area of triangle OPQ.

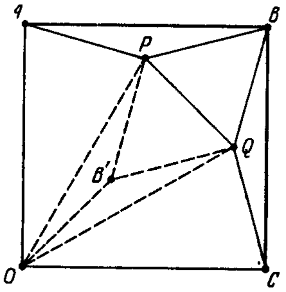


Figure 54 (Sol. 4.62)

b) Let the area of the regular 12-gon inscribed in a circle of radius 1 be equal to $12x$. Thanks to heading a) the area of the square circumscribed around this circle is equal to

$12x + 4x = 16x$; on the other hand, the area of the square is equal to 4; hence, $x = \frac{1}{4}$ and $12x = 3$.

CHAPTER 5. TRIANGLES

Background

1) The *inscribed circle* of a triangle is the circle tangent to all its sides. The *center* of an inscribed circle is the intersection point of the bisectors of the triangle's angles.

An *escribed circle* of triangle ABC is the circle tangent to one side of the triangle and extensions of the other two sides. For each triangle there are exactly three escribed circles. The *center* of an escribed circle tangent to side AB is the intersection point of the bisector of angle C and the bisectors of the *outer* angles A and B.

The *circumscribed circle* of a triangle is the circle that passes through the vertices of the triangle. The *center* of the circumscribed circle of a triangle is the intersection point of the midperpendiculars to the triangle's sides.

2) For elements of a triangle ABC the following notations are often used:

a, b and c are the lengths of sides BC, CA and AB, respectively;

α, β and γ are the values of angles at vertices A, B, C;

R is the radius of the circumscribed cirlce;

r is the radius of the inscribed circle;

r_a, r_b and r_c are the radii of the escribed circles tangent to sides BC, CA and AB, respectively;

h_a, h_b and h_c the lengths of the heights dropped from vertices A, B and C, respectively.

3) If AD is the bisector of angle A of triangle ABC (or the bisector of the outer angle A), then $BD : CD = AB : AC$ (cf. Problem 1.17).

4) In a right triangle, the median drawn from the vertex of the right angle is equal to a half the hypothenuse (cf. Problem 5.16).

5) To prove that the intersection points of certain lines lie on one line *Menelaus's theorem* (Problem 5.58) is often used.

6) To prove that certain lines intersect at one point *Ceva's theorem* (Problem 5.70) is often used.

Introductory problems

1. Prove that the triangle is an isosceles one if a) one of its medians coincides with a height;

b) if one of its bisectors coincides with a height.

2. Prove that the bisectors of a triangle meet at one point.

3. On height AH of triangle ABC a point M is taken. Prove that $AB^2 - AC^2 = MB^2 - MC^2$.

4. On sides AB, BC, CA of an equilateral triangle ABC points P, Q and R, respectively, are taken so that

$$AP : PB = BQ : QC = CR : RA = 2 : 1.$$

Prove that the sides of triangle PQR are perpendicular to the respective sides of triangle ABC.

1. The inscribed and the circumscribed circles

5.1. On sides BC, CA and AB of triangle ABC, points A_1, B_1 and C_1, respectively, are taken so that $AC_1 = AB_1$, $BA_1 = BC_1$ and $CA_1 = CB_1$. Prove that A_1, B and C_1 are the points at which the inscribed circle is tangent to the sides of the triangle.

5.2. Let O_a, O_b and O_c be the centers of the escribed circles of triangle ABC. Prove that points A, B and C are the bases of heights of triangle $O_aO_bO_c$.

5.3. Prove that side BC of triangle ABC subtends (1) an angle with the vertex at the center O of the inscribed circle; the value of the angle is equal to $90^\circ + \frac{1}{2}\angle A$ and (2) an angle with the vertex at the center O_a of the escribed circle; the value of the angle is equal to $90^\circ - \frac{1}{2}\angle A$.

5.4. Inside triangle ABC, point P is taken such that

$$\angle PAB : \angle PAC = \angle PCA : \angle PCB = \angle PBC : \angle PBA = x.$$

Prove that $x = 1$.

5.5. Let A_1, B_1 and C_1 be the projections of an inner point O of triangle ABC to the heights. Prove that if the lengths of segments AA_1, BB_1 and CC_1 are equal, then they are equal to $2r$.

5.6. An angle of value $\alpha = \angle BAC$ is rotated about its vertex O, the midpoint of the basis AC of an isosceles triangle ABC. The legs of this angle meet the segments AB and BC at points P and Q, respectively. Prove that the perimeter of triangle PBQ remains constant under the rotation.

5.7. In a scalene triangle ABC, line MO is drawn through the midpoint M of side BC and the center O of the inscribed circle. Line MO intersects height AH at point E. Prove that $AE = r$.

5.8. A circle is tangent to the sides of an angle with vertex A at points P and Q. The distances from points P, Q and A to a tangent to this circle are equal to u, v and w, respectively. Prove that $\frac{uv}{w^2} = \sin^2 \frac{1}{2}\angle A$.

* * *

5.9. Prove that the points symmetric to the intersection point of the heights of triangle ABC through its sides lie on the circumscribed circle.

5.10. From point P of arc BC of the circumscribed circle of triangle ABC perpendiculars PX, PY and PZ are dropped to BC, CA and AB, respectively. Prove that $\frac{BC}{PX} = \frac{AC}{PY} + \frac{AB}{PZ}$.

* * *

5.11. Let O be the center of the circumscribed circle of triangle ABC, let I be the center of the inscribed circle, I_a the center of the escribed circle tangent to side BC. Prove that

a) $d^2 = R^2 - 2Rr$, where $d = OI$;

b) $d_a^2 = R^2 + 2Rr_a$, where $d_a = OI_a$.

5.12. The extensions of the bisectors of the angles of triangle ABC intersect the circumscribed circle at points A_1, B_1 and C_1; let M be the intersection point of bisectors. Prove that a) $\frac{MA \cdot MC}{MB_1} = 2r$; b) $\frac{MA_1 \cdot MC_1}{MB} = R$.

5.13. The lengths of the sides of triangle ABC form an arithmetic progression: a, b, c, where $a < b < c$. The bisector of angle $\angle B$ intersects the circumscribed circle at point B_1. Prove that the center O of the inscribed circle divides segment BB_1 in halves.

5.14. In triangle ABC side BC is the shortest one. On rays BA and CA, segments BD and CE, respectively, each equal to BC, are marked. Prove that the radius of the circumscribed circle of triangle ADE is equal to the distance between the centers of the inscribed and circumscribed circles of triangle ABC.

§2. Right triangles

5.15. In triangle ABC, angle $\angle C$ is a right one. Prove that $r = \frac{a+b-c}{2}$ and $r_c = \frac{a+b+c}{2}$.

5.16. In triangle ABC, let M be the midpoint of side AB. Prove that $CM = \frac{1}{2}AB$ if and only if $\angle ACB = 90^\circ$.

5.17. Consider trapezoid $ABCD$ with base AD. The bisectors of the outer angles at vertices A and B meet at point P and the bisectors of the angles at vertices C and D meet at point Q. Prove that the length of segment PQ is equal to a half perimeter of the trapezoid.

5.18. In an isosceles triangle ABC with base AC bisector CD is drawn. The line that passes through point D perpendicularly to DC intersects AC at point E. Prove that $EC = 2AD$.

5.19. The sum of angles at the base of a trapezoid is equal to 90°. Prove that the segment that connects the midpoints of the bases is equal to a half difference of the bases.

5.20. In a right triangle ABC, height CK from the vertex C of the right angle is drawn and in triangle ACK bisector CE is drawn. Prove that $CB = BE$.

5.21. In a right triangle ABC with right angle $\angle C$, height CD and bisector CF are drawn; let DK and DL be bisectors in triangles BDC and ADC. Prove that $CLFK$ is a square.

5.22. On hypothenuse AB of right triangle ABC, square $ABPQ$ is constructed outwards. Let $\alpha = \angle ACQ$, $\beta = \angle QCP$ and $\gamma = \angle PCB$. Prove that $\cos\beta = \cos\alpha \cdot \cos\gamma$.

See also Problems 2.65, 5.62.

§3. The equilateral triangles

5.23. From a point M inside an equilateral triangle ABC perpendiculars MP, MQ and MR are dropped to sides AB, BC and CA, respectively. Prove that

$$AP^2 + BQ^2 + CR^2 = PB^2 + QC^2 + RA^2,$$
$$AP + BQ + CR = PB + QC + RA.$$

5.24. Points D and E divide sides AC and AB of an equilateral triangle ABC in the ratio of $AD : DC = BE : EA = 1 : 2$. Lines BD and CE meet at point O. Prove that $\angle AOC = 90^\circ$.

* * *

5.25. A circle divides each of the sides of a triangle into three equal parts. Prove that this triangle is an equilateral one.

5.26. Prove that if the intersection point of the heights of an acute triangle divides the heights in the same ratio, then the triangle is an equilateral one.

5.27. a) Prove that if $a + h_a = b + h_b = c + h_c$, then triangle ABC is a equilateral one.

b) Three squares are inscribed in triangle ABC: two vertices of one of the squares lie on side AC, those of another one lie on side BC, and those of the third lie one on AB. Prove that if all the three squares are equal, then triangle ABC is an equilateral one.

5.28. The circle inscribed in triangle ABC is tangent to the sides of the triangle at points A_1, B_1, C_1. Prove that if triangles ABC and $A_1B_1C_1$ are similar, then triangle ABC is an equilateral one.

5.29. The radius of the inscribed circle of a triangle is equal to 1, the lengths of the heights of the triangle are integers. Prove that the triangle is an equilateral one.

See also Problems 2.18, 2.26, 2.36, 2.44, 2.54, 4.46, 5.56, 7.45, 10.3, 10.77, 11.3, 11.5, 16.7, 18.9, 18.12, 18.15, 18.17-18.20, 18.22, 18.38, 24.1.

§4. Triangles with angles of 60° and 120°

5.30. In triangle ABC with angle A equal to 120° bisectors AA_1, BB_1 and CC_1 are drawn. Prove that triangle $A_1B_1C_1$ is a right one.

5.31. In triangle ABC with angle A equal to 120° bisectors AA_1, BB_1 and CC_1 meet at point O. Prove that $\angle A_1C_1O = 30°$.

5.32. a) Prove that if angle $\angle A$ of triangle ABC is equal to 120° then the center of the circumscribed circle and the orthocenter are symmetric through the bisector of the outer angle $\angle A$.

b) In triangle ABC, the angle $\angle A$ is equal to 60°; O is the center of the circumscribed circle, H is the orthocenter, I is the center of the inscribed circle and I_a is the center of the escribed circle tangent to side BC. Prove that $IO = IH$ and $I_aO = I_aH$.

5.33. In triangle ABC angle $\angle A$ is equal to 120°. Prove that from segments of lengths a, b and $b + c$ a triangle can be formed.

5.34. In an acute triangle ABC with angle $\angle A$ equal to 60° the heights meet at point H.

a) Let M and N be the intersection points of the midperpendiculars to segments BH and CH with sides AB and AC, respectively. Prove that points M, N and H lie on one line.

b) Prove that the center O of the circumscribed circle lies on the same line.

5.35. In triangle ABC, bisectors BB_1 and CC_1 are drawn. Prove that if $\angle CC_1B_1 = 30°$, then either $\angle A = 60°$ or $\angle B = 120°$.

See also Problem 2.33.

§5. Integer triangles

5.36. The lengths of the sides of a triangle are consecutive integers. Find these integers if it is known that one of the medians is perpendicular to one of the bisectors.

5.37. The lengths of all the sides of a right triangle are integers and the greatest common divisor of these integers is equal to 1. Prove that the legs of the triangle are equal to $2mn$ and $m^2 - n^2$ and the hypothenuse is equal to $m^2 + n^2$, where m and n are integers.

A right triangle the lengths of whose sides are integers is called a *Pythagorean triangle.*

5.38. The radius of the inscribed circle of a triangle is equal to 1 and the lengths of its sides are integers. Prove that these integers are equal to 3, 4, 5.

5.39. Give an example of an inscribed quadrilateral with pairwise distinct integer lengths of sides and the lengths of whose diagonals, the area and the radius of the circumscribed circle are all integers. (*Brakhmagupta.*)

5.40. a) Indicate two right triangles from which one can compose a triangle so that the lengths of the sides and the area of the composed triangle would be integers.

b) Prove that if the area of a triangle is an integer and the lengths of the sides are consecutive integers then this triangle can be composed of two right triangles the lengths of whose sides are integers.

5.41. a) In triangle ABC, the lengths of whose sides are rational numbers, height BB_1 is drawn.

Prove that the lengths of segments AB_1 and CB_1 are rational numbers.

b) The lengths of the sides and diagonals of a convex quadrilateral are rational numbers. Prove that the diagonals cut it into four triangles the lengths of whose sides are rational numbers.

See also Problem 26.7.

§6. Miscellaneous problems

5.42. Triangles ABC and $A_1B_1C_1$ are such that either their corresponding angles are equal or their sum is equal to 180°. Prove that the corresponding angles are equal, actually.

5.43. Inside triangle ABC an arbitrary point O is taken. Let points A_1, B_1 and C_1 be symmetric to O through the midpoints of sides BC, CA and AB, respectively. Prove that $\triangle ABC = \triangle A_1B_1C_1$ and, moreover, lines AA_1, BB_1 and CC_1 meet at one point.

5.44. Through the intersection point O of the bisectors of triangle ABC lines parallel to the sides of the triangle are drawn. The line parallel to AB meets AC and BC at points M and N, respectively, and lines parallel to AC and BC meet AB at points P and Q, respectively. Prove that $MN = AM + BN$ and the perimeter of triangle OPQ is equal to the length of segment AB.

5.45. a) Prove that the heigths of a triangle meet at one point.

b) Let H be the intersection point of heights of triangle ABC and R the radius of the circumscribed circle. Prove that

$$AH^2 + BC^2 = 4R^2 \quad \text{and} \quad AH = BC|\cot\alpha|.$$

5.46. Let $x = \sin 18^\circ$. Prove that $4x^2 + 2x = 1$.

5.47. Prove that the projections of vertex A of triangle ABC on the bisectors of the outer and inner angles at vertices B and C lie on one line.

5.48. Prove that if two bisectors in a triangle are equal, then the triangle is an isosceles one.

5.49. a) In triangles ABC and $A'B'C'$, sides AC and $A'C'$ are equal, the angles at vertices B and B' are equal, and the bisectors of angles $\angle B$ and $\angle B'$ are equal. Prove that these triangles are equal. (More precisely, either $\triangle ABC = \triangle A'B'C'$ or $\triangle ABC = \triangle C'B'A'$.)

b) Through point D on the bisector BB_1 of angle ABC lines AA_1 and CC_1 are drawn (points A_1 and C_1 lie on sides of triangle ABC). Prove that if $AA_1 = CC_1$, then $AB = BC$.

5.50. Prove that a line divides the perimeter and the area of a triangle in equal ratios if and only if it passes through the center of the inscribed circle.

5.51. Point E is the midpoint of arc $\smile AB$ of the circumscribed circle of triangle ABC on which point C lies; let C_1 be the midpoint of side AB. Perpendicular EF is dropped from point E to AC. Prove that:

a) line C_1F divides the perimeter of triangle ABC in halves;

b) three such lines constructed for each side of the triangle meet at one point.

5.52. On sides AB and BC of an acute triangle ABC, squares ABC_1D_1 and A_2BCD_2 are constructed outwards. Prove that the intersection point of lines AD_2 and CD_1 lies on height BH.

5.53. On sides of triangle ABC squares centered at A_1, B_1 and C_1 are constructed outwards. Let a_1, b_1 and c_1 be the lengths of the sides of triangle $A_1B_1C_1$; let S and S_1 be the areas of triangles ABC and $A_1B_1C_1$, respectively. Prove that:

a) $a_1^2 + b_1^2 + c_1^2 = a^2 + b^2 + c^2 + 6S$.

b) $S_1 - S = \frac{1}{8}(a^2 + b^2 + c^2)$.

5.54. On sides AB, BC and CA of triangle ABC (or on their extensions), points C_1, A_1 and B_1, respectively, are taken so that $\angle(CC_1, AB) = \angle(AA_1, BC) = \angle(BB_1, CA) =$

α. Lines AA_1 and BB_1, BB_1 and CC_1, CC_1 and AA_1 intersect at points C', A' and B', respectively. Prove that:

a) the intersection point of heights of triangle ABC coincides with the center of the circumscribed circle of triangle $A'B'C'$;

b) $\triangle A'B'C' \sim \triangle ABC$ and the similarity coefficient is equal to $2\cos\alpha$.

5.55. On sides of triangle ABC points A_1, B_1 and C_1 are taken so that $AB_1 : B_1C = c^n : a^n$, $BC_1 : CA = a^n : b^n$ and $CA_1 : A_1B = b^n : c^n$ (here a, b and c are the lengths of the triangle's sides). The circumscribed circle of triangle $A_1B_1C_1$ singles out on the sides of triangle ABC segments of length $\pm x, \pm y$ and $\pm z$, where the signs are chosen in accordance with the *orientation* of the triangle. Prove that

$$\frac{x}{a^{n-1}} + \frac{y}{b^{n-1}} + \frac{z}{c^{n-1}} = 0.$$

5.56. In triangle ABC *trisectors* (the rays that divide the angles into three equal parts) are drawn. The nearest to side BC trisectors of angles B and C intersect at point A_1; let us define points B_1 and C_1 similarly, (Fig. 55). Prove that triangle $A_1B_1C_1$ is an equilateral one. (*Morlie's theorem.*)

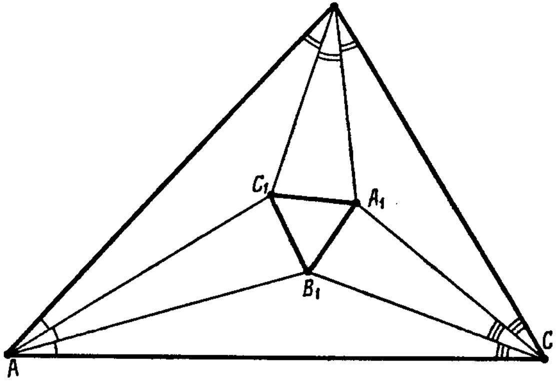


Figure 55 (5.56)

5.57. On the sides of an equilateral triangle ABC as on bases, isosceles triangles A_1BC, AB_1C and ABC_1 with angles α, β and γ at the bases such that $\alpha + \beta + \gamma = 60^\circ$ are constructed inwards. Lines BC_1 and B_1C meet at point A_2, lines AC_1 and A_1C meet at point B_2, and lines AB_1 and A_1B meet at point C_2. Prove that the angles of triangle $A_2B_2C_2$ are equal to 3α, 3β and 3γ.

§7. Menelaus's theorem

Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be colinear vectors. Denote by $\frac{\overline{AB}}{\overline{CD}}$ the quantity $\pm\frac{AB}{CD}$, where the plus sign is taken if the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are codirected and the minus sign if the vectors are directed opposite to each other.

5.58. On sides BC, CA and AB of triangle ABC (or on their extensions) points A_1, B_1 and C_1, respectively, are taken. Prove that points A_1, B_1 and C_1 lie on one line if and only if

$$\frac{\overline{BA_1}}{\overline{CA_1}} \cdot \frac{\overline{CB_1}}{\overline{AB_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}} = 1. \qquad \text{(\textit{Menelaus's theorem})}$$

5.59. Prove Problem 5.85 a) with the help of Menelaus's theorem.

5.60. A circle S is tangent to circles S_1 and S_2 at points A_1 and A_2, respectively. Prove that line A_1A_2 passes through the intersection point of either common outer or common inner tangents to circles S_1 and S_2.

5.61. a) The midperpendicular to the bisector AD of triangle ABC intersects line BC at point E. Prove that $BE : CE = c^2 : b^2$.

b) Prove that the intersection point of the midperpendiculars to the bisectors of a triangle and the extensions of the corresponding sides lie on one line.

5.62. From vertex C of the right angle of triangle ABC height CK is dropped and in triangle ACK bisector CE is drawn. Line that passes through point B parallel to CE meets CK at point F. Prove that line EF divides segment AC in halves.

5.63. On lines BC, CA and AB points A_1, B_1 and C_1, respectively, are taken so that points A_1, B_1 and C_1 lie on one line. The lines symmetric to lines AA_1, BB_1 and CC_1 through the corresponding bisectors of triangle ABC meet lines BC, CA and AB at points A_2, B_2 and C_2, respectively. Prove that points A_2, B_2 and C_2 lie on one line.

* * *

5.64. Lines AA_1, BB_1 and CC_1 meet at one point, O. Prove that the intersection points of lines AB and A_1B_1, BC and B_1C_1, AC and A_1C_1 lie on one line. (*Desargues's theorem.*)

5.65. Points A_1, B_1 and C_1 are taken on one line and points A_2, B_2 and C_2 are taken on another line. The intersection pointa of lines A_1B_2 with A_2B_1, B_1C_2 with B_2C_1 and C_1A_2 with C_2A_1 are C, A and B, respectively. Prove that points A, B and C lie on one line. (*Pappus' theorem.*)

5.66. On sides AB, BC and CD of quadrilateral $ABCD$ (or on their extensions) points K, L and M are taken. Lines KL and AC meet at point P, lines LM and BD meet at point Q. Prove that the intersection point of lines KQ and MP lies on line AD.

5.67. The extensions of sides AB and CD of quadrilateral $ABCD$ meet at point P and the extensions of sides BC and AD meet at point Q. Through point P a line is drawn that intersects sides BC and AD at points E and F. Prove that the intersection points of the diagonals of quadrilaterals $ABCD$, $ABEF$ and $CDFE$ lie on the line that passes through point Q.

5.68. a) Through points P and Q triples of lines are drawn. Let us denote their intersection points as shown on Fig. 56. Prove that lines KL, AC and MN either meet at one point or are parallel.

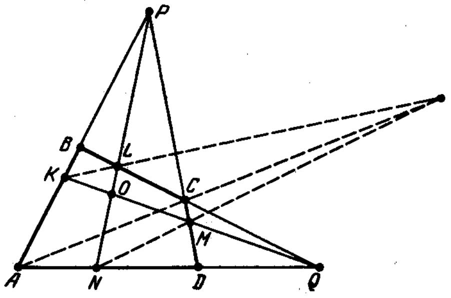


Figure 56 (5.68)

b) Prove further that if point O lies on line BD, then the intersection point of lines KL, AC and MN lies on line PQ.

5.69. On lines BC, CA and AB points A_1, B_1 and C_1 are taken. Let P_1 be an arbitrary point of line BC, let P_2 be the intersection point of lines P_1B_1 and AB, let P_3 be the intersection point of lines P_2A_1 and CA, let P_4 be the intersection point of P_3C_1 and BC, etc. Prove that points P_7 and P_1 coincide.

See also Problem 6.98.

§8. Ceva's theorem

5.70. Triangle ABC is given and on lines AB, BC and CA points C_1, A_1 and B_1, respectively, are taken so that k of them lie on sides of the triangle and $3-k$ on the extensions of the sides. Let

$$R = \frac{BA_1}{CA_1} \cdot \frac{CB_1}{AB_1} \cdot \frac{AC_1}{BC_1}.$$

Prove that

a) points A_1, B_1 and C_1 lie on one line if and only if $R=1$ and k is even. (*Menelaus's theorem.*)

b) lines AA_1, BB_1 and CC_1 either meet at one point or are parallel if and only if $R=1$ and k is odd. (*Ceva's theorem.*)

5.71. The inscribed (or an escribed) circle of triangle ABC is tangent to lines BC, CA and AB at points A_1, B_1 and C_1, respectively. Prove that lines AA_1, BB_1 and CC_1 meet at one point.

5.72. Prove that the heights of an acute triangle intersect at one point.

5.73. Lines AP, BP and CP meet the sides of triangle ABC (or their extensions) at points A_1, B_1 and C_1, respectively. Prove that:

a) lines that pass through the midpoints of sides BC, CA and AB parallel to lines AP, BP and CP, respectively, meet at one point;

b) lines that connect the midpoints of sides BC, CA and AB with the midpoints of segments AA_1, BB_1, CC_1, respectively, meet at one point.

5.74. On sides BC, CA, and AB of triangle ABC, points A_1, B_1 and C_1 are taken so that segments AA_1, BB_1 and CC_1 meet at one point. Lines A_1B_1 and A_1C_1 meet the line that passes through vertex A parallel to side BC at points C_2 and B_2, respectively. Prove that $AB_2 = AC_2$.

5.75. a) Let α, β and γ be arbitrary angles such that the sum of any two of them is not less than 180°. On sides of triangle ABC, triangles A_1BC, AB_1C and ABC_1 with angles at vertices A, B, and C equal to α, β and γ, respectively, are constructed outwards. Prove that lines AA_1, BB_1 and CC_1 meet at one point.

b) Prove a similar statement for triangles constructed on sides of triangle ABC inwards.

5.76. Sides BC, CA and AB of triangle ABC are tangent to a circle centered at O at points A_1, B_1 and C_1. On rays OA_1, OB_1 and OC_1 equal segments OA_2, OB_2 and OC_2 are marked. Prove that lines AA_2, BB_2 and CC_2 meet at one point.

5.77. Lines AB, BP and CP meet lines BC, CA and AB at points A_1, B_1 and C_1, respectively. Points A_2, B_2 and C_2 are selected on lines BC, CA and AB so that

$$\begin{aligned}\overline{BA_2} : \overline{A_2C} &= \overline{A_1C} : \overline{BA_1},\\ \overline{CB_2} : \overline{B_2A} &= \overline{B_1A} : \overline{CB_1},\\ \overline{AC_2} : \overline{C_2B} &= \overline{C_1B} : \overline{AC_1}.\end{aligned}$$

Prove that lines AA_2, BB_2 and CC_2 also meet at one point, Q (or are parallel).

Such points P and Q are called *isotomically conjugate with respect to triangle ABC.*

5.78. On sides BC, CA, AB of triangle ABC points A_1, B_1 and C_1 are taken so that lines AA_1, BB_1 and CC_1 intersect at one point, P. Prove that lines AA_2, BB_2 and CC_2 symmetric to these lines through the corresponding bisectors also intersect at one point, Q.

Such points P and Q are called *isogonally conjugate with respect to triangle ABC*.

5.80. The opposite sides of a convex hexagon are pairwise parallel. Prove that the lines that connect the midpoints of opposite sides intersect at one point.

5.81. From a point P perpendiculars PA_1 and PA_2 are dropped to side BC of triangle ABC and to height AA_3. Points B_1, B_2 and C_1, C_2 are similarly defined. Prove that lines A_1A_2, B_1B_2 and C_1C_2 either meet at one point or are parallel.

5.82. Through points A and D lying on a circle tangents that intersect at point S are drawn. On arc $\smile AD$ points B and C are taken. Lines AC and BD meet at point P, lines AB and CD meet at point Q. Prove that line PQ passes through point S.

5.83. a) On sides BC, CA and AB of an isosceles triangle ABC with base AB, points A_1, B_1 and C_1, respectively, are taken so that lines AA_1, BB_1 and CC_1 meet at one point. Prove that

$$\frac{AC_1}{C_1B} = \frac{\sin\angle ABB_1 \cdot \sin\angle CAA_1}{\sin\angle BAA_1 \cdot \sin\angle CBB_1}.$$

b) Inside an isosceles triangle ABC with base AB points M and N are taken so that $\angle CAM = \angle ABN$ and $\angle CBM = \angle BAN$. Prove that points C, M and N lie on one line.

5.84. In triangle ABC bisectors AA_1, BB_1 and CC_1 are drawn. Bisectors AA_1 and CC_1 intersect segments C_1B_1 and B_1A_1 at points M and N, respectively. Prove that $\angle MBB_1 = \angle NBB_1$.

See also Problems 10.56, 14.7, 14.38.

§9. Simson's line

5.85. a) Prove that the bases of the perpendiculars dropped from a point P of the circumscribed circle of a triangle to the sides of the triangle or to their extensions lie on one line.

This line is called *Simson's line* of point P with respect to the triangle.

b) The bases of perpendiculars dropped from a point P to the sides (or their extensions) of a triangle lie on one line. Prove that point P lies on the circumscribed circle of the triangle.

5.86. Points A, B and C lie on one line, point P lies outside this line. Prove that the centers of the circumscribed circles of triangles ABP, BCP, ACP and point P lie on one circle.

5.87. In triangle ABC the bisector AD is drawn and from point D perpendiculars DB' and DC' are dropped to lines AC and AB, respectively; point M lies on line $B'C'$ and $DM \perp BC$. Prove that point M lies on median AA_1.

5.88. a) From point P of the circumscribed circle of triangle ABC lines PA_1, PB_1 and PC_1 are drawn at a given (oriented) angle α to lines BC, CA and AB, respectively, so that points A_1, B_1 and C_1 lie on lines BC, CA and AB, respectively. Prove that points A_1, B_1 and C_1 lie on one line.

b) Prove that if in the definition of Simson's line we replace the angle 90° by an angle α, i.e., replace the perpendiculars with the lines that form angles of α, their intersection points with the sides lie on the line and the angle between this line and Simson's line becomes equal to $90^\circ - \alpha$.

5.89. a) From a point P of the circumscribed circle of triangle ABC perpendiculars PA_1 and PB_1 are dropped to lines BC and AC, respectively. Prove that $PA \cdot PA_1 = 2Rd$, where R is the radius of the circumscribed circle, d the distance from point P to line A_1B_1.

b) Let α be the angle between lines A_1B_1 and BC. Prove that $\cos\alpha = \frac{PA}{2R}$.

5.90. Let A_1 and B_1 be the projections of point P of the circumscribed circle of triangle ABC to lines BC and AC, respectively. Prove that the length of segment A_1B_1 is equal to the length of the projection of segment AB to line A_1B_1.

5.91. Points P and C on a circle are fixed; points A and B move along the circle so that angle $\angle ACB$ remains fixed. Prove that Simson's lines of point P with respect to triangle ABC are tangent to a fixed circle.

5.92. Point P moves along the circumscribed circle of triangle ABC. Prove that Simson's line of point P with respect to triangle ABC rotates accordingly through the angle equal to a half the angle value of the arc circumvent by P.

5.93. Prove that Simson's lines of two diametrically opposite points of the circumscribed circle of triangle ABC are perpendicular and their intersection point lies on the circle of 9 points, cf. Problem 5.106.

5.94. Points A, B, C, P and Q lie on a circle centered at O and the angles between vector $\overrightarrow{OP}$ and vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OQ}$ are equal to α, β, γ and $\frac{1}{2}(\alpha+\beta+\gamma)$, respectively. Prove that Simson's line of point P with respect to triangle ABC is parallel to OQ.

5.95. Chord PQ of the circumscribed circle of triangle ABC is perpendicular to side BC. Prove that Simson's line of point P with respect to triangle ABC is parallel to line AQ.

5.96. The heights of triangle ABC intersect at point H; let P be a point of its circumscribed circle. Prove that Simson's line of point P with respect to triangle ABC divides segment PH in halves.

5.97. Quadrilateral $ABCD$ is inscribed in a circle; l_a is Simson's line of point A with respect to triangle BCD; let lines l_b, l_c and l_d be similarly defined. Prove that these lines intersect at one point.

5.98. a) Prove that the projection of point P of the circumscribed circle of quadrilateral $ABCD$ onto Simson's lines of this point with respect to triangles BCD, CDA, DAB and BAC lie on one line. (*Simson's line of the inscribed quadrilateral.*)

b) Prove that by induction we can similarly define Simson's line of an inscribed n-gon as the line that contains the projections of a point P on Simson's lines of all $(n-1)$-gons obtained by deleting one of the vertices of the n-gon.

See also Problems 5.10, 5.59.

§10. The pedal triangle

Let A_1, B_1 and C_1 be the bases of the perpendiculars dropped from point P to lines BC, CA and AB, respectively. Triangle $A_1B_1C_1$ is called the *pedal* triangle of point P with respect to triangle ABC.

5.99. Let $A_1B_1C_1$ be the pedal triangle of point P with respect to triangle ABC. Prove that $B_1C_1 = \frac{BC\cdot AP}{2R}$, where R is the radius of the circumscribed circle of triangle ABC.

5.100. Lines AP, BP and CP intersect the circumscribed circle of triangle ABC at points A_2, B_2 and C_2; let $A_1B_1C_1$ be the pedal triangle of point P with respect to triangle ABC. Prove that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

5.101. Inside an acute triangle ABC a point P is given. If we drop from it perpendiculars PA_1, PB_1 and PC_1 to the sides, we get $\triangle A_1B_1C_1$. Performing for $\triangle A_1B_1C_1$ the same operation we get $\triangle A_2B_2C_2$ and then we similarly get $\triangle A_3B_3C_3$. Prove that $\triangle A_3B_3C_3 \sim \triangle ABC$.

5.102. A triangle ABC is inscribed in the circle of radius R centered at O. Prove that the area of the pedal triangle of point P with respect to triangle ABC is equal to $\frac{1}{4}\left|1-\frac{d^2}{R^2}\right| S_{ABC}$, where $d = |PO|$.

5.103. From point P perpendiculars PA_1, PB_1 and PC_1 are dropped on sides of triangle ABC. Line l_a connects the midpoints of segments PA and B_1C_1. Lines l_b and l_c are similarly defined. Prove that l_a, l_b and l_c meet at one point.

5.104. a) Points P_1 and P_2 are isogonally conjugate with respect to triangle ABC, cf. Problem 5.79. Prove that their pedal triangles have a common circumscribed circle whose center is the midpoint of segment P_1P_2.

b) Prove that the above statement remains true if instead of perpendiculars we draw from points P_1 and P_2 lines forming a given (oriented) angle to the sides.

See also Problems 5.132, 5.133, 14.19 b).

§11. Euler's line and the circle of nine points

5.105. Let H be the point of intersection of heights of triangle ABC, O the center of the circumscribed circle and M the point of intersection of medians. Prove that point M lies on segment OH and $OM : MH = 1 : 2$.

The line that contains points O, M and H is called *Euler's line*.

5.106. Prove that the midpoints of sides of a triangle, the bases of heights and the midpoints of segments that connect the intersection point of heights with the vertices lie on one circle and the center of this circle is the midpoint of segment OH.

The circle defined above is called the *circle of nine points*.

5.107. The heights of triangle ABC meet at point H.

a) Prove that triangles ABC, HBC, AHC and ABH have a common circle of 9 points.

b) Prove that Euler's lines of triangles ABC, HBC, AHC and ABH intersect at one point.

c) Prove that the centers of the circumscribed circles of triangles ABC, HBC, AHC and ABH constitute a quadrilateral symmetric to quadrilateral $HABC$.

5.108. What are the sides the Euler line intersects in an acute and an obtuse triangles?

5.109. a) Prove that the circumscribed circle of triangle ABC is the circle of 9 points for the triangle whose vertices are the centers of escribed circles of triangle ABC.

b) Prove that the circumscribed circle divides the segment that connects the centers of the inscribed and an escribed circles in halves.

5.110. Prove that Euler's line of triangle ABC is parallel to side BC if and only if $\tan B \tan C = 3$.

5.111. On side AB of acute triangle ABC the circle of 9 points singles out a segment. Prove that the segment subtends an angle of $2|\angle A - \angle B|$ with the vertex at the center.

5.112. Prove that if Euler's line passes through the center of the inscribed circle of a triangle, then the triangle is an isosceles one.

5.113. The inscribed circle is tangent to the sides of triangle ABC at points A_1, B_1 and C_1. Prove that Euler's line of triangle $A_1B_1C_1$ passes through the center of the circumscribed circle of triangle ABC.

5.114. In triangle ABC, heights AA_1, BB_1 and CC_1 are drawn. Let A_1A_2, B_1B_2 and C_1C_2 be diameters of the circle of nine points of triangle ABC. Prove that lines AA_2, BB_2 and CC_2 either meet at one point or are parallel.

See also Problems 3.65 a), 13.34 b).

§12. Brokar's points

5.115. a) Prove that inside triangle ABC there exists a point P such that $\angle ABP = \angle CAP = \angle BCP$.

b) On sides of triangle ABC, triangles CA_1B, CAB_1 and C_1AB similar to ABC are constructed outwards (the angles at the first vertices of all the four triangles are equal, etc.). Prove that lines AA_1, BB_1 and CC_1 meet at one point and this point coincides with the point found in heading a).

This point P is called *Brokar's point* of triangle ABC. The proof of the fact that there exists *another* Brokar's point Q for which $\angle BAQ = \angle ACQ = \angle CBQ$ is similar to the proof of existence of P given in what follows. We will refer to P and Q as the *first* and the *second* Brokar's points.

5.116. a) Through Brokar's point P of triangle ABC lines AB, BP and CP are drawn. They intersect the circumscribed circle at points A_1, B_1 and C_1, respectively. Prove that $\triangle ABC = \triangle B_1C_1A_1$.

b) Triangle ABC is inscribed into circle S. Prove that the triangle formed by the intersection points of lines PA, PB and PC with circle S can be equal to triangle ABC for no more than 8 distinct points P. (We suppose that the intersection points of lines PA, PB and PC with the circle are distinct from points A, B and C.)

5.117. a) Let P be Brokar's point of triangle ABC. Let $\varphi = \angle ABP = \angle BCP = \angle CAP$. Prove that $\cot\varphi = \cot\alpha + \cot\beta + \cot\gamma$.

The angle φ from Problem 5.117 is called *Brokar's angle* of triangle ABC.

b) Prove that Brokar's points of triangle ABC are isogonally conjugate to each other (cf. Problem 5.79).

c) The tangent to the circumscribed circle of triangle ABC at point C and the line passing through point B parallel to AC intersect at point A_1. Prove that Brokar's angle of triangle ABC is equal to angle $\angle A_1AC$.

5.118. a) Prove that Brokar's angle of any triangle does not exceed 30°.

b) Inside triangle ABC, point M is taken. Prove that one of the angles $\angle ABM$, $\angle BCM$ and $\angle CAM$ does not exceed 30°.

5.119. Let Q be the second Brokar's point of triangle ABC, let O be the center of its circumscribed circle; A_1, B_1 and C_1 the centers of the circumscribed circles of triangles CAQ, ABQ and BCQ, respectively. Prove that $\triangle A_1B_1C_1 \sim \triangle ABC$ and O is the first Brokar's point of triangle $A_1B_1C_1$.

5.120. Let P be Brokar's point of triangle ABC; let R_1, R_2 and R_3 be the radii of the circumscribed circles of triangles ABP, BCP and CAP, respectively. Prove that $R_1R_2R_3 = R^3$, where R is the radius of the circumscribed circle of triangle ABC.

5.121. Let P and Q be the first and the second Brokar's points of triangle ABC. Lines CP and BQ, AP and CQ, BP and AQ meet at points A_1, B_1 and C_1, respectively. Prove that the circumscribed circle of triangle $A_1B_1C_1$ passes through points P and Q.

5.122. On sides CA, AB and BC of an acute triangle ABC points A_1, B_1 and C_1, respectively, are taken so that $\angle AB_1A_1 = \angle BC_1B_1 = \angle CA_1C_1$. Prove that $\triangle A_1B_1C_1 \sim \triangle ABC$ and the center of the rotational homothety that sends one triangle into another coincides with the first Brokar's point of both triangles.

See also Problem 19.55.

§13. Lemoine's point

Let AM be a median of triangle ABC and line AS be symmetric to line AM through the bisector of angle A (point S lies on segment BC). Then segment AS is called a *simedian* of triangle ABC; sometimes the whole ray AS is referred to as a simedian.

Simedians of a triangle meet at the point *isogonally conjugate* to the intersection point of medians (cf. Problem 5.79). The intersection point of simedians of a triangle is called *Lemoine's point.*

5.123. Let lines AM and AN be symmetric through the bisector of angle $\angle A$ of triangle ABC (points M and N lie on line BC). Prove that $\frac{BM\cdot BN}{CM\cdot CN}=\frac{c^2}{b^2}$. In particular, if AS is a simedian, then $\frac{BS}{CS}=\frac{c^2}{b^2}$.

5.124. Express the length of simedian AS in terms of the lengths of sides of triangle ABC.

Segment B_1C_1, where points B_1 and C_1 lie on rays AC and AB, respectively, is said to be *antiparallel* to side BC if $\angle AB_1C_1=\angle ABC$ and $\angle AC_1B_1=\angle ACB$.

5.125. Prove that simedian AS divides any segment B_1C_1 antiparallel to side BC in halves.

5.126. The tangent at point B to the circumscribed circle S of triangle ABC intersects line AC at point K. From point K another tangent KD to circle S is drawn. Prove that BD is a simedian of triangle ABC.

5.127. Tangents to the circumscribed circle of triangle ABC at points B and C meet at point P. Prove that line AP contains simedian AS.

5.128. Circle S_1 passes through points A and B and is tangent to line AC, circle S_2 passes through points A and C and is tangent to line AB. Prove that the common chord of these circles is a simedian of triangle ABC.

5.129. Bisectors of the outer and inner angles at vertex A of triangle ABC intersect line BC at points D and E, respectively. The circle with diameter DE intersects the circumscribed circle of triangle ABC at points A and X. Prove that AX is a simedian of triangle ABC.

* * *

5.130. Prove that Lemoine's point of right triangle ABC with right angle $\angle C$ is the midpoint of height CH.

5.131. Through a point X inside triangle ABC three segments antiparallel to its sides are drawn, cf. Problem 5.125?. Prove that these segments are equal if and only if X is Lemoine's point.

5.132. Let A_1, B_1 and C_1 be the projections of Lemoine's point K to the sides of triangle ABC. Prove that K is the intersection point of medians of triangle $A_1B_1C_1$.

5.133. Let A_1, B_1 and C_1 be the projections of Lemoine's point K of triangle ABC on sides BC, CA and AB, respectively. Prove that median AM of triangle ABC is perpendicular to line B_1C_1.

5.134. Lines AK, BK and CK, where K is Lemoine's point of triangle ABC, intersect the circumscribed circle at points A_1, B_1 and C_1, respectively. Prove that K is Lemoine's point of triangle $A_1B_1C_1$.

5.135. Prove that lines that connect the midpoints of the sides of a triangle with the midpoints of the corresponding heights intersect at Lemoine's point.

See also Problems 11.22, 19.54, 19.55.

Problems for independent study

5.136. Prove that the projection of the diameter of a circumscribed circle perpendicular to a side of the triangle to the line that contains the second side is equal to the third side.

5.137. Prove that the area of the triangle with vertices in the centers of the escribed circles of triangle ABC is equal to $2pR$.

5.138. An isosceles triangle with base a and the lateral side b, and an isosceles triangle with base b and the lateral side a are inscribed in a circle of radius R. Prove that if $a \neq b$, then $ab = \sqrt{5}R^2$.

5.139. The inscribed circle of right triangle ABC is tangent to the hypothenuse AB at point P; let CH be a height of triangle ABC. Prove that the center of the inscribed circle of triangle ACH lies on the perpendicular dropped from point P to AC.

5.140. The inscribed circle of triangle ABC is tangent to sides CA and AB at points B_1 and C_1, respectively, and an escribed circle is tangent to the extension of sides at points B_2 and C_2. Prove that the midpoint of side BC is equidistant from lines B_1C_1 and B_2C_2.

5.141. In triangle ABC, bisector AD is drawn. Let O, O_1 and O_2 be the centers of the circumscribed circles of triangles ABC, ABD and ACD, respectively. Prove that $OO_1 = OO_2$.

5.142. The triangle constructed from a) medians, b) heights of triangle ABC is similar to triangle ABC. What is the ratio of the lengths of the sides of triangle ABC?

5.143. Through the center O of an equilateral triangle ABC a line is drawn. It intersects lines BC, CA and AB at points A_1, B_1 and C_1, respectively. Prove that one of the numbers $\frac{1}{OA_1}$, $\frac{1}{OB_1}$ and $\frac{1}{OC_1}$ is equal to the sum of the other two numbers.

5.144. In triangle ABC heights BB_1 and CC_1 are drawn. Prove that if $\angle A = 45°$, then B_1C_1 is a diameter of the circle of nine points of triangle ABC.

5.145. The angles of triangle ABC satisfy the relation $\sin^2\angle A + \sin^2\angle B + \sin^2\angle C = 1$. Prove that the circumscribed circle and the circle of nine points of triangle ABC intersect at a right angle.

Solutions

5.1. Let $AC_1 = AB_1 = x$, $BA_1 = BC_1 = y$ and $CA_1 = CB_1 = z$. Then

$$a = y + z, \quad b = z + x \quad \text{and} \quad c = x + y.$$

Subtracting the third equality from the sum of the first two ones we get $z = \frac{a+b-c}{2}$. Hence, if triangle ABC is given, then the position of points A_1 and B_1 is uniquely determined. Similarly, the position of point C_1 is also uniquely determined. It remains to notice that the tangency points of the inscribed circle with the sides of the triangle satisfy the relations indicated in the hypothesis of the problem.

5.2. Rays CO_a and CO_b are the bisectors of the outer angles at vertex C, hence, C lies on line O_aO_b and $\angle O_aCB = \angle O_bCA$. Since CO_c is the bisector of angle $\angle BCA$, it follows that $\angle BCO_c = \angle ACO_c$. Adding these equalities we get: $\angle O_aCO_c = \angle O_cCO_b$, i.e., O_cC is a height of triangle $O_aO_bO_c$. We similarly prove that O_aA and O_bB are heights of this triangle.

5.3. Clearly,

$$\angle BOC = 180° - \angle CBO - \angle BCO = 180° - \frac{\angle B}{2} - \frac{\angle C}{2} = 90° + \frac{\angle A}{2}$$

and $\angle BO_aC = 180° - \angle BOC$, because $\angle OBO_a = \angle OCO_a = 90°$.

5.4. Let AA_1, BB_1 and CC_1 be the bisectors of triangle ABC and O the intersection point of these bisectors. Suppose that $x > 1$. Then $\angle PAB > \angle PAC$, i.e., point P lies inside triangle AA_1C. Similarly, point P lies inside triangles CC_1B and BB_1A. But the only common point of these three triangles is point O. Contradiction. The case $x < 1$ is similarly treated.

5.5. Let d_a, d_b and d_c be the distances from point O to sides BC, CA and AB. Then $ad_a + bd_b + cd_c = 2S$ and $ah_a = bh_b = ch_c = 2S$. If $h_a - d_a = h_b - d_b = h_c - d_c = x$, then

$$(a+b+c)x = a(h_a - d_a) = b(h_b - d_b) + c(h_c - d_c) = 6S - 2S = 4S.$$

Hence, $x = \frac{4S}{2p} = 2r$.

5.6. Let us prove that point O is the center of the escribed circle of triangle PBQ tangent to side PQ. Indeed, $\angle POQ = \angle A = 90^\circ - \frac{1}{2}\angle B$. The angle of the same value with the vertex at the center of the escribed circle subtends segment PQ (Problem 5.3). Moreover, point O lies on the bisector of angle B. Hence, the semiperimeter of triangle PBQ is equal to the length of the projection of segment OB to line CB.

5.7. Let P be the tangent point of the inscribed circle with side BC, let PQ be a diameter of the inscribed circle, R the intersection point of lines AQ and BC. Since $CR = BP$ (cf. Problem 19.11 a)) and M is the midpoint of side BC, we have: $RM = PM$. Moreover, O is the midpoint of diameter PQ, hence, $MO \parallel QR$ and since $AH \parallel PQ$, we have $AE = OQ$.

5.8. The given circle can be the inscribed as well as the escribed circle of triangle ABC cut off by the tangent from the angle. Making use of the result of Problem 3.2 we can verify that in either case

$$\frac{uv}{w^2} = \frac{(p-b)(p-c)\sin\angle B \sin\angle C}{h_a^2}.$$

It remains to notice that $h_a = b\sin\angle C = c\sin\angle B$ and $\frac{(p-b)(p-c)}{bc} = \sin^2\frac{1}{2}\angle A$ (Problem 12.13).

5.9. Let A_1, B_1 and C_1 be points symmetric to point H through sides BC, CA and AB, respectively. Since $AB \perp CH$ and $BC \perp AH$, it follows that $\angle(AB, BC) = \angle(CH, HA)$ and since triangle AC_1H is an isosceles one, $\angle(CH, HA) = \angle(AC_1, C_1C)$. Hence, $\angle(AB, BC) = \angle(AC_1, C_1C)$, i.e., point C_1 lies on the circumscribed circle of triangle ABC. We similarly prove that points A_1 and B_1 lie on this same circle.

5.10. Let R be the radius of the circumscribed circle of triangle ABC. This circle is also the circumscribed circle of triangles ABP, APC and PBC. Clearly, $\angle ABP = 180^\circ - \angle ACP = \alpha$, $\angle BAP = \angle BCP = \beta$ and $\angle CAP = \angle CBP = \gamma$. Hence,

$$PX = PB\sin\gamma = 2R\sin\beta\sin\gamma,\ PY = 2R\sin\alpha\sin\gamma \text{ and } P = 2R\sin\alpha\sin\beta.$$

It is also clear that

$$BC = 2R\sin\angle BAC = 2R\sin(\beta+\gamma),\ AC = 2R\sin(\alpha-\gamma),\ AB = 2R\sin(\alpha+\beta).$$

It remains to verify the equality

$$\frac{\sin(\beta+\gamma)}{\sin\beta\sin\gamma} = \frac{\sin(\alpha-\gamma)}{\sin\alpha\sin\gamma} + \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$$

which is subject to a direct calculation.

5.11. a) Let M be the intersection point of line AI with the circumscribed circle. Drawing the diameter through point I we get

$$AI \cdot IM = (R+d)(R-d) = R^2 - d^2.$$

Since $IM = CM$ (by Problem 2.4 a)), it follows that $R^2 - d^2 = AI \cdot CM$. It remains to observe that $AI = \frac{r}{\sin\frac{1}{2}\angle A}$ and $CM = 2R\sin\frac{1}{2}\angle A$.

b) Let M be the intersection point of line AI_a with the circumscribed circle. Then $AI_a \cdot I_aM = d_a^2 - R^2$. Since $I_aM = CM$ (by Problem 2.4 a)), it follows that $d_a^2 - R^2 = AI_a \cdot CM$. It remains to notice that $AI_a = \frac{r_a}{\sin \frac{1}{2}\angle A}$ and $CM = 2R \sin \frac{1}{2}\angle A$.

5.12. a) Since B_1 is the center of the circumscribed circle of triangle AMC (cf. Problem 2.4 a)), $AM = 2MB_1 \sin \angle ACM$. It is also clear that $MC = \frac{r}{\sin \angle ACM}$. Hence, $\frac{MA \cdot MC}{MB_1} = 2r$.

b) Since

$$\angle MBC_1 = \angle BMC_1 = 180^\circ - \angle BMC \quad \text{and } \angle BC_1M = \angle A,$$

it follows that

$$\frac{MC_1}{BC} = \frac{BM}{BC} \cdot \frac{MC_1}{BM} = \frac{\sin \angle BCM}{\sin \angle BMC} \cdot \frac{\sin \angle MBC_1}{\sin \angle BC_1M} = \frac{\sin \angle BCM}{\sin \angle A}.$$

Moreover, $MB = 2MA_1 \sin \angle BCM$. Therefore, $\frac{MC_1 \cdot MA_1}{MB} = \frac{BC}{2 \sin \angle A} = R$.

5.13. Let M be the midpoint of side AC, and N the tangent point of the inscribed circle with side BC. Then $BN = p - b$ (see Problem 3.2), hence, $BN = AM$ because $p = \frac{3}{2}b$ by assumption. Moreover, $\angle OBN = \angle B_1AM$ and, therefore, $\triangle OBN = \triangle B_1AM$, i.e., $OB = B_1A$. But $B_1A = B_1O$ (see Problem 2.4 a)).

5.14. Let O and O_1 be the centers of the inscribed and circumscribed circles of triangle ABC. Let us consider the circle of radius $d = OO_1$ centered at O. In this circle, let us draw chords O_1M and O_1N parallel to sides AB and AC, respectively. Let K be the tangent point of the inscribed circle with side AB and L the midpoint of side AB. Since $OK \perp AB$, $O_1L \perp AB$ and $O_1M \parallel AB$, it follows that

$$O_1M = 2KL = 2BL - 2BK = c - (a + c - b) = b - a = AE.$$

Similarly, $O_1N = AD$ and, therefore, $\triangle MO_1N = \triangle EAD$. Consequently, the radius of the circumscribed circle of triangle EAD is equal to d.

5.15. Let the inscribed circle be tangent to side AC at point K and the escribed circle be tangent to the extension of side AC at point L. Then $r = CK$ and $r_c = CL$. It remains to make use of the result of Problem 3.2.

5.16. Since $\frac{1}{2}AB = AM = BM$, it follows that $CM = \frac{1}{2}AB$ if and only if point C lies on the circle with diameter AB.

5.17. Let M and N be the midpoints of sides AB and CD. Triangle APB is a right one; hence, $PM = \frac{1}{2}AB$ and $\angle MPA = \angle PAM$ and, therefore, $PM \parallel AD$. Similar arguments show that points P, M and Q lie on one line and

$$PQ = PM + MN + NQ = \frac{AB + (BC + AD) + CD}{2}.$$

5.18. Let F be the intersection point of lines DE and BC; let K be the midpoint of segment EC. Segment CD is simultaneosly a bisector and a height of triangle ECF, hence, $ED = DF$ and, therefore, $DK \parallel FC$. Median DK of right triangle EDC is twice shorter its hypothenuse EC (Problem 5.16), hence, $AD = DK = \frac{1}{2}EC$.

5.19. Let the sum of the angles at the base AD of trapezoid $ABCD$ be equal to 90°. Denote the intersection point of lines AB and CD by O. Point O lies on the line that passes through the midpoints of the bases. Let us draw through point C line CK parallel to this line and line CE parallel to line AB (points K and E lie on base AD). Then CK is a median of right triangle ECD, hence, $CK = \frac{ED}{2} = \frac{AD - BC}{2}$ (cf. Problem 5.16).

5.20. It is clear that $\angle CEB = \angle A + \angle ACE = \angle BCK + \angle KCE = \angle BCE$.

5.21. Segments CF and DK are bisectors in similar triangles ACB and CDB and, therefore, $AB : FB = CB : KB$. Hence, $FK \parallel AC$. We similarly prove that $LF \parallel CB$.

Therefore, $CLFK$ is a rectangle whose diagonal CF is the bisector of angle LCK, i.e., the rectangle is a square.

5.22. Since $\frac{\sin\angle ACQ}{AQ} = \frac{\sin\angle AQC}{AC}$, it follows that

$$\frac{\sin\alpha}{a} = \frac{\sin(180^\circ - \alpha - 90^\circ - \varphi)}{a\cos\varphi} = \frac{\cos(\alpha+\varphi)}{a\cos\varphi},$$

where a is the (length of the) side of square $ABPQ$ and $\varphi = \angle CAB$. Hence, $\cot\alpha = 1+\tan\varphi$. Similarly,

$$\cot\gamma = 1 + \tan(90^\circ - \varphi) = 1 + \cot\varphi.$$

It follows that

$$\tan\alpha + \tan\gamma = \frac{1}{1+\tan\varphi} + \frac{1}{1+\cot\varphi} = 1$$

and, therefore,

$$\cos\alpha\cos\gamma = \cos\alpha\sin\gamma + \cos\gamma\sin\alpha = \sin(\alpha+\gamma) = \cos\beta.$$

5.23. By Pythagoras theorem

$$AP^2 + BQ^2 + CR^2 + (AM^2 - PM^2) + (BM^2 - QM^2) + (CM^2 - RM^2)$$

and

$$PB^2 + QC^2 + RA^2 = (BM^2 - PM^2) + (CM^2 - QM^2) + (AM^2 - RM^2).$$

These equations are equal.

Since

$$AP^2 + BQ^2 + CR^2 = (a - PB)^2 + (a - QC)^2 + (a - RA)^2 =$$
$$3a^2 - 2a(PB + QC + RA) + PB^2 + QC^2 + RA^2,$$

where $a = AB$, it follows that $PB + QC + RA = \frac{3}{2}a$.

5.24. Let point F divide segment BC in the ratio of $CF : FB = 1 : 2$; let P and Q be the intersection points of segment AF with BD and CE, respectively. It is clear that triangle OPQ is an equilateral one. Making use of the result of Problem 1.3 it is easy to verify that $AP : PF = 3 : 4$ and $AQ : QF = 6 : 1$. Hence, $AP : PQ : QF = 3 : 3 : 1$ and, therefore, $AP = PQ = OP$. Hence, $\angle AOP = \frac{180^\circ - \angle APO}{2} = 30^\circ$ and $\angle AOC = \angle AOP + \angle POQ = 90^\circ$.

5.25. Let A and B, C and D, E and F be the intersection points of the circle with sides PQ, QR, RP, respectively, of triangle PQR. Let us consider median PS. It connects the midpoints of parallel chords FA and DC and, therefore, is perpendicular to them. Hence, PS is a height of triangle PQR and, therefore, $PQ = PR$. Similarly, $PQ = QR$.

5.26. Let H be the intersection point of heights AA_1, BB_1 and CC_1 of triangle ABC. By hypothesis, $A_1H \cdot BH = B_1H \cdot AH$. On the other hand, since points A_1 and B_1 lie on the circle with diameter AB, then $AH \cdot A_1H = BH \cdot B_1H$. It follows that $AH = BH$ and $A_1H = B_1H$ and, therefore, $AC = BC$. Similarly, $BC = AC$.

5.27. a) Suppose that triangle ABC is not an equilateral one; for instance, $a \neq b$. Since $a + h_a = a + b\sin\gamma$ and $b + h_b = b + a\sin\gamma$, it follows that $(a-b)(1-\sin\gamma) = 0$; hence, $\sin\gamma = 0$, i.e., $\gamma = 90^\circ$. But then $a \neq c$ and similar arguments show that $\beta = 90^\circ$. Contradiction.

b) Let us denote the (length of the) side of the square two vertices of which lie on side BC by x. The similarity of triangles ABC and APQ, where P and Q are the vertices of the square that lie on AB and AC, respectively, yields $\frac{x}{a} = \frac{h_a - x}{h_a}$, i.e., $x = \frac{ah_a}{a+h_a} = \frac{2S}{a+h_a}$.

Similar arguments for the other squares show that $a + h_a = b + h_b = c + h_c$.

5.28. If α, β and γ are the angles of triangle ABC, then the angles of triangle $A_1B_1C_1$ are equal to $\frac{\beta+\gamma}{2}$, $\frac{\gamma+\alpha}{2}$ and $\frac{\alpha+\beta}{2}$. Let, for definiteness, $\alpha \geq \beta \geq \gamma$. Then $\frac{\alpha+\beta}{2} \geq \frac{\alpha+\gamma}{2} \geq \frac{\beta+\gamma}{2}$. Hence, $\alpha = \frac{\alpha+\beta}{2}$ and $\gamma = \frac{\beta+\gamma}{2}$, i.e., $\alpha = \beta$ and $\beta = \gamma$.

5.29. In any triangle a height is longer than the diameter of the inscribed circle. Therefore, the lengths of heights are integers greater than 2, i.e., all of them are not less than 3. Let S be the area of the triangle, a the length of its longest side and h the corresponding height.

Suppose that the triangle is not an equilateral one. Then its perimeter P is shorter than $3a$. Therefore, $3a > P = \mathrm{Pr} = 2S = ha$, i.e., $h < 3$. Contradiction.

5.30. Since the outer angle at vertex A of triangle ABA_1 is equal to 120° and $\angle A_1AB_1 = 60^\circ$, it follows that AB_1 is the bisector of this outer angle. Moreover, BB_1 is the bisector of the outer angle at vertex B, hence, A_1B_1 is the bisector of angle $\angle AA_1C$. Similarly, A_1C_1 is the bisector of angle $\angle AA_1B$. Hence,

$$\angle B_1A_1C_1 = \frac{\angle AA_1C + \angle AA_1B}{2} = 90^\circ.$$

5.31. Thanks to the solution of the preceding problem ray A_1C_1 is the bisector of angle $\angle AA_1B$. Let K be the intersection point of the bisectors of triangle A_1AB. Then

$$\angle C_1KO = \angle A_1KB = 90^\circ + \frac{\angle A}{2} = 120^\circ.$$

Hence, $\angle C_1KO + \angle C_1AO = 180^\circ$, i.e., quadrilateral $AOKC_1$ is an inscribed one. Hence, $\angle A_1C_1O = \angle KC_1O = \angle KAO = 30^\circ$.

5.32. a) Let S be the circumscribed circle of triangle ABC, let S_1 be the circle symmetric to S through line BC. The orthocenter H of triangle ABC lies on circle S_1 (Problem 5.9) and, therefore, it suffices to verify that the center O of circle S also belongs to S_1 and the bisector of the outer angle A passes through the center of circle S_1. Then $POAH$ is a rhombus, because $PO \parallel HA$.

Let PQ be the diameter of circle S perpendicular to line BC; let points P and A lie on one side of line BC. Then AQ is the bisector of angle A and AP is the bisector of the outer angle $\angle A$. Since $\angle BPC = 120^\circ = \angle BOC$, point P is the center of circle S_1 and point O belongs to circle S_1.

b) Let S be the circumscribed circle of triangle ABC and Q the intersection point of the bisector of angle $\angle BAC$ with circle S. It is easy to verify that Q is the center of circle S_1 symmetric to circle S through line BC. Moreover, points O and H lie on circle S_1 and since $\angle BIC = 120^\circ$ and $\angle BI_aC = 60^\circ$ (cf. Problem 5.3), it follows that II_a is a diameter of circle S_1. It is also clear that $\angle OQI = \angle QAH = \angle AQH$, because $OQ \parallel AH$ and $HA = QO = QH$. Hence, points O and H are symmetric through line II_a.

5.33. On side AC of triangle ABC, construct outwards an equilateral triangle AB_1C. Since $\angle A = 120^\circ$, point A lies on segment BB_1. Therefore, $BB_1 = b + c$ and, moreover, $BC = a$ and $B_1C = b$, i.e., triangle BB_1C is the desired one.

5.34. a) Let M_1 and N_1 be the midpoints of segments BH and CH, respectively; let BB_1 and CC_1 be heights. Right triangles ABB_1 and BHC_1 have a common acute angle — the one at vertex B; hence, $\angle C_1HB = \angle A = 60^\circ$. Since triangle BMH is an isosceles one, $\angle BHM = \angle HBM = 30^\circ$. Therefore, $\angle C_1HM = 60^\circ - 30^\circ = 30^\circ = \angle BHM$, i.e., point M lies on the bisector of angle $\angle C_1HB$. Similarly, point N lies on the bisector of angle $\angle B_1HC$.

b) Let us make use of the notations of the preceding problem and, moreover, let B' and C' be the midpoints of sides AC and AB. Since $AC_1 = AC\cos\angle A = \frac{1}{2}AC$, it follows that $C_1C' = \frac{1}{2}|AB - AC|$. Similarly, $B_1B' = \frac{1}{2}|AB - AC|$, i.e., $B_1B' = C_1C'$. It follows that the parallel lines BB_1 and $B'O$, CC_1 and $C'O$ form not just a parallelogram but a rhombus. Hence, its diagonal HO is the bisector of the angle at vertex H.

5.35. Since

$$\angle BB_1C = \angle B_1BA + \angle B_1AB > \angle B_1BA = \angle B_1BC,$$

it follows that $BC > B_1C$. Hence, point K symmetric to B_1 through bisector CC_1 lies on side BC and not on its extension. Since $\angle CC_1B = 30^\circ$, we have $\angle B_1C_1K = 60^\circ$ and, therefore, triangle B_1C_1K is an equilateral one. In triangles BC_1B_1 and BKB_1 side BB_1 is a common one and sides C_1B_1 and KB_1 are equal; the angles C_1BB_1 and KBB_1 are also equal but these angles are not the ones between equal sides. Therefore, the following two cases are possible:

1) $\angle BC_1B_1 = \angle BKB_1$. Then $\angle BB_1C_1 = \angle BB_1K = \frac{60^\circ}{2} = 30^\circ$. Therefore, if O is the intersection point of bisectors BB_1 and CC_1, then

$$\angle BOC = \angle B_1OC_1 = 180^\circ - \angle OC_1B_1 - \angle OB_1C_1 = 120^\circ.$$

On the other hand, $\angle BOC = 90^\circ + \frac{\angle A}{2}$ (cf. Problem 5.3), i.e., $\angle A = 60^\circ$.

2) $\angle BC_1B_1 + \angle BKB_1 = 180^\circ$. Then quadrilateral BC_1B_1K is an inscribed one and since triangle B_1C_1K is an equilateral one, $\angle B = 180^\circ - \angle C_1B_1K = 120^\circ$.

5.36. Let BM be a median, AK a bisector of triangle ABC and $BM \perp AK$. Line AK is a bisector and a height of triangle ABM, hence, $AM = AB$, i.e., $AC = 2AM = 2AB$. Therefore, $AB = 2$, $BC = 3$ and $AC = 4$.

5.37. Let a and b be legs and c the hypothenuse of the given triangle. If numbers a and b are odd, then the remainder after division of $a^2 + b^2$ by 4 is equal to 2 and $a^2 + b^2$ cannot be a perfect square. Hence, one of the numbers a and b is even and another one is odd; let, for definiteness, $a = 2p$. The numbers b and c are odd, hence, $c + b = 2q$ and $c - b = 2r$ for some q and r. Therefore, $4p^2 = a^2 = c^2 - b^2 = 4qr$. If d is a common divisor of q and r, then $a = 2\sqrt{qr}$, $b = q - r$ and $c = q + r$ are divisible by d. Therefore, q and r are relatively prime, ??? since $p^2 = qr$, it follows that $q = m^2$ and $r = n^2$. As a result we get $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$.

It is also easy to verify that if $a = 2mn$, $b = m^2 - n^2$ and $c = m^2 + n^2$, then $a^2 + b^2 = c^2$.

5.38. Let p be the semiperimeter of the triangle and a, b, c the lengths of the triangle's sides. By Heron's formula $S^2 = p(p-a)(p-b)(p-c)$. On the other hand, $S^2 = p^2r^2 = p^2$ since $r = 1$. Hence, $p = (p-a)(p-b)(p-c)$. Setting $x = p - a$, $y = p - b$, $z = p - c$ we rewrite our equation in the form

$$x + y + z = xyz.$$

Notice that p is either integer or half integer (i.e., of the form $\frac{2n+1}{2}$, where n is an integer) and, therefore, all the numbers x, y, z are simultaneously either integers or half integers. But if they are half integers, then $x + y + z$ is a half integer and xyz is of the form $\frac{m}{8}$, where m is an odd number. Therefore, numbers x, y, z are integers. Let, for definiteness, $x \leq y \leq z$. Then $xyz = x + y + z \leq 3z$, i.e., $xy \leq 3$. The following three cases are possible:

1) $x = 1$, $y = 1$. Then $2 + z = z$ which is impossible.

2) $x = 1$, $y = 2$. Then $3 + z = 2z$, i.e., $z = 3$.

3) $x = 1$, $y = 3$. Then $4 + z = 3z$, i.e., $z = 2 < y$ which is impossible.

Thus, $x = 1$, $y = 2$, $z = 3$. Therefore, $p = x + y + z = 6$ and $a = p - x = 5$, $b = 4$, $c = 3$.

5.39. Let a_1 and b_1, a_2 and b_2 be the legs of two distinct Pythagorean triangles, c_1 and c_2 their hypothenuses. Let us take two perpendicular lines and mark on them segments $OA = a_1a_2$, $OB = a_1b_2$, $OC = b_1b_2$ and $OD = a_2b_1$ (Fig. 57). Since $OA \cdot OC = OB \cdot OD$, quadrilateral $ABCD$ is an inscribed one. By Problem 2.71

$$4R^2 = OA^2 + OB^2 + OC^2 + OD^2 = (c_1c_2)^2,$$

i.e., $R = \frac{c_1c_2}{2}$. Magnifying, if necessary, quadrilateral $ABCD$ twice, we get the quadrilateral to be found.

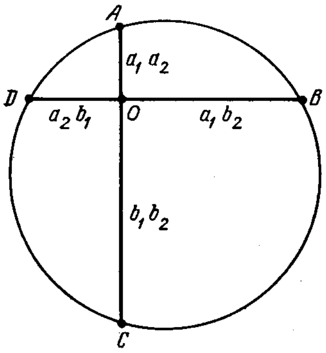


Figure 57 (Sol. 5.39)

5.40. a) The lengths of hypothenuses of right triangles with legs 5 and 12, 9 and 12 are equal to 13 and 15, respectively. Identifying the equal legs of these triangles we get a triangle whose area is equal to $\frac{12(5+9)}{2} = 84$.

b) First, suppose that the length of the shortest side of the given triangle is an even number, i.e., the lengths of the sides of the triangle are equal to $2n$, $2n+1$, $2n+2$. Then by Heron's formula

$$16S^2 = (6n+3)(2n+3)(2n+1)(2n-1) = 4(3n^2+6n+2)(4n^2-1) + 4n^2 - 1.$$

We have obtained a contradiction since the number in the right-hand side is not divisible by 4. Consecutively, the lengths of the sides of the triangle are equal to $2n-1$, $2n$ and $2n+1$, where $S^2 = 3n^2(n^2-1)$. Hence, $S = nk$, where k is an integer and $k^2 = 3(n^2-1)$. It is also clear that k is the length of the height dropped to the side of length $2n$. This height divides the initial triangle into two right triangles with a common leg of length k and hypothenuses of length $2n+1$ and $2n-1$ the squares of the lengths of the other legs of these triangles are equal to

$$(2n \pm 1)^2 - k^2 = 4n^2 \pm 4n + 1 - 3n^2 + 3 = (n \pm 2)^2.$$

5.41. a) Since $AB^2 - AB_1^2 = BB_1^2 = BC^2 - (AC \pm AB_1)^2$, we see that $AB_1 = \pm\frac{AB^2+AC^2-BC^2}{2AC}$.

b) Let diagonals AC and BD meet at point O. Let us prove, for example, that the number $q = \frac{BO}{OD}$ is a rational one (then the number $OD = \frac{BD}{q+1}$ is also a rational one). In triangles ABC and ADC draw heights BB_1 and DD_1. By heading a) the numbers AB_1 and CD_1 — the lengths of the corresponding sides — are rational and, therefore, the number B_1D_1 is also rational.

Let E be the intersection point of line BB_1 and the line that passes through point D parallel to AC. In right triangle BDE, we have $ED = B_1D_1$ and the lengths of leg ED and hypothenuse BD are rational numbers; hence, BE^2 is also a rational number. From triangles ABB_1 and CDD_1 we derive that numbers BB_1^2 and DD_1^2 are rational. Since

$$BE^2 = (BB_1 + DD_1)^2 = BB_1^2 + DD_1^2 + 2BB_1 \cdot DD_1,$$

number $BB_1 \cdot DD_1$ is rational. It follows that the number

$$\frac{BO}{OD} = \frac{BB_1}{DD_1} = \frac{BB_1 \cdot DD_1}{DD_1^2}$$

is a rational one.

5.42. Triangles ABC and $A_1B_1C_1$ cannot have two pairs of corresponding angles whose sum is equal to 180° since otherwise their sum would be equal to 360° and the third angles of these triangles should be equal to zero. Now, suppose that the angles of the first triangle are equal to α, β and γ and the angles of the second one are equal to $180^\circ - \alpha$, β and γ. The sum of the angles of the two triangles is equal to 360°, hence, $180^\circ + 2\beta + 2\gamma = 360^\circ$, i.e., $\beta + \gamma = 90^\circ$. It follows that $\alpha = 90^\circ = 180^\circ - \alpha$.

5.43. Clearly, $\overrightarrow{A_1C} = \overrightarrow{BO}$ and $\overrightarrow{CB_1} = \overrightarrow{OA}$, hence, $\overrightarrow{A_1B_1} = \overrightarrow{BA}$. Similarly, $\overrightarrow{B_1C_1} = \overrightarrow{CB}$ and $\overrightarrow{C_1A_1} = \overrightarrow{AC}$, i.e., $\triangle ABC = \triangle A_1B_1C_1$. Moreover, ABA_1B_1 and ACA_1C_1 are parallelograms. It follows that segments BB_1 and CC_1 pass through the midpoint of segment AA_1.

5.44. Since $\angle MAO = \angle PAO = \angle AOM$, it follows that $AMOP$ is a rhombus. Similarly, $BNOQ$ is a rhombus. It follows that

$$MN = MO + ON = AM + BN \quad \text{and} \quad OP + PQ + QO = AP + PQ + QB = AB.$$

5.45. a) Through vertices of triangle ABC let us draw lines parallel to the triangle's opposite sides. As a result we get triangle $A_1B_1C_1$; the midpoints of the sides of the new triangle are points A, B and C. The heights of triangle ABC are the midperpendiculars to the sides of triangle $A_1B_1C_1$ and, therefore, the center of the circumscribed circle of triangle $A_1B_1C_1$ is the intersection point of heights of triangle ABC.

b) Point H is the center of the circumscribed circle of triangle $A_1B_1C_1$, hence,

$$4R^2 = B_1H^2 = B_1A^2 + AH^2 = BC^2 + AH^2.$$

Therefore,

$$AH^2 = 4R^2 - BC^2 = \left(\frac{1}{\sin^2\alpha} - 1\right) BC^2 = (BC\cot\alpha)^2.$$

5.46. Let AD be the bisector of an equilateral triangle ABC with base AB and angle 36° at vertex C. Then triangle ACD is an isosceles one and $\triangle ABC \sim \triangle BDA$. Therefore, $CD = AD = AB = 2xBC$ and $DB = 2xAB = 4x^2BC$; hence,

$$BC = CD + DB = (2x + 4x^2)BC.$$

5.47. Let B_1 and B_2 be the projections of point A to bisectors of the inner and outer angles at vertex B; let M the midpoint of side AB. Since the bisectors of the inner and outer angles are perpendicular, it follows that AB_1BB_2 is a rectangular and its diagonal B_1B_2 passes through point M. Moreover,

$$\angle B_1MB = 180^\circ - 2\angle MBB_1 = 180^\circ - \angle B.$$

Hence, $B_1B_2 \parallel BC$ and, therefore, line B_1B_2 coincides with line l that connects the midpoints of sides AB and AC.

We similarly prove that the projections of point A to the bisectors of angles at vertex C lie on line l.

5.48. Suppose that the bisectors of angles A and B are equal but $a > b$. Then $\cos\frac{1}{2}\angle A < \cos\frac{1}{2}\angle B$ and $\frac{1}{c} + \frac{1}{b} > \frac{1}{c} + \frac{1}{a}$, i.e., $\frac{bc}{b+c} < \frac{ac}{a+c}$. By multiplying these inequalities we get a contradiction, since $l_a = \frac{2bc\cos\frac{\angle A}{2}}{b+c}$ and $l_b = \frac{2ac\cos\frac{\angle B}{2}}{a+c}$ (cf. Problem 4.47).

5.49. a) By Problem 4.47 the length of the bisector of angle $\angle B$ of triangle ABC is equal to $\frac{2ac\cos\frac{\angle B}{2}}{a+c}$ and, therefore, it suffices to verify that the system of equations

$$\frac{ac}{a+c} = p, \quad a^2 + c^2 - 2ac\cos\angle B = q$$

has (up to a transposition of a with c) a unique positive solution. Let $a+c=u$. Then $ac=pu$ and $q=u^2-2pu(1+\cos\beta)$. The product of the roots of this quadratic equation for u is equal to $-q$ and, therefore, it has one positive root. Clearly, the system of equations

$$a+c=u, \quad ac=pu$$

has a unique solution.

b) In triangles AA_1B and CC_1B, sides AA_1 and CC_1 are equal; the angles at vertex B are equal, and the bisectors of the angles at vertex B are also equal. Therefore, these triangles are equal and either $AB=BC$ or $AB=BC_1$. The second equality cannot take place.

5.50. Let points M and N lie on sides AB and AC. If r_1 is the radius of the circle whose center lies on segment MN and which is tangent to sides AB and AC, then $S_{AMN}=qr_1$, where $q=\frac{AM+AN}{2}$. Line MN passes through the center of the inscribed circle if and only if $r_1=r$, i.e., $\frac{S_{AMN}}{q}=\frac{S_{ABC}}{p}=\frac{S_{BCNM}}{p-q}$.

5.51. a) On the extension of segment AC beyond point C take a point B' such that $CB'=CB$. Triangle BCB' is an isosceles one; hence, $\angle AEB=\angle ACB=2\angle CBB'$ and, therefore, E is the center of the circumscribed circle of triangle ABB'. It follows that point F divides segment AB' in halves; hence, line C_1F divides the perimeter of triangle ABC in halves.

b) It is easy to verify that the line drawn through point C parallel to BB' is the bisector of angle ACB. Since $C_1F \parallel BB'$, line C_1F is the bisector of the angle of the triangle with vertices at the midpoints of triangle ABC. The bisectors of this new triangle meet at one point.

5.52. Let X be the intersection point of lines AD_2 and CD_1; let M, E_1 and E_2 be the projections of points X, D_1 and D_2, respectively, to line AC. Then $CE_2=CD_2\sin\gamma=a\sin\gamma$ and $AE_1=c\sin\alpha$. Since $a\sin\gamma=c\sin\alpha$, it follows that $CE_2=AE_1=q$. Hence,

$$\frac{XM}{AM}=\frac{D_2E_2}{AE_2}=\frac{a\cos\gamma}{b+q} \quad\text{and}\quad \frac{XM}{CM}=\frac{c\cos\alpha}{b+q}.$$

Therefore, $AM:CM=c\cos\alpha : a\cos\gamma$. Height BH divides side AC in the same ratio.

5.53. a) By the law of cosines

$$B_1C_1^2=AC_1^2+AB_1^2-2AC_1\cdot AB_1\cdot\cos(90^\circ+\alpha),$$

i.e.,

$$a_1^2=\frac{c^2}{2}+\frac{b^2}{2}+bc\sin\alpha=\frac{b^2+c^2}{2}+2S.$$

Writing similar equalities for b_1^2 and c_1^2 and taking their sum we get the statement desired.

b) For an acute triangle ABC, add to S the areas of triangles ABC_1, AB_1C and A_1BC; add to S_1 the areas of triangles AB_1C_1, A_1BC_1 and A_1B_1C. We get equal quantities (for a triangle with an obtuse angle $\angle A$ the area of triangle AB_1C_1 should be taken with a minus sign). Hence,

$$S_1=S+\frac{a^2+b^2+c^2}{4}-\frac{ab\cos\gamma+ac\cos\beta+bc\cos\alpha}{4}.$$

It remains to notice that

$$ab\cos\gamma+bc\cos\alpha+ac\cos\beta=2S(\cot\gamma+\cot\alpha+\cot\beta)=\frac{a^2+b^2+c^2}{2};$$

cf. Problem 12.44 a).

5.54. First, let us prove that point B' lies on the circumscribed circle of triangle AHC, where H is the intersection point of heights of triangle ABC. We have

$$\angle(AB', B'C) = \angle(AA_1, CC_1) = \angle(AA_1, BC) + \angle(BC, AB) + \angle(AB, CC_1) = \angle(BC, AB).$$

But as follows from the solution of Problem 5.9 $\angle(BC, AB) = \angle(AH, HC)$ and, therefore, points A, B', H and C lie on one circle and this circle is symmetric to the circumscribed circle of triangle ABC through line AC. Hence, both these circles have the same radius, R, consequently,

$$B'H = 2R\sin B'AH = 2R\cos\alpha.$$

Similarly, $A'H = 2R\cos\alpha = C'H$. This completes solution of heading a); to solve heading b) it remains to notice that $\triangle A'B'C' \sim \triangle ABC$ since after triangle $A'B'C'$ is rotated through an angle of α its sides become parallel to the sides of triangle ABC.

5.55. Let $a_1 = BA_1$, $a_2 = A_1C$, $b_1 = CB_1$, $b_2 = B_1A$, $c_1 = AC_1$ and $c_2 = C_1B$. The products of the lengths of segments of intersecting lines that pass through one point are equal and, therefore, $a_1(a_1 + x) = c_2(c_2 - z)$, i.e.,

$$a_1x + c_2z = c_2^2 - a_1^2.$$

We similarly get two more equations for x, y and z:

$$b_1y + a_2x = a_2^2 - b_1^2 \quad \text{and} \quad c_1z + b_2y = b_2^2 - c_1^2.$$

Let us multiply the first equation by b^{2n}; multiply the second and the third ones by c^{2n} and a^{2n}, respectively, and add the equations obtained. Since, for instance, $c_2b^n - c_1a^n = 0$ by the hypothesis, we get zero in the right-hand side. The coefficient of, say, x in the left-hand side is equal to

$$a_1b^{2n} + a_2c^{2n} = \frac{ac^nb^{2n} + ab^nc^{2n}}{b^n + c^n} = ab^nc^n.$$

Hence,

$$ab^nc^nx + ba^nc^ny + ca^nb^nz = 0.$$

Dividing both sides of this equation by $(abc)^n$ we get the statement desired.

5.56. Let in the initial triangle $\angle A = 3\alpha$, $\angle B = 3\beta$ and $\angle C = 3\gamma$. Let us take an equilateral triangle $A_2B_2C_2$ and construct on its sides as on bases isosceles triangles A_2B_2R, B_2C_2P and C_2A_2Q with angles at the bases equal to $60^\circ - \gamma$, $60^\circ - \alpha$, $60^\circ - \beta$, respectively (Fig. 58).

Let us extend the lateral sides of these triangles beyond points A_2, B_2 and C_2; denote the intersection point of the extensions of sides RB_2 and QC_2 by A_3, that of PC_2 and RA_2 by B_3, that of QA_2 and PB_2 by C_3.Through point B_2 draw the line parallel to A_2C_2 and denote by M and N the its intersection points with lines QA_3 and QC_3, respectively. Clearly, B_2 is the midpoint of segment MN. Let us compute the angles of triangles B_2C_3N and B_2A_3M:

$$\angle C_3B_2N = \angle PB_2M = \angle C_2B_2M = \angle C_2B_2P = \alpha;$$
$$\angle B_2NC_3 = 180^\circ - \angle C_2A_2Q = 120^\circ + \beta;$$

hence, $\angle B_2C_3N = 180^\circ - \alpha - (120^\circ + \beta) = \gamma$. Similarly, $\angle A_3B_2M = \gamma$ and $\angle B_2A_3M = \alpha$. Hence, $\triangle B_2C_3N \sim \triangle A_3B_2M$. It follows that $C_3B_2 : B_2A_3 = C_3N : B_2M$ and since $B_2M = B_2N$ and $\angle C_3B_2A_3 = \angle C_3NB_2$, it follows that $C_3B_2 : B_2A_3 = C_3N : NB_2$ and $\triangle C_3B_2A_3 \sim \triangle C_3NB_2$; hence, $\angle B_2C_3A_3 = \gamma$.

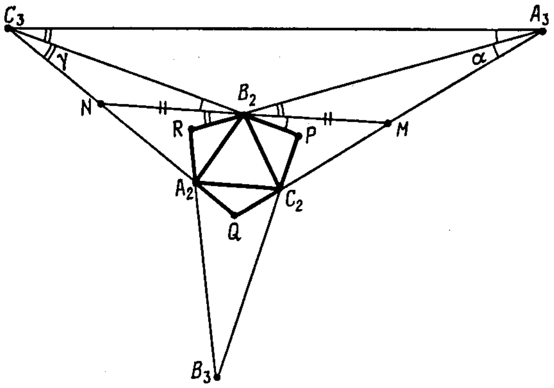


Figure 58 (Sol. 5.56)

Similarly, $\angle A_2C_3B_3 = \gamma$ and, therefore, $\angle A_3C_3B_3 = 3\gamma = \angle C$ and C_3B_3, C_3A_2 are the trisectors of angle C_3 of triangle $A_3B_3C_3$. Similar arguments for vertices A_3 and B_3 show that $\triangle ABC \sim \triangle A_3B_3C_3$ and the intersection points of the trisectors of triangle $A_3B_3C_3$ are vertices of an equilateral triangle $A_2B_2C_2$.

5.57. Point A_1 lies on the bisector of angle $\angle BAC$, hence, point A lies on the extension of the bisector of angle $\angle B_2A_1C_2$. Moreover, $\angle B_2AC_2 = \alpha = \frac{180^\circ - \angle B_2A_1C_2}{2}$. Hence, A is the center of an escribed circle of triangle $B_2A_1C_2$ (cf. Problem 5.3). Let D be the intersection point of lines AB and CB_2. Then

$$\angle AB_2C_2 = \angle AB_2D = 180^\circ - \angle B_2AD - \angle ADB_2 = 180^\circ - \gamma - (60^\circ + \alpha) = 60^\circ + \beta.$$

Since

$$\angle AB_2C = 180^\circ - (\alpha + \beta) - (\beta + \gamma) = 120^\circ - \beta,$$

it follows that

$$\angle CB_2C_2 = \angle AB_2C - \angle AB_2C_2 = 60^\circ - 2\beta.$$

Similarly, $\angle AB_2A_2 = 60^\circ - 2\beta$. Hence,

$$\angle A_2B_2C_2 = \angle AB_2C - \angle AB_2A_2 - \angle CB_2C_2 = 3\beta.$$

Similarly, $\angle B_2A_2C_2 = 3\alpha$ and $\angle A_2C_2B_2 = 3\gamma$.

5.58. Let the projection to a line perpendicular to line A_1B_1 send points A, B and C to A', B' and C', respectively; point C_1 to Q and points A_1 and B_1 into one point, P. Since

$$\frac{\overline{A_1B}}{\overline{A_1C}} = \frac{\overline{PB'}}{\overline{PC'}}, \quad \frac{\overline{B_1C}}{\overline{B_1A}} = \frac{\overline{PC'}}{\overline{PA'}} \quad \text{and} \quad \frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{QA'}}{\overline{QB'}},$$

it follows that

$$\frac{\overline{A_1B}}{\overline{A_1C}} \cdot \frac{\overline{B_1C}}{\overline{B_1A}} \cdot \frac{\overline{C_1A}}{\overline{C_1B}} = \frac{\overline{PB'}}{\overline{PC'}} \cdot \frac{\overline{PC'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{\overline{PB'}}{\overline{PA'}} \cdot \frac{\overline{QA'}}{\overline{QB'}} = \frac{b'}{a'} \cdot \frac{a'+x}{b'+x},$$

where $|x| = PQ$. The equality $\frac{b'}{a'} \cdot \frac{a'+x}{b'+x} = 1$ is equivalent to the fact that $x = 0$. (We have to take into account that $a' \neq b'$ since $A' \neq B'$.) But the equality $x = 0$ means that $P = Q$, i.e., point C_1 lies on line A_1B_1.

5.59. Let point P lie on arc $\smile BC$ of the circumscribed circle of triangle ABC. Then

$$\frac{\overline{BA_1}}{\overline{CA_1}} = -\frac{BP\cos\angle PBC}{CP\cos\angle PCB}, \quad \frac{\overline{CB_1}}{\overline{AB_1}} = -\frac{CP\cos\angle PCA}{AP\cos PAC}, \quad \frac{\overline{AC_1}}{\overline{BC_1}} = -\frac{AP\cos\angle PAB}{PB\cos\angle PBA}.$$

By multiplying these equalities and taking into account that

$$\angle PAC = \angle PBC, \quad \angle PAB = \angle PCB \quad \text{and} \quad \angle PAC + \angle PBA = 180^\circ$$

we get

$$\frac{\overline{BA_1}}{\overline{CA_1}} \cdot \frac{\overline{CB_1}}{\overline{AB_1}} \cdot \frac{\overline{AC_1}}{\overline{BC_1}} = 1.$$

5.60. Let O, O_1 and O_2 be the centers of circles S, S_1 and S_2; let X be the intersection point of lines O_1O_2 and A_1A_2. By applying Menelaus's theorem to triangle OO_1O_2 and points A_1, A_2 and X we get

$$\frac{O_1X}{O_2X} \cdot \frac{O_2A_2}{OA_2} \cdot \frac{OA_1}{O_1A_1} = 1$$

and, therefore, $O_1X : O_2X = R_1 : R_2$, where R_1 and R_2 are the radii of circles S_1 and S_2, respectively. It follows that X is the intersection point of the common outer or common inner tangents to circles S_1 and S_2.

5.61. a) Let, for definiteness, $\angle B < \angle C$. Then $\angle DAE = \angle ADE = \angle B + \frac{\angle A}{2}$; hence, $\angle CAE = \angle B$. Since

$$\frac{BE}{AB} = \frac{\sin \angle BAE}{\sin \angle AEB} \quad \text{and} \quad \frac{AC}{CE} = \frac{\sin \angle AEC}{\sin \angle CAE},$$

it follows that

$$\frac{BE}{CE} = \frac{c \sin \angle BAE}{b \sin \angle CAE} = \frac{c \sin(\angle A + \angle B)}{b \sin \angle B} = \frac{c \sin \angle C}{b \sin \angle B} = \frac{c^2}{b^2}.$$

b) In heading a) point E lies on the extension of side BC since $\angle ADC = \angle BAD + \angle B > \angle CAD$. Therefore, making use of the result of heading a) and Menelaus's theorem we get the statement desired.

5.62. Since $\angle BCE = 90^\circ - \frac{\angle B}{2}$, we have: $\angle BCE = \angle BEC$ and, therefore, $BE = BC$. Hence,

$$CF : KF = BE : BK = BC : BK \quad \text{and} \quad AE : KE = CA : CK = BC : BK.$$

Let line EF intersect AC at point D. By Menelaus's theorem $\frac{AD}{CD} \cdot \frac{CF}{KF} \cdot \frac{KE}{AE} = 1$. Taking into account that $CF : KF = AE : KE$ we get the statement desired.

5.63. Proof is similar to that of Problem 5.79; we only have to consider the ratio of *oriented* segments and angles.

5.64. Let A_2, B_2 and C_2 be the intersection points of lines BC with B_1C_1, AC with A_1C_1, AB with A_1B_1, respectively. Let us apply Menelaus's theorem to the following triangles and points on their sides: OAB and (A_1, B_1, C_2), OBC and (B_1, C_1, A_2), OAC and (A_1, C_1, B_2). Then

$$\frac{\overline{AA_1}}{\overline{OA_1}} \cdot \frac{\overline{OB_1}}{\overline{BB_1}} \cdot \frac{\overline{BC_2}}{\overline{AC_2}} = 1, \quad \frac{\overline{OC_1}}{\overline{CC_1}} \cdot \frac{\overline{BB_1}}{\overline{OB_1}} \cdot \frac{\overline{CA_2}}{\overline{BA_2}} = 1, \quad \frac{\overline{OA_1}}{\overline{AA_1}} \cdot \frac{\overline{CC_1}}{\overline{OC_1}} \cdot \frac{\overline{AB_2}}{\overline{CB_2}} = 1.$$

By multiplying these equalities we get

$$\frac{\overline{BC_2}}{\overline{AC_2}} \cdot \frac{\overline{AB_2}}{\overline{CB_2}} \cdot \frac{\overline{CA_2}}{\overline{BA_2}} = 1.$$

Menelaus's theorem implies that points A_2, B_2, C_2 lie on one line.

5.65. Let us consider triangle $A_0B_0C_0$ formed by lines A_1B_2, B_1C_2 and C_1A_2 (here A_0 is the intersection point of lines A_1B_2 and A_2C_1, etc), and apply Menelaus's theorem to this triangle and the following five triples of points:

$$(A, B_2, C_1), \ (B, C_2, A_1), \ (C, A_2, B_1), \ (A_1, B_1, C_1) \ \text{and} \ (A_2, B_2, C_2).$$

As a result we get

$$\frac{\overline{B_0A}}{\overline{C_0A}}\cdot\frac{\overline{A_0B_2}}{\overline{B_0B_2}}\cdot\frac{\overline{C_0C_1}}{\overline{A_0C_1}}=1,\quad \frac{\overline{C_0B}}{\overline{A_0B}}\cdot\frac{\overline{B_0C_2}}{\overline{C_0C_2}}\cdot\frac{\overline{A_0A_1}}{\overline{B_0A_1}}=1,$$

$$\frac{\overline{A_0C}}{\overline{B_0C}}\cdot\frac{\overline{C_0A_2}}{\overline{A_0A_2}}\cdot\frac{\overline{B_0B_1}}{\overline{C_0B_1}}=1,\quad \frac{\overline{B_0A_1}}{\overline{A_0A_1}}\cdot\frac{\overline{C_0B_1}}{\overline{B_0B_1}}\cdot\frac{\overline{A_0C_1}}{\overline{C_0C_1}}=1, \tag{2}$$

$$\frac{\overline{A_0A_2}}{\overline{C_0A_2}}\cdot\frac{\overline{B_0B_2}}{\overline{A_0B_2}}\cdot\frac{\overline{C_0C_2}}{\overline{B_0C_2}}=1. \tag{3}$$

By multiplying these equalities we get $\frac{\overline{B_0A}}{\overline{C_0A}}\cdot\frac{\overline{C_0B}}{\overline{A_0B}}\cdot\frac{\overline{A_0C}}{\overline{B_0C}}=1$ and, therefore, points A, B and C lie on one line.

5.66. Let N be the intersection point of lines AD and KQ, P' the intersection point of lines KL and MN. By Desargue's theorem applied to triangles KBL and NDM we derive that P', A and C lie on one line. Hence, $P'=P$.

5.67. It suffices to apply Desargues's theorem to triangles AED and BFC and Pappus' theorem to triples of points (B,E,C) and (A,F,D).

5.68. a) Let R be the intersection point of lines KL and MN. By applying Pappus' theorem to triples of points (P,L,N) and (Q,M,K), we deduce that points A, C and R lie on one line.

b) By applying Desargues's theorem to triangles NDM and LBK we see that the intersection points of lines ND with LB, DM with BK, and NM with LK lie on one line.

5.69. Let us make use of the result of Problem 5.68 a). For points P and Q take points P_2 and P_4, for points A and C take points C_1 and P_1 and for K, L, M and N take points P_5, A_1, B_1 and P_3, respectively. As a result we see that line P_6C_1 passes through point P_1.

5.70. a) This problem is a reformulation of Problem 5.58 since the number $\overline{BA_1}:\overline{CA_1}$ is negative if point A_1 lies on segment BC and positive otherwise.

b) First, suppose that lines AA_1, BB_1 and CC_1 meet at point M. Any three (nonzero) vectors in plane are linearly dependent, i.e., there exist numbers λ, μ and ν (not all equal to zero) such that $\lambda\overrightarrow{AM}+\mu\overrightarrow{BM}+\nu\overrightarrow{CM}=0$. Let us consider the projection to line BC parallel to line AM. This projection sends points A and M to A_1 and points B and C into themselves. Therefore, $\mu\overrightarrow{BA_1}+\nu\overrightarrow{CA_1}=0$, i.e.,

$$\frac{\overline{BA_1}}{\overline{CA_1}}=-\frac{\nu}{\mu}.$$

Similarly,

$$\frac{\overline{CB_1}}{\overline{AB_1}}=-\frac{\lambda}{\nu}\quad\text{and}\quad\frac{\overline{AC_1}}{\overline{BC_1}}=-\frac{\mu}{\lambda}.$$

By multiplying these three equalities we get the statement desired.

If lines AA_1, BB_1 and CC_1 are parallel, in order to get the proof it suffices to notice that

$$\frac{\overline{BA_1}}{\overline{CA_1}}=\frac{\overline{BA}}{\overline{C_1A}}\quad\text{and}\quad\frac{\overline{CB_1}}{\overline{AB_1}}=\frac{\overline{C_1B}}{\overline{AB}}.$$

Now, suppose that the indicated relation holds and prove that then lines AA_1, BB_1 and CC_1 intersect at one point. Let C_1^* be the intersection point of line AB with the line that passes through point C and the intersection point of lines AA_1 and BB_1. For point C_1^* the same relation as for point C_1 holds. Therefore, $C_1^*A:C_1^*B=C_1A:C_1B$. Hence, $C_1^*=C_1$, i.e., lines AA_1, BB_1 and CC_1 meet at one point.

It is also possible to verify that if the indicated relation holds and two of the lines AA_1, BB_1 and CC_1 are parallel, then the third line is also parallel to them.

5.71. Clearly, $AB_1 = AC_1$, $BA_1 = BC_1$ and $CA_1 = CB_1$, and, in the case of the inscribed circle, on sides of triangle ABC, there are three points and in the case of an escribed circle there is just one point on sides of triangle ABC. It remains to make use of Ceva's theorem.

5.72. Let AA_1, BB_1 and CC_1 be heights of triangle ABC. Then

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{b\cos\angle A}{a\cos\angle B} \cdot \frac{c\cos\angle B}{b\cos\angle C} \cdot \frac{a\cos\angle C}{c\cos\angle A} = 1.$$

5.73. Let A_2, B_2 and C_2 be the midpoints of sides BC, CA and AB. The considered lines pass through the vertices of triangle $A_2B_2C_2$ and in heading a) they divide its sides in the same ratios in which lines AP, BP and CP divide sides of triangle ABC whereas in heading b) they divide them in the inverse ratios. It remains to make use of Ceva's theorem.

5.74. Since $\triangle AC_1B_2 \sim \triangle BC_1A_1$ and $\triangle AB_1C_2 \sim \triangle CB_1A_1$, it follows that $AB_2 \cdot C_1B = AC_1 \cdot BA_1$ and $AC_2 \cdot CB_1 = A_1C \cdot B_1A$. Hence,

$$\frac{AB_2}{AC_2} = \frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

5.75. Let lines AA_1, BB_1 and CC_1 intersect lines BC, CA and AB at points A_1, B_2 and C_2.

a) If $\angle B + \beta < 180^\circ$ and $\angle C + \gamma < 180^\circ$, then

$$\frac{BA_2}{A_2C} = \frac{S_{ABA_1}}{S_{ACA_1}} = \frac{AB \cdot BA_1 \sin(\angle B + \beta)}{AC \cdot CA_1 \sin(\angle C + \gamma)} = \frac{AB}{AC} \cdot \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin(\angle B + \beta)}{\sin(\angle C + \gamma)}.$$

The latter expression is equal to $\overline{BA_2} : \overline{A_2C}$ in all the cases. Let us write similar expressions for $\overline{CB_2} : \overline{B_2A}$ and $\overline{AC_2} : \overline{C_2B}$ and multiply them. Now it remains to make use of Ceva's theorem.

b) Point A_2 lies outside segment BC only if precisely one of the angles β and γ is greater than the corresponding angle $\angle B$ or $\angle C$. Hence,

$$\frac{\overline{BA_2}}{\overline{A_2C}} = \frac{AB}{AC} \cdot \frac{\sin\gamma}{\sin\beta} \cdot \frac{\sin(\angle B - \beta)}{\sin(\angle C - \gamma)}.$$

5.76. It is easy to verify that this problem is a particular case of Problem 5.75.

Remark. A similar statement is also true for an escribed circle.

5.77. The solution of the problem obviously follows from Ceva's theorem.

5.78. By applying the sine theorem to triangles ACC_1 and BCC_1 we get

$$\frac{AC_1}{C_1C} = \frac{\sin\angle ACC_1}{\sin\angle A} \quad\text{and}\quad \frac{CC_1}{C_1B} = \frac{\sin\angle B}{\sin\angle C_1CB},$$

i.e.,

$$\frac{AC_1}{C_1B} = \frac{\sin\angle ACC_1}{\sin\angle C_1CB} \cdot \frac{\sin\angle B}{\sin\angle A}.$$

Similarly,

$$\frac{BA_1}{A_1C} = \frac{\sin\angle BAA_1}{\sin\angle A_1AC} \cdot \frac{\sin\angle C}{\sin\angle B} \quad\text{and}\quad \frac{CB_1}{B_1A} = \frac{\sin\angle CBB_1}{\sin\angle B_1BA} \cdot \frac{\sin\angle A}{\sin\angle C}.$$

To complete the proof it remains to multiply these equalities.

Remark. A similar statement is true for the ratios of oriented segments and angles in the case when the points are taken on the extensions of sides.

5.79. We may assume that points A_2, B_2 and C_2 lie on the sides of triangle ABC. By Problem 5.78

$$\frac{AC_2}{C_2B}\cdot\frac{BA_2}{A_2C}\cdot\frac{CB_2}{B_2A}=\frac{\sin\angle ACC_2}{\sin\angle C_2CB}\cdot\frac{\sin\angle BAA_2}{\sin\angle A_2AC}\cdot\frac{\sin\angle CBB_2}{\sin\angle B_2BA}.$$

Since lines AA_2, BB_2 and CC_2 are symmetric to lines AA_1, BB_1 and CC_1, respectively, through the bisectors, it follows that $\angle ACC_2=\angle C_1CB$, $\angle C_2CB=\angle ACC_1$ etc., hence,

$$\frac{\sin\angle ACC_2}{\sin\angle C_2CB}\cdot\frac{\sin\angle BAA_2}{\sin\angle A_2AC}\cdot\frac{\sin\angle CBB_2}{\sin\angle B_2BA}=\frac{\sin\angle C_1CB}{\sin\angle ACC_1}\cdot\frac{\sin\angle A_1AC}{\sin\angle BAA_1}\cdot\frac{\sin\angle B_1BA}{\sin\angle CBB_1}=\frac{C_1B}{AC_1}\cdot\frac{A_1C}{BA_1}\cdot\frac{B_1A}{CB_1}=1.$$

Therefore,

$$\frac{AC_2}{C_2B}\cdot\frac{BA_2}{A_2C}\cdot\frac{CB_2}{B_2A}=1,$$

i.e., lines AA_2, BB_2 and CC_2 meet at one point.

Remark. The statement holds also in the case when points A_1, B_1 and C_1 are taken on the extensions of sides if only point P does not lie on the circumscribed circle S of triangle ABC; if P does lie on S, then lines AA_2, BB_2 and CC_2 are parallel (cf. Problem 2.90).

5.80. Let diagonals AD and BE of the given hexagon $ABCDEF$ meet at point P; let K and L be the midpoints of sides AB and ED, respectively. Since $ABDE$ is a trapezoid, segment KL passes through point P (by Problem 19.2). By the law of sines

$$\sin\angle APK:\sin\angle AKP=AK:AP\quad\text{and}\quad\sin\angle BPK:\sin\angle BKP=BK:BP.$$

Since $\sin\angle AKP=\sin\angle BKP$ and $AK=BK$, we have

$$\sin\angle APK:\sin\angle BPK=BP:AP=BE:AD.$$

Similar relations can be also written for the segments that connect the midpoints of the other two pairs of the opposite sides. By multiplying these relations and applying the result of Problem 5.78 to the triangle formed by lines AD, BE and CF, we get the statement desired.

5.81. Let us consider the homothety with center P and coefficient 2. Since $PA_1A_3A_2$ is a rectangle, this homothety sends line A_1A_2 into line l_a that passes through point A_3; lines l_a and A_3P are symmetric through line A_3A. Line A_3A divides the angle $B_3A_3C_3$ in halves (Problem 1.56 a)).

We similarly prove that lines l_b and l_c are symmetric to lines B_3P and C_3P, respectively, through bisectors of triangle $A_3B_3C_3$. Therefore, lines l_a, l_b and l_c either meet at one point or are parallel (Problem 1.79) and, therefore, lines A_1A_2, B_1B_2 and C_1C_2 meet at one point.

5.82. By Problems 5.78 and 5.70 b)) we have

$$\frac{\sin\angle ASP}{\sin\angle PSD}\cdot\frac{\sin\angle DAP}{\sin\angle PAS}\cdot\frac{\sin\angle SDP}{\sin\angle PDA}=1=\frac{\sin\angle ASQ}{\sin\angle QSD}\cdot\frac{\sin\angle DAQ}{\sin\angle QAS}\cdot\frac{\sin\angle SDQ}{\sin\angle QDA}.$$

But

$$\angle DAP=\angle SDQ,\ \ \angle SDP=\angle DAQ,\ \ \angle PAS=\angle QDA\ \text{ and }\ \angle PDA=\angle QAS.$$

Hence,

$$\frac{\sin\angle ASP}{\sin\angle PSD}=\frac{\sin\angle ASQ}{\sin\angle QSD}.$$

This implies that points S, P and Q lie on one line, since the function $\frac{\sin(\alpha-x)}{\sin x}$ is monotonous with respect to x: indeed,

$$\frac{d}{dx}\left(\frac{\sin(\alpha-x)}{\sin x}\right)=-\frac{\sin\alpha}{\sin^2 x}.$$

5.83. a) By Ceva's theorem

$$\frac{AC_1}{C_1B}=\frac{CA_1}{A_1B}\cdot\frac{AB_1}{B_1C}$$

and by the law of sines

$$\begin{array}{rclrcl} CA_1 & = & \frac{CA\sin\angle CAA_1}{\sin\angle AA_1B}, & A_1B & = & \frac{AB\sin\angle BAA_1}{\sin\angle AA_1B}, \\ AB_1 & = & \frac{AB\sin\angle ABB_1}{\sin\angle AB_1B}, & B_1C & = & \frac{BC\sin\angle CBB_1}{\sin\angle AB_1B}. \end{array}$$

Substituting the last four identities in the first identity and taking into account that $AC=BC$, we get the statement desired.

b) Let us denote the intersection points of lines CM and CN with base AB by M_1 and N_1, respectively. We have to prove that $M_1=N_1$. From heading a) it follows that $AM_1:M_1B=AN_1:N_1B$, i.e., $M_1=N_1$.

5.84. Let segments BM and BN meet side AC at points P and Q, respectively. Then

$$\frac{\sin\angle PBB_1}{\sin PBA}=\frac{\sin\angle PBB_1}{\sin\angle BPB_1}\cdot\frac{\sin\angle APB}{\sin\angle PBA}=\frac{PB}{BB_1}\cdot\frac{AB}{PA}.$$

If O is the intersection point of bisectors of triangle ABC, then $\frac{AP}{PB_1}\cdot\frac{B_1O}{OB}\cdot\frac{BC_1}{C_1A}=1$ and, therefore,

$$\frac{\sin\angle PBB_1}{\sin\angle PBA}=\frac{AB}{BB_1}\cdot\frac{B_1O}{OB}\cdot\frac{BC_1}{C_1A}.$$

Observe that $BC_1:C_1A=BC:CA$ and perform similar calculations for $\sin\angle QBB_1:\sin\angle QBC$; we deduce that

$$\frac{\sin\angle PBB_1}{\sin\angle PBA}=\frac{\sin\angle QBB_1}{\sin\angle QBC}.$$

Since $\angle ABB_1=\angle CBB_1$, we have: $\angle PBB_1=\angle QBB_1$.

5.85. a) Let point P lie on arc $\smile AC$ of the circumscribed circle of triangle ABC; let A_1, B_1 and C_1 be the bases of perpendiculars dropped from point P to lines BC, CA and AB. The sum of angles at vertices A_1 and C_1 of quadrilateral A_1BC_1P is equal to 180°, hence, $\angle A_1PC_1=180^\circ-\angle B=\angle APC$. Therefore, $\angle APC_1=\angle A_1PC$, where one of points A_1 and C_1 (say, A_1) lies on a side of the triangle and the other point lies on the extension of a side. Quadrilaterals AB_1PC_1 and A_1B_1PC are inscribed ones, hence,

$$\angle AB_1C_1=\angle APC_1=\angle A_1PC=\angle A_1B_1C$$

and, therefore, point B_1 lies on segment A_1C_1.

b) By the same arguments as in heading a) we get

$$\angle(AP,PC_1)=\angle(AB_1,B_1C)=\angle(CB_1,B_1A_1)=\angle(CP,PA_1).$$

Add $\angle(PC_1,PC)$ to $\angle(AP,PC_1)$; we get

$$\angle(AP,PC)=\angle(PC_1,PA_1)=\angle(BC_1,BA_1)=\angle(AB,BC),$$

i.e., point P lies on the circumscribed circle of triangle ABC.

5.86. Let A_1, B_1 and C_1 be the midpoints of segments PA, PB and PC, respectively; let O_a, O_b and O_c be the centers of the circumscribed circles of triangles BCP, ACP and ABP, respectively. Points A_1, B_1 and C_1 are the bases of perpendiculars dropped from point

P to sides of triangle $O_aO_bO_c$ (or their extensions). Points A_1, B_1 and C_1 lie on one line, hence, point P lies on the circumscribed circle of triangle $O_aO_cO_c$ (cf. Problem 5.85, b).

5.87. Let the extension of the bisector AD intersect the circumscribed circle of triangle ABC at point P. Let us drop from point P perpendiculars PA_1, PB_1 and PC_1 to lines BC, CA and AB, respectively; clearly, A_1 is the midpoint of segment BC. The homothety centered at A that sends P to D sends points B_1 and C_1 to B' and C' and, therefore, it sends point A_1 to M, because M(???) lies on line B_1C_1 and $PA_1 \parallel DM$.

5.88. a) The solution of Problem 5.85 can be adapted without changes to this case.

b) Let A_1 and B_1 be the bases of perpendiculars dropped from point P to lines BC and CA, respectively, and let points A_2 and B_2 from lines BC and AC, respectively, be such that $\angle(PA_2, BC) = \alpha = \angle(PB_2, AC)$. Then $\triangle PA_1A_2 \sim \triangle PB_1B_2$ hence, points A_1 and B_1 turn under a rotational homothety centered at P into A_2 and B_2 and $\angle A_1PA_2 = 90^\circ - \alpha$ is the angle of the rotation.

5.89. a) Let the angle between lines PC and AC be equal to φ. Then $PA = 2R\sin\varphi$. Since points A_1 and B_1 lie on the circle with diameter PC, the angle between lines PA_1 and A_1B_1 is also equal to φ. Hence, $PA_1 = \frac{d}{\sin\varphi}$ and, therefore, $PA \cdot PA_1 = 2Rd$.

b) Since $PA_1 \perp BC$, it follows that $\cos\alpha = \sin\varphi = \frac{d}{PA_1}$. It remains to notice that $PA_1 = \frac{2Rd}{PA}$.

5.90. Points A_1 and B_1 lie on the circle with diameter PC, hence, $A_1B_1 = PC\sin\angle A_1CB_1 = PC\sin\angle C$. Let the angle between lines AB and A_1B_1 be equal to γ and C_1 be the projection of point P to line A_1B_1. Lines A_1B_1 and B_1C_1 coincide, hence, $\cos\gamma = \frac{PC}{2R}$ (cf. Problem 5.89). Therefore, the length of the projection of segment AB to line A_1B_1 is equal to

$$AB\cos\gamma = \frac{(2R\sin\angle C)PC}{2R} = PC\sin\angle C.$$

5.91. Let A_1 and B_1 be the bases of perpendiculars dropped from point P to lines BC and AC. Points A_1 and B_1 lie on the circle with diameter PC. Since $\sin\angle A_1CB_1 = \sin\angle ACB$, the chords A_1B_1 of this circle are of the same length. Therefore, lines A_1B_1 are tangent to a fixed circle.

5.92. Let A_1 and B_1 be the bases of perpendiculars dropped from point P to lines BC and CA. Then

$$\angle(A_1B_1, PB_1) = \angle(A_1C, PC) = \frac{\smile BP}{2}.$$

It is also clear that for all points P lines PB_1 have the same direction.

5.93. Let P_1 and P_2 be diametrically opposite points of the circumscribed circle of triangle ABC; let A_i and B_i be the bases of perpendiculars dropped from point P_i to lines BC and AC, respectively; let M and N be the midpoints of sides AC and BC, respectively; let X be the intersection point of lines A_1B_1 and A_2B_2, respectively. By Problem 5.92 $A_1B_1 \perp A_2B_2$. It remains to verify that $\angle(MX, XN) = \angle(BC, AC)$. Since $AB_2 = B_1C$, it follows that XM is a median of right triangle B_1XB_2. Hence, $\angle(XM, XB_2) = \angle(XB_2, B_2M)$.

Similarly, $\angle(XA_1, XN) = \angle(A_1N, XA_1)$. Therefore,

$$\angle(MX, XN) = \angle(XM, XB_2) + \angle(XB_2, XA_1) + \angle(XA_1, XN) = \\ \angle(XB_2, B_2M) + \angle(A_1N, XA_1) + 90^\circ.$$

Since

$$\angle(XB_2, B_2M) + \angle(AC, CB) + \angle(NA_1, A_1X) + 90^\circ = 0^\circ,$$

we have: $\angle(MN, XN) + \angle(AC, CB) = 0^\circ$.

5.94. If point R on the given circle is such that $\angle(\overrightarrow{OP},\overrightarrow{OR})=\frac{1}{2}(\beta+\gamma)$, then $OR\perp BC$. It remains to verify that $\angle(OR,OQ)=\angle(PA_1,A_1B_1)$. But $\angle(OR,OQ)=\frac{1}{2}\alpha$ and

$$\angle(PA_1,A_1B_1)=\angle(PB,BC_1)=\frac{\angle(\overrightarrow{OP},\overrightarrow{OA})}{2}=\frac{\alpha}{2}.$$

5.95. Let lines AC and PQ meet at point M. In triangle MPC draw heights PB_1 and CA_1. Then A_1B_1 is Simson's line of point P with respect to triangle ABC. Moreover, by Problem 1.52 $\angle(MB_1,B_1A_1)=\angle(CP,PM)$. It is also clear that $\angle(CP,PM)=\angle(CA,AQ)=\angle(MB_1,AQ)$. Hence, $A_1B_1\parallel AQ$.

5.96. Let us draw chord PQ perpendicular to BC. Let points H' and P' be symmetric to points H and P, respectively, through line BC; point H' lies on the circumscribed circle of triangle ABC (Problem 5.9). First, let us prove that $AQ\parallel P'H$. Indeed, $\angle(AH',AQ)=\angle(PH',PQ)=\angle(AH',P'H)$. Simson's line of point P is parallel to AQ (Problem 5.95), i.e., it passes through the midpoint of side PP' of triangle $PP'H$ and is parallel to side $P'H$; hence, it passes through the midpoint of side PH.

5.97. Let H_a, H_b, H_c and H_d be the orthocenters of triangles BCD, CDA, DAB and ABC, respectively. Lines l_a, l_b, l_c and l_d pass through the midpoints of segments AH_a, BH_b, CH_c and DH_d, respectively (cf. Problem 5.96). The midpoints of these segments coincide with point H such that $2\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}$, where O is the center of the circle (cf. Problem 13.33).

5.98. a) Let B_1, C_1 and D_1 be the projections of point P to lines AB, AC and AD, respectively. Points B_1, C_1 and D_1 lie on the circle with diameter AP. Lines B_1C_1, C_1D_1 and D_1B_1 are Simson's lines of point P with respect to triangles ABC, ACD and ADB, respectively. Therefore, projections of point P to Simson's lines of these triangles lie on one line — Simson's line of triangle $B_1C_1D_1$.

We similarly prove that any triple of considered points lies on one line.

b) Let P be a point of the circumscribed circle of n-gon $A_1\dots A_n$; let B_2, B_3, $\dots$, B_n be the projections of point P to lines A_1A_2, $\dots$, A_1A_n, respectively. Points B_2, $\dots$, B_n lie on the circle with diameter A_1P.

Let us prove by induction that Simson's line of point P with respect to n-gon $A_1\dots A_n$ coincides with Simson's line of point P with respect to $(n-1)$-gon $B_2\dots B_n$ (for $n=4$ this had been proved in heading a)). By the inductive hypothesis Simson's line of the $(n-1)$-gon $A_1A_3\dots A_n$ coincides with Simson's line of $(n-2)$-gon $B_3\dots B_n$. Hence, the projections of point P to Simson's line of $(n-1)$-gons whose vertices are obtained by consecutive deleting points $A_2,\dots,A_n$ from the collection $A_1,\dots,A_n$????? lie on Simson's line of the $(n-1)$-gon $B_2\dots B_n$. The projection of point P to Simson's line of the $(n-1)$-gon $A_2\dots A_n$ lies on the same line, because our arguments show that any $n-1$ of the considered n points of projections lie on one line.

5.99. Points B_1 and C_1 lie on the circle with diameter AP. Hence, $B_1C_1=AP\sin\angle B_1AC_1=AP\left(\frac{BC}{2R}\right)$.

5.100. This problem is a particular case of Problem 2.43.

5.101. Clearly,

$$\angle C_1AP=\angle C_1B_1P=\angle A_2B_1P=\angle A_2C_2P=\angle B_3C_2P=\angle B_3A_3P.$$

(The first, third and fifth equalities are obtained from the fact that the corresponding quadrilaterals are inscribed ones; the remaining equalities are obvious.) Similarly, $\angle B_1AP=\angle C_3A_3P$. Hence,

$$\angle B_3A_3C_3=\angle B_3A_3P+\angle C_3A_3P=\angle C_1AP+\angle BAP=\angle BAC.$$

Similarly, the equalities of the remaining angles of triangles ABC and $A_3B_3C_3$ are similarly obtained.

5.102. Let A_1, B_1 and C_1 be the bases of perpendiculars dropped from point P to lines BC, CA and AB, respectively; let A_2, B_2 and C_2 be the intersection points of lines PA, PB and PC, respectively, with the circumscribed circle of triangle ABC. Further, let S, S_1 and S_2 be areas of triangles ABC, $A_1B_1C_1$ and $A_2B_2C_2$, respectively. It is easy to verify that $a_1 = \frac{a\cdot AP}{2R}$ (Problem 5.99) and $a_2 = \frac{a\cdot B_2P}{CP}$. Triangles $A_1B_1C_1$ and $A_2B_2C_2$ are similar (Problem 5.100); hence, $\frac{S_1}{S_2} = k^2$, where $k = \frac{a_1}{a_2} = \frac{AP\cdot CP}{2R\cdot B_2P}$. Since $B_2P \cdot BP = |d^2 - R^2|$, we have:

$$\frac{S_1}{S_2} = \frac{(AP \cdot BP \cdot CP)^2}{4R^2(d^2 - R^2)^2}.$$

Triangles $A_2B_2C_2$ and ABC are inscribed in one circle, hence, $\frac{S_2}{S} = \frac{a_2b_2c_2}{abc}$ (cf. Problem 12.1). It is also clear that, for instance,

$$\frac{a_2}{a} = \frac{B_2P}{CP} = \frac{|d^2 - R^2|}{BP \cdot CP}.$$

Therefore,

$$S_2 : S = |d^2 - R^2|^3 : (AP \cdot BP \cdot CP)^2.$$

Hence,

$$\frac{S_1}{S} = \frac{S_1}{S_2} \cdot \frac{S_2}{S} = \frac{|d^2 - R^2|}{4R^2}.$$

5.103. Points B_1 and C_1 lie on the circle with diameter PA and, therefore, the midpoint of segment PA is the center of the circumscribed circle of triangle AB_1C_1. Consequently, l_a is the midperpendicular to segment B_1C_1. Hence, lines l_a, l_b and l_c pass through the center of the circumscribed circle of triangle $A_1B_1C_1$.

5.104. a) Let us drop from points P_1 and P_2 perpendiculars P_1B_1 and P_2B_2, respectively, to AC and perpendiculars P_1C_1 and P_2C_2 to AB. Let us prove that points B_1, B_2, C_1 and C_2 lie on one circle. Indeed,

$$\angle P_1B_1C_1 = \angle P_1AC_1 = \angle P_2AB_2 = \angle P_2C_2B_2;$$

and, since $\angle P_1B_1A = \angle P_2C_2A$, it follows that $\angle C_1B_1A = \angle B_2C_2A$. The center of the circle on which the indicated points lie is the intersection point of the midperpendiculars to segments B_1B_2 and C_1C_2; observe that both these perpendiculars pass through the midpoint O of segment P_1P_2, i.e., O is the center of this circle. In particular, points B_1 and C_1 are equidistant from point O. Similarly, points A_1 and B_1 are equidistant from point O, i.e., O is the center of the circumscribed circle of triangle $A_1B_1C_1$. Moreover, $OB_1 = OB_2$.

b) The preceding proof passes virtually without changes in this case as well.

5.105. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB. Triangles $A_1B_1C_1$ and ABC are similar and the similarity coefficient is equal to 2. The heights of triangle $A_1B_1C_1$ intersect at point O; hence, $OA_1 : HA = 1 : 2$. Let M' be the intersection point of segments OH and AA_1. Then $OM' : M'H = OA_1 : HA = 1 : 2$ and $AM' : M'A_1 = OA_1 : HA = 1 : 2$, i.e., $M' = M$.

5.106. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively; let A_2, B_2 and C_2 the bases of heights; A_3, B_3 and C_3 the midpoints of segments that connect the intersection point of heights with vertices. Since $A_2C_1 = C_1A = A_1B_1$ and $A_1A_2 \parallel B_1C_1$, point A_2 lies on the circumscribed circle of triangle $A_1B_1C_1$. Similarly, points B_2 and C_2 lie on the circumscribed circle of triangle $A_1B_1C_1$.

Now, consider circle S with diameter A_1A_3. Since $A_1B_3 \parallel CC_2$ and $A_3B_3 \parallel AB$, it follows that $\angle A_1B_3A_3 = 90°$ and, therefore, point B_3 lies on S. We similarly prove that points C_1,

B_1 and C_3 lie on S. Circle S passes through the vertices of triangle $A_1B_1C_1$; hence, it is its circumscribed circle.

The homothety with center H and coefficient $\frac{1}{2}$ sends the circumscribed circle of triangle ABC into the circumscribed circle of triangle $A_3B_3C_3$, i.e., into the circle of 9 points. Therefore, this homothety sends point O into the center of the circle of nine points.

5.107. a) Let us prove that, for example, triangles ABC and HBC share the same circle of nine points. Indeed, the circles of nine points of these triangles pass through the midpoint of side BC and the midpoints of segments BH and CH.

b) Euler's line passes through the center of the circle of 9 points and these triangles share one circle of nine points.

c) The center of symmetry is the center of the circle of 9 points of these triangles.

5.108. Let $AB > BC > CA$. It is easy to verify that for an acute and an obtuse triangles the intersection point H of heights and the center O of the circumscribed circle are positioned precisely as on Fig. 59 (i.e., for an acute triangle point O lies inside triangle BHC_1 and for an acute triangle points O and B lie on one side of line CH).

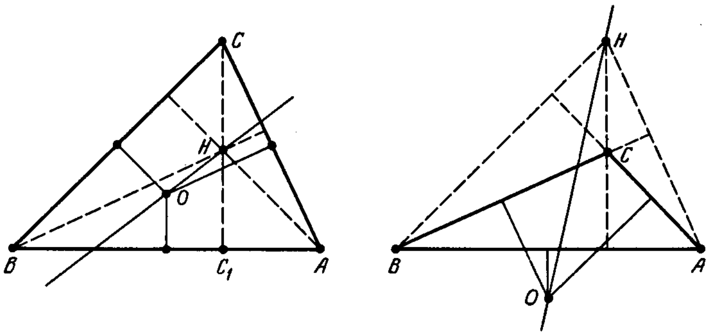


Figure 59 (Sol. 5.108)

Therefore, in an acute triangle Euler's line intersects the longest side AB and the shortest side AC, whereas in an acute triangle it intersects the longest side AB, and side BC of intermediate length.

5.109. a) Let O_a, O_b and O_c be the centers of the escribed circles of triangle ABC. The vertices of triangle ABC are the bases of the heights of triangle $O_aO_bO_c$ (Problem 5.2) and, therefore, the circle of 9 points of triangle $O_aO_bO_c$ passes through point A, B and C.

b) Let O be the intersection point of heights of triangle $O_aO_bO_c$, i.e., the intersection point of the bisectors of triangle ABC. The circle of 9 points of triangle $O_aO_bO_c$ divides segment OO_a in halves.

5.110. Let AA_1 be an height, H the intersection point of heights. By Problem 5.45 b) $AH = 2R|\cos\angle A|$. The medians are divided by their intersection point in the ratio of 1:2, hence, Euler's line is parallel to BC if and only if $AH : AA_1 = 2 : 3$ and vectors $\overrightarrow{AH}$ and $\overrightarrow{AA_1}$ are codirected, i.e.,

$$2R\cos\angle A : 2R\sin\angle B\sin\angle C = 2 : 3.$$

Taking into account that

$$\cos\angle A = -\cos(\angle B + \angle C) = \sin\angle B\sin\angle C - \cos\angle B\cos\angle C$$

we get

$$\sin\angle B\sin\angle C = 3\cos\angle B\cos\angle C.$$

5.111. Let CD be a height, O the center of the circumscribed circle, N the midpoint of side AB and let point E divide the segment that connects C with the intersection point of the heights in halves. Then $CENO$ is a parallelogram, hence, $\angle NED = \angle OCH = |\angle A - \angle B|$ (cf. Problem 2.88). Points N, E and D lie on the circle of 9 points, hence, segment ND is seen from its center under an angle of $2\angle NED = 2|\angle A - \angle B|$.

5.112. Let O and I be the centers of the circumscribed and inscribed circles, respectively, of triangle ABC, let H be the intersection point of the heights; lines AI and BI intersect the circumscribed circle at points A_1 and B_1. Suppose that triangle ABC is not an isosceles one. Then $OI : IH = OA_1 : AH$ and $OI : IH = OB_1 : BH$. Since $OB_1 = OA_1$, we see that $AH = BH$ and, therefore, $AC = BC$. Contradiction.

5.113. Let O and I be the centers of the circumscribed and inscribed circles, respectively, of triangle ABC, H the orthocenter of triangle $A_1B_1C_1$. In triangle $A_1B_1C_1$, draw heights A_1A_2, B_1B_2 and C_1C_2. Triangle $A_1B_1C_1$ is an acute one (e.g., $\angle B_1A_1C_1 = \frac{\angle B + \angle C}{2} < 90^\circ$), hence, H is the center of the inscribed circle of triangle $A_2B_2C_2$ (cf. Problem 1.56, a). The corresponding sides of triangles ABC and $A_2B_2C_2$ are parallel (cf. Problem 1.54 a) and, therefore, there exists a homothety that sends triangle ABC to triangle $A_2B_2C_2$. This homothety sends point O to point I and point I to point H; hence, line IH passes through point O.

5.114. Let H be be the intersection point of the heights of triangle ABC, let E and M be the midpoints of segments CH and AB, see Fig. 60. Then C_1MC_2E is a rectangle.

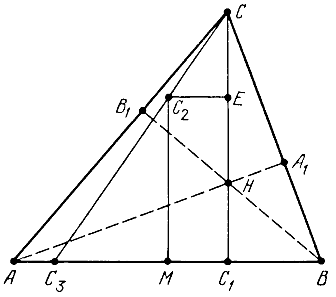


Figure 60 (Sol. 5.114)

Let line CC_2 meet line AB at point C_3. Let us prove that $\overline{AC_3} : \overline{C_3B} = \tan 2\alpha : \tan 2\beta$. It is easy to verify that

$$\begin{gathered}\overline{C_3M} : \overline{C_2E} = \overline{MC_2} : \overline{EC}, \quad \overline{EC} = R\cos\gamma,\\ \overline{MC_2} = \overline{C_1E} = 2R\sin\alpha\sin\beta - R\cos\gamma\\ \text{and}\quad \overline{C_2E} = \overline{MC_1} = R\sin(\beta-\alpha)\end{gathered}$$

Hence,

$$\overline{C_3M} = \frac{R\sin(\beta-\alpha)(2\sin\beta\sin\alpha - \cos\gamma)}{\cos\gamma} = \frac{R\sin(\beta-\alpha)\cos(\beta-\alpha)}{\cos\gamma}.$$

Therefore,

$$\frac{\overline{AC_3}}{\overline{C_3B}} = \frac{\overline{AM} + \overline{MC_3}}{\overline{C_3M} + \overline{MB}} = \frac{\sin 2\gamma + \sin 2(\alpha-\beta)}{\sin 2\gamma - \sin 2(\alpha-\beta)} = \frac{\tan 2\alpha}{\tan 2\beta}.$$

Similar arguments show that

$$\frac{\overline{AC_3}}{\overline{C_3B}} \cdot \frac{\overline{BA_3}}{\overline{A_3C}} \cdot \frac{\overline{CB_3}}{\overline{B_3A}} = \frac{\tan 2\alpha}{\tan 2\beta} \cdot \frac{\tan 2\beta}{\tan 2\gamma} \cdot \frac{\tan 2\gamma}{\tan 2\alpha} = 1.$$

5.115. Let us solve a more general heading b). First, let us prove that lines AA_1, BB_1 and CC_1 meet at one point. Let the circumscribed circles of triangles A_1BC and AB_1C

intersect at point O. Then

$$\begin{aligned}\angle(BO, OA) &= \angle(BO, OC) + \angle(OC, OA) = \angle(BA_1, A_1C) + \angle(CB_1, B_1A) = \\ &= \angle(BA, AC_1) + \angle(C_1B, BA) = \angle(C_1B, AC_1),\end{aligned}$$

i.e., the circumscribed circle of triangle ABC_1 also passes through point O. Hence,

$$\angle(AO, OA_1) = \angle(AO, OB) + \angle(BO, OA_1) = \angle(AC_1, C_1B + \angle(BC, CA_1) = 0^\circ,$$

i.e., line AA_1 passes through point O. We similarly prove that lines BB_1 and CC_1 pass through point O.

Now, let us prove that point O coincides with point P we are looking for. Since $\angle BAP = \angle A - \angle CAP$, the equality $\angle ABP = \angle CAP$ is equivalent to the equality $\angle BAP + \angle ABP = \angle A$, i.e., $\angle APB = \angle B + \angle C$. For point O the latter equality is obvious since it lies on the circumscribed circle of triangle ABC_1.

5.116. a) Let us prove that $\smile AB = \smile B_1C_1$, i.e., $AB = B_1C_1$. Indeed, $\smile AB = \smile AC_1 + \smile C_1B$ and $\smile C_1B = \smile AB_1$; hence, $\smile AB = \smile AC_1 + \smile AB_1 = \smile B_1C_1$.

b) Let us assume that triangles ABC and $A_1B_1C_1$ are inscribed in one circle, where triangle ABC is fixed and triangle $A_1B_1C_1$ rotates. Lines AA_1, BB_1 and CC_1 meet at one point for not more than one position of triangle $A_1B_1C_1$, see Problem 7.20 b). We can obtain 12 distinct families of triangles $A_1B_1C_1$: triangles ABC and $A_1B_1C_1$ can be identified after a rotation or an axial symmetry; moreover, there are 6 distinct ways to associate symbols A_1, B_1 and C_1 to the vertices of the triangle.

From these 12 families of triangles 4 families can never produce the desired point P. For similarly oriented triangles the cases

$$\triangle ABC = \triangle A_1C_1B_1, \quad \triangle ABC = \triangle C_1B_1A_1, \quad \triangle ABC = \triangle B_1A_1C_1$$

are excluded: for example, if $\triangle ABC = \triangle A_1C_1B_1$, then point P is the intersection point of line $BC = B_1C_1$ with the tangent to the circle at point $A = A_1$; in this case triangles ABC and $A_1B_1C_1$ coincide.

For differently oriented triangles the case $\triangle ABC = \triangle A_1B_1C_1$ is excluded: in this case $AA_1 \parallel BB_1 \parallel CC_1$.

Remark. Brokar's points correspond to differently oriented triangles; for the first Brokar's point $\triangle ABC = \triangle B_1C_1A_1$ and for the second Brokar's point we have $\triangle ABC = \triangle C_1A_1B_1$.

5.117. a) Since $PC = \frac{AC \sin\angle CAP}{\sin\angle APC}$ and $PC = \frac{BC \sin\angle CBP}{\sin\angle BPC}$, it follows that

$$\frac{\sin\varphi\sin\beta}{\sin\gamma} = \frac{\sin(\beta-\varphi)\sin\alpha}{\sin\beta}.$$

Taking into account that

$$\sin(\beta-\gamma) = \sin\beta\cos\varphi - \cos\beta\sin\varphi$$

we get $\cot\varphi = \cot\beta + \frac{\sin\beta}{\sin\alpha\sin\gamma}$. It remains to notice that

$$\sin\beta = \sin(\alpha+\gamma) = \sin\alpha\cos\gamma + \sin\gamma\cos\alpha.$$

b) For the second Brokar's angle we get precisely the same expression as in heading a). It is also clear that both Brokar's angles are acute ones.

c) Since $\angle A_1BC = \angle BCA$ and $\angle BCA_1 = \angle CAB$, it follows that $\triangle CA_1B \sim \triangle ABC$. Therefore, Brokar's point P lies on segment AA_1 (cf. Problem 5.115 b)).

5.118. a) By Problem 10.38 a)

$$\cot\varphi = \cot\alpha + \cot\beta + \cot\gamma \geq \sqrt{3} = \cot 30^\circ;$$

hence, $\varphi \leq 30^\circ$.

b) Let P be the first Brokar's point of triangle ABC. Point M lies inside (or on the boundary of) one of the triangles ABP, BCP and CAP. If, for example, point M lies inside triangle ABP, then $\angle ABM \leq \angle ABP \leq 30^\circ$.

5.119. Lines A_1B_1, B_1C_1 and C_1A_1 are the midperpendiculars to segments AQ, BQ and CQ, respectively. Therefore, we have, for instance, $\angle B_1A_1C_1 = 180^\circ - \angle AQC = \angle A$. For the other angles the proof is similar.

Moreover, lines A_1O, B_1O and C_1O are the midperpendiculars to segments CA, AB and BC, respectively. Hence, acute angles $\angle OA_1C_1$ and $\angle ACQ$, for example, have pairwise perpendicular sides and, consecutively, they are equal. Similar arguments show that $\angle OA_1C_1 = \angle OB_1A_1 = \angle OC_1B_1 = \varphi$, where φ is the Brokar's angle of triangle ABC.

5.120. By the law of sines

$$R_1 = \frac{AB}{2\sin\angle APB}, \quad R_2 = \frac{BC}{2\sin\angle BPC} \quad \text{and} \quad R_3 = \frac{CA}{2\sin\angle CPA}.$$

It is also clear that

$$\sin\angle APB = \sin\angle A, \quad \sin\angle BPC = \sin\angle B \quad \text{and} \quad \sin\angle CPA = \sin\angle C.$$

5.121. Triangle ABC_1 is an isosceles one and the angle at its base AB is equal to Brokar's angle φ. Hence, $\angle(PC_1, C_1Q) = \angle(BC_1, C_1A) = 2\varphi$. Similarly

$$\angle(PA_1, A_1Q) = \angle(PB_1, B_1Q) = \angle(PC_1, C_1Q) = 2\varphi.$$

5.122. Since $\angle CA_1B_1 = \angle A + \angle AB_1A_1$ and $\angle AB_1A_1 = \angle CA_1C_1$, we have $\angle B_1A_1C_1 = \angle A$. We similarly prove that the remaining angles of triangles ABC and $A_1B_1C_1$ are equal.

The circumscribed circles of triangles AA_1B_1, BB_1C_1 and CC_1A_1 meet at one point O. (Problem 2.80 a). Clearly, $\angle AOA_1 = \angle AB_1A_1 = \varphi$. Similarly, $\angle BOB_1 = \angle COC_1 = \varphi$. Hence, $\angle AOB = \angle A_1OB_1 = 180^\circ - \angle A$. Similarly, $\angle BOC = 180^\circ - \angle B$ and $\angle COA = 180^\circ - \angle C$, i.e., O is the first Brokar's point of both triangles. Hence, the rotational homothety by angle φ with center O and coefficient $\frac{AO}{A_1O}$ sends triangle $A_1B_1C_1$ to triangle ABC.

5.123. By the law of sines $\frac{AB}{BM} = \frac{\sin\angle AMB}{\sin\angle BAM}$ and $\frac{AB}{BN} = \frac{\sin\angle ANB}{\sin\angle BAN}$. Hence,

$$\frac{AB^2}{BM \cdot BN} = \frac{\sin\angle AMB \sin\angle ANB}{\sin\angle BAM \sin\angle BAN} = \frac{\sin\angle AMC \sin\angle ANC}{\sin\angle CAN \sin\angle CAM} = \frac{AC^2}{CM \cdot CN}.$$

5.124. Since $\angle BAS = \angle CAM$, we have

$$\frac{BS}{CM} = \frac{S_{BAS}}{S_{CAM}} = \frac{AB \cdot AS}{AC \cdot AM},$$

i.e., $\frac{AS}{AM} = \frac{2b \cdot BS}{ac}$. It remains to observe that, as follows from Problems 5.123 and 12.11 a), $BS = \frac{ac^2}{b^2+c^2}$ and $2AM = \sqrt{2b^2 + 2c^2 - a^2}$.

5.125. The symmetry through the bisector of angle A sends segment B_1C_1 into a segment parallel to side BC, it sends line AS to line AM, where M is the midpoint of side BC.

5.126. On segments BC and BA, take points A_1 and C_1, respectively, so that $A_1C_1 \parallel BK$. Since $\angle BAC = \angle CBK = \angle BA_1C_1$, segment A_1C_1 is antiparallel to side AC. On the other hand, by Problem 3.31 b) line BD divides segment A_1C_1 in halves.

5.127. It suffices to make use of the result of Problem 3.30.

5.128. Let AP be the common chord of the considered circles, Q the intersection point of lines AP and BC. Then

$$\frac{BQ}{AB} = \frac{\sin\angle BAQ}{\sin\angle AQB} \quad \text{and} \quad \frac{AC}{CQ} = \frac{\sin\angle AQC}{\sin\angle CAQ}.$$

Hence, $\frac{BQ}{CQ} = \frac{AB\sin\angle BAP}{AC\sin\angle CAP}$. Since AC and AB are tangents to circles S_1 and S_2, it follows that $\angle CAP = \angle ABP$ and $\angle BAP = \angle ACP$ and, therefore, $\angle APB = \angle APC$. Hence,

$$\frac{AB}{AC} = \frac{AB}{AP}\cdot\frac{AP}{AC} = \frac{\sin\angle APB}{\sin\angle ABP}\cdot\frac{\sin\angle ACP}{\sin\angle APC} = \frac{\sin\angle ACP}{\sin\angle ABP} = \frac{\sin\angle BAP}{\sin\angle CAP}.$$

It follows that $\frac{BQ}{CQ} = \frac{AB^2}{AC^2}$.

5.129. Let S be the intersection point of lines AX and BC. Then $\frac{AS}{AB} = \frac{CS}{CX}$ and $\frac{AS}{AC} = BSBX$ and, therefore,

$$\frac{CS}{BS} = \frac{AC}{AB}\cdot\frac{XC}{XB}.$$

It remains to observe that $\frac{XC}{XB} = \frac{AC}{AB}$ (see the solution of Problem 7.16 a)).

5.130. Let L, M and N be the midpoints of segments CA, CB and CH. Since $\triangle BAC \sim \triangle CAH$, it follows that $\triangle BAM \sim \triangle CAN$ and, therefore, $\angle BAM = \angle CAN$. Similarly, $\angle ABL = \angle CBN$.

5.131. Let B_1C_1, C_2A_2 and A_3B_3 be given segments. Then triangles A_2XA_3, B_1XB_3 and C_1XC_2 are isosceles ones; let the lengths of their lateral sides be equal to a, b and c. Line AX divides segment B_1C_1 in halves if and only if this line contains a simedian. Hence, if X is Lemoin's point, then $a = b$, $b = c$ and $c = a$. And if $B_1C_1 = C_2A_2 = A_3B_3$, then $b + c = c + a = a + b$ and, therefore, $a = b = c$.

5.132. Let M be the intersection point of medians of triangle ABC; let a_1, b_2, c_1 and a_2, b_2, c_2 be the distances from points K and M, respectively, to the sides of the triangle. Since points K and M are isogonally conjugate, $a_1a_2 = b_1b_2 = c_1c_2$. Moreover, $aa_2 = bb_2 = cc_2$ (cf. Problem 4.1). Therefore, $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$. Making use of this equality and taking into account that areas of triangles A_1B_1K, B_1C_1K and C_1A_1K are equal to $\frac{a_1b_1c}{4R}$, $\frac{b_1c_1a}{4R}$ and $\frac{c_1a_1b}{4R}$, respectively, where R is the radius of the circumscribed circle of triangle ABC, we deduce that the areas of these triangles are equal. Moreover, point K lies inside triangle $A_1B_1C_1$. Therefore, K is the intersection point of medians of triangle $A_1B_1C_1$ (cf. Problem 4.2).

5.133. Medians of triangle $A_1B_1C_1$ intersect at point K (Problem 5.132); hence, the sides of triangle ABC are perpendicular to the medians of triangle $A_1B_1C_1$. After a rotation through an angle of 90° the sides of triangle ABC become pairwise parallel to the medians of triangle $A_1B_1C_1$ and, therefore, the medians of triangle ABC become parallel to the corresponding sides of triangle $A_1B_1C_1$ (cf. Problem 13.2). Hence, the medians of triangle ABC are perpendicular to the corresponding sides of triangle $A_1B_1C_1$.

5.134. Let A_2, B_2 and C_2 be the projections of point K to lines BC, CA and AB, respectively. Then $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$ (Problem 5.100) and K is the intersection point of medians of triangle $A_2B_2C_2$ (Problem 5.132). Hence, the similarity transformation that sends triangle $A_2B_2C_2$ to triangle $A_1B_1C_1$ sends point K to the intersection point M of medians of triangle $A_1B_1C_1$. Moreover, $\angle KA_2C_2 = \angle KBC_2 = \angle B_1A_1K$, i.e., points K and M are isogonally conjugate with respect to triangle $A_1B_1C_1$ and, therefore, K is Lemoin's point of triangle $A_1B_1C_1$.

5.135. Let K be Lemoin's point of triangle ABC; let A_1, B_1 and C_1 be the projections of point K on the sides of triangle ABC; let L be the midpoint of segment B_1C_1 and N the intersection point of line KL and median AM; let O be the midpoint of segment AK (Fig. 61). Points B_1 and C_1 lie on the circle with diameter AK, hence, by Problem 5.132 $OL \perp B_1C_1$. Moreover, $AN \perp B_1C_1$ (Problem 5.133) and O is the midpoint of segment AK, consequently, OL is the midline of triangle AKN and $KL = LN$. Therefore, K is the midpoint of segment A_1N. It remains to notice that the homothety with center M that sends N to A sends segment NA_1 to height AH.

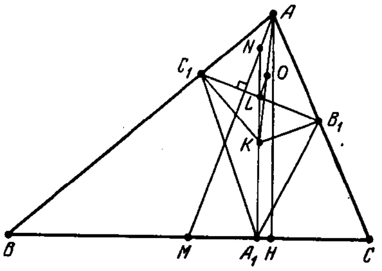


Figure 61 (Sol. 5.135)

Chapter 6. POLYGONS

Background

1) A polygon is called *a convex* one if it lies on one side of any line that connects two of its neighbouring vertices.

2) A convex polygon is called *a circumscribed* one if all its sides are tangent to a circle. A convex quadrilateral is a circumscribed one if and only if $AB + CD = BC + AD$.

A convex polygon is called *an inscribed* one if all its vertices lie on one circle. *A convex quadrilateral is an inscribed one if and only if*

$$\angle ABC + \angle CDA = \angle DAB + \angle BCD.$$

3) A convex polygon is called *a regular* one if all its sides are equal and all its angles are also equal.

A convex n-gon is a regular one if and only if under a rotation by the angle of $\frac{2\pi}{n}$ with center at point O it turns into itself. This point O is called *the center* of the regular polygon.

Introductory problems

1. Prove that a convex quadrilateral $ABCD$ can be inscribed into a circle if and only if $\angle ABC + \angle CDA = 180^\circ$.

2. Prove that a circle can be inscribed in a convex quadrilateral $ABCD$ if and only if $AB + CD = BC + AD$.

3. a) Prove that the axes of symmetry of a regular polygon meet at one point.

b) Prove that a regular $2n$-gon has a center of symmetry.

4. a) Prove that the sum of the angles at the vertices of a convex n-gon is equal to $(n-2)\cdot 180^\circ$.

b) A convex n-gon is divided by nonintersecting diagonals into triangles. Prove that the number of these triangles is equal to $n - 2$.

§1. The inscribed and circumscribed quadrilaterals

6.1. Prove that if the center of the circle inscribed in a quadrilateral coincides with the intersection point of the quadrilateral's diagonals, then this quadrilateral is a rhombus.

6.2. Quadrilateral $ABCD$ is circumscribed about a circle centered at O. Prove that $\angle AOB + \angle COD = 180^\circ$.

6.3. Prove that if there exists a circle tangent to all the sides of a convex quadrilateral $ABCD$ and a circle tangent to the extensions of all its sides then the diagonals of such a quadrilateral are perpendicular.

6.4. A circle singles out equal chords on all the four sides of a quadrilateral. Prove that a circle can be inscribed into this quadrilateral.

6.5. Prove that if a circle can be inscribed into a quadrilateral, then the center of this circle lies on one line with the centers of the diagonals.

6.6. Quadrilateral $ABCD$ is circumscribed about a circle centered at O. In triangle AOB heights AA_1 and BB_1 are drawn. In triangle COD heights CC_1 and DD_1 are drawn. Prove that points A_1, B_1, C_1 and D_1 lie on one line.

6.7. The angles at base AD of trapezoid $ABCD$ are equal to 2α and 2β. Prove that the trapezoid is a circumscribed one if and only if $\frac{BC}{AD} = \tan\alpha\tan\beta$.

6.8. In triangle ABC, segments PQ and RS parallel to side AC and a segment BM are drawn as plotted on Fig. 62. Trapezoids $RPKL$ and $MLSC$ are circumscribed ones. Prove that trapezoid $APQC$ is also a circumscribed one.

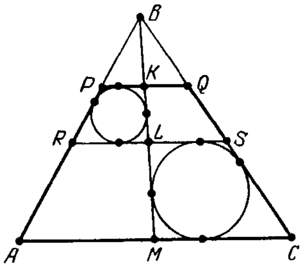


Figure 62 (6.8)

6.9. Given convex quadrilateral $ABCD$ such that rays AB and CD intersects at a point P and rays BC and AD intersect at a point Q. Prove that quadrilateral $ABCD$ is a circumscribed one if and only if one of the following conditions hold:

$$AB + CD = BC + AD, \quad AP + CQ = AQ + CP \quad BP + BQ = DP + DQ.$$

6.10. Through the intersection points of the extension of sides of convex quadrilateral $ABCD$ two lines are drawn that divide it into four quadrilaterals. Prove that if the quadrilaterals adjacent to vertices B and D are circumscribed ones, then quadrilateral $ABCD$ is also a circumscribed one.

6.11. Prove that the intersection point of the diagonals of a circumscribed quadrilateral coincides with the intersection point of the diagonals of the quadrilateral whose vertices are the tangent points of the sides of the initial quadrilateral with the inscribed circle.

* * *

6.12. Quadrilateral $ABCD$ is an inscribed one; H_c and H_d are the orthocenters of triangles ABD and ABC respectively. Prove that CDH_cH_d is a parallelogram.

6.13. Quadrilateral $ABCD$ is an inscribed one. Prove that the centers of the inscribed circles of triangles ABC, BCD, CDA and DAB are the vertices of a rectangle.

6.14. The extensions of the sides of quadrilateral $ABCD$ inscribed in a circle centered at O intersect at points P and Q and its diagonals intersect at point S.

a) The distances from points P, Q and S to point O are equal to p, q and s, respectively, and the radius of the circumscribed circle is equal to R. Find the lengths of the sides of triangle PQS.

b) Prove that the heights of triangle PQS intersect at point O.

* * *

6.15. Diagonal AC divides quadrilateral $ABCD$ into two triangles whose inscribed circles are tangent to diagonal AC at one point. Prove that the inscribed circles of triangle ABD and BCD are also tangent to diagonal BD at one point and their tangent points with the sides of the quadrilateral lie on one circle.

6.16. Prove that the projections of the intersection point of the diagonals of the inscribed quadrilateral to its sides are vertices of a circumscribed quadrilateral only if the projections do not lie on the extensions of the sides.

6.17. Prove that if the diagonals of a quadrilateral are perpendicular, then the projections of the intersection points of the diagonals on its sides are vertices of an inscribed quadrilateral.

See also Problem 13.33, 13.34, 16.4.

§2. Quadrilaterals

6.18. The angle between sides AB and CD of quadrilateral $ABCD$ is equal to φ. Prove that

$$AD^2 = AB^2 + BC^2 + CD^2 - 2(AB \cdot BC \cos B + BC \cdot CD \cos C + CD \cdot AB \cos \varphi).$$

6.19. In quadrilateral $ABCD$, sides AB and CD are equal and rays AB and DC intersect at point O. Prove that the line that connects the midpoints of the diagonals is perpendicular to the bisector of angle AOD.

6.20. On sides BC and AD of quadrilateral $ABCD$, points M and N, respectively, are taken so that $BM : MC = AN : ND = AB : CD$. Rays AB and DC intersect at point O. Prove that line MN is parallel to the bisector of angle AOD.

6.21. Prove that the bisectors of the angles of a convex quadrilateral form an inscribed quadrilateral.

6.22. Two distinct parallelograms $ABCD$ and $A_1B_1C_1D_1$ with corresponding parallel sides are inscribed into quadrilateral $PQRS$ (points A and A_1 lie on side PQ, points B and B_1 lie on side QR, etc.). Prove that the diagonals of the quadrilateral are parallel to the corresponding sides of the parallelograms.

6.23. The midpoints M and N of diagonals AC and BD of convex quadrilateral $ABCD$ do not coincide. Line MN intersects sides AB and CD at points M_1 and N_1. Prove that if $MM_1 = NN_1$, then $AD \parallel BC$.

6.24. Prove that two quadrilaterals are similar if and only if four of their corresponding angles are equal and the corresponding angles between the diagonals are also equal.

6.25. Quadrilateral $ABCD$ is a convex one; points A_1, B_1, C_1 and D_1 are such that $AB \parallel C_1D_1$ and $AC \parallel B_1D_1$, etc. for all pairs of vertices. Prove that quadrilateral $A_1B_1C_1D_1$ is also a convex one and $\angle A + \angle C_1 = 180^\circ$.

6.26. From the vertices of a convex quadrilateral perpendiculars are dropped on the diagonals. Prove that the quadrilateral with vertices at the basis of the perpendiculars is similar to the initial quadrilateral.

6.27. A convex quadrilateral is divided by the diagonals into four triangles. Prove that the line that connects the intersection points of the medians of two opposite triangles is perpendicular to the line that connects the intersection points of the heights of the other two triangles.

6.28. The diagonals of the circumscribed trapezoid $ABCD$ with bases AD and BC intersect at point O. The radii of the inscribed circles of triangles AOD, AOB, BOC and COD are equal to r_1, r_2, r_3 and r_4, respectively. Prove that $\frac{1}{r_1} + \frac{1}{r_3} = \frac{1}{r_2} + \frac{1}{r_4}$.

6.29. A circle of radius r_1 is tangent to sides DA, AB and BC of a convex quadrilateral $ABCD$; a circle of radius r_2 is tangent to sides AB, BC and CD; the radii r_3 and r_4 are similarly defined. Prove that $\frac{AB}{r_1} + \frac{CD}{r_3} = \frac{BC}{r_2} + \frac{AD}{r_4}$.

6.30. A quadrilateral $ABCD$ is convex and the radii of the circles inscribed in triangles ABC, BCD, CDA and DAB are equal. Prove that $ABCD$ is a rectangle.

6.31. Given a convex quadrilateral $ABCD$ and the centers A_1, B_1, C_1 and D_1 of the circumscribed circles of triangles BCD, CDA, DAB and ABC, respectively. For quadrilateral $A_1B_1C_1D_1$ points A_2, B_2, C_2 and D_2 are similarly defined. Prove that quadrilaterals $ABCD$ and $A_2B_2C_2D_2$ are similar and their similarity coefficient is equal to

$$\frac{1}{4}\left|(\cot A + \cot C)(\cot B + \cot D)\right|.$$

6.32. Circles whose diameters are sides AB and CD of a convex quadrilateral $ABCD$ are tangent to sides CD and AB, respectively. Prove that $BC \parallel AD$.

6.33. Four lines determine four triangles. Prove that the orthocenters of these triangles lie on one line.

§3. Ptolemy's theorem

6.34. Quadrilateral $ABCD$ is an inscribed one. Prove that

$$AB \cdot CD + AD \cdot BC = AC \cdot BD \quad \text{(Ptolemy's theorem)}.$$

6.35. Quadrilateral $ABCD$ is an inscribed one. Prove that

$$\frac{AC}{BD} = \frac{AB \cdot AD + CB \cdot CD}{BA \cdot BC + DA \cdot DC}.$$

6.36. Let $\alpha = \frac{\pi}{7}$. Prove that

$$\frac{1}{\sin\alpha} = \frac{1}{\sin 2\alpha} + \frac{1}{\sin 3\alpha}.$$

6.37. The distances from the center of the circumscribed circle of an acute triangle to its sides are equal to d_a, d_b and d_c. Prove that $d_a + d_b + d_c = R + r$.

6.38. The bisector of angle $\angle A$ of triangle ABC intersects the circumscribed circle at point D. Prove that $AB + AC \leq 2AD$.

6.39. On arc $\smile CD$ of the circumscribed circle of square $ABCD$ point P is taken. Prove that $PA + PC = \sqrt{2}PB$.

6.40. Parallelogram $ABCD$ is given. A circle passing through point A intersects segments AB, AC and AD at points P, Q and R, respectively. Prove that

$$AP \cdot AB + AR \cdot AD = AQ \cdot AC.$$

6.41. On arc $\smile A_1A_{2n+1}$ of the circumscribed circle S of a regular $(2n+1)$-gon $A_1 \dots A_{2n+1}$ a point A is taken. Prove that:

a) $d_1 + d_3 + \dots + d_{2n+1} = d_2 + d_4 + \dots + d_{2n}$, where $d_i = AA_i$;

b) $l_1 + \dots + l_{2n+1} = l_2 + \dots + l_{2n}$, where l_i is the length of the tangent drawn from point A to the circle of radius r tangent to S at point A_i (all the tangent points are simultaneously either inner or outer ones).

6.42. Circles of radii x and y are tangent to a circle of radius R and the distance between the tangent points is equal to a. Calculate the length of the following common tangent to the first two circles:

a) the outer one if both tangents are simultaneously either outer or inner ones;

b) the inner one if one tangent is an inner one and the other one is an outer one.

6.43. Circles α, β, γ and δ are tangent to a given circle at vertices A, B, C and D, respectively, of convex quadrilateral $ABCD$. Let $t_{\alpha\beta}$ be the length of the common tangent to circles α and β (the outer one if both tangent are simultaneously either inner or outer ones and the inner one if one tangent is an inner one and the other one is an outer one); $t_{\beta\gamma}$, $t_{\gamma\delta}$, etc. are similarly determined. Prove that

$$t_{\alpha\beta}t_{\gamma\delta} + t_{\beta\gamma}t_{\delta\alpha} = t_{\alpha\gamma}t_{\beta\delta} \qquad \text{(The \textit{generalized Ptolemy's theorem})}$$

See also Problem 9.67.

§4. Pentagons

6.44. In an equilateral (non-regular) pentagon $ABCDE$ we have angle $\angle ABC = 2\angle DBE$. Find the value of angle $\angle ABC$.

6.45. a) Diagonals AC and BE of a regular pentagon $ABCDE$ intersect at point K. Prove that the inscribed circle of triangle CKE is tangent to line BC.

b) Let a be the length of the side of a regular pentagon, d the length of its diagonal. Prove that $d^2 = a^2 + ad$.

6.46. Prove that a square can be inscribed in a regular pentagon so that the vertices of the square would lie on four sides of the pentagon.

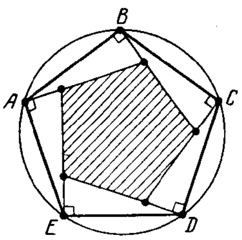


Figure 63 (6.46)

6.47. Regular pentagon $ABCDE$ with side a is inscribed in circle S. The lines that pass through the pentagon's vertices perpendicularly to the sides form a regular pentagon with side b (Fig. 63). A side of a regular pentagon circumsribed about circle S is equal to c. Prove that $a^2 + b^2 = c^2$.

See also Problems 2.59, 4.9, 9.23, 9.44, 10.63, 10.67, 13.10, 13.56, 20.11.

§5. Hexagons

6.48. The opposite sides of a convex hexagon $ABCDEF$ are pairwise parallel. Prove that:

a) the area of triangle ACE constitutes not less than a half area of the hexagon.

b) the areas of triangles ACE and BDF are equal.

6.49. All the angles of a convex hexagon $ABCDEF$ are equal. Prove that

$$|BC - EF| = |DE - AB| = |AF - CD|.$$

6.50. The sums of the angles at vertices A, C, E and B, D, F of a convex hexagon $ABCDEF$ with equal sides are equal. Prove that the opposite sides of this hexagon are parallel.

6.51. Prove that if in a convex hexagon each of the three diagonals that connect the opposite vertices divides the area in halves then these diagonals intersect at one point.

6.52. Prove that if in a convex hexagon each of the three segments that connect the midpoints of the opposite sides divides the area in halves then these segments intersect at one point.

See also problems 2.11, 2.20, 2.46, 3.66, 4.6, 4.28, 4.31, 5.80, 9.45 a), 9.76–9.78, 13.3, 14.6, 18.22, 18.23.

§6. Regular polygons

6.53. The number of sides of a polygon $A_1 \dots A_n$ is odd. Prove that:

a) if this polygon is an inscribed one and all its angles are equal, then it is a regular polygon;

b) if this polygon is a circumscribed one and all its sides are equal, then it is a regular polygon.

6.54. All the angles of a convex polygon $A_1 \dots A_n$ are equal; an inner point O of the polygon is the vertex of equal angles that subtend all the polygon's sides. Prove that the polygon is a regular one.

6.55. A paper band of constant width is tied in a simple knot and then tightened in order to make the knot flat, cf. Fig. 64. Prove that the knot is of the form of a regular pentagon.

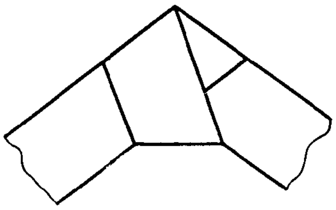


Figure 64 (6.55)

6.56. On sides AB, BC, CD and DA of square $ABCD$ equilateral triangles ABK, BCL, CDM and DAN are constructed inwards. Prove that the midpoints of sides of these triangles (which are not the sides of a square) and the midpoints of segments KL, LM, MN and NK form a regular 12-gon.

* * *

6.57. Does there exist a regular polygon the length of one of whose diagonal is equal to the sum of lengths of some other two diagonals?

6.58. A regular $(4k+2)$-gon is inscribed in a circle of radius R centered at O. Prove that the sum of the lengths of segments singled out by the legs of angle $\angle A_kOA_{k+1}$ on lines A_1A_{2k}, $A_2A_{2k-1}, \dots, A_kA_{k+1}$ is equal to R.

6.59. In regular 18-gon $A_1 \dots A_{18}$, diagonals A_aA_d, A_bA_e and A_cA_f are drawn. Let $k = a - b$, $p = b - c$, $m = c - d$, $q = d - e$, $n = e - f$ and $r = f - a$. Prove that the indicated diagonals intersect at one point in any of the following cases and only in these cases:

a) $\overrightarrow{k, m, n} = \overrightarrow{p, q, r}$;

b) $\overrightarrow{k, m, n} = \overrightarrow{1, 2, 7}$ and $\overrightarrow{p, q, r} = \overrightarrow{1, 3, 4}$;

c) $\overrightarrow{k, m, n} = \overrightarrow{1, 2, 8}$ and $\overrightarrow{p, q, r} = \overrightarrow{2, 2, 3}$.

Remark. The equality $\overrightarrow{k, m, n} = \overrightarrow{x, y, z}$ means that the indicated tuples of numbers coincide; the order in which they are written in not taken into account.

6.60. In a regular 30-gon three diagonals are drawn. For them define tuples $\overrightarrow{k, m, n}$ and $\overrightarrow{p, q, r}$ as in the preceding problem. Prove that if $\overrightarrow{k, m, n} = \overrightarrow{1, 3, 14}$ and $\overrightarrow{p, q, r} = \overrightarrow{2, 2, 8}$, then the diagonals intersect at one point.

6.61. In a regular n-gon ($n \geq 3$) the midpoints of all its sides and the diagonals are marked. What is the greatest number of marked points that lie on one circle?

6.62. The vertices of a regular n-gon are painted several colours so that the points of one colour are the vertices of a regular polygon. Prove that among these polygons there are two equal ones.

6.63. Prove that for $n \geq 6$ a regular $(n-1)$-gon is impossible to inscribe in a regular n-gon so that on every side of the n-gon except one there lies exactly one vertex of the $(n-1)$-gon.

* * *

6.64. Let O be the center of a regular n-gon $A_1 \dots A_n$ and X an arbitrary point. Prove that

$$\overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \overrightarrow{0} \quad \text{and} \quad \overrightarrow{XA_1} + \dots + \overrightarrow{XA_n} = n\overrightarrow{XO}.$$

6.65. Prove that it is possible to place real numbers $x_1, \dots, x_n$ all distinct from zero in the vertices of a regular n-gon so that for any regular k-gon all vertices of which are vertices of the initial n-gon the sum of the numbers at the vertices of the k-gon is equal to zero.

6.66. Point A lies inside regular 10-gon $X_1 \dots X_{10}$ and point B outside it. Let $\mathbf{a} = \overrightarrow{AX_1} + \dots \overrightarrow{AX_{10}}$ and $\mathbf{b} = \overrightarrow{BX_1} + \dots \overrightarrow{BX_{10}}$. Is it possible that $|\mathbf{a}| > |\mathbf{b}|$?

6.67. A regular polygon $A_1 \dots A_n$ is inscribed in the circle of radius R centered at O; let X be an arbitrary point. Prove that

$$A_1X^2 + \dots + A_nX^2 = n(R^2 + d^2), \quad \text{where} \quad d = OX.$$

6.68. Find the sum of squares of the lengths of all the sides and diagonals of a regular n-gon inscribed in a circle of radius R.

6.69. Prove that the sum of distances from an arbitrary X to the vertices of a regular n-gon is the least if X is the center of the n-gon.

6.70. A regular n-gon $A_1 \dots A_n$ is inscribed in the circle of radius R centered at O; let $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{x} = \overrightarrow{OX}$ be an arbitrary vector. Prove that

$$\sum (\mathbf{e}_i, \mathbf{x})^2 = \frac{nR^2 \cdot OX^2}{2}.$$

6.71. Find the sum of the squared distances from the vertices of a regular n-gon inscribed in a circle of radius R to an arbitrary line that passes through the center of the n-gon.

6.72. The distance from point X to the center of a regular n-gon is equal to d and r is the radius of the inscribed circle of the n-gon. Prove that the sum of squared distances from point X to the lines that contain the sides of the n-gon is equal to $n\left(r^2 + \frac{d^2}{2}\right)$.

6.73. Prove that the sum of squared lengths of the projections of the sides of a regular n-gon to any line is equal to $\frac{1}{2}na^2$, where a is the length of the side of the n-gon.

6.74. A regular n-gon $A_1 \dots A_n$ is inscribed in a circle of radius R; let X be a point on this circle. Prove that

$$XA_1^4 + \dots + XA_n^4 = 6nR^4.$$

6.75. a) A regular n-gon $A_1 \dots A_n$ is inscribed in the circle of radius 1 centered at 0, let $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{u}$ an arbitrary vector. Prove that $\sum (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i = \frac{1}{2}n\mathbf{u}$.

b) From an arbitrary point X perpendiculars $XA_1, \dots, XA_n$ are dropped to the sides (or their extensions) of a regular n-gon. Prove that $\sum \overrightarrow{XA_i} = \frac{1}{2}n\overrightarrow{XO}$, where O is the center of the n-gon.

6.76. Prove that if the number n is not a power of a prime, then there exists a convex n-gon with sides of length 1, 2, …, n, all the angles of which are equal.

See also Problems 2.9, 4.59, 4.62, 6.36, 6.41, 6.45–6.47, 9.83, 9.84, 11.46, 11.48, 17.31, 18.30, 19.47, 23.8, 24.2.

§7. The inscribed and circumscribed polygons

6.77. On the sides of a triangle three squares are constructed outwards. What should be the values of the angles of the triangle in order for the six vertices of these squares distinct from the vertices of the triangle belong to one circle?

6.78. A $2n$-gon $A_1 \dots A_{2n}$ is inscribed in a circle. Let $p_1, \dots, p_{2n}$ be the distances from an arbitrary point M on the circle to sides A_1A_2, A_2A_3, $\dots$, $A_{2n}A_1$. Prove that $p_1p_3 \dots p_{2n-1} = p_2p_4 \dots p_{2n}$.

6.79. An inscribed polygon is divided by nonintersecting diagonals into triangles. Prove that the sum of radii of all the circles inscribed in these triangles does not depend on the partition.

6.80. Two n-gons are inscribed in one circle and the collections of the length of their sides are equal but the corresponding sides are not necessarily equal. Prove that the areas of these polygons are equal.

6.81. Positive numbers $a_1, \dots, a_n$ are such that $2a_i < a_1 + \dots + a_n$ for all $i = 1, \dots, n$. Prove that there exists an inscribed n-gon the lengths of whose sides are equal to $a_1, \dots, a_n$.

* * *

6.82. A point inside a circumscribed n-gon is connected by segments with all the vertices and tangent points. The triangles formed in this way are alternately painted red and blue. Prove that the product of areas of red triangles is equal to the product of areas of blue triangles.

6.83. In a $2n$-gon (n is odd) $A_1 \dots A_{2n}$ circumscribed about a circle centered at O the diagonals A_1A_{n+1}, A_2A_{n+2}, $\dots$, $A_{n-1}A_{2n-1}$ pass through point O. Prove that the diagonals A_nA_{2n} also passes through point O.

6.84. A circle of radius r is tangent to the sides of a polygon at points $A_1, \dots, A_n$ and the length of the side on which point A_i lies is equal to a_i. The distance from point X to the center of the circle is equal to d. Prove that

$$a_1XA_1^2 + \dots + a_nXA_n^2 = P(r^2 + d^2),$$

where P is the perimeter of the polygon.

6.85. An n-gon $A_1 \dots A_n$ is circumscribed about a circle; l is an arbitrary tangent to the circle that does not pass through any vertex of the n-gon. Let a_i be the distance from vertex A_i to line l and b_i the distance from the tangent point of side A_iA_{i+1} with the circle to line l. Prove that:

a) the value $\frac{b_1 \dots b_n}{a_1 \dots a_n}$ does not depend on the choice of line l;

b) the value $\frac{a_1a_3 \dots a_{2m-1}}{a_2a_4 \dots a_{2m}}$ does not depend on the choice of line l if $n = 2m$.

6.86. Certain sides of a convex polygon are red; the other ones are blue. The sum of the lengths of the red sides is smaller than the semiperimeter and there is no pair of neighbouring blue sides. Prove that it is impossible to inscribe this polygon in a circle.

See also Problems 2.12, 4.39, 19.6.

§8. Arbitrary convex polygons

6.87. What is the greatest number of acute angles that a convex polygon can have?

6.88. How many sides whose length is equal to the length of the longest diagonal can a convex polygon have?

6.89. For which n there exists a convex n-gon one side of which is of length 1 and the lengths of the diagonals are integers?

6.90. Can a convex non-regular pentagon have exactly four sides of equal length and exactly four diagonals of equal lengths? Can the fifth side of such a pentagon have a common point with the fifth diagonal?

6.91. Point O that lies inside a convex polygon forms, together with each two of its vertices, an isosceles triangle. Prove that point O is equidistant from the vertices of this polygon.

See also Problems 4.49, 4.50, 9.82, 9.85, 9.86, 11.35, 13.14, 14.26, 16.8, 17.33, 17.34, 19.9, 23.13, 23.15.

§9. Pascal's theorem

6.92. Prove that the intersection points of the opposite sides (if these sides are not parallel) of an inscribed hexagon lie on one line. (*Pascal's theorem.*)

6.93. Point M lies on the circumscribed cirlce of triangle ABC; let R be an arbitrary point. Lines AR, BR and CR intersect the circumscribed circle at points A_1, B_1 and C_1, respectively. Prove that the intersection points of lines MA_1 and BC, MB_1 and CA, MC_1 and AB lie on one line and this line passes through point R.

6.94. In triangle ABC, heights AA_1 and BB_1 and bisectors AA_2 and BB_2 are drawn; the inscribed circle is tangent to sides BC and AC at points A_3 and B_3, respectively. Prove that lines A_1B_1, A_2B_2 and A_3B_3 either intersect at one point or are parallel.

6.95. Quadrilateral $ABCD$ is inscribed in circle S; let X be an arbitrary point, M and N be the other intersection points of lines XA and XD with circle S. Lines DC and AX, AB and DX intersect at points E and F, respectively. Prove that the intersection point of lines MN and EF lies on line BC.

6.96. Points A and A_1 that lie inside a circle centered at O are symmetric through point O. Rays AP and A_1P_1 are codirected, rays AQ and A_1Q_1 are also codirected. Prove that the intersection point of lines P_1Q and PQ_1 lies on line AA_1. (Points P, P_1, Q and Q_1 lie on the circle.)

6.97. On a circle, five points are given. With the help of a ruler only construct a sixth point on this circle.

6.98. Points $A_1, \dots, A_6$ lie on one circle and points K, L, M and N lie on lines A_1A_2, A_3A_4, A_1A_6 and A_4A_5, respectively, so that $KL \parallel A_2A_3$, $LM \parallel A_3A_6$ and $MN \parallel A_6A_5$. Prove that $NK \parallel A_5A_2$.

Problems for independent study

6.99. Prove that if $ABCD$ is a rectangle and P is an arbitrary point, then $AP^2+CP^2 = DP^2 + BP^2$.

6.100. The diagonals of convex quadrilateral $ABCD$ are perpendicular. On the sides of the quadrilateral, squares centered at P, Q, R and S are constructed outwards. Prove that segment PR passes through the intersection point of diagonals AC and BD so that $PR = \frac{1}{2}(AC + BD)$.

6.101. On the longest side AC of triangle ABC, points A_1 and C_1 are taken so that $AC_1 = AB$ and $CA_1 = CB$ and on sides AB and BC points A_2 and C_2 are taken so that $AA_1 = AA_2$ and $CC_1 = CC_2$. Prove that quadrilateral $A_1A_2C_2C_1$ is an inscribed one.

6.102. A convex 7-gon is inscribed in a circle. Prove that if certain three of its angles are equal to 120° each, then some two of its sides are equal.

6.103. In plane, there are given a regular n-gon $A_1 \dots A_n$ and point P. Prove that from segments $A_1P, \dots , A_nP$ a closed broken line can be constructed.

6.104. Quadrilateral $ABCD$ is inscribed in circle S_1 and circumscribed about circle S_2; let K, L, M and N be tangent points of its sides with circle S_2. Prove that $KM \perp LN$.

6.105. Pentagon $ABCDE$ the lengths of whose sides are integers and $AB = CD = 1$ is circumscribed about a circle. Find the length of segment BK, where K is the tangent point of side BC with the circle.

6.106. Prove that in a regular $2n$-gon $A_1 \dots A_{2n}$ the diagonals A_1A_{n+2}, $A_{2n-1}A_3$ and $A_{2n}A_5$ meet at one point.

6.107. Prove that in a regular 24-gon $A_1 \dots A_{24}$ diagonals A_1A_7, A_3A_{11} and A_5A_{21} intersect at a point that lies on diameter A_4A_{16}.

Solutions

6.1. Let O be the center of the inscribed circle and the intersection point of the diagonals of quadrilateral $ABCD$. Then $\angle ACB = \angle ACD$ and $\angle BAC = \angle CAD$. Hence, triangles ABC and ADC are equal, since they have a common side AC. Therefore, $AB = DA$. Similarly, $AB = BC = CD = DA$.

6.2. Clearly,

$$\angle AOB = 180^\circ - \angle BAO - \angle ABO = 180^\circ - \frac{\angle A + \angle B}{2}$$

and $\angle COD = 180^\circ - \frac{\angle C + \angle D}{2}$. Hence,

$$\angle AOB + \angle COD = 360^\circ - \frac{\angle A + \angle B + \angle C + \angle D}{2} = 180^\circ.$$

6.3. Let us consider two circles tangent to the sides of the given quadrilateral and their extensions. The lines that contain the sides of the quadrilateral are the common inner and outer tangents to these circles. The line that connects the midpoints of the circles contains a diagonal of the quadrilateral and besides it is an axis of symmetry of the quadrilateral. Hence, the other diagonal is perpendicular to this line.

6.4. Let O be the center of the given circle, R its radius, a the length of chords singled out by the circle on the sides of the quadrilateral. Then the distances from point O to the sides of the quadrilateral are equal to $\sqrt{R^2 - \frac{a^2}{4}}$, i.e., point O is equidistant from the sides of the quadrilateral and is the center of the inscribed circle.

6.5. For a parallelogram the statement of the problem is obvious therefore, we can assume that lines AB and CD intersect. Let O be the center of the inscribed circle of quadrilateral $ABCD$; let M and N be the midpoints of diagonals AC and BD. Then

$$S_{ANB} + S_{CND} = S_{AMB} + S_{CMD} = S_{AOB} + S_{COD} = \frac{S_{ABCD}}{2}.$$

It remains to make use of the result of Problem 7.2.

6.6. Let the inscribed circle be tangent to sides DA, AB and BC at points M, H and N, respectively. Then OH is a height of triangle AOB and the symmetries through lines AO and BO sends point H into points M and N, respectively. Hence, by Problem 1.57 points A_1 and B_1 lie on line MN. Similarly, points C_1 and D_1 lie on line MN.

6.7. Let r be the distance from the intersection point of bisectors of angles A and D to the base AD, let r' be the distance from the intersection point of bisectors of angles B and

C to base BC. Then $AD = r(\cot\alpha + \cot\beta)$ and $BC = r'(\tan\alpha + \tan\beta)$. Hence, $r = r'$ if and only if

$$\frac{BC}{AD} = \frac{\tan\alpha + \tan\beta}{\cot\alpha\cot\beta} = \tan\alpha \cdot \tan\beta.$$

6.8. Let $\angle A = 2\alpha$, $\angle C = 2\beta$ and $\angle BMA = 2\varphi$. By Problem 6.7, $\frac{PK}{RL} = \frac{\tan\alpha}{\tan\varphi}$ and $\frac{LS}{MC} = \cot\varphi\tan\beta$. Since $\frac{PQ}{RS} = \frac{PK}{RL}$ and $\frac{RS}{AC} = \frac{LS}{MC}$, it follows that

$$\frac{PQ}{AC} = \frac{PK}{RL}\frac{LS}{MC} = \tan\alpha\tan\beta.$$

Hence, trapezoid $APQC$ is a circumscribed one.

6.9. First, let us prove that if quadrilateral $ABCD$ is a circumscribed one, then all the conditions take place. Let K, L, M and N be the tangent points of the inscribed circle with sides AB, BC, CD and DA. Then

$$\begin{gathered}
AB + CD = AK + BK + CM + DM = AN + BL + CL + DN = BC + AD,\\
AP + CQ = AK + PK + QL - CL = AN + PM + QN - CM = AQ + CP,\\
BP + BQ = AP - AB + BC + CQ = (AP + CQ) + (BC - AB) =\\
AQ + CP + CD - AD = DP + DQ.
\end{gathered}$$

Now, let us prove, for instance, that if $BP + BQ = DP + DQ$, then quadrilateral $ABCD$ is a circumscribed one. For this let us consider the circle tangent to side BC and rays BA and CD. Assume that line AD is not tangent to this circle; let us shift this line in order for it to touch the circle (Fig. 65).

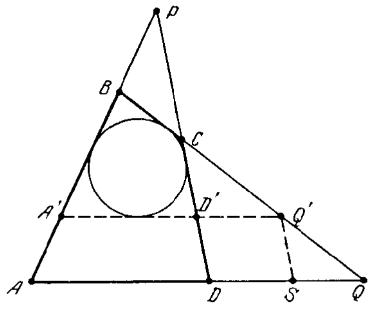


Figure 65 (Sol. 6.9)

Let S be a point on line AQ such that $Q'S \parallel DD'$. Since $BP + BQ = DP + DQ$ and $BP + BQ' = D'P + D'Q'$, it follows that $QS + SQ' = QQ'$. Contradiction.

In the other two cases the proof is similar.

6.10. Let rays AB and DC intersect at point P, let rays BC and AD intersect at point Q; let the given lines passing through points P and Q intersect at point O. By Problem 6.9 we have $BP + BQ = OP + OQ$ and $OP + OQ = DP + DQ$. Hence, $BP + BQ = DP + DQ$ and, therefore, quadrilateral $ABCD$ is a circumscribed one.

6.11. Let sides AB, BC, CD and DA of quadrilateral $ABCD$ be tangent of the inscribed circle at points E, F, G and H, respectively. First, let us show that lines FH, EG and AC intersect at one point. Denote the points at which lines FH and EG intersect line AC by M and M', respectively. Since $\angle AMH = \angle BFM$ as angles between the tangents and chord HF, it follows that $\sin\angle AHM = \sin\angle CFM$. Hence,

$$\frac{AM \cdot MH}{FM \cdot MC} = \frac{S_{AMH}}{S_{FMC}} = \frac{AH \cdot MH}{FC \cdot FM},$$

i.e., $\frac{AM}{MC} = \frac{AH}{FC}$. Similarly,

$$\frac{AM'}{M'C} = \frac{AE}{CG} = \frac{AH}{FC} = \frac{AM}{MC};$$

hence, $M = M'$, i.e., lines FH, EG and AC intersect at one point.

Similar arguments show that lines FH, EG and BD intersect at one point and therefore, lines AC, BD, FH and EG intersect at one point.

6.12. Segments CH_d and DH_c are parallel because they are perpendicular to line BC. Moreover, since $\angle BCA = \angle BDA = \varphi$, the lengths of these segments are equal to $AB|\cot\varphi|$, cf. Problem 5.45 b).

6.13. Let O_a, O_b, O_c and O_d be the centers of the inscribed circles of triangles BCD, ACD, ABD and ABC, respectively. Since $\angle ADB = \angle ACB$, it follows that

$$\angle AO_cB = 90^\circ + \frac{\angle ADB}{2} = 90^\circ + \frac{\angle ACB}{2} = \angle AO_dB,$$

cf. Problem 5.3. Therefore, quadrilateral ABO_dO_c is an inscribed one, i.e.,

$$\angle O_cO_dB = 180^\circ - \angle O_cAB = 180^\circ - \frac{\angle A}{2}.$$

Similarly, $\angle O_aO_dB = 180^\circ - \frac{\angle C}{2}$. Since $\angle A + \angle C = 180^\circ$, it follows that $\angle O_cO_dB + \angle O_aO_bB = 270^\circ$ and, therefore, $\angle O_aO_dO_c = 90^\circ$. We similarly prove that the remaining angles of quadrilateral $O_aO_bO_cO_d$ are equal to 90°.

6.14. a) Let rays AB and DC intersect at point P and rays BC and AD intersect at point Q. Let us prove that point M at which the circumscribed circles of triangles CBP and CDQ intersect lies on segment PQ. Indeed,

$$\angle CMP + \angle CMQ = \angle ABC + \angle ADC = 180^\circ.$$

Hence, $PM + QM = PQ$ and since

$$PM \cdot PQ = PD \cdot PC = p^2 - R^2 \quad \text{and } QM \cdot PQ = QD \cdot QA = q^2 - R^2,$$

it follows that $PQ^2 = PM \cdot PQ + QM \cdot PQ = p^2 + q^2 - 2R^2$. Let N be the intersection point of the circumscribed circles of triangles ACP and ABS. Let us prove that point S lies on segment PN. Indeed,

$$\angle ANP = \angle ACP = 180^\circ - \angle ACD = 180^\circ - \angle ABD = \angle ANS.$$

Hence, $PN - SN = PS$ and since

$$PN \cdot PS = PA \cdot PB = p^2 - R^2 \quad \text{and } SN \cdot PS = SA \cdot SC = R^2 - s^2,$$

it follows that

$$PS^2 = PN \cdot PS - SN \cdot PS = p^2 + s^2 - 2R^2.$$

Similarly, $QS^2 = q^2 + s^2 - 2R^2$.

b) By heading a)

$$PQ^2 - PS^2 = q^2 - s^2 = OQ^2 - OS^2.$$

Hence, $OP \perp QS$, cf. Problem 7.6. We similarly prove that $OQ \perp PS$ and $OS \perp PQ$.

6.15. Let the inscribed circles of triangles ABC and ACD be tangent to diagonal AC at points M and N, respectively. Then

$$AM = \frac{AC + AB - BC}{2} \quad \text{and} quad AN = \frac{AC + AD - CD}{2},$$

cf. Problem 3.2. Points M and N coincide if and only if $AM = AN$, i.e., $AB + CD = BC + AD$. Thus, if points M and N coincide, then quadrilateral $ABCD$ is a circumscribed one and similar arguments show that the tangent points of the inscribed circles of triangles ABD and BCD with the diagonal BD coincide.

Let the inscribed circle of triangle ABC be tangent to sides AB, BC and CA at points P, Q and M, respectively and the inscribed circle of triangle ACD be tangent to sides AC, CD and DA at points M, R and S, respectively. Since $AP = AM = AS$ and $CQ = CM = CR$, it follows that triangles APS, BPQ, CQR and DRS are isosceles ones; let α, β, γ and δ be the angles at the bases of these isosceles triangles. The sum of the angles of these triangles is equal to

$$2(\alpha+\beta+\gamma+\delta)+\angle A+\angle B+\angle C+\angle D;$$

hence, $\alpha+\beta+\gamma+\delta = 180^\circ$. Therefore,

$$\angle SPQ+\angle SRQ = 360^\circ-(\alpha+\beta+\gamma+\delta) = 180^\circ,$$

i.e., quadrilateral $PQRS$ is an inscribed one.

6.16. Let O be the intersection point of diagonals AC and BD; let A_1, B_1, C_1 and D_1 be the projections of O to sides AB, BC, CD and DA, respectively. Points A_1 and D_1 lie on the circle with diameter AO, hence, $\angle OA_1D_1 = \angle OAD_1$. Similarly, $\angle OA_1B_1 = \angle OBB_1$. Since $\angle CAD = \angle CBD$, we have: $\angle OA_1D_1 = \angle OA_1B_1$.

We similarly prove that B_1O, C_1O and D_1O are the bisectors of the angles of quadrilateral $A_1B_1C_1D_1$, i.e., O is the center of its inscribed circle.

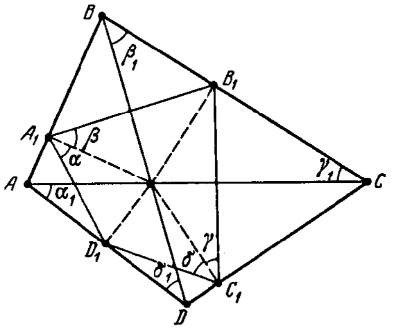


Figure 66 (Sol. 6.17)

6.17. Let us make use of the notations on Fig. 66. The condition that quadrilateral $A_1B_1C_1D_1$ is an inscribed one is equivalent to the fact that $(\alpha+\beta)+(\gamma+\delta) = 180^\circ$ and the the fact that AC and BD are perpendicular is equivalent to the fact that $(\alpha_1+\delta_1)+(\beta_1+\gamma_1) = 180^\circ$. It is also clear that $\alpha = \alpha_1$, $\beta = \beta_1, \gamma = \gamma_1$ and $\delta = \delta_1$.

6.18. By the law of cosines

$$AD^2 = AC^2+CD^2-2AC\cdot CD\cos ACD, \quad AC^2 = AB^2+BC^2-2AB\cdot BC\cos B.$$

Since the length of the projection of segment AC to line l perpendicular to CD is equal to the sum of the lengths of projections of segments AB and BC to line l,

$$AC\cos ACD = AB\cos\varphi+BC\cos C.$$

6.19. Let $\angle AOD = 2\alpha$; then the distances from point O to the projections of the midpoints of diagonals AC and BD to the bisector of angle $\angle AOD$ are equal to $\frac{OA+OC}{2}\cos\alpha$ and $\frac{OB+OD}{2}\cos\alpha$, respectively. Since

$$OA+OC = AB+OB+OC = CD+OB+OC = OB+OD,$$

these projections coincide.

6.20. Let us complement triangles ABM and DCM to parallelograms $ABMM_1$ and $DCMM_2$. Since $AM_1 : DM_2 = BM : MC = AN : DN$, it follows that $\triangle ANM_1 \sim \triangle DNM_2$. Hence, point N lies on segment M_1M_2 and

$$MM_1 : MM_2 = AB : CD = AN : ND = M_1N : M_2N,$$

i.e., MN is the bisector of angle M_1MM_2.

6.21. Let a, b, c and d be (the lengths of) the bisectors of the angles at vertices A, B, C and D. We have to verify that $\angle(a,b) + \angle(c,d) = 0^\circ$. Clearly,

$$\angle(a,b) = \angle(a,AB) + \angle(AB,b) \quad \text{and} \quad \angle(c,d) = \angle(c,CD) + \angle(CD,d).$$

Since quadrilateral $ABCD$ is a convex one and

$$\angle(a,AB) = \frac{\angle(AD,AB)}{2},\ \ \angle(AB,b) = \frac{\angle(AB,BC)}{2},$$
$$\angle(c,CD) = \frac{\angle(CB,CD)}{2},\ \ \angle(CD,d) = \frac{\angle(CD,DA)}{2},$$

it follows that

$$\angle(a,b) + \angle(c,d) = \frac{\angle(AD,AB) + \angle(AB,BC) + \angle(CB,CD) + \angle(CD,DA)}{2} = \frac{360^\circ}{2} = 0^\circ$$

(see Background to Chapter 2).

6.22. Let, for definiteness, $AB > A_1B_1$. The parallel translation by vector $\overrightarrow{CB}$ sends triangle SD_1C_1 to $S'D_1'C_1'$ and segment CD to BA. Since $QA_1 : QA = A_1B_1 : AB = S'D_1' : S'A$, we see that $QS' \parallel A_1D_1'$. Hence, $QS \parallel AD$. Similarly, $PR \parallel AB$.

6.23. Suppose that lines AD and BC are not parallel. Let M_2, K, N_2 be the midpoints of sides AB, BC, CD, respectively. If $MN \parallel BC$, then $BC \parallel AD$, because $AM = MC$ and $BN = ND$. Therefore, let us assume that lines MN and BC are not parallel, i.e., $M_1 \neq M_2$ and $N_1 \neq N_2$. Clearly, $\overrightarrow{M_2M} = \frac{1}{2}\overrightarrow{BC} = \overrightarrow{NN_2}$ and $\overrightarrow{M_1M} = \overrightarrow{NN_1}$. Hence, $M_1M_2 \parallel N_1N_2$. Therefore, $KM \parallel AB \parallel CD \parallel KN$, i.e., $M = N$. Contradiction.

6.24. By a similarity transformation we can identify one pair of the corresponding sides of quadrilaterals, therefore, it suffices to consider quadrilaterals $ABCD$ and ABC_1D_1 whose points C_1 and D_1 lie on rays BC and AD and such that $CD \parallel C_1D_1$. Denote the intersection points of diagonals of quadrilaterals $ABCD$ and ABC$_1D_1$ by O and O_1, respectively.

Suppose that points C and D lie closer to points B and A, then points C_1 and D_1, respectively. Let us prove then that $\angle AOB > \angle AO_1B$. Indeed, $\angle C_1BA > \angle CAB$ and $\angle D_1BA > \angle DBA$, hence,

$$\angle AO_1B = 180^\circ - \angle C_1AB - \angle D_1BA < 180^\circ - \angle CAB - \angle DBA = \angle AOB.$$

We have obtained a contradiction and, therefore, $C_1 = C$, $D_1 = D$.

6.25. Any quadrilateral is determined up to similarity by the directions of its sides and diagonals. Therefore, it suffices to construct one example of a quadrilateral $A_1B_1C_1D_1$ with the required directions of sides and diagonals. Let O be the intersection point of diagonals AC and BD. On ray OA, take an arbitrary point D_1 and draw $D_1A_1 \parallel BC$, $A_1B_1 \parallel CD$ and $B_1C_1 \parallel DA$ (Fig. 67).

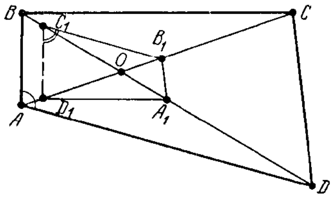


Figure 67 (Sol. 6.25)

Since

$$OC_1 : OB_1 = OD : OA, \quad OB_1 : OA_1 = OC : OD \quad \text{and} \quad OA_1 : OD_1 = OB : OC,$$

it follows that $OC_1 : OD_1 = OB : OA$, consequently, $C_1D_1 \parallel AB$. The obtained plot shows that $\angle A + \angle C_1 = 180^\circ$.

6.26. Let O be the intersection point of the diagonals of quadrilateral $ABCD$. Without loss of generality we may assume that $\alpha = \angle AOB < 90^\circ$. Let us drop perpendiculars AA_1, BB_1, CC_1, DD_1 to the diagonals of quadrilateral $ABCD$. Since

$$OA_1 = OA\cos\alpha, \quad OB_1 = OB\cos\alpha, \quad OC_1 = OC\cos\alpha, \quad OD_1 = OD\cos\alpha,$$

it follows that the symmetry through the bisector of angle AOB sends quadrilateral $ABCD$ into a quadrilateral homothetic to quadrilateral $A_1B_1C_1D_1$ with coefficient $\overrightarrow{BC}1\cos\alpha$.

6.27. Let the diagonals of quadrilateral $ABCD$ intersect at point O; let H_a and H_b be the orthocentres of triangles AOB and COD; let K_a and K_b be the midpoints of sides BC and AD; let P be the midpoint of diagonal AC. The intersection point of medians of triangles AOD and BOC divide segments K_aO and K_bO in the ratio of $1 : 2$ and, therefore, we have to prove that $H_aH_b \perp K_aK_b$.

Since $OH_a = AB|\cot\varphi|$ and $OH_b = CD|\cot\varphi|$, where $\varphi = \angle AOB$, cf. Problem 5.45 b), then $OH_a : OH_b = PK_a : PK_b$. The correspondiong legs of angles $\angle H_aOH_b$ and $\angle K_aPK_b$ are perpendicular; moreover, vectors $\overrightarrow{OH_a}$ and $\overrightarrow{OH_b}$ are directed *towards* lines AB and CD for $\varphi < 90^\circ$ and *away from* these lines for $\varphi > 90^\circ$. Hence, $\angle H_aOH_b = \angle K_aPK_b$ and $\triangle H_aOH_b \sim \triangle K_aPK_b$. It follows that $H_aH_b \perp K_aK_b$.

6.28. Let $S = S_{AOD}$, $x = AO$, $y = DO$, $a = AB$, $b = BC$, $c = CD$, $d = DA$ and k the similarity coefficient of triangles BOC and AOD. Then

$$2\left(\frac{1}{r_1} + \frac{1}{r_3}\right) = \frac{d + x + y}{S} + \frac{kd + kx + ky}{k^2S},$$

$$2\left(\frac{1}{r_2} + \frac{1}{r_4}\right) = \frac{a + x + ky}{kS} + \frac{c + kx + y}{kS}$$

because $S_{BOC} = k^2S$ and $S_{AOB} = S_{COD} = kS$. Since

$$\frac{x+y}{S} + \frac{x+y}{k^2S} = \frac{x+ky}{kS} + \frac{kx+y}{kS},$$

it remains to notice that $a + c = b + d = kd + d$.

6.29. It is easy to verify that

$$AB = r_1\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right) \quad \text{and} \quad CD = r_3\left(\cot\frac{C}{2} + \cot\frac{D}{2}\right).$$

Hence,

$$\frac{AB}{r_1} + \frac{CD}{r_3} = \cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} + \cot\frac{D}{2} = \frac{BC}{r_2} + \frac{AD}{r_4}.$$

6.30. Let us complete triangles ABD and DBC to parallelograms $ABDA_1$ and $DBCC_1$. The segments that connect point D with the vertices of parallelogram ACC_1A_1 divide it into four triangles equal to triangles DAB, CDA, BCD and ABC and, therefore, the radii of the inscribed circles of these triangles are equal.

Let us prove that point D coincides with the intersection point O of the diagonals of the parallelogram. If $D \neq 0$, then we may assume that point D lies inside triangle AOC. Then $r_{ADC} < r_{AOC} = r_{A_1OC_1} < r_{A_1DC_1} = r_{ABC}$, cf. Problem 10.86. We have obtained a contradiction, hence, $D = O$.

Since $p = \overrightarrow{BC}Sr$ and the areas and radii of the inscribed circles of triangles into which the diagonals divide the parallelogram ACC_1A_1 are equal, the triangles' perimeters are equal. Hence, ACC_1A_1 is a rhombus and $ABCD$ is a rectangular.

6.31. Points C_1 and D_1 lie on the midperpendicular to segment AB, hence, $AB \perp C_1D_1$. Similarly, $C_1D_1 \perp A_2B_2$ and, therefore, $AB \parallel A_2B_2$. We similarly prove that the remaining corresponding sides and the diagonals of quadrilaterals $ABCD$ and $A_2B_2C_2D_2$ are parallel. Therefore, these quadrilaterals are similar.

Let M be the midpoint of segment AC. Then $B_1M = |AM \cot D|$ and $D_1M = |AM \cot B|$, where $B_1D_1 = |\cot B + \cot D| \cdot \frac{1}{2}AC$. Let us rotate quadrilateral $A_1B_1C_1D_1$ by 90°. Then making use of the result of Problem 6.25 we see that this quadrilateral is a convex one and $\cot A = -\cot C_1$, etc. Therefore,

$$A_2C_2 = |\cot A + \cot C| \cdot \frac{1}{2}B_1D_1 = \frac{1}{4}|(\cot A + \cot C)(\cot B + \cot D)| \cdot AC.$$

6.32. Let M and N be the midpoints of sides AB and CD, respectively. Let us drop from point D perpendicular DP to line MN and from point M perpendicular MQ to line CD. Then Q is the tangent point of line CD and a circle with diameter AB. Right triangles PDN and OMN are similar, hence,

$$DP = \frac{ND \cdot MQ}{MN} = \frac{ND \cdot MA}{MN}.$$

Similarly, the distance from point A to line MN is equal to $\overrightarrow{BC}ND \cdot MAMN$. Therefore, $AD \parallel MN$. Similarly, $BC \parallel MN$.

6.33. It suffices to verify that the orthocentres of any three of the four given triangles lie on one line. Let a certain line intersect lines B_1C_1, C_1A_1 and A_1B_1 at points A, B and C, respectively; let A_2, B_2 and C_2 be the orthocentres of triangles A_1BC, AB_1C and ABC_1, respectively. Lines AB_2 and A_2B are perpendicular to line A_1B_1 and, therefore, they are parallel. Similarly, $BC_2 \parallel B_2C$ and $CA_2 \parallel C_2A$. Points A, B and C lie on one line and, therefore, points A_2, B_2 and C_2 also lie on one line, cf. Problem 1.12 b).

6.34. On diagonal BD, take point M so that $\angle MCD = \angle BCA$. Then $\triangle ABC \sim \triangle DMC$, because angles $\angle BAC$ and $\angle BDC$ subtend the same arc. Hence, $AB \cdot CD = AC \cdot MD$. Since $\angle MCD = \angle BCA$, then $\angle BCM = \angle ACD$ and $\triangle BCM \sim \triangle ACD$ because angles $\angle CBD$ and $\angle CAD$ subtend one arc. Hence, $BC \cdot AD = AC \cdot BM$. It follows that

$$AB \cdot CD + AD \cdot BC = AC \cdot MD + AC \cdot BM = AC \cdot BD.$$

6.35. Let S be the area of quadrilateral $ABCD$, let R be the radius of its circumscribed circle. Then

$$S = S_{ABC} + S_{ADC} = \frac{AC(AB \cdot BC + AD \cdot DC)}{4R},$$

cf. Problem 12.1. Similarly,

$$S = \frac{BD(AB \cdot AD + BC \cdot CD)}{4R}.$$

By equating these equations for S we get the desired statement.

6.36. Let regular hexagon $A_1 \dots A_7$ be inscribed in a circle. By applying Ptolemey's theorem to qudrilateral $A_1A_3A_4A_5$ we get

$$A_1A_3 \cdot A_5A_4 + A_3A_4 \cdot A_1A_5 = A_1A_4 \cdot A_3A_5,$$

i.e.,

$$\sin 2\alpha \sin \alpha + \sin \alpha \sin 3\alpha = \sin 3\alpha \sin 2\alpha.$$

6.37. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively. By Ptolemy's theorem

$$AC_1 \cdot OB_1 + AB_1 \cdot OC_1 = AO \cdot B_1C_1,$$

where O is the center of the circumscribed circle. Hence, $cd_b + bd_c = aR$. Similarly, $ad_c + cd_a = bR$ and $ad_b + bd_a = cR$. Moreover, $ad_a + bd_b + cd_c = 2S = (a + b + c)r$. By adding all these equalities and dividing by $a + b + c$ we get the desired statement.

6.38. By Ptolemy's theorem

$$AB \cdot CD + AC \cdot BD = AD \cdot BC.$$

Taking into account that $CD = BD \geq \frac{1}{2}BC$ we get the desired statement.

6.39. By applying Ptolemy's theorem to quadrilateral $ABCP$ and dividing by the lengths of the square's side we get the desired statement.

6.40. By applying Ptolemy's theorem to quadrilateral $APQR$ we get

$$AP \cdot RQ + AR \cdot QP = AQ \cdot PR.$$

Since $\angle ACB = \angle RAQ = \angle RPQ$ and $\angle RQP = 180° - \angle PAR = \angle ABC$, it follows that $\triangle RQP \sim \triangle ABC$ and, therefore, $RQ : QP : PR = AB : BC : CA$. It remains to notice that $BC = AD$.

6.41. a) Let us express Ptolemy's theorem for all quadrilaterals with vertices at point A and three consecutive vertices of the given polygon; then let us group in the obtained equalities the factors in which d_i with even indices enter in the right-hand side. By adding these equalities we get

$$(2a + b)(d_1 + \dots + d_{2n+1}) = (2a + b)(d_2 + \dots + d_{2n}),$$

where a is the side of the given polygon and b is its shortest diagonal.

b) Let R be the radius of circle S. Then $l_i = d_i\sqrt{\frac{R \pm r}{R}}$, cf. Problem 3.20. It remains to make use of the result of heading a).

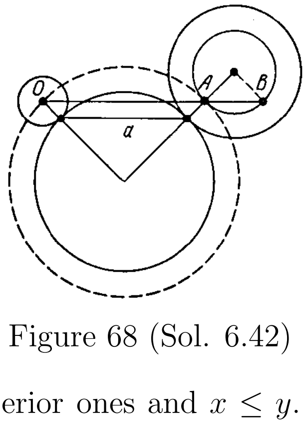


Figure 68 (Sol. 6.42)

6.42. Let both tangent be exterior ones and $x \leq y$. The line that passes through the center O of the circle of radius x parallel to the segment that connects the tangent points intersects the circle of radius $y - x$ (centered in the center of the circle of radius y) at points A and B (Fig. 68).

Then $OA = \frac{a(R+x)}{R}$ and

$$OB = OA + \frac{a(y-x)}{R} = \frac{a(R+y)}{R}.$$

The square of the length to be found of the common outer tangent is equal to

$$OA \cdot OB = \left(\frac{a}{R}\right)^2 (R+x)(R+y).$$

Similar arguments show that if both tangent are inner ones, then the square of the lengths of the outer tangent is equal to $\left(\frac{a}{R}\right)^2 (R-x)(R-y)$ and if the circle of radius x is tangent from the outside and the circle of radius y from the inside, then the square of the length of the inner tangent is equal to $\left(\frac{a}{R}\right)^2 (R-y)(R+x)$.

Remark. In the case of an inner tangency of the circles we assume that $R > x$ and $R > y$.

6.43. Let R be the radius of the circumscribed circle of quadrilateral $ABCD$; let r_a, r_b, r_c and r_d be the radii of circles α, β, γ and δ, respectively. Further, let $a = \sqrt{R \pm r_a}$, where the plus sign is taken if the tangent is an outer one and the minus sign if it is an inner one; numbers b, c and d are similarly defined. Then $t_{\alpha\beta} = \frac{ab \cdot AB}{R}$, cf. Problem 6.42, etc. Therefore, by multiplying the equality

$$AB \cdot CD + BC \cdot DA = AC \cdot BD$$

by $\frac{abcd}{R}$ we get the desired statement.

6.44. Since $\angle EBD = \angle ABE + \angle CBD$, it is possible to take a point P on side ED so that $\angle EBP = \angle ABE = \angle AEB$, i.e., $BP \parallel AE$. Then $\angle PBD = \angle EBD - \angle EBP = \angle CBD = \angle BDC$, i.e., $BP \parallel CD$. Therefore, $AE \parallel CD$ and since $AE = CD$, $CDEA$ is a parallelogram. Hence, $AC = ED$, i.e., triangle ABC is an equilateral one and $\angle ABC = 60^\circ$.

6.45. a) Let O be the center of the circumscribed circle of triangle CKE. It suffices to verify that $\angle COK = 2\angle KCB$. It is easy to calculate both these angles:

$$\angle COK = 180^\circ - 2\angle OKC = 180^\circ - \angle EKC = 180^\circ - \angle EDC = 72^\circ$$

and $\angle KCB = \frac{180^\circ - \angle ABC}{2} = 36^\circ$.

b) Since BC is a tangent to the circumscribed circle of triangle CKE, then $BE \cdot BK = BC^2$, i.e., $d(d-a) = a^2$.

6.46. Let the perpendiculars erected to line AB at points A and B intersect sides DE and CD at points P and Q, respectively. Any point of segment CQ is a vertex of a rectangle inscribed in pentagon $ABCDE$ (the respective sides of this pentagon are parallel to AB and AP); as this point moves from Q to C the ratio of the lengths of the sides of the rectangles varies from $\frac{AP}{AB}$ to 0. Since angle $\angle AEP$ is an obtuse one, $AP > AE = AB$. Therefore, for a certain point of segment QC the ratio of the lengths of the sides of the rectangle is equal to 1.

6.47. Let points $A_1, \dots, E_1$ be symmetric to points $A, \dots, E$ through the center of circle S; let P, Q and R be the intersection points of lines BC_1 and AB_1, AE_1 and BA_1, BA_1 and CB_1, see Fig. 69.

Then $PQ = AB = a$ and $QR = b$. Since $PQ \parallel AB$ and $\angle ABA_1 = 90^\circ$, it follows that $PR^2 = PQ^2 + QR^2 = a^2 + b^2$. Line PR passes through the center of circle S and $\angle AB_1C = 4 \cdot 18^\circ = 72^\circ$, hence, PR is a side of a regular pentagon circumscribed about the circle with center B_1 whose radius B_1O is equal to the radius of circle S.

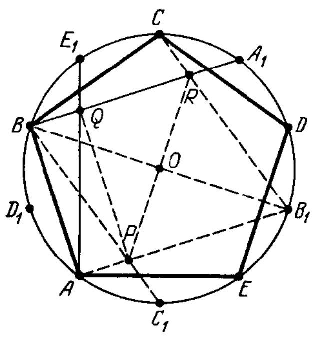


Figure 69 (Sol. 6.47)

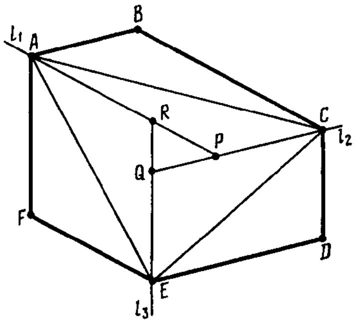


Figure 70 (Sol. 6.48)

6.48. Through points A, C and E draw lines l_1, l_2 and l_3 parallel to lines BC, DE and FA, respectively. Denote the intersection points of lines l_1 and l_2, l_2 and l_3, l_3 and l_1 by P, Q, R, see Fig. 70. Then

$$S_{ACE}=\frac{S_{ABCDEF}-S_{PQR}}{2}+S_{PQR}=\frac{S_{ABCDEF}+S_{PQR}}{2}\geq\frac{S_{ABCDEF}}{2}.$$

Similarly, $S_{BDF}=\frac{1}{2}(S_{ABCDEF}+S_{P'Q'R'})$. Clearly,

$$PQ=|AB-DE|,\quad QR=|CD-AF|,\quad PR=|EF-BC|,$$

hence, triangles PQR and $P'Q'R'$ are equal. Therefore, $S_{ACE}=S_{BDF}$.

6.49. Let us construct triangle PQR as in the preceding problem. This triangle is an equilateral one and

$$PQ=|AB-DE|,\quad QR=|CD-AF|,\quad RP=|EF-BC|.$$

Hence, $|AB-DE|=|CD-AF|=|EF-BC|$.

6.50. The sum of the angles at vertices A, C and E is equal to 360°, hence, from isosceles triangles ABF, CBD and EDF we can construct a triangle by juxtaposing AB to CB, ED to CD and EF to AF. The sides of the obtained triangle are equal to the respective sides of triangle BDF. Therefore, the symmetry through lines FB, BD and DF sends points A, C and E, respectively, into the center O of the circumscribed circle of triangle BDF, and, therefore, $AB \parallel OF \parallel DE$.

6.51. Let us suppose that the diagonals of the hexagon form triangle PQR. Denote the vertices of the hexagon as follows: vertex A lies on ray QP, vertex B on RP, vertex C on RQ, etc. Since lines AD and BE divide the area of the hexagon in halves, then

$$S_{APEF}+S_{PED}=S_{PDCB}+S_{ABP}\quad\text{and}\quad S_{APEF}+S_{ABP}=S_{PDCB}+S_{PED}.$$

Hence, $S_{ABP} = S_{PED}$, i.e.,

$$AP \cdot BP = EP \cdot DP = (ER + RP)(DQ + QP) > ER \cdot DQ.$$

Similarly, $CQ \cdot DQ > AP \cdot FR$ and $FR \cdot ER > BP \cdot CQ$. By multiplying these inequalities we get

$$AP \cdot BP \cdot CQ \cdot DQ \cdot FR \cdot ER > ER \cdot DQ \cdot AP \cdot FR \cdot BP \cdot CQ$$

which is impossible. Hence, the diagonals of the hexagon intersect at one point.

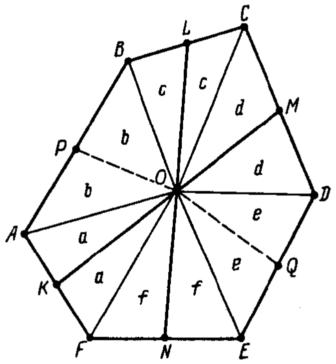


Figure 71 (Sol. 6.52)

6.52. Denote the midpoints of the sides of convex hexagon $ABCDEF$ as plotted on Fig. 71. Let O be the intersection point of segments KM and LN. Let us denote the areas of triangles into which the segments that connect point O with the vertices and the midpoints of the sides divide the hexagon as indicated on the same figure. It is easy to verify that $S_{KONF} = S_{LOMC}$, i.e., $a + f = c + d$. Therefore, the broken line POQ divides the hexagon into two parts of equal area; hence, segment PQ passes through point O.

6.53. a) Let O be the center of the circumscribed circle. Since

$$\angle A_kOA_{k+2} = 360^\circ - 2\angle A_kA_{k+1}A_{k+2} = \varphi$$

is a constant, the rotation through an angle of φ with center O sends point A_k into A_{k+2}. For n odd this implies that all the sides of polygon $A_1 \dots A_n$ are equal.

b) Let a be the length of the side of the given polygon. If one of its sides is divided by the tangent point with the inscribed circle into segments of length x and $a - x$, then its neighbouring sides are also divided into segments of length x and $a - x$ (the neighbouring segments of neighbouring sides are equal), etc. For n odd this implies that all the sides of polygon $A_1 \dots A_n$ are divided by the tangent points with the inscribed circle in halves; therefore, all the angles of the polygon are equal.

6.54. The sides of polygon $A_1 \dots A_n$ are parallel to respective sides of a regular n-gon. On rays $OA_1, \dots, OA_n$ mark equal segments $OB_1, \dots, OB_n$. Then polygon $B_1 \dots B_n$ is a regular one and the sides of polygon $A_1 \dots A_n$ form equal angles with the respective sides of polygon $B_1 \dots B_n$. Therefore,

$$OA_1 : OA_2 = OA_2 : OA_3 = \dots = OA_n : OA_1 = k,$$

i.e.,

$$OA_1 = kOA_2 = k^2OA_3 = \dots = k^nOA_1;$$

thus, $k = 1$.

6.55. Denote the vertices of the pentagon as indicated on Fig. 72. Notice that if in a triangle two heights are equal, then the sides on which these heights are dropped are also equal.

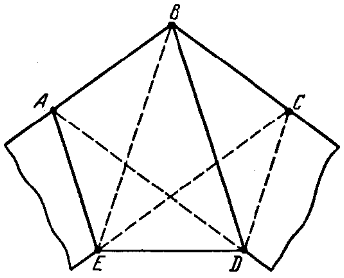


Figure 72 (Sol. 6.55)

From consideration of triangles EAB, ABC and BCD we deduce that $EA = AB$, $AB = BC$ and $BC = CD$. Therefore, trapezoids $EABC$ and $ABCD$ are isosceles ones, i.e., $\angle A = \angle B = \angle C$. By considering triangles ABD and BCE we get $AD = BD$ and $BE = CE$. Since triangles EAB, ABC, BCD are equal, it follows that $BE = AC = BD$. Hence, $AD = BE$ and $BD = CE$, i.e., trapezoids $ABDE$ and $CDEB$ are isosceles ones. Therefore, $ED = AB = BC = CD = AE$ and $\angle E = \angle A = \angle B = \angle C = \angle D$, i.e., $ABCDE$ is a regular pentagon.

6.56. Triangles BAM and BCN are isosceles ones with angle 15° at the base, cf. Problem 2.26, and, therefore, triangle BMN is an equilateral one. Let O be the centre of the square, P and Q the midpoints of segments MN and BK (Fig. 73). Since OQ is the midline of triangle MBK, it follows that $OQ = \frac{1}{2}BM = MP = OP$ and $\angle QON = \angle MBA = 15^\circ$. Therefore, $\angle POQ = \angle PON - \angle QON = 30^\circ$.

The remaining part of the proof is carried out similarly.

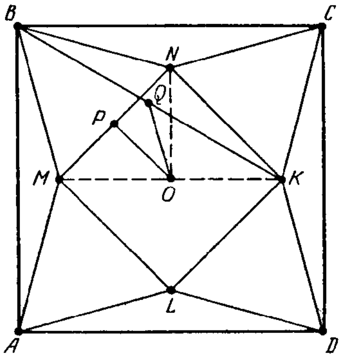


Figure 73 (Sol. 6.56)

6.57. Let us consider a regular 12-gon $A_1 \dots A_{12}$ inscribed in a circle of radius R. Clearly, $A_1A_7 = 2R$, $A_1A_3 = A_1A_{11} = R$. Hence, $A_1A_7 = A_1A_3 + A_1A_{11}$.

6.58. For $k = 3$ the solution of the problem is clear from Fig. 74. Indeed, $A_3A_4 = OQ$, $KL = QP$ and $MN = PA_{14}$ and, therefore,

$$A_3A_4 + KL + MN = OQ + QP + PA_{14} = OA_{14} = R.$$

Proof is carried out in a similar way for any k.

6.59. In the proof if suffices to apply the result of Problems 5.78 and 5.70 b) to triangle $A_aA_cA_e$ and lines A_aA_d, A_cA_f and A_eA_b. Solvling heading b) we have to notice additionally that

$$\sin 20^\circ \sin 70^\circ = \sin 20^\circ \cos 20^\circ = \frac{\sin 40^\circ}{2} = \sin 30^\circ \sin 40^\circ$$

and in the solution of heading c) that $\sin 10^\circ \sin 80^\circ = \sin 30^\circ \sin 20^\circ$.

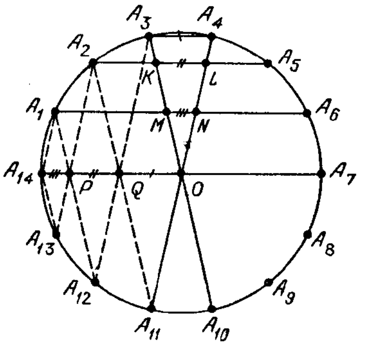


Figure 74 (Sol. 6.58)

6.60. As in the preceding problem we have to verify the equality

$$\sin 2\alpha \sin 2\alpha \sin 8\alpha = \sin\alpha \sin 3\alpha \sin 14\alpha, \text{ where } \alpha = \frac{180^\circ}{30} = 6^\circ.$$

Clearly, $\sin 14\alpha = \cos\alpha$, hence, $2\sin\alpha\sin 3\alpha \sin 14\alpha = \sin 2\alpha \sin 3\alpha$. It remains to verify that

$$\sin 3\alpha = 2\sin 2\alpha \sin 8\alpha = \cos 6\alpha - \cos 10\alpha = 1 - 2\sin^2 3\alpha - \frac{1}{2},$$

i.e., $4\sin^2 18^\circ + 2\sin 18^\circ = 1$, cf. Problem 5.46.

6.61. First, let $n = 2m$. The diagonals and sides of a regular $2m$-gon have m distinct lengths. Therefore, the marked points lie on $m-1$ concentric circles (having n points each) or in the common center of these circles. Since distinct circles have not more than two common points, the circle that does not belong to this family of concentric circles contains not more than $1 + 2(m-1) = 2m - 1 = n - 1$ of marked points.

Now, let $n = 2m+1$. There are m distinct lengths among the lengths of the diagonals and sides of a regular $(2m+1)$-gon. Hence, the marked points lie on m concentric circles (n points on each). A circle that does not belong to this family of concentric circles contains not more than $2m = n - 1$ marked points.

In either case the greatest number of marked points that lie on one circle is equal to n.

6.62. Denote the center of the polygon by O and the vertices of the polygon by $A_1, \dots, A_n$. Suppose that there are no equal polygons among the polygons of the same colour, i.e., they have $m = m_1 < m_2 < m_3 < \dots < m_k$ sides, respectively. Let us consider a transformation f defined on the set of vertices of the n-gon as the one that sends vertex A_k to vertex A_{mk} : $f(A_k) = A_{mk}$ (we assume that $A_{p+qn} = A_p$). This transformation sends the vertices of a regular m-gon into one point, B, hence, the sum of vectors $\overrightarrow{Of(A_i)}$, where A_i are the vertices of an m-gon, is equal to $m\overrightarrow{OB} \neq \overrightarrow{0}$.

Since $\angle A_{mi}OA_{mj} = m\angle A_iOA_j$, the vertices of any regular polygon with the number of sides greater than m pass under the considered transformation into the vertices of a regular polygon. Therefore, the sum of vectors $\overrightarrow{Of(A_i)}$ over all vertices of an n-gon and similar sums over the vertices of m_2-, m_3-, $\dots, m_k$-gons are equal to zero. We have obtained a contradiction with the fact that the sum of vectors $\overrightarrow{Of(A_i)}$ over the vertices of an m-gon is not equal to zero.

Therefore, among the polygons of one color there are two equal ones.

6.63. Let a regular $(n-1)$-gon $B_1 \dots B_{n-1}$ be inscribed into a regular n-gon $A_1 \dots A_n$. We may assume that A_1 and B_1 are the least distant from each other vertices of these polygons and points B_2, B_3, B_4 and B_5 lie on sides A_2A_3, A_3A_4, A_4A_5 and A_5A_6. Let $\alpha_i = \angle A_{i+1}B_iB_{i+1}$ and $\beta_i = \angle B_iB_{i+1}A_{i+1}$, where $i = 1, 2, 3, 4$. By the sine theorem

$A_2B_2 : B_1B_2 = \sin\alpha_1 : \sin\varphi$ and $B_2A_3 : B_2B_3 = \sin\beta_2 : \sin\varphi$, where φ is the angle at a vertex of a regular n-gon. Therefore, $\sin\alpha_1 + \sin\beta_2 = \frac{a_n \sin\varphi}{a_{n-1}}$, where a_n and a_{n-1} are the (lengths of the) sides of the given polygons.

Similar arguments show that

$$\sin\alpha_1 + \sin\beta_2 = \sin\alpha_2 + \sin\beta_3 = \sin\alpha_3 + \sin\beta_4.$$

Now, observe that

$$\sin\alpha_i + \sin\beta_{i+1} = 2\sin\frac{\alpha_i + \beta_{i+1}}{2}\cos\frac{\alpha_i - \beta_{i+1}}{2}$$

and compute $\alpha_i + \beta_{i+1}$ and $\alpha_i - \beta_{i+1}$. Since $\alpha_i + \beta_i = \frac{2\pi}{n}$ and $\alpha_{i+1} + \beta_i = \frac{2\pi}{n-1}$, it follows that $\alpha_{i+1} = \alpha_i + \frac{2\pi}{n(n-1)}$ and $\beta_{i+1} = \beta_i - \frac{2\pi}{n(n-1)}$; therefore,

$$\alpha_i + \beta_{i+1} = \frac{2\pi}{n} - \frac{2\pi}{n(n-1)}$$

is a constant and

$$\alpha_i - \beta_{i+1} = \alpha_{i-1} - \beta_i + \frac{4\pi}{n(n-1)}.$$

Hence,

$$\cos\theta = \cos\left(\theta + \frac{2\pi}{n(n-1)}\right) = \cos\left(\theta + \frac{4\pi}{(n-1)n}\right) \quad \text{for} \quad \theta = \frac{\alpha_1 - \beta_2}{2}.$$

We have obtained a contradiction because on an interval shorter than 2π the cosine cannot attain the same value at three distinct points.

Remark. A square *can* be inscribed in a regular pentagon, cf. Problem 6.46.

6.64. Let $\mathbf{a} = \overrightarrow{OA_1} + \dots + \overrightarrow{OA_n}$. A rotation about point O by $\frac{360^\circ}{n}$ sends point A_i to A_{i+1} and, therefore, sends vector $\mathbf{a}$ into itself, i.e., $\mathbf{a} = \mathbf{0}$.

Since $\overrightarrow{XA_i} = \overrightarrow{XO} + \overrightarrow{OA_i}$ and $\overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \overrightarrow{0}$, it follows that $\overrightarrow{XA_1} + \dots + \overrightarrow{XA_n} = n\overrightarrow{XO}$.

6.65. Through the center of a regular polygon $A_1 \dots A_n$, draw line l that does not pass through the vertices of the polygon. Let x_i be equal to the length of the projection of vector $\overrightarrow{OA_i}$ to a line perpendicular to l. Then all the x_i are nonzero and the sum of numbers x_i assigned to the vertices of a regular k-gon is equal to zero since the corresponding sum of vectors $\overrightarrow{OA_i}$ vanishes, cf. Problem 6.64.

6.66. By Problem 6.64 $\mathbf{a} = 10\overrightarrow{AO}$ and $\mathbf{b} = 10\overrightarrow{BO}$, where O is the center of polygon $X_1 \dots X_{10}$. Clearly, if point A is situated rather close to a vertex of the polygon and point B rather close to the midpoint of a side, then $AO > BO$.

6.67. Since

$$A_iX^2 = |\overrightarrow{A_iO} + \overrightarrow{OX}|^2 = A_iO^2 + OX^2 + 2(\overrightarrow{A_iO}, \overrightarrow{OX}) = R^2 + d^2 + 2(\overrightarrow{A_iO}, \overrightarrow{OX}),$$

it follows that

$$\sum A_iX^2 = n(R^2 + d^2) + 2(\sum \overrightarrow{A_iO},\ \overrightarrow{OX}) = n(R^2 + d^2),$$

cf. Problem 6.64.

6.68. Denote by S_k the sum of squared distances from vertex A_k to all the other vertices. Then

$$S_k = A_kA_1^2 + A_kA_2^2 + \dots + A_kA_n^2 = A_kO^2 + 2(\overrightarrow{A_kO}, \overrightarrow{OA_1}) + A_1O^2 + \dots \\ + A_kO^2 + 2(\overrightarrow{A_kO}, \overrightarrow{OA_n}) + A_nO^2 = 2nR^2$$

because $\sum_{i=1}^n \overrightarrow{OA_i} = \overrightarrow{0}$. Hence, $\sum_{i=1}^n S_k = 2n^2R^2$. Since each squared side and diagonal enters this sum twice, the sum to be found is equal to n^2R^2.

6.69. Consider the rotation of the given n-gon about the n-gon's center O that sends A_k to A_1. Let X_k be the image of point X under the rotation. This rotation sends segment A_kX to A_1X_k. Therefore,

$$A_1X + \dots + A_nX = A_1X_1 + \dots + A_1X_n.$$

Since n-gon $X_1 \dots X_n$ is a regular one,

$$\overrightarrow{A_1X_1} + \dots + \overrightarrow{A_1X_n} = n\overrightarrow{A_1O},$$

cf. Problem 6.64. Therefore, $A_1X_1 + \dots + A_nX_n \geq n\overrightarrow{A_1O}$.

6.70. Let B_i be the projection of point X to line OA_i. Then

$$(\mathbf{e}_i, \mathbf{x}) = (\overrightarrow{OA_i},\ \overrightarrow{OB_i} + \overrightarrow{B_iX}) = (\overrightarrow{OA_i},\ \overrightarrow{OB_i}) = \pm R \cdot OB_i.$$

Points $B_1, \dots, B_n$ lie on the circle with diameter OX and are vertices of a regular n-gon for n odd and vertices of an $\frac{n}{2}$-gon counted twice for n even, cf. Problem 2.9. Therefore, $\sum OB_i^2 = \frac{1}{2}n \cdot OX^2$, cf. Problem 6.67.

6.71. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the vectors that go from the center of the given n-gon into its vertices; $\mathbf{x}$ a unit vector perpendicular to line l. The sum to be found is equal to $\sum(\mathbf{e}_i, \mathbf{x})^2 = \frac{1}{2}n \cdot R^2$, cf. Problem 6.70.

6.72. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the unit vectors directed from the center O of a regular n-gon into the midpoints of its sides; $\mathbf{x} = \overrightarrow{OX}$. Then the distance from point X to the i-th side is equal to $|(\mathbf{x}, \mathbf{e}_i) - r|$. Hence, the sum to be found is equal to

$$\sum((\mathbf{x}, \mathbf{e}_i)^2 - 2r(\mathbf{x}, \mathbf{e}_i) + r^2) = \sum(\mathbf{x}, \mathbf{e}_i)^2 + nr^2.$$

By Problem 6.70 $\sum(\mathbf{x}, \mathbf{e}_i)^2 = \frac{1}{2}nd^2$.

6.73. Let $\mathbf{x}$ be the unit vector parallel to line l and $\mathbf{e}_i = \overrightarrow{A_iA_{i+1}}$. Then the squared length of the projection of side A_iA_{i+1} to line l is equal to $(\mathbf{x}, \mathbf{e}_i)^2$. By Problem 6.70 $\sum(\mathbf{x}, \mathbf{e}_i)^2 = \frac{1}{2}na^2$.

6.74. Let $\mathbf{a} = \overrightarrow{OX}, \mathbf{e}_i = \overrightarrow{OA_i}$. Then

$$XA_i^4 = |\mathbf{a} + \mathbf{e}_i|^4 = (|\mathbf{a}|^2 + 2(\mathbf{a}, \mathbf{e}_i) + |\mathbf{e}_i|^2)^2 = \\ 4(R^2 + (\mathbf{a}, \mathbf{e}_i))^2 = 4(R^4 + 2R^2(\mathbf{a}, \mathbf{e}_i) + (\mathbf{a}, \mathbf{e}_i)^2).$$

Clearly, $\sum(\mathbf{a}, \mathbf{e}_i) = (\mathbf{a}, \sum \mathbf{e}_i) = \mathbf{0}$. By Problem 6.70 $\sum(\mathbf{a}, \mathbf{e}_i)^2 = \frac{1}{2}nR^4$; hence, the sum to be found is equal to $4\left(nR^4 + \frac{nR^4}{2}\right) = 6nR^4$.

6.75. a) First, let us prove the required relation for $\mathbf{u} = \mathbf{e}_1$. Let $\mathbf{e}_i = (\sin\varphi_i, \cos\varphi_i)$, where $\cos\varphi_1 = 1$. Then

$$\sum(\mathbf{e}_1, \mathbf{e}_i)\mathbf{e}_i = \sum\cos\varphi_i\mathbf{e}_i = \sum(\sin\varphi_i\cos\varphi_i, \cos^2\varphi_i) = \sum\left(\frac{\sin 2\varphi_i}{2}, \frac{1+\cos 2\varphi_i}{2}\right) = \left(0, \frac{n}{2}\right) = \frac{n\mathbf{e}_1}{2}.$$

For $\mathbf{u} = \mathbf{e}_2$ the proof is similar.

It remains to notice that any vector $\mathbf{u}$ can be represented in the form $\mathbf{u} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$.

b) Let $B_1, \dots, B_n$ be the midpoints of sides of the given polygon, $\mathbf{e}_i = \frac{\overrightarrow{OB_i}}{OB_i}$, $\mathbf{u} = \overrightarrow{XO}$. Then $\overrightarrow{XA_i} = \overrightarrow{OB_i} + (\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i$. Since $\sum\overrightarrow{OB_i} = \overrightarrow{0}$, it follows that

$$\sum\overrightarrow{XA_i} = \sum(\mathbf{u}, \mathbf{e}_i)\mathbf{e}_i = \frac{n\mathbf{u}}{2} = \frac{n\overrightarrow{XO}}{2}.$$

6.76. Let $\mathbf{e}_0, \dots, \mathbf{e}_{n-1}$ be the vectors of sides of a regular n-gon. It suffices to prove that by reordering these vectors we can get a set of vectors $\overrightarrow{\mathbf{a}_1, \dots, \mathbf{a}_n}$ such that $\sum_{k=1}^n k\mathbf{a}_k = \mathbf{0}$. A number n which is not a power of a prime can be represented in the form $n = pq$, where p and q are relatively prime. Now, let us prove that the collection

$$\begin{aligned}&\mathbf{e}_0, \mathbf{e}_p, \dots, \mathbf{e}_{(q-1)p}; \mathbf{e}_q, \mathbf{e}_{q+p}, \dots, \mathbf{e}_{q+(q-1)p};\\ &\qquad\qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\qquad\qquad\qquad\qquad \mathbf{e}_{(p-1)q}, \mathbf{e}_{(p-1)q+p}, \dots, \mathbf{e}_{(p-1)q+(q-1)p}\end{aligned}$$

is the one to be found. First, notice that if

$$x_1q + y_1p \equiv x_2q + y_2p \pmod{pq},$$

then $x_1 \equiv x_2 \pmod p$ and $y_1 \equiv y_2 \pmod q$; therefore, in the considered collection each of the vectors $\mathbf{e}_0, \dots, \mathbf{e}_{n-1}$ is encountered exactly once.

The endpoints of vectors $\mathbf{e}_q$, $\mathbf{e}_{q+p}, \dots, \mathbf{e}_{q+(q-1)p}$ with a common beginning point destinguish a regular q-gon and, therefore, their sum is equal to zero. Moreover, vectors $\mathbf{e}_0$, $\mathbf{e}_p, \dots, \mathbf{e}_{(q-1)p}$ turn into $\mathbf{e}_q$, $\mathbf{e}_{q+p}, \dots, \mathbf{e}_{q+(p-1)q}$ under the rotation by an angle of $\varphi = \frac{2\pi}{p}$. Hence, if $\mathbf{e}_0 + 2\mathbf{e}_p + \dots + q\mathbf{e}_{(q-1)p} = \mathbf{b}$, then

$$(q+1)\mathbf{e}_q + (q+2)\mathbf{e}_{q+p} + \dots + 2q\mathbf{e}_{q+(q-1)p} = q(\mathbf{e}_q + \dots + \mathbf{e}_{q+(q-1)p}) + \mathbf{e}_q + 2\mathbf{e}_{q+p} + \dots + q\mathbf{e}_{q+(q-1)p} = R^\varphi\mathbf{b},$$

where $R^\varphi\mathbf{b}$ is the vector obtained from $\mathbf{b}$ after the rotation by $\varphi = \frac{2\pi}{p}$. Similar arguments show that for the considered set of vectors we have

$$\sum_{k=1}^n k\mathbf{a}_k = \mathbf{b} + R^\varphi\mathbf{b} + \dots + R^{(p-1)\varphi}\mathbf{b} = \mathbf{0}.$$

6.77. Suppose that on the sides of triangle ABC squares ABB_1A_1, BCC_2B_2, ACC_3A_3 are constructed outwards and vertices A_1, B_1, B_2, C_2, C_3, A_3 lie on one circle S. The midperpendiculars to segments A_1B_1, B_2C_2, A_3C_3 pass through the center of circle S. It is clear that the midperpendiculars to segments A_1B_1, B_2C_2, A_3C_3 coincide with the midperpendiculars to sides of triangle ABC and therefore, the center of circle S coincides with the center of the circumscribed circle of the triangle.

Denote the center of the circumscribed circle of triangle ABC by O. The distance from O to line B_2C_2 is equal to $R\cos\angle A+2R\sin\angle A$, where R is the radius of the circumscribed circle of triangle ABC. Hence,

$$OB_2^2=(R\sin\angle A)^2+(R\cos\angle A+2R\sin\angle A)^2=$$
$$R^2(3+2(\sin\angle 2A-\cos 2\angle A))=R^2(3-2\sqrt{2}\cos(45^\circ+2\angle A)).$$

Clearly, in order for the triangle to possess the desired property, it is necessary and sufficient that $OB_2^2=OC_3^2=OA_1^2$, i.e.,

$$\cos(45^\circ+2\angle A)=\cos(45^\circ+2\angle B)=\cos(45^\circ+2\angle C).$$

This equality holds for $\angle A=\angle B=\angle C=60^\circ$. If, contrarywise, $\angle A\neq\angle B$, then $(45^\circ+2\angle A)+(45^\circ+2\angle B)=360^\circ$, i.e., $\angle A+\angle B=135^\circ$. Hence, $\angle C=45^\circ$ and $\angle A=\angle C=45^\circ$, $\angle B=90^\circ$ (or $\angle B=45^\circ$, $\angle A=90^\circ$). We see that the triangle should be either an equilateral or an isosceles one.

6.78. In any triangle we have $h_c=\frac{ab}{2R}$ (Problem 12.33); hence, $p_k=\frac{MA_k\cdot MA_{k+1}}{2R}$. Therefore,

$$p_1p_3\dots p_{2n-1}=\frac{MA_1\cdot MA_2\dots MA_{2n}}{(2R)^n}=p_2p_4\dots p_{2n}.$$

6.79. Let ABC be a triangle inscribed in circle S. Denote the distances from the center O of S to sides BC, CA and AB by a, b and c, respectively. Then $R+r=a+b+c$ if point O lies inside triangle ABC and $R+r=-a+b+c$ if points A and O lie on various sides of line BC, cf. Problem 12.38.

Each of the diagonals of the partition belongs to two triangles of the partition. For one of these triangles point O and the remaining vertex lie on one side of the diagonal, for the other one the points lie on different sides.

A partition of an n-gon by nonintersecting diagonals into triangles consists of $n-2$ triangles. Therefore, the sum $(n-2)R+r_1+\dots+r_{n-2}$ is equal to the sum of distances from point O to the sides of an n-gon (the distances to the sides are taken with the corresponding signs). This implies that the sum $r_1+\dots+r_{n-2}$ does not depend on the partition.

6.80. Let polygon $A_1\dots A_n$ be inscribed in a circle. Let us consider point A_2' symmetric to point A_2 through the midperpendicular to segment A_1A_3. Then polygon $A_1A_2'A_3\dots A_n$ is an inscribed one and its area is equal to the area of polygon $A_1\dots A_n$. Therefore, we can transpose any two sides. Therefore, we can make any side, call it X, a neighbouring side of any given side, Y; next, make any of the remaining sides a neighbour of X, etc. Therefore, the area of an n-gon inscribed into the given circle only depends on the set of lengths of the sides but not on their order.

6.81. Without loss of generality we may assume that a_n is the greatest of the numbers $a_1,\dots,a_n$. Let n-gon $A_1\dots A_n$ be inscribed into a circle centered at O. Then

$$A_iA_{i+1}:A_1A_n=\sin\frac{\angle A_iOA_{i+1}}{2}:\sin\frac{\angle A_1OA_n}{2}.$$

Therefore, let us proceed as follows. From the relation $\sin\frac{\varphi_i}{2}:\sin\frac{\varphi}{2}=a_i:a_n$ the angle φ_i is uniquely determined in terms of φ if $\varphi_i<\pi$. On a circle of radius 1, fix a point A_n and consider variable points $A_1,\dots,A_{n-1}$, A_n' such that

$$\smile A_nA_1=\varphi,\ \smile A_1A_2=\varphi_1,\ \dots,\smile A_{n-2}A_{n-1}=\varphi_{n-2}\text{ and }\smile A_{n-1}A_n'=\varphi_{n-1}.$$

Denote these points in two distinct ways as plotted on Fig. 75. (The first way — Fig. 75 a) — corresponds to an n-gon that contains the center of the circle, and the second way — Fig.

75 b) — corresponds to an n-gon that does not contain the center of the circle). It remains to prove that as φ varies from 0 to π, then in one of these cases point A_n' coincides with A_n (indeed, then up to a similarity we get the required n-gon). Suppose that in the first case points A_n' and A_n never coincide for $0 \leq \varphi \leq \pi$, i.e., for $\varphi = \pi$ we have $\varphi_1 + \dots + \varphi_{n-1} < \pi$.

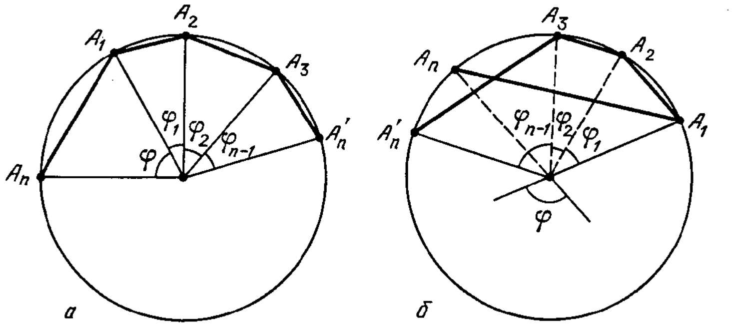


Figure 75 (Sol. 6.81)

Fig. 75 b) requires certain comments: $\sin\alpha \approx \alpha$ for small values of α; hence, the conditions of the problem imply that for small angles point A_n does indeed lie on arc $\smile A_1A_n'$ because $\varphi_1 + \dots + \varphi_{n-1} > \varphi$. Thus, for small angles $\varphi_1 + \dots + \varphi_{n-1} > \varphi$ and if $\varphi = \pi$, then by the hypothesis $\varphi_1 + \dots + \varphi_{n-1} < \pi = \varphi$. Hence, at certain moment $\varphi = \varphi_1 + \dots + \varphi_{n-1}$, i.e., points A_n and A_n' coincide.

6.82. Let $h_1, \dots, h_n$ be the distances from the given point to the corresponding sides; let $a_1, \dots, a_n$ be the distances from the vertices of the polygon to tangent points. Then the product of areas of red as well as blue triangles is equal to $\frac{a_1 \dots a_n h_1 \dots h_n}{2^n}$.

6.83. Let OH_i be a height of triangle OA_iA_{i+1}. Then $\angle H_{i-1}OA_i = \angle H_iOA_i = \varphi_i$. The conditions of the problem imply that

$$\begin{gathered}
\varphi_1 + \varphi_2 = \varphi_{n+1} + \varphi_{n+2},\\
\varphi_{n+2} + \varphi_{n+3} = \varphi_2 + \varphi_3,\\
\varphi_3 + \varphi_4 = \varphi_{n+3} + \varphi_{n+4},\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
\varphi_{n-2} + \varphi_{n-1} = \varphi_{2n-2} + \varphi_{2n-1}
\end{gathered}$$

(expressing the last equality we have taken into account that n is odd) and

$$\varphi_{n-1} + 2\varphi_n + \varphi_{n+1} = \varphi_{2n-1} + 2\varphi_{2n} + \varphi_1.$$

Adding all these equalities we get

$$\varphi_{n-1} + \varphi_n = \varphi_{2n-1} + \varphi_{2n},$$

as required.

6.84. Let O be the center of the given circle. Then $\overrightarrow{XA_i} = \overrightarrow{XO} + \overrightarrow{OA_i}$ and, therefore,

$$XA_i^2 = XO^2 + OA_i^2 + 2(\overrightarrow{OX}, \overrightarrow{OA_i}) = d^2 + r^2 + 2(\overrightarrow{XO}, \overrightarrow{OA_i}).$$

Since $a_1\overrightarrow{OA_1} + \dots + A_n\overrightarrow{OA_n} = \overrightarrow{0}$ (cf. Problem 13.4), it follows that

$$a_1XA_1^2 + \dots + a_nXA_n^2 = (a_1 + \dots + a_n)(d^2 + r^2).$$

6.85. By Problem 5.8 $\frac{b_{i-1}b_i}{a_i^2} = \sin^2\frac{\angle A_i}{2}$. To solve heading a) it suffices to multiply all these equalities and to solve heading b) we have to divide the product of all equalities with even index i by the product of all equalities with odd index i.

6.86. Let BC be a blue side, AB and CD be the sides neighbouring with BC. By the hypothesis sides AB and CD are red ones. Suppose that the polygon is a circumscribed one;

let P, Q, R be the tangent points of sides AB, BC, CD, respectively, with the inscribed circle. Clearly, $BP = BQ$, $CR = CQ$ and segments BP, CR only neighbour one blue segment. Therefore, the sum of the lengths of the red sides is not smaller than the sum of the lengths of the blue sides. We have obtained a contradiction with the fact that the sum of the lengths of red sides is smaller than the semiperimeter. Therefore, a circle cannot be inscribed into the polygon.

6.87. Let the given n-gon have k acute angles. Then the sum of its angles is smaller than $k\cdot 90^\circ + (n-k)\cdot 180^\circ$. On the other hand, the sum of the angles of the n-gon is equal to $(n-2)\cdot 180^\circ$. Hence,

$$(n-2)\cdot 180^\circ < k\cdot 90^\circ + (n-k)\cdot 180^\circ, \quad \text{i.e., } k < 4.$$

Since k is an integer, $k \leq 3$.

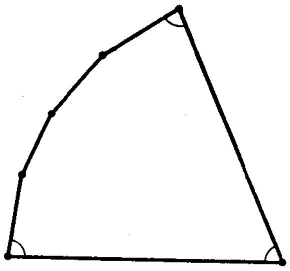


Figure 76 (Sol. 6.87)

For any $n \geq 3$ there exists a convex n-gon with three acute angles (Fig. 76).

6.88. Suppose that the lengths of nonadjacent sides AB and CD are equal to the length of the greatest diagonal. Then $AB + CD \geq AC + BD$. But by Problem 9.14 $AB + CD < AC + BD$. We have obtained a contradiction and therefore, the sides whose length is equal to the length of the longest diagonal should be adjacent ones, i.e., there are not more than two of such sides.

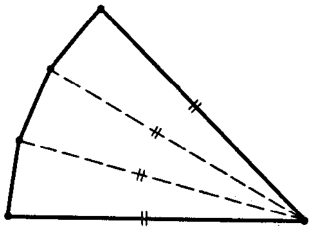


Figure 77 (Sol. 6.88)

An example of a polygon with two sides whose lengths are equal to the length of the longest diagonal is given on Fig. 77. Clearly, such an n-gon exists for any $n > 3$.

6.89. Let us prove that $n \leq 5$. Let $AB = 1$ and C the vertex not adjacent to either A or B. Then $|AC - BC| < AB = 1$. Hence, $AC = BC$, i.e., point C lies on the midperpendicular to side AB. Therefore, in addition to vertices A, B, C the polygon can have only two more vertices.

An example of a pentagon with the required property is given on Fig. 78. Let us elucidate its construction. Clearly, $ACDE$ is a rectangle, $AC = ED = 1$ and $\angle CAD = 60^\circ$. Point B is determined from the condition $BE = BD = 3$.

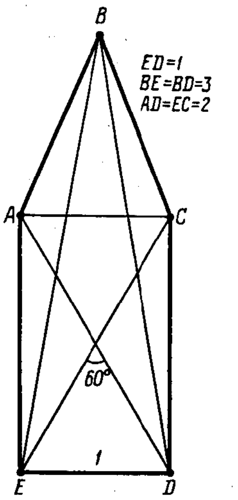


Figure 78 (Sol. 6.89)

An example of a quadrilateral with the desired property is rectangle $ACDE$ on the same figure.

6.90. An example of a pentagon satisfying the conditions of the problem is plotted on Fig. 79. Let us clarify its construction. Take an equilateral right triangle EAB and draw midperpendiculars to sides EA, AB; on them construct points C and D, respectively, so that $ED = BC = AB$ (i.e., lines BC and ED form angles of 30° with the corresponding midperpendiculars). Clearly,

$$DE = BC = AB = EA < EB < DC \quad \text{and} \quad DB = DA = CA = CE > EB.$$

Now, let us prove that the fifth side and the fifth diagonal cannot have a common point. Suppose that the fifth side AB has a common point A with the fifth diagonal. Then the fifth diagonal is either AC or AD. Let us consider these two cases.

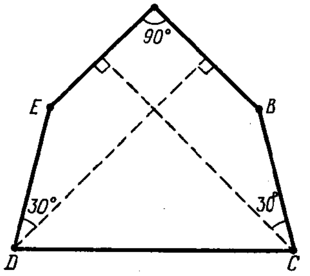


Figure 79 (Sol. 6.90)

In the first case $\triangle AED = \triangle CDE$; hence, under the symmetry through the midperpendicular to segment ED point A turns into point C. This symmetry preserves point B because $BE = BD$. Therefore, segment AB turns into CB, i.e., $AB = CB$. Contradiction.

In the second case $\triangle ACE = \triangle EBD$; hence, under the symmetry through the bisector of angle $\angle AED$ segment AB turns into DC, i.e., $AB = CD$. Contradiction.

6.91. Let us consider two neighbouring vertices A_1 and A_2. If $\angle A_1OA_2 \geq 90°$, then $OA_1 = OA_2$ because neither right nor acute angle can be adjacent to the base of an isosceles triangle.

Now, let $\angle A_1OA_2 < 90°$. Let us draw through point O lines l_1 and l_2 perpendicular to lines OA_1 and OA_2, respectively. Denote the regions into which these lines divide the

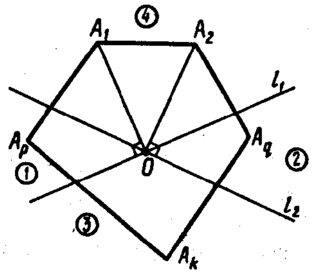


Figure 80 (Sol. 6.91)

plane as indicated on Fig. 80. If in region 3 there is a vertex, A_k, then $A_1O = A_kO = A_2O$ because $\angle A_1OA_k \geq 90°$ and $\angle A_2OA_k \geq 90°$. If region 3 has no vertices of the polygon, then in region 1 there is a vertex A_p and in region 2 there is a vertex A_q (if neither of the regions 1 or 2 would have contained vertices of the polygon, then point O would have been outside the polygon). Since $\angle A_1OA_q \geq 90°$, $\angle A_2OA_p \geq 90°$ and $\angle A_pOA_q \geq 90°$, it follows that $A_1O = A_qO = A_pO = A_2O$.

It remains to notice that if the distances from point O to any pair of the neighbouring vertices of the polygon are equal, then all the distances from point O to the vertices of the polygon are equal.

6.92. Let us prove that if A, B, C, D, E, F are points on the circle placed in an arbitrary order; lines AB and DE, BC and EF, CD and FA, intersect at points G, H, K, respectively. Then points G, H and K lie on one line.

Let a, b, $\dots$, f be oriented angles between a fixed line and lines OA, OB, $\dots$, OF, respectively, where O is the center of the circumscribed circle of the hexagon. Then

$$\angle(AB, DE) = \frac{a+b-d-e}{2}, \quad \angle(CD, FA) = \frac{c+d-f-a}{2},$$
$$\angle(EF, BC) = \frac{e+f-b-c}{2}$$

and, therefore, the sum of these angles is equal to 0.

Let Z be the intersection point of circumscribed circles of triangles BDG and DFK. Let us prove that point B, F, Z and H lie on one circle. For this we have to verify that $\angle(BZ, ZF) = \angle(BH, HF)$. Clearly,

$$\begin{gathered}\angle(BZ, ZF) = \angle(BZ, ZD) + \angle(DZ, ZF),\\ \angle(BZ, ZD) = \angle(BG, GD) = \angle(AB, DE),\\ \angle(DZ, ZF) = \angle(DK, KF) = \angle(CD, FA)\end{gathered}$$

and, as we have just proved,

$$\angle(AB, DE) + \angle(CD, FA) = -\angle(EF, BC) = \angle(BC, EF) = \angle(BH, HF).$$

Now, let us prove that points H, Z and G lie on one line. For this it suffices to verify that $\angle(GZ, ZB) = \angle(HZ, ZB)$. Clearly,

$$\angle(GZ, ZB) = \angle(GD, DB) = \angle(ED, DB), \quad \angle(HZ, ZB) = \angle(HF, FB) = \angle(ED, DB).$$

We similarly prove that points K, Z and G lie on one line:

$$\begin{gathered}\angle(DZ, ZG) = \angle(DB, BG) = \angle(DB, BA);\\ \angle(DZ, ZK) = \angle(DF, FK) = \angle(DB, BA)\end{gathered}$$

We have deduced that points H and K lie on line GZ, consequently, points G, H and K lie on one line.

6.93. Let A_2, B_2 and C_2 be the indicated intersection points of lines. By applying Pascal's theorem to points M, A_1, A, C, B, B_1 we deduce that A_2, B_2 and R lie on one line. Similarly, points A_2, C_2 and R lie on one line. Hence, points A_2, B_2, C_2 and R lie on one line.

6.94. Points A_1 and B_1 lie on circle S of diameter AB. Let A_4 and B_4 be the intersection points of lines AA_2 and BB_2 with line A_3B_3. By Problem 2.41 a) these points lie on circle S. Lines A_1B and A_4A intersect at point A_2 and lines BB_4 and AB_1 at point B_2. Therefore, applying Pascal's theorem to points B_1, A_1, B, B_4, A_4, A we see that the intersection point of lines B_1A_1 and B_4A_4 (the latter line coincides with A_3B_3) lies on line A_2B_2.

6.95. Let K be the intersection point of lines BC and MN. Apply Pascal's theorem to points A, M, N, D, C, B. We see that points E, K, F lie on one line and, therefore, K is the intersection point of lines MN and EF.

6.96. Let rays PA and QA intersect the circle at points P_2 and Q_2, i.e., P_1P_2 and Q_1Q_2 are diameters of the given circle. Let us apply Pascal's theorem to hexagon $PP_2P_1QQ_2Q_1$. Lines PP_2 and QQ_2 intersect at point A and lines P_1P_2 and Q_1Q_2 intersect at point O, hence, the intersection point of lines P_1Q and Q_1P lies on line AO.

6.97. Let given points A, B, C, D, E lie on one line. Suppose that we have constructed point F of the same circle. Denote by K, L, M the intersection points of lines AB and DE, BC and EF, CD and FA, respectively. Then by Pascal's theorem points K, L, M lie on one line.

The above implies the following construction. Let us draw through point E an arbitrary line a and denote its intersection point with line BC by L. Then construct the intersection point K of lines AB and DE and the intersection point M of lines KL and CD. Finally, let F be the intersection point of lines AM and a. Let us prove that F lies on our circle. Let F_1 be the intersection point of the circle and line a. From Pascal's theorem it follows that F_1 lies on line AM, i.e., F_1 is the intersection point of a and AM. Hence, $F_1 = F$.

6.98. Let P and Q be the intersection points of line A_3A_4 with A_1A_2 and A_1A_6, respectively, and R and S be the intersection points of line A_4A_5 with A_1A_6 and A_1A_2, respectively. Then

$$A_2K : A_3L = A_2P : A_3P, \quad A_3L : A_6M = A_3Q : A_6Q, \quad A_6M : A_5N = A_6R : A_5R.$$

Therefore, the desired relation $A_2K : A_5N = A_2S : A_5S$ takes the form

$$\frac{A_2P}{A_3P} \cdot \frac{A_3Q}{A_6Q} \cdot \frac{A_6R}{A_5R} \cdot \frac{A_5S}{A_2S} = 1.$$

Let T be the intersection point of lines A_2A_3 and A_5A_6; by Pascal's theorem points S, Q and T lie on one line. By applying Menelau's theorem (cf. Problem 5.58) to triangle PQS and points T, A_2, A_3 and also to triangle RQS and points T, A_5, A_6 we get

$$\frac{A_2P}{A_2S} \cdot \frac{A_3Q}{A_3P} \cdot \frac{TS}{TQ} = 1 \quad \text{and} \quad \frac{TQ}{TS} \cdot \frac{A_5S}{A_5R} \cdot \frac{A_6R}{A_6Q} = 1.$$

By multiplying these equalities we get the statement desired. (The ratio of segments should be considered oriented ones.)

Chapter 7. LOCI

Background

1) A *locus* is a figure consisting of all points having a cirtain property.

2) A solution of a problem where a locus is to be found should contain the proof of the following facts:

a) the points with a required property belong to figure Φ which is the answer to the problem;

b) All points of Φ have the required property.

3) A locus possessing two properties is the intersection of two figures: (1) the locus of points possessing the first property and (2) the locus of points possessing the other property.

4) Three most important loci:

a) *The locus of points equidistant from points A and B is the midperpendicular to segment AB*;

b) *The locus of points whose distance from a given point O is equal to R is the circle of radius R centered at O*;

c) *The locus of vertices of a given angle that subtend given segment AB is the union of two arcs of circles symmetric through line AB* (*points A and B do not belong to the locus*).

Introductory problems

1. a) Find the locus of points equidistant from two parallel lines.

b) Find the locus of points equidistant from two intersecting lines.

2. Find the locus of the midpoints of segments with the endpoints on two given parallel lines.

3. Given triangle ABC, find the locus of points X satisfying inequalities $AX \leq BX \leq$CX.

4. Find the locus of points X such that the tangents drawn from X to the given circle have a given length.

5. A point A on a circle is fixed. Find the locus of points X that divide chords with A as an endpoint in the ratio of $1:2$ counting from point A.

§1. The locus is a line or a segment of a line

7.1. Two wheels of radii r_1 and r_2 roll along line l. Find the set of intersection points M of their common inner tangents.

7.2. Sides AB and CD of quadrilateral $ABCD$ of area S are not parallel. Inside the quadrilateral find the locus of points X for which $S_{ABX} + S_{CDX} = \frac{1}{2}S$.

7.3. Given two lines that meet at point O. Find the locus of points X for which the sum of the lengths of projections of segments OX to these lines is a constant.

7.4. Given rectangle $ABCD$, find the locus of points X for which $AX+BX = CX+DX$.

7.5. Find the locus of points M that lie inside rhombus $ABCD$ and with the property that $\angle AMD + \angle BMC = 180^\circ$.

* * *

7.6. Given points A and B in plane, find the locus of points M for which the difference of the squared lengths of segments AM and PM is a constant.

7.7. A circle S and a point M outside it are given. Through point M all possible circles S_1 that intersect S are drawn; X is the intersection point of the tangent at M to S_1 with the extension of the common chord of circles S and S_1. Find the locus of points X.

7.8. Given two nonintersecting circles, find the locus of the centers of circles that divide the given circles in halves (i.e., that intersect the given circles in diametrically opposite points).

7.9. A point A inside a circle is taken. Find the locus of the intersection points of tangents to circles drawn through the endpoints of possible chords that contain point A.

7.10. a) Parallelogram $ABCD$ is given. Prove that the quantity

$$AX^2 + CX^2 - BX^2 - DX^2$$

does not depend on the choice of point X.

b) Quadrilateral $ABCD$ is not a parallelogram. Prove that all points X that satisfy the relation $AX^2 + CX^2 = BX^2 + DX^2$ lie on one line perpendicular to the segment that connects the midpoints of the diagonals.

See also Problems 6.14, 15.14.

§2. The locus is a circle or an arc of a circle

7.11. A segment moves along the plane so that its endpoints slide along the legs of a right angle $\angle ABC$. What is the trajectory traversed by the midpoint of this segment? (We naturally assume that the length of the segment does not vary while it moves.)

7.12. Find the locus of the midpoints of the chords of a given circle, provided the chords pass through a given point.

7.13. Given two points, A and B and two circles that are tangent to line AB: one circle is tangent at A and the other one at B, and the circles are tangent to each other at point M. Find the locus of points M.

* * *

7.14. Two points, A and B in plane are given. Find the locus of points M for which $AM : BM = k$. (*Apollonius's circle.*)

7.15. Let S be Apollonius's circle for points A and B where point A lies outside circle S. From point A tangents AP and AQ to circle S are drawn. Prove that B is the midpoint of segment PQ.

7.16. Let AD and AE be the bisectors of the inner and outer angles of triangle ABC and S_a be the circle with diameter DE; circles S_b and S_c are similarly defined. Prove that:

a) circles S_a, S_b and S_c have two common points, M and N, such that line MN passes through the center of the circumscribed circle of triangle ABC;

b) The projections of point M (and N) to the sides of triangle ABC distinguish an equilateral triangle.

7.17. Triangle ABC is an equilateral one, M is a point. Prove that if the lengths of segments AM, BM and CM form a geometric progression, then the quotient of this progression is smaller than 2.

See also Problems 14.19 a), 18.14.

§3. The inscribed angle

7.18. Points A and B on a circle are fixed and a point C runs along the circle. Find the set of the intersection points of a) heights; b) bisectors of triangles ABC.

7.19. Point P runs along the circumscribed circle of square $ABCD$. Lines AP and BD intersect at point Q and the line that passes through point Q parallel to AC intersects line BP at point X. Find the locus of points X.

7.20. a) Points A and B on a circle are fixed and points A_1 and B_1 run along the same circle so that the value of arc $\smile A_1B_1$ remains a constant; let M be the intersection point of lines AA_1 and BB_1. Find the locus of points M.

b) Triangles ABC and $A_1B_1C_1$ are inscribed in a circle; triangle ABC is fixed and triangle $A_1B_1C_1$ rotates. Prove that lines AA_1, BB_1 and CC_1 intersect at one point for not more than one position of triangle $A_1B_1C_1$.

7.21. Four points in the plane are given. Find the locus of the centers of rectangles formed by four lines that pass through the given points.

7.22. Find the locus of points X that lie inside equilateral triangle ABC and such that $\angle XAB + \angle XBC + \angle XCA = 90^\circ$.

See also Problems 2.5, 2.37.

§4. Auxiliary equal triangles

7.23. A semicircle centered at O is given. From every point X on the extension of the diameter of the semicircle a ray tangent to the semicircle is drawn. On the ray segment XM equal to segment XO is marked. Find the locus of points M obtained in this way.

7.24. Let A and B be fixed points in plane. Find the locus of points C with the following property: height h_b of triangle ABC is equal to b.

7.25. A circle and a point P inside it are given. Through every point Q on the circle the tangent is drawn. The perpendicular dropped from the center of the circle to line PQ and the tangent intersect at a point M. Find the locus of points M.

§5. The homothety

7.26. Points A and B on a circle are fixed. Point C runs along the circle. Find the set of the intersection points of the medians of triangles ABC.

7.27. Triangle ABC is given. Find the locus of the centers of rectangles $PQRS$ whose vertices Q and P lie on side AC and vertices R and S lie on sides AB and BC, respectively.

7.28. Two circles intersect at points A and B. Through point A a line passes. It intersects the circles for the second time at points P and Q. What is the line plotted by the midpoint of segment PQ while the intersecting line rotates about point A.

7.29. Points A, B and C lie on one line; B is between A and C. Find the locus of points M such that the radii of the circumscribed circles of triangles AMB and CMB are equal.

See also Problems 19.10, 19.21, 19.38.

§6. A method of loci

7.30. Points P and Q move with the same constant speed v along two lines that intersect at point O. Prove that there exists a fixed point A in plane such that the distances from A to P and Q are equal at all times.

7.31. Through the midpoint of each diagonal of a convex quadrilateral a line is drawn parallel to the other diagonal. These lines meet at point O. Prove that segments that connect

O with the midpoints of the sides of the quadrilateral divide the area of the quadrilateral into equal parts.

7.32. Let D and E be the midpoints of sides AB and BC of an acute triangle ABC and point M lies on side AC. Prove that if $MD < AD$, then $ME > EC$.

7.33. Inside a convex polygon points P and Q are taken. Prove that there exists a vertex of the polygon whose distance from Q is smaller than that from P.

7.34. Points A, B and C are such that for any fourth point M either $MA \leq MB$ or $MA \leq MC$. Prove that point A lies on segment BC.

7.35. Quadrilateral $ABCD$ is given; in it $AB < BC$ and $AD < DC$. Point M lies on diagonal BD. Prove that $AM < MC$.

§7. The locus with a nonzero area

7.36. Let O be the center of rectangle $ABCD$. Find the locus of points M for which $AM \geq OM$, $BM \geq OM$, $CM \geq OM$ and $DM \geq OM$.

7.37. Find the locus of points X from which tangents to a given arc AB of a circle can be drawn.

7.38. Let O be the center of an equilateral triangle ABC. Find the locus of points M satisfying the following condition: any line drawn through M intersects either segment AB or segment CO.

7.39. In plane, two nonintersecting disks are given. Does there necessarily exist a point M outside these disks that satisfies the following condition: each line that passes through M intersects at least one of these disks?

Find the locus of points M with this property.

See also Problem 18.11.

§8. Carnot's theorem

7.40. Prove that the perpendiculars dropped from points A_1, B_1 and C_1 to sides BC, CA, AB of triangle ABC intersect at one point if and only if

$$A_1B^2 + C_1A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1B^2. \qquad (\textit{Carnot's formula})$$

7.41. Prove that the heights of a triangle meet at one point.

7.42. Points A_1, B_1 and C_1 are such that $AB_1 = AC_1$, $BC = BA_1$ and $CA_1 = CB_1$. Prove that the perpendiculars dropped from points A_1, B_1 and C_1 to lines BC, CA and AB meet at one point.

7.43. a) The perpendiculars dropped from the vertices of triangle ABC to the corresponding sides of triangle $A_1B_1C_1$ meet at one point. Prove that the perpendiculars dropped from the vertices of triangle $A_1B_1C_1$ to the corresponding sides of triangle ABC also meet at one point.

b) Lines drawn through vertices of triangle ABC parallelly to the corresponding sides of triangle $A_1B_1C_1$ intersect at one point. Prove that the lines drawn through the vertices of triangle $A_1B_1C_1$ parallelly to the corresponding sides of triangle ABC also intersect at one point.

7.44. On line l points A_1, B_1 and C_1 are taken and from the vertices of triangle ABC perpendiculars AA_2, BB_2 and CC_2 are dropped to this line. Prove that the perpendiculars dropped from points A_1, B_1 and C_1 to lines BC, CA and AB, respectively, intersect at one point if and only if

$$\overline{A_1B_1} : \overline{B_1C_1} = \overline{A_2B_2} : \overline{B_2C_2}.$$

The ratios of segments are oriented ones.

7.45. Triangle ABC is an equilateral one, P an arbitrary point. Prove that the perpendiculars dropped from the centers of the inscribed circles of triangles PAB, PBC and PCA to lines AB, BC and CA, respectively, meet at one point.

7.46. Prove that if perpendiculars raised at the bases of bisectors of a triangle meet at one point, then the triangle is an isosceles one.

§9. Fermat-Apollonius's circle

7.47. Prove that the set of points X such that

$$k_1A_1X^2 + \dots + k_nA_nX^2 = c$$

is either

a) a circle or the empty set if $k_1 + \dots + k_n \neq 0$;

b) a line, a plane or the empty set if $k_1 + \dots + k_n = 0$.

7.48. Line l intersects two circles at four points. Prove that the quadrilateral formed by the tangents at these points is a circumscribed one and the center of its circumscribed circle lies on the line that connects the centers of the given circles.

7.49. Points M and N are such that $AM : BM : CM = AN : BN : CN$. Prove that line MN passes through the center O of the circumscribed circle of triangle ABC.

See also Problems 7.6, 7.14, 8.59–8.63.

Problems for independent study

7.50. On sides AB and BC of triangle ABC, points D and E are taken. Find the locus of the midpoints of segments DE.

7.51. Two circles are tangent to a given line at two given points A and B; the circles are also tangent to each other. Let C and D be the tangent points of these circles with another outer tangent. Both tangent lines to the circles are outer ones. Find the locus of the midpoints of segments CD.

7.52. The bisector of one of the angles of a triangle has inside the triangle a common point with the perpendicular erected from the midpoint of the side opposite the angle. Prove that the triangle is an isosceles one.

7.53. Triangle ABC is given. Find the locus of points M of this triangle for which the condition $AM \geq BM \geq CM$ holds. When the obtained locus is a) a pentagon; b) a triangle?

7.54. Square $ABCD$ is given. Find the locus of the midpoints of the sides of the squares inscribed in the given square.

7.55. An equilateral triangle ABC is given. Find the locus of points M such that triangles AMB and BCM are isosceles ones.

7.56. Find the locus of the midpoints of segments of length $\frac{2}{\sqrt{3}}$ whose endpoints lie on the sides of a unit square.

7.57. On sides AB, BC and CA of a given triangle ABC points P, Q and R, respectively, are taken, so that $PQ \parallel AC$ and $PR \parallel BC$. Find the locus of the midpoints of segments QR.

7.58. Given a semicircle with diameter AB. For any point X on this semicircle, point Y on ray XA is taken so that $XY = XB$. Find the locus of points Y.

7.59. Triangle ABC is given. On its sides AB, BC and CA points C_1, A_1 and B_1, respectively, are selected. Find the locus of the intersection points of the circumscribed circles of triangles AB_1C_1, A_1BC_1 and A_1B_1C.

Solutions

7.1. Let O_1 and O_2 be the centers of the wheels of radii r_1 and r_2, respectively. If M is the intersection point of the inner tangents, then $OM : O_2M = r_1 : r_2$. It is easy to derive from this condition that the distance from point M to line l is equal to $\frac{2r_1r_2}{r_1+r_2}$. Hence, all the intersection points of the common inner tangents lie on the line parallel to l and whose distance from l is equal to $\frac{2r_1r_2}{r_1+r_2}$.

7.2. Let O be the intersection point of lines AB and CD. On rays OA and OD, mark segments OK and OL equal to AB and CD, respectively. Then

$$S_{ABX} + S_{CDX} = S_{KOX} + S_{LOX} = S_{KOL} \pm S_{KXL}.$$

Therefore, the area of triangles KXL is a constant, i.e., point X lies on a line parallel to KL.

7.3. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors parallel to the given lines; $\mathbf{x} = \overrightarrow{OX}$. The sum of the lengths of the projections of vector $\mathbf{x}$ to the given lines is equal to $|(\mathbf{a}, \mathbf{x})| + |(\mathbf{b}, \mathbf{x})| = |(\mathbf{a} \pm \mathbf{b}, \mathbf{x})|$, where the change of sign occurs on the perpendiculars to the given lines erected at point O. Therefore, the locus to be found is a rectangle whose sides are parallel to the bisectors of the angles between the given lines and the vertices lie on the indicated perpendiculars.

7.4. Let l be the line that passes through the midpoints of sides BC and AD. Suppose that point X does not lie on l; for instance, points A and X lie on one side of l. Then $AX < DX$ and $BX < CX$ and, therefore, $AX + BX < CX + DX$. Hence, l is the locus to be found.

7.5. Let N be a point such that $\overrightarrow{MN} = \overrightarrow{DA}$. Then $\angle NAM = \angle DMA$ and $\angle NBM = \angle BMC$ and, therefore, quadrilateral $AMBN$ is an inscribed one. The diagonals of the inscribed quadrilateral $AMBN$ are equal, hence, either $AM \parallel BN$ or $BM \parallel AN$. In the first case $\angle AMD = \angle MAN = \angle AMB$ and in the second case $\angle BMC = \angle MBN = \angle BMA$. If $\angle AMB = \angle AMD$, then $\angle AMB + \angle BMC = 180^\circ$ and point M lies on diagonal AC and if $\angle BMA = \angle BMC$, then point M lies on diagonal BD. It is also clear that if point M lies on one of the diagonals, then $\angle AMD + \angle BMC = 180^\circ$.

7.6. Introduce a coordinate system selecting point A as the origin and directing Ox-axis along ray AB. Let (x, y) be the coordinates of M. Then $AM^2 = x^2 + y^2$ and $BM^2 = (x-a)^2 + y^2$, where $a = AB$. Hence, $AM^2 - BM^2 = 2ax - a^2$. This quantity is equal to k for points M whose coordinates are $(\frac{a^2+k}{2a}, y)$. All such points lie on a line perpendicular to AB.

7.7. Let A and B be the intersection points of circles S and S_1. Then $XM^2 = XA \cdot XB = XO^2 - R^2$, where O and R are the center and the radius, respectively, of circle S. Hence, $XO^2 - XM^2 = R^2$ and, therefore, points X lie on the perpendicular to line OM (cf. Problem 7.6).

7.8. Let O_1 and O_2 be the centers of the given circles, R_1 and R_2 their respective radii. The circle of radius r centered at X intersects the first circle in the diametrically opposite points if and only if $r^2 = XO_1^2 + R_1^2$; hence, the locus to be found consists of points X such that $XO_1^2 + R_1^2 = XO_2^2 + R_2^2$. All such points X lie on a line perpendicular to O_1O_2, cf. Problem 7.6.

7.9. Let O be the center of the circle, R its radius, M the intersection point of the tangents drawn through the endpoints of the chord that contains point A, and P the midpoint of this chord. Then $OP \cdot OM = R^2$ and $OP = OA\cos\varphi$, where $\varphi = \angle AOP$. Hence,

$$AM^2 = OM^2 + OA^2 - 2OM \cdot OA\cos\varphi = OM^2 + OA^2 - 2R^2,$$

and, therefore, the quantity

$$OM^2 - AM^2 = 2R^2 - OA^2$$

is a constant. It follows that all points M lie on a line perpendicular to OA, cf. Problem 7.6.

7.10. Let P and Q be the midpoints of diagonals AC and BD. Then

$$AX^2 + CX^2 = 2PX^2 + \frac{AC^2}{2} \quad \text{and} \quad BX^2 + DX^2 = 2QX^2 + \frac{BD^2}{2}$$

(cf. Problem 12.11 a)) and, therefore, in heading b) the locus to be found consists of points X such that $PX^2 - QX^2 = \frac{1}{4}(BD^2 - AC^2)$ and in heading a) $P = Q$ and, therefore, the considered quantity is equal to $\frac{1}{2}(BD^2 - AC^2)$.

7.11. Let M and N be the midpoints of the given segment, O its midpoint. Point B lies on the circle with diameter MN, hence, $OB = \frac{1}{2}MN$. The trajectory of point O is the part of the circle of radius $\frac{1}{2}MN$ centered at B confined inside angle $\angle ABC$.

7.12. Let M be the given point, O the center of the given circle. If X is the midpoint of chord AB, then $XO \perp$AB. Therefore, the locus to be found is the circle with diameter MO.

7.13. Let us draw through point M a common tangent to the circles. Let O be the intersection point of this tangent with line AB. Then $AO = MO = BO$, i.e., O is the midpoint of segment AB. Point M lies on the circle with center O and radius $\frac{1}{2}AB$. The locus of points M is the circle with diameter AB (points A and B excluded).

7.14. For $k = 1$ we get the midperpendicular to segment AB. In what follows we will assume that $k \neq 1$.

Let us introduce a coordinate system in plane so that the coordinates of A and B are $(-a, 0)$ and $(a, 0)$, respectively. If the coordinates of point M are (x, y), then

$$\frac{AM^2}{BM^2} = (x + a)^2 + \frac{y^2}{(x - a)^2} + y^2.$$

The equation $\frac{AM^2}{BM^2} = k^2$ takes the form

$$x + 1 - \frac{k^2}{1 - k^2 a} + y^2 = {\frac{2ka}{1 - k^2}}^{2}.$$

This is an equation of the circle with center $(-1 + \frac{k^2}{1-k^2 a}, 0)$ and radius $\frac{2ka}{|1-k^2|}$.

7.15. Let line AB intersect circle S at points E and F so that point E lies on segment AB. Then PE is the bisector of triangle APB, hence, $\angle EPB = \angle EPA = \angle EFP$. Since $\angle EPF = 90^\circ$, it follows that $PB \perp EF$.

7.16. a) The considered circles are Apollonius's circles for the pairs of vertices of triangle ABC and, therefore, if X is a common point of circles S_a and S_b, then $XB : XC = AB : AC$ and $XC : XA = BC : BA$, i.e., $XB : XA = CB : CA$ and, therefore, point X belongs to circle S_c. It is also clear that if $AB > BC$, then point D lies inside circle S_b and point A outside it. It follows that circles S_a and S_b intersect at two distinct points.

To complete the proof, it remains to make use of the result of Problem 7.49.

b) According to heading a) $MA = \frac{\lambda}{a}$, $MB = \frac{\lambda}{b}$ and $MC = \frac{\lambda}{c}$. Let B_1 and C_1 be the projections of point M on lines AC and AB, respectively. Points B_1 and C_1 lie on the circle with diameter MA, hence,

$$B_1C_1 = MA \sin \angle B_1AC_1 = \frac{\lambda}{a}\frac{a}{2R} = \frac{\lambda}{2R},$$

where R is the radius of the circumscribed circle of triangle ABC. Similarly, $A_1C_1 = A_1B_1 = \frac{\lambda}{2R}$.

7.17. Let O_1 and O_2 be points such that $\overrightarrow{BO_1} = \frac{4}{3}\overrightarrow{BA}$ and $\overrightarrow{\mathrm{CO}_2} = \frac{4}{3}\overrightarrow{CB}$. It is easy to verify that if $BM > 2AM$, then point M lies inside circle S_1 of radius $\frac{2}{3}AB$ with center O_1 (cf. Problem 7.14) and if $CM > 2BM$, then point M lies inside circle S_2 of radius $\frac{2}{3}AB$ centered at O_2. Since $O_1O_2 > BO_1 = \frac{4}{3}AB$ and the sum of the radii of circles S_1 and S_2 is equal to $\frac{4}{3}AB$, it follows that these circles do not intersect. Therefore, if $BM = qAM$ and $CM = qBM$, then $q < 2$.

7.18. a) Let O be the intersection point of heights AA_1 and BB_1. The points A_1 and B_1 lie on the circle with diameter CO. Therefore, $\angle AOB = 180^\circ - \angle C$. Hence, the locus to be found is the circle symmetric to the given one through line AB (points A and B should be excluded).

b) If O is the intersection point of the bisectors of triangle ABC, then $\angle AOB = 90^\circ + \frac{1}{2}\angle C$. On each of the two arcs $\smile AB$ the angles C are constant and, therefore, the desired locus of the vertices of angles of $90^\circ + \frac{1}{2}\angle C$ that subtend segment AB is the union of two arcs (points A and B should be excluded).

7.19. Points P and Q lie on the circle with diameter DX, hence,

$$\angle(QD, DX) = \angle(QP, PX) = \angle(AP, PB) = 45^\circ,$$

i.e., point X lies on line CD.

7.20. a) If point A_1 traverses along the circle an arc of value 2φ, then point B_1 also traverses an arc of value 2φ, consequently, lines AA_1 and BB_1 turn through an angle of φ and the angle between them will not change.

Hence, point M moves along a circle that contains points A and B.

b) Let at some moment lines AA_1, BB_1 and CC_1 meet at point P. Then, for instance, the intersection point of lines AA_1 and BB_1 moves along the circumscribed circle of triangle ABP. It is also clear that the circumscribed circles of triangles ABP, BCP and CAP have a unique common point, P.

7.21. Suppose that points A and C lie on opposite sides of a rectangle. Let M and N be the midpoints of segments AC and BD, respectively. Let us draw through point M line l_1 parallel to the sides of the rectangle on which points A and C lie and through point N line l_2 parallel to the sides of the rectangle on which points B and D lie. Let O be the intersection point of lines l_1 and l_2. Clearly, point O lies on circle S constructed on segment MN as on a diameter.

On the other hand, point O is the center of the rectangle. Clearly, the rectangle can be constructed for any point O that lies on circle S.

It remains to notice that on the opposite sides of the rectangle points A and B or A and D can also lie. Hence, the locus to be found is the union of three circles.

7.22. It is easy to verify that the points of heights of triangle ABC possess the required property. Suppose that a point X not belonging to any of the heights of triangle ABC possesses the required property. Then line BX intersects heights AA_1 and CC_1 at points X_1 and X_2. Since

$$\angle XAB + \angle XBC + \angle XCA = 90^\circ = \angle X_1AB + \angle X_1BC + \angle X_1CA,$$

it follows that

$$\angle XAB - \angle X_1AB = \angle X_1CA - \angle XCA,$$

i.e., $\angle(XA, AX_1) = \angle(X_1C, CX)$. Therefore, point X lies on the circumscribed circle of triangle AXC', where point C' is symmetric to C through line BX. We similarly prove that point X_2 lies on the circle and, therefore, line BX intersects this circle at three distinct points. Contradiction.

7.23. Let K be the tangent point of line MX with the given semicircle and P the projection of point M to the diameter. In right triangles MPX and OKX, the hypothenuses are equal and $\angle PXM = \angle OXK$; hence, these triangles are equal. In particular, $MP = KO = R$, where R is the radius of the given semicircle. It follows that point M lies on line l parallel to the diameter of the semicircle and tangent to the semicircle. Let AB be the segment of line l whose projection is the diameter of the semicircle. From a point on l that does not belong to segment AB a tangent to the given semicircle cannot be drawn because the tangent drawn to the circle should be tangent to the other semicircle as well.

The locus to be found is punctured segment AB: without points A, B, and the midpoint.

7.24. Let H be the base of height h_b of triangle ABC and $h_b = b$. Denote by B' the intersection point of the perpendicular to line AB drawn through point A and the perpendicular to line AH drawn through point C. Right triangles $AB'C$ and BAH are equal, becuase $\angle AB'C = \angle BAH$ and $\angle AC = BH$. Therefore, $AB' = AB$, i.e., point C lies on the circle with diameter AB'.

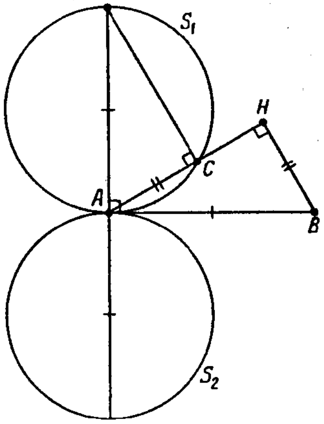


Figure 81 (Sol. 7.24)

Let S_1 and S_2 be the images of circle S with diameter AB under the rotations through angles of ±90° with center at A (Fig. 81). We have proved that point $C \neq A$ belongs to the union of circles S_1 and S_2.

Conversely, let a point C, $C \neq A$, belong to either of the circles S_1 or S_2; let AB' be a diameter of the corresponding circle. Then $\angle AB'C = \angle HAB$ and $A'B = AB$; hence, $AC = HB$.

7.25. Let O be the center of the circle, N the intersection point of lines OM and QP. Let us drop from point M perpendicular MS to line OP. Since $\triangle ONQ \sim \triangle OQM$ and $\triangle OPN \sim \triangle OMS$, we derive that

$$ON : OQ = OQ : OM \quad \text{and} \quad OP : ON = OM : OS.$$

By multiplying these equalities we get $OP : OQ = OQ : OS$. Hence, $OS = OQ^2 : OP$ is a constant. Since point S lies on line OP, its position does not depend on the choice of point Q. The locus to be found is the line perpendicular to line OP and passing through point S.

7.26. Let O be the midpoint of segment AB, and M the intersection point of the medians of triangle ABC. The homothety with center O and coefficient $\frac{1}{3}$ sends point C to point M. Therefore, the intersection point of the medians of triangle ABC lies on circle S which is the image of the initial circle under the homothety with center O and coefficient $\frac{1}{3}$. To get the desired locus we have to delete from S the images of points A and B.

7.27. Let O be the midpoint of height BH; let M, D and E be the midpoints of segment AC, and sides RQ and PS, respectively (Fig. 82).

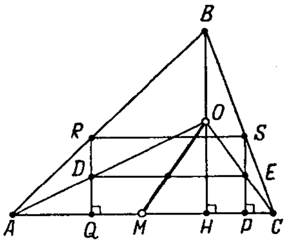


Figure 82 (Sol. 7.27)

Points D and E lie on lines AO and CO, respectively. The midpoint of segment DE is the center of rectangle $PQRS$. Clearly, this midpoint lies on segment OM. The locus in question is segment OM without its endpoints.

7.28. Let O_1 and O_2 be the centers of the given circles (point P lies on the circle centered at O_1); O the midpoint of segment O_1O_2; P', Q' and O' the projections of points O_1, O_2 and O to line PQ. As line PQ rotates, point O' runs the circle S with diameter AO. Clearly, the homothety with center A and coefficient 2 sends segment $P'Q'$ to segment PQ, i.e., point O' turns into the midpoint of segment PQ. Hence, the locus in question is the image of circle S under this homothety.

7.29. Let P and Q be the centers of the circumscribed circles of triangles AMB and CMB. Point M belongs to the locus to be found if $BPMQ$ is a rhombus, i.e., point M is the image of the midpoint of segment PQ under the homothety with center B and coefficient 2. Since the projections of points P and Q to line AC are the midpoints of segments AB and BC, respectively, the midpoints of all segments PQ lie on one line. (The locus to be found is the above-obtained line without the intersection point with line AC.)

7.30. Point P passes through point O at time t_1, it passes point Q at time t_2. At time $\frac{1}{2}(t_1 + t_2)$ the distances from O to points P and Q are equal to $\frac{1}{2}|t_1 + t_2|v$. At this moment erect the perpendiculars to the lines at points P and Q. It is easy to verify that the intersection point of these perpendiculars is the required one.

7.31. Denote the midpoints of diagonals AC and BD of quadrilateral $ABCD$ by M and N, respectively. Clearly, $S_{AMB} = S_{BMC}$ and $S_{AMD} = S_{DMC}$, i.e., $S_{DABM} = S_{BCDM}$. Since the areas of quadrilaterals $DABM$ and $BCDM$ do not vary as point M moves parallelly to BD, it follows that $S_{DABO} = S_{CDAO}$. Similar arguments for point N show that $S_{ABCO} = S_{CDAO}$. Hence,

$$S_{ADO} + S_{ABO} = S_{BCO} + S_{CDO} \text{ and } S_{ABO} + S_{BCO} = S_{CDO} + S_{ADO}$$

and, therefore,

$$S_{ADO} = S_{BCO} = S_1 \text{ and } S_{ABO} = S_{CDO} = S_2,$$

i.e., the area of each of the four parts into which the segments that connect point O with the midpoints of sides of the quadrilateral divide it is equal to $\frac{1}{2}(S_1 + S_2)$.

7.32. Let us drop height BB_1 from point B. Then $AD = B_1D$ and $CE = B_1E$. Clearly, if $MD < AD$, then point M lies on segment AB_1, i.e., outside segment B_1C. Therefore, $ME > EC$.

7.33. Suppose that the distance from any vertex of the polygon to point Q is not shorter than to point P. Then all the vertices of the polygon lie in the same half plane determined by the perpendicular to segment PQ at point P; point Q lies in the other half plane. Therefore, point Q lies outside the polygon. This contradicts the hypothesis.

7.34. Let us find the locus of points M for which $MA > MB$ and $MA > MC$. Let us draw midperpendiculars l_1 and l_2 to segments AB and AC. We have $MA > MB$ for the points that lie inside the half-plane bounded by line l_1 and the one without point A. Therefore, the locus in question is the intersection of half-planes (without boundaries) bounded by lines l_1 and l_2 and not containing point A.

If points A, B and C do not lie on one line, then this locus is always nonempty. If A, B, C lie on one line but A does not lie on segment BC, then this locus is also nonempty. If point A lies on segment BC, then this locus is empty, i.e., for any point M either $MA \leq MB$ or $MA \leq MC$.

7.35. Let O be the midpoint of diagonal AC. The projections of points B and D to line AC lie on segment AO, hence, the projection of point M also lies on segment AO.

7.36. Let us draw the midperpendicular l to segment AO. Clearly, $AM \geq OM$ if and only if point M lies on the same side of line l as O (or lies on line l itself). Therefore, the locus in question is the rhombus formed by the midperpendiculars to segments OA, OB, OC and OD.

7.37. The locus to be found is shaded on Fig. 83 (the boundary belongs to the locus).

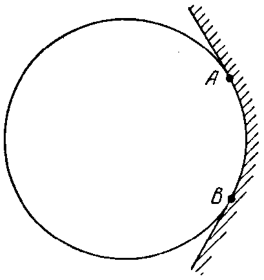


Figure 83 (Sol. 7.37)

7.38. Let A_1 and B_1 be the midpoints of sides CB and AC, respectively. The locus to be found is the interior of quadrilateral OA_1CB_1.

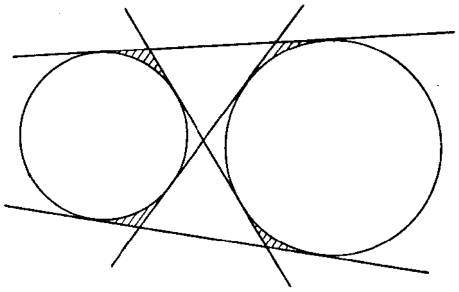


Figure 84 (Sol. 7.39)

7.39. Let us draw the common tangents to given disks (Fig. 84). It is easy to verify that the points that belong to the shaded domains (but not to their boundaries) satisfy the required condition and the points that do not belong to these domains do not satisfy this condition.

7.40. Let the perpendiculars dropped from points A_1, B_1, C_1 to lines BC, CA, AB, respectively, intersect at point M. Since points B_1 and M lie on one perpendicular to line AC, we have

$$B_1A^2 - B_1C^2 = MA^2 - MC^2.$$

Similarly,

$$C_1B^2 - C_1A^2 = MB^2 - MA^2 \text{ and } A_1C^2 - A_1B^2 = MC^2 - MB^2.$$

Adding these equalities we get

$$(*) \qquad A_1B^2 + C_1A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1B^2.$$

Conversely, let $(*)$ hold. Denote the intersection point of the perpendiculars dropped from points A_1 and B_1 to lines BC and AC, respectively, by M. Let us draw through point M line l perpendicular to line AB. If C_1' is a point on line l, then by the above

$$A_1B^2 + C_1'A^2 + B_1C^2 = B_1A^2 + A_1C^2 + C_1'B^2.$$

Hence, $C_1'A^2 - C_1'B^2 = C_1A^2 - C_1B^2$. By Problem 7.6 the locus of points X for which $XA^2 - XB^2 = k$ is a line perpendicular to segment AB. Therefore, the perpendicular dropped from point C_1 to line AB passes through point M, as required.

7.41. Set $A_1 = A$, $B_1 = B$ and $C_1 = C$. From the obvious identity

$$AB^2 + CA^2 + BC^2 = BA^2 + AC^2 + CB^2$$

we derive that the heights dropped from points A, B and C to sides BC, CA and AB, respectively, intersect at one point.

7.42. It suffices to make use of the result of Problem 7.40.

7.43. a) This problem is an obvious corollary of Problem 7.40.

b) Let the rotation by 90° about a point send triangle $A_1B_1C_1$ to triangle $A_2B_2C_2$. The perpendiculars to sides of triangle $A_2B_2C_2$ are parallel to the corresponding sides of triangle $A_1B_1C_1$ and, therefore, the perpendiculars dropped from the vertices of triangle ABC to the corresponding sides of triangle $A_2B_2C_2$ intersect at one point. It follows that the perpendiculars dropped from the vertices of triangle $A_2B_2C_2$ to the corresponding sides of triangle ABC intersect at one point. It remains to notice that the rotation by 90° that sends triangle $A_2B_2C_2$ to triangle $A_1B_1C_1$ sends these perpendiculars to the lines that pass through the sides of triangle $A_1B_1C_1$ parallelly the corresponding sides of triangle ABC.

7.44. We have to find out when the identity

$$AB_1^2 + BC_1^2 + CA_1^2 = BA_1^2 + CB_1^2 + AC_1^2$$

holds. By subtracting $AA_2^2 + BB_2^2 + CC_2^2$ from both sides of this identity we get

$$A_2B_1^2 + B_2C_1^2 + C_2A_1^2 = B_2A_1^2 + C_2B_1^2 + A_2C_1^2,$$

i.e.,

$$(b_1 - a_2)^2 + (c_1 - b_2)^2 + (a_1 - c_2)^2 = (a_1 - b_2)^2 + (b_1 - c_2)^2 + (c_1 - a_2)^2,$$

where a_i, b_i and c_i are the coordinates of points A_i, B_i and C_i on line l. After simplification we get

$$a_2b_1 + b_2c_1 + c_2a_1 = a_1b_2 + b_1c_2 + c_1a_2$$

and, therefore,

$$(b_2 - a_2)(c_1 - b_1) = (b_1 - a_1)(c_2 - b_2), \quad \text{i.e., } A_2B_2 : B_2C_2 = A_1B_1 : B_1C_1.$$

7.45. We may assume that the length of a side of the given equilateral triangle is equal to 2. Let $PA = 2a$, $PB = 2b$ and $PC = 2c$; let A_1, B_1 and C_1 be the projections of the centers of the inscribed circles of triangles PBC, PCA and PAB to lines BC, CA and AB, respectively. By Problem 3.2 we have

$$AB_1^2 + BC_1^2 + CA_1^2 = (1 + a - c)^2 + (1 + b - a)^2 + (1 + c - b)^2 = \\ 3 + (a - c)^2 + (b - a)^2 + (c - b)^2 = BA_1^2 + CB_1^2 + AC_1^2.$$

7.46. The segments into which the bisectors divide the sides of the triangle are easy to calculate. As a result we see that if the perpendiculars raised from the bases of the bisectors intersect, then

$$(\frac{ac}{b} + c)^2 + (\frac{ab}{a} + c)^2 + (\frac{bc}{a} + b)^2 = (\frac{ab}{b} + c)^2 + (\frac{bc}{a} + c)^2 + (\frac{ac}{a} + b)^2,$$

i.e.,

$$0 = a^2\frac{c-b}{b+c} + b^2\frac{a-c}{a+c} + c^2\frac{b-a}{a+b} = -(b-a)(a-c)\frac{a^2+b^2+c^2}{(a+b)(a+c)(b+c)}.$$

7.47. Let (a_i, b_i) be the coordinates of point A_i and (x, y) the coordinates of point X. Then the equation satisfied by point X takes the form

$$c = \sum k_i((x - a_i)^2 + (x - b_i)^2) = \\ (\sum k_i)(x^2 + y^2) - (2\sum k_i a_i)x - (2\sum k_i b_i)y + \sum k_i(a_i^2 + b_i^2).$$

If the coefficient of $x^2 + y^2$ is nonzero, then this equation determines either a circle or the empty set and if it is zero, then the equation determines either a line, or a plane, or the empty set.

Remark. If in case a) points $A_1, \dots, A_n$ lie on one line l, then this line can be taken for Ox-axis. Then $b_i = 0$ and, therefore, the coefficient of y is equal to zero, i.e., the center of the circle lies on l.

7.48. Let line l cut on the given circles arcs $\smile A_1B_1$ and $\smile A_2B_2$ whose values are $2\alpha_1$ and $2\alpha_2$, respectively; let O_1 and O_2 be the centers of the circles, R_1 and R_2 their respective radii. Let K be the intersection point of the tangents at points A_1 and A_2. By the law of sines $KA_1 : KA_2 = \sin\alpha_2 : \sin\alpha_1$, i.e., $KA_1 \sin\alpha_1 = KA_2 \sin\alpha_2$. Since

$$KO_1^2 = KA_1^2 + R_1^2 \quad \text{and } KO_2^2 = KA_2^2 + R_2^2,$$

it follows that

$$(\sin^2\alpha_2)KO_1^2 - (\sin^2\alpha_2)KO_2^2 = (R_1\sin\alpha_1)^2 - (R_2\sin\alpha_2)^2 = q.$$

We similarly prove that the other intersection points of the tangents belong to the locus of points X such that

$$(\sin^2\alpha_1)XO_1^2 - (\sin^2\alpha_2)XO_2^2 = q.$$

This locus is a circle whose center lies on line O_1O_2 (cf. Remark to Problem 7.47).

7.49. Let $AM : BM : CM = p : q : r$. All the points X that satisfy

$$(q^2 - r^2)AX^2 + (r^2 - p^2)BX^2 + (p^2 - q^2)CX^2 = 0$$

lie on one line (cf. Problem 7.47) and points M, N and O satisfy this relation.

Chapter 8. CONSTRUCTIONS

§1. The method of loci

8.1. Construct triangle ABC given a, h_a and R.

8.2. Inside triangle ABC construct point M so that $S_{ABM} : S_{BCM} : S_{ACM} = 1 : 2 : 3$.

8.3. Through given point P inside a given circle draw a chord so that the difference of the lengths of the segments into which P divides the chord would be equal to the given value a.

8.4. Given a line and a circle without common points, construct a circle of a given radius r tangent to them.

8.5. Given point A and circle S draw a line through point A so that the chord cut by circle S on this line would be of given length d.

8.6. Quadrilateral $ABCD$ is given. Inscribe in it a parallelogram with given directions of sides.

§2. The inscribed angle

8.7. Given a, m_c and angle $\angle A$, construct triangle ABC.

8.8. A circle and two points A and B inside it are given. Inscribe a right triangle in the circle so that the legs would pass through the given points.

8.9. The extensions of sides AB and CD of rectangle $ABCD$ intersect a line at points M and N, respectively, and the extensions of sides AD and BC intersect the same line at points P and Q, respectively. Construct rectangle $ABCD$ given points M, N, P, Q and the length a of side AB.

8.10. Construct a triangle given its bisector, median and height drawn from one vertex.

8.11. Construct triangle ABC given side a, angle $ConstructatriangleA$ and the radius r of the inscribed circle.

§3. Similar triangles and a homothety

8.12. Construct a triangle given two angles $\angle A$, $\angle B$ and the perimeter P.

8.13. Construct triangle ABC given m_a, m_b and m_c.

8.14. Construct triangle ABC given h_a, h_b and h_c.

8.15. In a given acute triangle ABC inscribe square $KLMN$ so that vertices K and N lie on sides AB and AC and vertices L and M lie on side BC.

8.16. Construct triangle ABC given h_a, $b - c$ and r.

Cf. also Problems 19.15-19.20, 19.39, 19.40.

§4. Construction of triangles from various elements

In the problems of this section it is necessary to construct triangle ABC given the elements indicated below.

8.17. c, m_a and m_b.

8.18. a, b and h_a.

8.19. h_b, h_c and m_a.
8.20. $\angle A$, h_b and h_c.
8.21. a, h_b and m_b.
8.22. h_a, m_a and h_b.
8.23. a, b and m_c.
8.24. h_a, m_a and $\angle A$.
8.25. a, b and l_c.
8.26. $\angle A$, h_a and p.

See also Problems 17.6-17.8.

§5. Construction of triangles given various points

8.27. Construct triangle ABC given (1) line l containing side AB and (2) bases A_1 and B_1 of heights dropped on sides BC and AC, respectively.

8.28. Construct an equilateral triangle given the bases of its bisectors.

8.29. a) Construct triangle ABC given three points A', B', C' at which the bisectors of the angles of triangle ABC intersect the circumscribed circle (both triangles are supposed to be acute ones).

b) Construct triangle ABC given three points A', B', C' at which the heights of the triangle intersect the circumscribed circle (both triangles are supposed to be acute ones).

8.30. Construct triangle ABC given three points A', B', C' symmetric to the center O of the circumscribed circle of this triangle through sides BC, CA, AB, respectively.

8.31. Construct triangle ABC given three points A', B', C' symmetric to the intersection point of the heights of the triangle through sides BC, CA, AB, respectively (both triangles are supposed to be acute ones).

8.32. Construct triangle ABC given three points P, Q, R at which the height, the bisector and the median drawn from vertex C, respectively, intersect the circumscribed circle.

8.33. Construct triangle ABC given the position of points A_1, B_1, C_1 that are the centeres of the escribed circles of triangle ABC.

8.34. Construct triangle ABC given the center of the circumscribed circle O, the intersection point of medians, M, and the base H of height CH.

8.35. Construct triangle ABC given the centers of the inscribed, the circumscribed, and one of the escribed circles.

§6. Triangles

8.36. Construct points X and Y on sides AB and BC, respectively, of triangle ABC so that $AX = BY$ and $XY \parallel AC$.

8.37. Construct a triangle from sides a and b if it is known that the angle opposite one of the sides is three times the angle opposite the other side.

8.38. In given triangle ABC inscribe rectangle $PRQS$ (vertices R and Q lie on sides AB and BC and vertices P and S lie on side AC) so that its diagonal would be of a given length.

8.39. Through given point M draw a line so that it would cut from the given angle with vertex A a triangle ABC of a given perimeter $2p$.

8.40. Construct triangle ABC given its median m_c and bisector l_c if $\angle C = 90°$.

8.41. Given triangle ABC such that $AB < BC$, construct on side AC point D so that the perimeter of triangle ABD would be equal to the length of side BC.

8.42. Construct triangle ABC from the radius of its circumscribed circle and the bisector of angle $\angle A$ if it is known that $\angle B - \angle C = 90^\circ$.

8.43. On side AB of triangle ABC point P is given. Draw a line (distinct from AB) through point P that cuts rays CA and CB at points M and N, respectively, such that $AM = BN$.

8.44. Construct triangle ABC from the radius of the inscribed circle r and (nonzero) lengths of segments AO and AH, where O is the center of the inscribed circle and H the orthocenter.

See also Problems 15.12 b), 17.12-17.15, 18.10, 18.29.

§7. Quadrilaterals

8.45. Construct a rhombus two sides of which lie on two given parallel lines and two other sides pass through two given points.

8.46. Construct quadrilateral $ABCD$ given the lengths of the four sides and the angle between AB and CD.

8.47. Through vertex A of convex quadrilateral $ABCD$ draw a line that divides $ABCD$ into two parts of equal area.

8.48. In a convex quadrilateral three sides are equal. Given the midpoints of the equal sides construct the quadrilateral.

8.49. A quadrilateral is both inscribed and circumscribed. Given three of its vertices, construct its fourth vertex.

8.50. Given vertices A and C of an isosceles circumscribed trapezoid $ABCD$ ($AD \parallel BC$) and the directions of its bases, construct vertices B and D.

8.51. On the plane trapezoid $ABCD$ is drawn ($AD \parallel BC$) and perpendicular OK from the intersection point O is dropped on base AD; the midpoint EF is drawn. Then the trapezoid itself was erased. How to recover the plot of the trapezoid from the remaining segments OK and EF?

8.52. Construct a convex quadrilateral given the lengths of all its sides and one of the midlines.

8.53. (*Brachmagupta.*) Construct an inscribed quadrilateral given its four sides.

See also Problems 15.10, 15.13, 16.17, 17.4, 17.5.

§8. Circles

8.54. Inside an angle two points A and B are given. Construct a circle that passes through these points and intercepts equal segments on the sides of the angle.

8.55. Given circle S, point A on it and line l. Construct a circle tangent to the given circle at point A and tangent to the given line.

8.56. a) Two points, A, B and line l are given. Construct a circle that passes through point A, B and is tangent to l.

b) Two points A, B and circle S are given. Construct a circle that passes through points A and B and is tangent to S.

8.57. Three points A, B and C are given. Construct three circles that are pairwise tangent at these points.

8.58. Construct a circle the tangents to which drawn from three given points A, B and C have given lengths a, b and c, respectively.

See also Problems 15.8, 15.9, 15.11, 15.12 a), 16.13, 16.14, 16.18–16.20, 18.24.

§9. Apollonius' circle

8.59. Construct triangle ABC given a, h_a and $\frac{b}{c}$.

8.60. Construct triangle ABC given the length of bisector CD and the lengths of segments AD and BD into which the bisector divides side AB.

8.61. On a line four points A, B, C, D are given in the indicated order. Construct point M — the vertex of equal angles that subtend segments AB, BC, CD.

8.62. Two segments AB and $A'B'$ are given in plane. Construct point O so that triangles AOB and $A'OB'$ would be similar (equal letters stand for the corresponding vertices of similar triangles).

8.63. Points A and B lie on a diameter of a given circle. Through A and B draw two equal chords with a common endpoint.

§10. Miscellaneous problems

8.64. a) On parallel lines a and b, points A and B are given. Through a given point C draw line l that intersects lines a and b at points A_1 and B_1, respectively, and such that $AA_1 = BB_1$.

b) Through point C draw a line equidistant from given points A and B.

8.65. Construct a regular decagon.

8.66. Construct a rectangle with the given ratio of sides knowing one point on each of its sides.

8.67. Given diameter AB of a circle and point C on the diameter. On this circle, construct points X and Y symmetric through line AB and such that lines AX and YC are perpendicular.

See also Problems 15.7, 16.15, 16.16, 16.21, 17.9–17.11, 17.27–17.29, 18.41.

§11. Unusual constructions

8.68. With the help of a ruler and a compass divide the angle of 19° into 19 equal parts.

8.69. Prove that an angle of value n°, where n is an integer not divisible by 3, can be divided into n equal parts with the help of a compass and ruler.

8.70. On a piece of paper two lines are drawn. They form an angle whose vertex lies outside this piece of paper. With the help of a ruler and a compass draw the part of the bisector of the angle that lies on this piece of paper.

8.71. With the help of a two-sided ruler construct the center of the given circle whose diameter is greater than the width of the ruler.

8.72. Given points A and B; the distance between them is greater than 1 m. The length of a ruler is 10 cm. With the help of the ruler only construct segment AB. (Recall that with the help of a ruler one can only draw straight lines.)

8.73. On a circle of radius a a point is given. With the help of a coin of radius a construct the point diametrically opposite to the given one.

§12. Construction with a ruler only

In the problems of this section we have to perform certain constructions with the help of a ruler only, without a compass or anything else. With the help of one ruler it is almost impossible to construct anything. For example, it is even impossible to construct the midpoint of a segment (Problem 30.59).

But if certain additional lines are drawn on the plane, it is possible to perform certain constructions. In particular, if an additional circle is drawn on the plane and its center is marked, then with the help of a ruler one can perform all the constructions that can be performed with the help of a ruler and a compass. One has, however, to convene that a circle is "constructed" whenever its center and one of its points are marked.

Remark. If a circle is drawn on the plane but its center is not marked then to construct its center with the help of a ruler only is impossible (Problem 30.60).

8.74. Given two parallel lines and a segment that lies on one of the given lines. Divide the segment in halves.

8.75. Given two parallel lines and a segment that lies on one of the given lines. Double the segment.

8.76. Given two parallel lines and a segment that lies on one of the given lines. Divide the segment into n equal parts.

8.77. Given two parallel lines and point P, draw a line through P parallel to the given lines.

8.78. A circle, its diameter AB and point P are given. Through point P draw the perpendicular to line AB.

8.79. In plane circle S and its center O are given. Then with the help of a ruler only one can:

a) additionally given a line, draw a line through any point parallel to the given line and drop the perpendicular to the given line from this point;

b) additionally given a line a point on it and a length of a segment, on the given line, mark a segment of length equal to the given one and with one of the endpoints in the given point;

c) additionally given lengths of a, b, c of segments, construct a segment of length $\frac{ab}{c}$;

d) additionally given line l, point A and the length r of a segment, construct the intersection points of line l with the circle whose center is point A and the radius is equal to r;

e) additionally given two points and two segments, construct the intersection points of the two circles whose centers are the given points and the radii are the given segments.

See also Problem 6.97.

§13. Constructions with the help of a two-sided ruler

In problems of this section we have to perform constructions with the help of a ruler with two parallel sides (without a compass or anything else). *With the help of a two-sided ruler one can perform all the constructions that are possible to perform with the help of a compass and a ruler.*

Let a be the width of a two-sided ruler. By *definition* of the two-sided ruler with the help of it one can perform the following elementary constructions:

1) draw the line through two given points;

2) draw the line parallel to a given one and with the distance between the lines equal to a;

3) through two given points A and B, where $AB \geq a$, draw a pair of parallel lines the distance between which is equal to a (there are two pairs of such lines).

8.80. a) Construct the bisector of given angle $\angle AOB$.

b) Given acute angle $\angle AOB$, construct angle $\angle BOC$ whose bisector is ray OA.

8.81. Erect perpendicular to given line l at given point A.

8.82. a) Given a line and a point not on the line. Through the given point draw a line parallel to the given line.

b) Construct the midpoint of a given segment.

8.83. Given angle $\angle AOB$, line l and point P on it, draw through P lines that form together with l an angle equal to angle $\angle AOB$.

8.84. Given segment AB, a non-parallel to it line l and point M on it, construct the intersection points of line l with the circle of radius AB centered at M.

8.85. Given line l and segment OA, parallel to l, construct the intersection points of l with the circle of radius OA centered at O.

8.86. Given segments O_1A_1 and O_2A_2, construct the radical axis of circles of radii O_1A_1 and O_2A_2 centered at O_1 and O_2, respectively.

§14. Constructions using a right angle

In problems of this section we have to perform the constructions indicated using a *right angle*. A right angle enables one to perform the following elementary constructions:

a) given a line and a point not on it, place the right angle so that one of its legs lies on the given line and the other leg runs through the given point;

b) given a line and two points not on it, place the right angle so that its vertex lies on the given line and thelegs pass through two given points (if, certainly, for the given line and points such a position of the right angle exists).

Placing the right angle in one of the indicated ways we can draw rays corresponding to its sides.

8.87. Given line l and point A not on it, draw a line parallel to l.

8.88. Given segment AB, construct

a) the midpoint of AB;

b) segment AC whose midpoint is point B.

8.89. Given angle $\angle AOB$, construct

a) an angle of value $2\angle AOB$;

b) an angle of value $\frac{1}{2}\angle AOB$.

8.90. Given angle $\angle AOB$ and line l, draw line l_1 so that the angle between lines l and l_1 is equal to $\angle AOB$.

8.91. Given segment AB, line l and point O on it, construct on l point X such that $OX = AB$.

8.92. Given segment OA parallel to line l, construct the locus of points in which the disc segment of radius OA centered at O intersects l.

Problems for independent study

8.93. Construct a line tangent to two given circles (consider all the possible cases).

8.94. Construct a triangle given (the lengths of) the segments into which a height divides the base and a median drawn to a lateral side.

8.95. Construct parallelogram $ABCD$ given vertex A and the midpoints of sides BC and CD.

8.96. Given 3 lines, a line segment and a point. Construct a trapezoid whose lateral sides lie on the given lines, the diagonals intersect at the given point and one of the bases is of the given length.

8.97. Two circles are given. Draw a line so that it would be tangent to one of the circles and the other circle would intersept on it a chord of a given length.

8.98. Through vertex C of triangle ABC draw line l so that the areas of triangles AA_1C and BB_1C, where A_1 and B_1 are projections of points A and B on line l, are equal.

8.99. Construct triangle ABC given sides AB and AC if it is given that bisector AD, median BM, and height CH meet at one point.

8.100. Points A_1, B_1 and C_1 that divide sides BC, CA and AB, respectively, of triangle ABC in the ratio of $1 : 2$ are given. Recover triangle ABC from this data.

Solutions

8.1. Let us construct segment BC of length a. The center O of the circumscribed circle of triangle ABC is the intersection point of two circles of radius R centered at B and C. Select one of these intersection points and construct the circumscribed circle S of triangle ABC. Point A is the intersection point of circle S and a line parallel line BC and whose distance from BC is equal to h_a (there are two such lines).

8.2. Let us construct points A_1 and B_1 on sides BC and AC, respectively, so that $BA_1 : A_1C = 1 : 3$ and $AB_1 : B_1C = 1 : 2$. Let point X lie inside triangle ABC. Clearly, $S_{ABX} : S_{BCX} = 1 : 2$ if and only if point X lies on segment BB_1 and $S_{ABX} : S_{ACX} = 1 : 3$ if and only if point X lies on segment AA_1. Therefore, the point M to be constucted is the intersection point of segments AA_1 and BB_1.

8.3. Let O be the center of the given circle, AB a chord that passes through point P and M the midpoint of AB. Then $|AP - BP| = 2PM$. Since $\angle PMO = 90^\circ$, point M lies on circle S with diameter OP. Let us construct chord PM of circle S so that $PM = \frac{1}{2}a$ (there are two such chords). The chord to be constructed is determined by line PM.

8.4. Let R be the radius of the given circle, O its center. The center of the circle to be constructed lies on circle S of radius $R + r$ centered at O. On the other hand, the center to be constructed lies on line l passing parallelly to the given line at distance r (there are two such lines). Any intersection point of S with l can serve as the center of the circle to be constructed.

8.5. Let R be the radius of circle S and O its center. If circle S intersepts on the line that passes through point A chord PQ and M is the midpoint of PQ, then

$$OM^2 + OQ^2 - NQ^2 = R^2 - \frac{d^2}{4}.$$

Therefore, the line to be constructed is tangent to the circle of radius $\sqrt{R^2 - \frac{d^2}{4}}$ centered at O.

8.6. On lines AB and CD take points E and F so that lines BF and CE would have had prescribed directions. Let us considered all possible parallelograms $PQRS$ with prescribed directions of sides whose vertices P and R lie on rays BA and CD and vertex Q lies on side BC (Fig. 85).

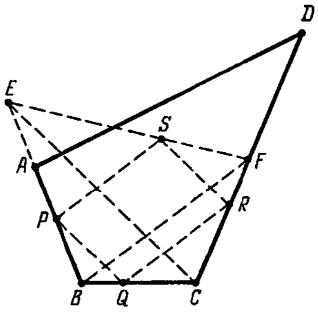


Figure 85 (8.6)

Let us prove that the locus of vertices S is segment EF. Indeed, $\frac{SR}{EC}=\frac{PQ}{EC}=\frac{BQ}{BC}=\frac{FR}{FC}$, i.e., point S lies on segment EF. Conversely if point S' lies on segment EF then let us draw lines $S'P'$, $P'Q'$ and $Q'R'$ so that $S'P' \parallel BF$, $P'Q' \parallel EC$ and $Q'R' \parallel BF$, where P', Q' and R' are some points on lines AB, BC, CD, respectively. Then $\frac{S'P'}{BF}=\frac{P'E}{BE}=\frac{Q'C}{BC}=\frac{Q'R'}{BF}$, i.e., $S'P'=Q'R'$ and $P'Q'R'S'$ is a parallelogram.

This implies the following construction. First, construct points E and F. Vertex S is the intersection point of segments AD and EF. The continuation of construction is obvious.

8.7. Suppose that triangle ABC is constructed. Let A_1 and C_1 be the midpoints of sides CD and AB, respectively. Since $C_1A_1 \parallel AC$, it follows that $\angle A_1C_1B = \angle A$. This implies the following construction.

First, let us construct segment CD of length a and its midpoint, A_1. Point C_1 is the intersection point of the circle of radius m_c centered at C and the arcs of the circles whose points are vertices of the angles equal to $\angle A$ that segment A_1B subtends. Construct point C_1, then mark on ray BC_1 segment $BA = 2BC_1$. Then A is the vertex of the triangle to be constructed.

8.8. Suppose that the desired triangle is constructed and C is the vertex of its right angle. Since $\angle ACB = 90^\circ$, point C lies on circle S with diameter AB. Hence, point C is the intersection point of circle S and the given circle. Constructing point C and drawing lines CA and AB, we find the remaining vertices of the triangle to be constructed.

8.9. Suppose that rectangle $ABCD$ is constructed. Let us drop perpendicular PR from point P to line BC. Point R can be constructed because it lies on the circle with diameter PQ and $PR = AB = a$. Constructing point R, let us construct lines BC and AD and drop on them perpendiculars from points M and N, respectively.

8.10. Suppose that triangle ABC is constructed, AH is its height, AD its bisector, AM its median. By Problem 2.67 point D lies between M and H. Point E, the intersection point of line AD with the perpendicular drawn from point M to side BC, lies on the circumscribed circle of triangle ABC. Hence, the center O of the circumscribed circle lies on the intersection of the midperpendicular to segment AE and the perpendicular to side BC drawn through point M.

The sequence of constructions is as follows: on an arbitrary line (which in what follows turns out to be line BC) construct point H, then consecutively construct points A, D, M, E, O. The desired vertices B and C of triangle ABC are intersection points of the initial line with the circle of radius OA centered at O.

8.11. Suppose that triangle ABC is constructed and O is the center of its inscribed circle. Then $\angle BOC = 90^\circ + \frac{1}{2}\angle A$ (Problem 5.3). Point O is the vertex of an angle of $90^\circ + \frac{1}{2}\angle A$ that subtends segment BC; the distance from O to line BC is equal to r, hence, BC(??) can be constructed. Further, let us construct the inscribed circle and draw the tangents to it from points B and C.

8.12. Let us construct any triangle with angles $\angle A$ and $\angle B$ and find its perimeter P_1. The triangle to be found is similar to the constructed triangle with coefficient $\frac{P}{P_1}$.

8.13. Suppose that triangle ABC is constructed. Let AA_1, BB_1 and CC_1 be its medians, M their intersection point, M' the point symmetric to M through point A_1. Then $MM' = \frac{2}{3}m_a$, $MC = \frac{2}{3}m_c$ and $M'C = \frac{2}{3}m_b$; hence, triangle $MM'C$ can be constructed. Point A is symmetric to M' through point M and point B is symmetric to C through the midpoint of segment MM'.

8.14. Clearly,

$$BC : AC : AB = \frac{S}{h_a} : \frac{S}{h_b} : \frac{S}{h_c} = \frac{1}{h_a} : \frac{1}{h_b} : \frac{1}{h_c}.$$

Let us take an arbitrary segment $B'C'$ and construct triangle $A'B'C'$ so that $B'C' : A'C' = h_b : h_a$ and $B'C' : A'B' = h_c : h_a$. Let h'_a be the height of triangle $A'B'C'$ dropped from vertex A'. The triangle to be found is similar to triangle $A'B'C'$ with coefficient $\frac{h_a}{h'_a}$.

8.15. On side AB, take an arbitrary point K' and drop from it perpendicular $K'L'$ to side BC; then construct square $K'L'M'N'$ that lies inside angle $\angle ABC$. Let line BN' intersect side AC at point N. Clearly, the square to be constructed is the image of square $K'L'M'N'$ under the homothety with center B and coefficient $BN : BN'$.

8.16. Suppose that the desired triangle ABC is constructed. Let Q be the tangent point of the inscribed circle with side BC; let PQ be a diameter of the circle, R the tangent point of an escribed circle with side BC. Clearly,

$$BR = \frac{a+b+c}{2} - c = \frac{a+b-c}{2} \quad \text{and} \quad BQ = \frac{a+c-b}{2}.$$

Hence, $RQ = |BR - BQ| = |b - c|$. The inscribed circle of triangle ABC and the escribed circle tangent to side BC are homothetic with A being the center of homothety. Hence, point A lies on line PR (Fig. 86).

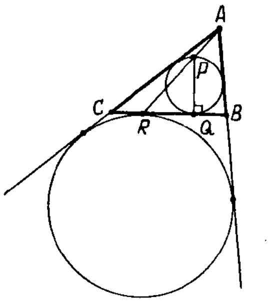


Figure 86 (Sol. 8.16)

This implies the following construction. Let us construct right triangle PQR from the known legs $PQ = 2r$ and $RQ = |b - c|$. Then draw two lines parallel to line RQ and whose distances from RQ are equal to h_a. Vertex A is the intersection point of one of these lines with ray RP. Since the length of diameter PQ of the inscribed circle is known, it can be constructed. The intersection points of the tangents to this circle drawn from point A with line RQ are vertices B and C of the triangle.

8.17. Suppose that triangle ABC is constructed. Let M be the intersection point of medians AA_1 and BB_1. Then $AM = \frac{2}{3}m_a$ and $BM = \frac{2}{3}m_b$. Triangle ABM can be constructed from the lengths of sides $AB = c$, AM and BM. Then on rays AM and BM segments $AA_1 = m_a$ and $BB_1 = m_b$ should be marked. Vertex C is the intersection point of lines AB_1 and A_1B.

8.18. Suppose triangle ABC is constructed. Let H be the base of the height dropped from vertex A. Right triangle ACH can be constructed from its hypothenuse $AC = b$ and leg $AH = h_a$. Then on line CH construct point B so that $CB = a$.

8.19. Suppose that triangle ABC is constructed. Let us draw from the midpoint A_1 of side BC perpendiculars A_1B' and A_1C' to lines AC and AB, respectively. Clearly, $AA_1 = m_a$, $A_1B' = \frac{1}{2}h_b$ and $A_1C' = \frac{1}{2}h_c$. This implies the following construction.

First, let us construct segment AA_1 of length m_a. Then construct right triangles AA_1B' and AA_1C' from the known legs and hypothenuse so that they would lie on distinct sides of line AA_1. It remains to construct points B and C on sides AC' and AB' of angle $C'AB'$ so that segment BC would be divided by points A_1 in halves. For this let us mark on ray AA_1 segment $AD = 2AA_1$ and then draw through point D the lines parallel to the legs of

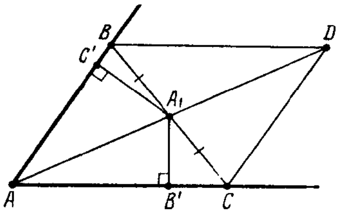


Figure 87 (Sol. 8.19)

angle $\angle C'AB'$. The intersection points of these lines with the legs of angle $\angle C'AB'$ are the vertices of the triangle to be constructed (Fig. 87).

8.20. Let us construct angle $\angle B'AC'$ equal to $\angle A$. Point B is constructed as the intersection of ray AB' with a line parallel to ray AC' and passsing at distance h_b from it. Point C is similarly constructed.

8.21. Suppose that triangle ABC is constructed. Let us drop height BH from point B and draw median BB_1. In right triangles CBH and B_1BH, leg BH and hypothenuses CB and B_1B are known; hence, these segments can be constructed. Then on ray CB_1 we mark segment $CA = 2CB_1$. The problem has two solutions because we can construct triangles CBH and B_1BH either on one or on distinct sides of line BH.

8.22. Suppose that triangle ABC is constructed. Let M be the midpoint of segment BC. From point A drop height AH and from point M drop perpendicular MD to side AC. Clearly, $MD = \frac{1}{2}h_b$. Hence, triangles AMD and AMH can be constructed.

Vertex C is the intersection point of lines AD and MH. On ray CM, mark segment $CB = 2CM$. The problem has two solutions because triangles AMD and AMH can be constructed either on one or on distinct sides of line AM.

8.23. Suppose that triangle ABC is constructed. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively. In triangle CC_1B_1 all the sides are known: $CC_1 = m_c$, $C_1B_1 = \frac{1}{2}a$ and $CB_1 = \frac{1}{2}b$; hence, it can be constructed. Point A is symmetric to C through point B_1 and point B is symmetric to A through C_1.

8.24. Suppose that triangle ABC is constructed, AM is its median, AH its height. Let point A' be symmetric to A through point M.

Let us construct segment $AA' = 2m_a$. Let M be the midpoint of AA'. Let us construct right triangle AMH with hypothenuse AM and leg $AH = h_a$. Point C lies on an arc of the circle whose points are the vertices of the angles that subtend segment AA'; the values of these angles are equal to $180^\circ - \angle A$ because $\angle ACA' = 180^\circ - \angle CAB$. Hence, point C is the intersection point of this arc and line MH. Point B is symmetric to C through point M.

8.25. Suppose triangle ABC is constructed. Let CD be its bisector. Let us draw line MD parallel to side BC (point M lies on side AC). Triangle CMD is an isosceles one because $\angle MCD = \angle DCB = \angle MDC$. Since

$$MC : AM = DB : AD = CB : AC = a : b \quad \text{and} \quad AM + MC = b,$$

it follows that $MC = \frac{ab}{a+b}$. Let us construct an isosceles triangle CMD from its base $CD = l_c$ and lateral sides $MD = MC = \frac{ab}{a+b}$. Further, on ray CM, mark segment $CA = b$ and on the ray symmetric to ray CM through line CD mark segment $CB = a$.

8.26. Suppose that triangle ABC is constructed. Let S_1 be the escribed circle tangent to side BC. Denote the tangent points of circle S_1 with the extensions of sides AB and AC by K and L, respectively, and the tangent point of S_1 with side BC by M. Since

$$AK = AL, AL = AC + CM \quad \text{and} \quad AK = AB + BM,$$

it follows that $AK = AL = p$. Let S_2 be the circle of radius h_a centered at A. Line BC is a common inner tangent to circles S_1 and S_2.

This implies the following construction. Let us construct angle $\angle KAL$ whose value is equal to that of A so that $KA = LA = p$. Next, construct circle S_1 tangent to the sides of angle $\angle KAL$ at points K and L and circle S_2 of radius h_a centered at A. Then let us draw a common inner tangent to circles S_1 and S_2. The intersection points of this tangent with the legs of angle $\angle KAL$ are vertices B and C of the triangle to be constructed.

8.27. Points A_1 and B_1 lie on the circle S with diameter AB. The center O of this circle lies on the midperpendicular to chord A_1B_1. This implies the following construction. First, let us construct point O which is the intersection point of the midperpendicular to segment A_1B_1 with line l. Next, construct the circle of radius $OA_1 = OB_1$ centered at O. The vertices A and B are the intersection points of circle S with line l. Vertex C is the intersection point of lines AB_1 and BA_1.

8.28. Let $AB = BC$ and A_1, B_1, C_1 the bases of the bisectors of triangle ABC. Then $\angle A_1C_1C = \angle C_1CA = \angle C_1CA_1$, i.e., triangle CA_1C_1 is an isosceles one and $A_1C = A_1C_1$.

This implies the following construction.

Let us draw through point B_1 line l parallel to A_1C_1. On l, construct point C such that $CA_1 = C_1A_1$ and $\angle C_1A_1C > 90^\circ$. Point A is symmetric to point C through point B_1 and vertex B is the intersection point of lines AC_1 and A_1C.

8.29. a) By Problem 2.19 a) points A, B and C are the intersection points of the extensions of heights of triangle $A'B'C'$ with its circumscribed circle.

b) By Problem 2.19 b) points A, B and C are the intersection points of the extensions of bisectors of the angles of triangle $A'B'C'$ with its circumscribed circle.

8.30. Denote the midpoints of sides BC, CA, AB of the triangle by A_1, B_1, C_1, respectively. Since $BC \parallel B_1C_1 \parallel B'C'$ and $OA_1 \perp BC$, it follows that $OA' \perp B'C'$. Similarly, $OB' \perp A'C'$ and $OC' \perp A'B'$, i.e., O is the intersection point of the heights of triangle $A'B'C'$. Constructing point O, let us draw the midperpendiculars to segments OA', OB', OC'. These lines form triangle ABC.

8.31. Thanks to Problem 5.9 our problem coincides with Problem 8.29 b).

8.32. Let O be the center of the circumscribed circle, M the midpoint of side AB and H the base of the height dropped from point C. Point Q is the midpoint of arc $\smile AB$, therefore, $OQ \perp AB$. This implies the following construction. First, the three given points determine the circumscribed circle S of triangle PQR. Point C is the intersection point of circle S and the line drawn parallelly to OQ through point P. Point M is the intersection point of line OQ and line RC. Line AB passes through point M and is perpendicular to OQ.

8.33. By Problem 5.2, points A, B and C are the bases of the heights of triangle $A_1B_1C_1$.

8.34. Let H_1 be the intersection point of heights of triangle ABC. By Problem 5.105, $OM : MH_1 = 1 : 2$ and point M lies on segment OH_1. Therefore, we can construct point H_1. Then let us draw line H_1H and erect at point H of this line perpendicular l. Dropping perpendicular from point O to line l we get point C_1 (the midpoint of segment AB). On ray C_1M, construct point C so that $CC_1 : MC_1 = 3 : 1$. Points A and B are the intersection points of line l with the circle of radius CO centered at O.

8.35. Let O and I be the centers of the circumscribed and inscribed circles, I_c the center of the escribed circle tangent to side AB. The circumscribed circle of triangle ABC divides segment II_c (see Problem 5.109 b)) in halves and segment II_c divides arc $\smile AB$ in halves. It is also clear that points A and B lie on the circle with diameter II_c. This implies the following construction.

Let us construct circle S with diameter II_c and circle S_1 with center O and radius OD, where D is the midpoint of segment II_c. Circles S and S_1 intersect at points A and B. Now, we can construct the inscribed circle of triangle ABC and draw tangents to it at points A and B.

8.36. Suppose that we have constructed points X and Y on sides AB and BC, respectively, of triangle ABC so that $AX = BY$ and $XY \parallel AC$. Let us draw YY_1 parallel to AB and Y_1C_1 parallel to BC (points Y_1 and C_1 lie on sides AC and AB, respectively). Then $Y_1Y = AX = BY$, i.e., BYY_1C is a rhombus and BY_1 is the bisector of angle $\angle B$.

This implies the following construction. Let us draw bisector BY_1, then line Y_1Y parallel to side AB (we assume that Y lies on BC). Now, it is obvious how to construct point X.

8.37. Let, for definiteness, $a < b$. Suppose that triangle ABC is constructed. On side AC, take point D such that $\angle ABD = \angle BAC$. Then $\angle BDC = 2\angle BAC$ and

$$\angle CBD = 3\angle BAC - \angle BAC = 2\angle BAC,$$

i.e., $CD = CB = a$. In triangle BCD all the sides are known: $CD = CB = a$ and $DB = AD = b - a$. Constructing triangle BCD, draw ray BA that does not intersect side CD so that $\angle DBA = \frac{1}{2}\angle DBC$. Vertex A to be constructed is the intersection point of line CD and this ray.

8.38. Let point B' lie on line l that passes through point B parallelly to AC. Sides of triangles ABC and $AB'C$ intersept equal segments on l. Hence, rectangles $P'R'Q'S'$ and $PRQS$ inscribed in triangles ABC and $AB'C$, respectively, are equal if points R, Q, R' and Q' lie on one line.

On line l, take point B' so that $\angle B'AC = 90^\circ$. It is obvious how to inscribe rectangle $P'R'Q'S'$ with given diagonal $P'Q'$ in triangle $AB'C$ (we assume that $P' = A$). Draw line $R'Q'$; we thus find vertices R and Q of the rectangle to be found.

8.39. Suppose that triangle ABC is constructed. Let K and L be points at which the escribed circle tangent to side BC is tangent to the extensions of sides AB and AC, respectively. Since $AK = AL = p$, this escribed circle can be constructed; it remains to draw the tangent through the given point M to the constructed circle.

8.40. Let the extension of the bisector CD intersect the circumscribed circle of triangle ABC (with right angle $\angle C$) at point P, let PQ be a diameter of the circumscribed circle and O its center. Then $PD : PO = PQ : PC$, i.e., $PD \cdot PC = 2R^2 = 2m_c^2$. Therefore, drawing a tangent of length $\sqrt{2}m_c$ to the circle with diameter CD, it is easy to construct a segment of length PC. Now, the lengths of all the sides of triangle OPC are known.

8.41. Let us construct point K on side AC so that $AK = BC - AB$. Let point D lie on segment AC. The equality $AD + BD + AB = BC$ is equivalent to the equality $AD + BD = AK$. For point D that lies on segment AK the latter equality takes the form $AD + BD = AD + DK$ and for point D outside segment AK it takes the form $AD + BD = AD - DK$. In the first case $BD = DK$ and the second case is impossible. Hence, point D is the intersection point of the midperpendicular to segment BK and segment AC.

8.42. Suppose that triangle ABC is constructed. Let us draw diameter CD of the circumscribed circle. Let O be the center of the circumscribed circle, L the intersection point of the extension of the bisector AK with the circumscribed circle (Fig. 88). Since $\angle ABC - \angle ACB = 90^\circ$, it follows that $\angle ABD = \angle ACB$; hence, $\smile DA = \smile AB$. It is also clear that $\smile BL = \smile LC$. Therefore, $\angle AOL = 90^\circ$.

This implies the following construction. Let us construct circle S with center O and a given radius. On circle S select an arbitrary point A. Let us construct a point L on circle S so that $\angle AOL = 90^\circ$. On segment AL, construct segment AK whose length is equal to

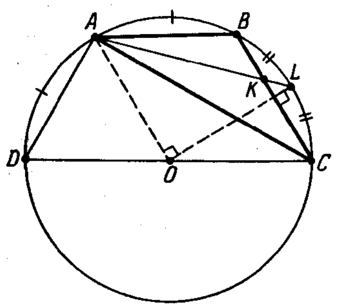


Figure 88 (Sol. 8.42)

that of the given bisector. Let us draw through point K line l perpendicular to OL. The intersection points of l with circle S are vertices B and C of triangle ABC to be constructed.

8.43. On sides BC and AC, take points A_1 and B_1 such that $PA_1 \parallel AC$ and $PB_1 \parallel BC$. Next, on rays A_1B and B_1A mark segments $A_1B_2 = AB_1$ and $B_1A_2 = BA_1$. Let us prove that line A_2B_2 is the one to be found. Indeed, let $k = \frac{AP}{AB}$. Then

$$\frac{BA_2}{BP} = \frac{(1-k)a}{ka} = \frac{(1-k)a + (1-k)b}{ka + kb} = \frac{CA_2}{CB_2},$$

i.e., $\triangle A_2B_1P \sim \triangle A_2CB_2$ and line A_2B_2 passes through point P. Moreover, $AA_2 = |(1-k)a - kb| = BB_2$.

8.44. Suppose that triangle ABC is constructed. Let B_1 be the tangent point of the inscribed circle with side AC. In right triangle AOB_1 leg $OB_1 = r$ and hypothenuse AO are known, therefore, we can construct angle $\angle OAB_1$, hence, angle $\angle BAC$. Let O_1 be the center of the circumscribed circle of triangle ABC, let M be the midpoint of side BC. In right triangle BO_1M leg $O_1M = \frac{1}{2}AH$ is known (see solution to Problem 5.105) and angle $\angle BO_1M$ is known (it is equal to either $\angle A$ or $180^\circ - \angle A$); hence, it can be constructed. Next, we can determine the length of segment $OO_1 = \sqrt{R(R-2r)}$, cf. Problem 5.11 a). Thus, we can construct segments of length R and $OO_1 = d$.

After this take segment AO and construct point O_1 for which $AO_1 = R$ and $OO_1 = d$ (there could be two such points). Let us draw from point A tangents to the circle of radius r centered at O. Points B and C to be found lie on these tangents and their distance from point O_1 is equal to R; obviously, points B and C are distinct from point A.

8.45. Let the distance between the given parallel lines be equal to a. We have to draw parallel lines through points A and B so that the distance between the lines is equal to a. To this end, let us construct the circle with segment AB as its diameter and find the intersection points C_1 and C_2 of this circle with the circle of radius a centered at B. A side of the rhombus to be constructed lies on line AC_1 (another solution: it lies on AC_2). Next, let us draw through point B the line parallel to AC_1 (resp. AC_2).

8.46. Suppose that quadrilateral $ABCD$ is constructed. Let us denote the midpoints of sides AB, BC, CD and DA by P, Q, R and S, respectively, and the midpoints of diagonals AC and BD by K and L, respectively. In triangle KSL we know $KS = \frac{1}{2}CD$, $LS = \frac{1}{2}AB$ and angle $\angle KSL$ equal to the angle between the sides AB and CD.

Having constructed triangle KSL, we can construct triangle KRL because the lengths of all its sides are known. After this we complement triangles KSL and KRL to parallelograms $KSLQ$ and $KRLP$, respectively. Points A, B, C, D are vertices of parallelograms $PLSA$, $QKPB$, $RLQC$, $SKRD$ (Fig. 89).

8.47. Let us drop perpendiculars BB_1 and DD_1 from vertices B and D, respectively, to diagonal AC. Let, for definiteness, $DD_1 > BB_1$. Let us construct a segment of length

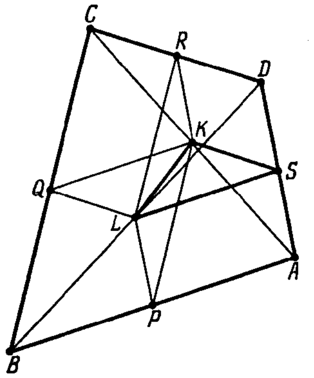


Figure 89 (Sol. 8.46)

$a = DD_1 - BB_1$; draw a line parallel to line AC and such that the distance between this line and AC is equal to a and which intersects side CD at a point, E. Clearly,

$$S_{AED} = \frac{ED}{CD} S_{ACD} = \frac{BB_1}{DD_1} S_{ACD} = S_{ABC}.$$

Therefore, the median of triangle AEC lies on the line to be constructed.

8.48. Let P, Q, R be the midpoints of equal sides AB, BC, CD of quadrilateral $ABCD$. Let us draw the midperpendiculars l_1 and l_2 to segments PQ and QR. Since $AB = BC = CD$, it is clear that points B and C lie on lines l_1 and l_2 and $BQ = QC$.

This implies the following construction. Let us draw the midperpendiculars l_1 and l_2 to segments PQ and QR, respectively. Then through point Q we draw a segment with endpoints on lines l_1 and l_2 so that Q were its midpoint, cf. Problem 16.15.

8.49. Let vertices A, B and C of quadrilateral $ABCD$ which is both inscribed and circumscribed be given and $AB \geq BC$. Then $AD - CD = AB - BC \geq 0$. Hence, on side AD we can mark segment DC_1 equal to DC. In triangle AC_1C the lengths of sides AC and $AC_1 = AB - BC$ are known and $\angle AC_1C = 90^\circ + \frac{1}{2}\angle D = 180^\circ - \frac{1}{2}\angle B$. Since angle $\angle AC_1C$ is an obtuse one, triangle AC_1C is uniquely recoverable from these elements. The remaining part of the construction is obvious.

8.50. Let $ABCD$ be a circumscribed equilateral trapezoid with bases AD and BC such that $AD > BC$; let C_1 be the projection of point C to line AD. Let us prove that $AB = AC_1$. Indeed, if P and Q are the tangent points of sides AB and AD with the inscribed circle, then $AB = AP + PB = AQ + \frac{1}{2}BC = AQ + QC_1 = AC_1$.

This implies the following construction. Let C_1 be the projection of point C to base AD. Then B is the intersection point of line BC and the circle of radius AC_1 centered at A. A trapezoid with $AD < BC$ is similarly constructed.

8.51. Let us denote the midpoints of bases AD and BC by L and N and the midpoint of segment EF by M. Points L, O, N lie on one line (by Problem 19.2). Clearly, point M also lies on this line. This implies the following construction.

Let us draw through point K line l perpendicular to line OK. Base AD lies on l. Point L is the intersection point of l and line OM. Point N is symmetric to point L through point M. Let us draw lines through point O parallel to lines EN and FN. The intersection points of the lines we have just drawn are vertices A and D of the trapezoid. Vertices B and C are symmetric to vertices A and D through points E and F, respectively.

8.52. Suppose that we have constructed quadrilateral $ABCD$ with given lengths of sides and a given midline KP (here K and P are the midpoints of sides AB and CD, respectively). Let A_1 and B_1 be the points symmetric to points A and B, respectively, through point P. Triangle A_1BC can be constructed because its sides BC, $CA_1 = AD$ and $BA_1 = 2KP$ are known. Let us complement triangle A_1BC to parallelogram A_1EBC. Now we can construct point D because CD and $ED = BA$ are known. Making use of the fact that $\overrightarrow{DA} = \overrightarrow{A_1C}$ we construct point A.

8.53. Making use of the formulas of Problems 6.34 and 6.35 it is easy to express the lengths of the diagonals of the inscribed triangle in terms of the lengths of its sides. The obtained formulas can be applied for the construction of the diagonals (for convenience it is advisable to introduce an arbitrary segment e as the measure of unit length and construct segments of length pq, $\frac{p}{q}$ and $\sqrt{p}$ as $\frac{pq}{e}$, $\frac{pe}{q}$ and $\sqrt{pe}$).

8.54. A circle intercepts equal segments on the legs of an angle if and only if the center of the circle lies on the bisector of the angle. Therefore, the center of the circle to be found is the intersection point of the midperpendicular to segment AB and the bisector of the given angle.

8.55. Let us suppose that we have constructed circle S' tangent to the given circle S at point A and the given line l at a point, B. Let O and O' be the centers of circles S and S', respectively (Fig. 90). Clearly, points O, O' and A lie on one line and $O'B = O'A$. Hence, we have to construct point O' on line OA so that $O'A = O'B$, where B is the base of the perpendicular dropped from point O' to line l.

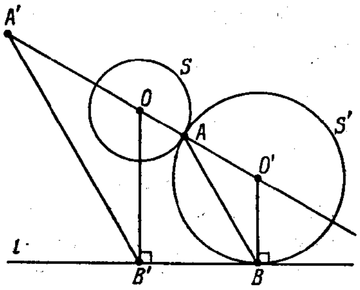


Figure 90 (Sol. 8.55)

To this end let us drop perpendiculr OB' on line l. Next, on line AO mark segment OA' of length OB'. Let us draw through point A line AB parallel to $A'B'$ (point B lies on line l). Point O' is the intersection point of line OA and the perpendicular to l drawn through point B.

8.56. a) Let l_1 be the midperpendicular to segment AB, let C be the intercection point of lines l_1 and l; let l' be the line symmetric to l through line l_1. The problem reduces to the necessity to construct a circle that passes through point A and is tangent to lines l and l', cf. Problem 19.15.

b) We may assume that the center of circle S does not lie on the midperpendicular to segment AB (otherwise the construction is obvious). Let us take an arbitrary point C on circle S and construct the circumscribed circle of triangle ABC; this circle intersects S at a point D. Let M be the intersection point of lines AB and CD. Let us draw tangents MP and MQ to circle S. Then the circumscribed circles of triangles ABP and ABQ are the ones to be found since $MP^2 = MQ^2 = MA \cdot MB$.

8.57. Suppose we have constructed circles S_1, S_2 and S_3 tangent to each other pairwise at given points: S_1 and S_2 are tangent at point C; circles S_1 and S_3 are tangent at point B; circles S_2 and S_3 are tangent at point A. Let O_1, O_2 and O_3 be the centers of circles S_1, S_2 and S_3, respectively. Then points A, B and C lie on the sides of triangle $O_1O_2O_3$ and $O_1B = O_1C$, $O_2C = O_2A$ and $O_3A = O_3B$. Hence, points A, B and C are the tangent points of the inscribed circle of triangle $O_1O_2O_3$ with its sides.

This implies the following construction. First, let us construct the circumscribed circle of triangle ABC and draw tangents to it at points A, B and C. The intersection points of these tangents are the centers of circles to be found.

8.58. Suppose that we have constructed circle S whose tangents AA_1, BB_1 and CC_1, where A_1, B_1 and C_1 are the tangent points, are of length a, b and c, respectively. Let us construct circles S_a, S_b and S_c with the centers A, B and C and radii a, b and c, respectively (Fig. 91). If O is the center of circle S, then segments OA_1, OB_1 and OC_1 are radii of circle S and tangents to circles S_a, S_b and S_c as well. Hence, point O is the radical center (cf. §3.10) of circles S_a, S_b and S_c.

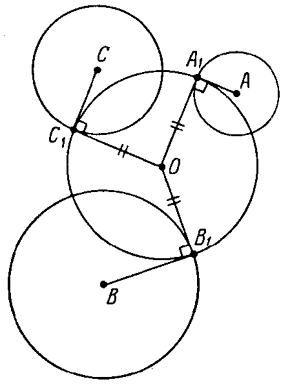


Figure 91 (Sol. 8.58)

This implies the following construction. First, construct circles S_a, S_b and S_c. Then let us construct their radical center O. The circle to be found is the circle with center O and the radius whose length is equal to that of the tangent drawn from point O to circle S_a.

8.59. First, let us construct segment BC of length a. Next, let us construct the locus of points X for which $CX : BX = b : c$, cf. Problem 7.14. For vertex A we can take any of the intersection points of this locus with a line whose distance from line BC is equal to h_a.

8.60. Given the lengths of segments AD' and BD, we can construct segment AB and point D on this segment. Point C is the intersection point of the circle of radius CD centered at D and the locus of points X for which $AX : BX = AD : BD$.

8.61. Let X be a point that does not lie on line AB. Clearly, $\angle AXB = \angle BCX$ if and only if $AX : CX = AB : CB$. Hence, point M is the intersection point of the locus of points X for which $AX : CX = AB : CB$ and the locus of points Y for which $BY : DY = BC : DC$ (it is possible for these loci not to intersect).

8.62. We have to construct a point O for which $AO : A'O = AB : A'B'$ and $BO : B'O = AB : A'B'$. Point O is the intersection point of the locus of points X for which $AX : A'X = AB : A'B'$ and the locus of points Y for which $BY : B'Y = AB : A'B'$.

8.63. Let O be the center of the given circle. Chords XP and XQ that pass through points A and B are equal if and only if XO is the bisector of angle PXO, i.e., $AX : BX = AO : BO$. The point X to be found is the intersection point of the corresponding Apollonius's circle with the given circle.

8.64. a) If line l does not intersect segment AB, then ABB_1A_1 is a parallelogram and $l \parallel AB$. If line l intersects segment AB, then AA_1BB_1 is a parallelogram and l passes through the midpoint of segment AB.

b) One of the lines to be found is parallel to line AB and another one passes through the midpoint of segment AB.

8.65. Let us construct a circle of radius 1 and in it draw two perpendicular diameters, AB and CD. Let O be the center of the circle, M the midpoint of segment OC, P the intersection point of line AM and the circle with diameter OC (Fig. 92). Then $AM^2 = 1 + \frac{1}{4} = \frac{5}{4}$ and, therefore, $AP = AM - PM = \frac{\sqrt{5}-1}{2} = 2\sin 18^\circ$ (cf. Problem 5.46), i.e., AP is the length of a side of a regular decagon inscribed in the given circle.

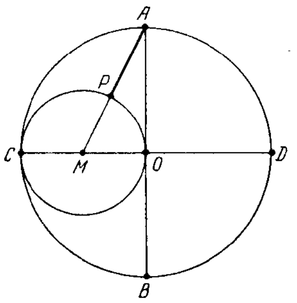


Figure 92 (Sol. 8.65)

8.66. Suppose we have constructed rectangle $PQRS$ so that the given points A, B, C, D lie on sides PQ, QR, RS, SP, respectively, and $PQ : QR = a$, where a is the given ratio of sides. Let F be the intersection point of the line drawn through point D perpendicularly to line AC and line QR. Then $DF : AC = a$.

This implies the following construction. From point D draw a ray that intersects segment AC at a right angle and on this ray construct a point F so that $DF = a \cdot AC$. Side QR lies on line BF. The continuation of the construction is obvious.

8.67. Suppose that points X and Y with the required properties are constructed. Denote the intersection point of lines AX and YC by M, that of lines AB and XY by K. Right triangles AXK and YXM have a common acute angle $\angle X$, hence, $\angle XAK = \angle XYM$. Angles $\angle XAB$ and $\angle XYB$ subtend the same arc, hence, $\angle XAB = \angle XYB$. Therefore, $\angle XYM = \angle XYB$. Since $XY \perp AB$, it follows that K is the midpoint of segment CB.

Conversely, if K is the midpoint of segment CB, then $\angle MYX = \angle BYX = \angle XAB$. Triangles AXK and YXM have a common angle $\angle X$ and $\angle XAK = \angle XYM$; hence, $\angle YMX = \angle AKX = 90^\circ$.

This implies the following construction. Through the midpoint K of segment CB draw line l perpendicular to line AB. Points X and Y are the intersection points of line l with the given circle.

8.68. If we have an angle of value α, then we can construct angles of value 2α, 3α, etc. Since $19 \cdot 19^\circ = 361^\circ$, we can construct an angle of 361° that coincides with the angle of 1°.

8.69. First, let us construct an angle of 36°, cf. Problem 8.65. Then we can construct the angle of $\frac{36^\circ - 30^\circ}{2} = 3^\circ$. If n is not divisible by 3, then having at our disposal angles of n° and 3° we can construct an angle of 1°. Indeed, if $n = 3k + 1$, then $1^\circ = n^\circ - k \cdot 3^\circ$ and if $n = 3k + 2$, then $1^\circ = 2n^\circ - (2k + 1) \cdot 3^\circ$.

8.70. The sequence of constructions is as follows. On the piece of paper take an arbitrary point O and perform the homothety with center O and sufficiently small coefficient k so that this homothety sends the image of the intersection point of the given lines on the piece of paper. Then we can construct the bisector of the angle between the images of the lines. Next, let us perform the homothety with the same center and coefficient $\frac{1}{k}$ which yields the desired segment of the bisector.

8.71. Let us construct with the help of a two-sided ruler two parallel chords AB and CD. Let P and Q be the intersection points of lines AC with BD and AD with BC, respectively. Then line PQ passes through the center of the given circle. Constructing similarly one more such line we find the center of the circle.

8.72. Let us draw through point A two rays p and q that form a small angle inside which point B lies (the rays can be constructed by replacing the ruler). Let us draw through point B segments PQ_1 and P_1Q (Fig. 93). If $PQ < 10$ cm and $P_1Q_1 < 10$ cm, then we can construct point O at which lines PQ and P_1Q_1 intersect.

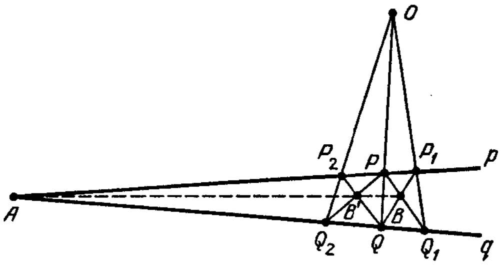


Figure 93 (Sol. 8.72)

Through point O draw line P_2Q_2. If $PQ_2 < 10$ cm and $P_2Q < 10$ cm; then we can construct point B' at which lines PQ_2 and P_2Q intersect. If $BB' < 10$ cm, then by Problem 5.67 we can construct line BB'; this line passes through point A.

8.73. The construction is based on the fact that if A and B are the intersection points of equal circles centered at P and Q, then $\overrightarrow{PA} = \overrightarrow{BQ}$. Let S_1 be the initial circle, A_1 the given point. Let us draw circle S_2 through point A_1 and circle S_3 through the intersection point A_2 of circles S_1 and S_2; circle S_4 through the intersection point A_3 of circles S_2 and S_3 and, finally, circle S_5 through the intersection points B_1 and A_4 of circles S_1 and S_3, respectively, with circle S_4. Let us prove that the intersection point B_2 of circles S_5 and S_1 is the one to be found.

Let O_i be the center of circle S_i. Then

$$\overrightarrow{A_1O_1} = \overrightarrow{O_2A_2} = \overrightarrow{A_3O_3} = \overrightarrow{O_4A_4} = \overrightarrow{B_1O_5} = \overrightarrow{O_1B_2}.$$

Remark. There are two intersection points of circles S_1 and S_4; for point B_1 we can take any of them.

8.74. Let AB be the given segment, P an arbitrary point not on the given lines. Let us construct the intersection points C and D of the second of the given lines with lines PA and PB, respectively, and the intersection point Q of lines AD and BC. By Problem 19.2 line PQ passes through the midpoint of segment AB.

8.75. Let AB be the given segment; let C and D be arbitrary points on the second of given lines. By the preceding problem we can construct the midpoint, M, of segment CD. Let P be the intersection point of lines AM and BD; let E be the intersection point of lines PC and AB. Let us prove that EB is the segment to be found.

Since $\triangle PMC \sim \triangle PAE$ and $\triangle PMD \sim \triangle PAB$, it follows that

$$\frac{AB}{AE} = \frac{AB}{AP} : \frac{AE}{AP} = \frac{MD}{MP} : \frac{MC}{MP} = \frac{MD}{MC} = 1.$$

8.76. Let AB be the given segment; let C and D be arbitrary points on the second of the given lines. By the preceding problem we can construct points $D_1 = D, D_2, \dots, D_n$ such that all the segments D_iD_{i+1} are equal to segment CD. Let P be the intersection point of lines AC and BD_n and let $B_1, \dots, B_{n-1}$ be the intersection points of line AB with lines $PD_1, \dots, PD_{n-1}$, respectively. Clearly, points $B_1, \dots, B_{n-1}$ divide segment AB in n equal parts.

8.77. On one of the given lines take segment AB and construct its midpoint, M (cf. Problem 8.74). Let A_1 and M_1 be the intersection points of lines PA and PM with the second of the given lines, Q the intersection point of lines BM_1 and MA_1. It is easy to verify that line PQ is parallel to the given lines.

8.78. In the case when point P does not lie on line AB, we can make use of the solution of Problem 3.36. If point P lies on line AB, then we can first drop perpendiculars l_1 and l_2 from some other points and then in accordance with Problem 8.77 draw through point P the line parallel to lines l_1 and l_2.

8.79. a) Let A be the given point, l the given line. First, let us consider the case when point O does not lie on line l. Let us draw through point O two arbitrary lines that intersect line l at points B and C. By Problem 8.78, in triangle OBC, heights to sides OB and OC can be dropped. Let H be their intersection point. Then we can draw line OH perpendicular to l. By Problem 8.78 we can drop the perpendicular from point A to OH. This is the line to be constructed that passes through A and is parallel to l. In order to drop the perpendicular from A to l we have to erect perpendicular l' to OH at point O and then drop the perpendicular from A to l'.

If point O lies on line l, then by Problem 8.78 we can immediately drop the perpendicular l' from point A to line l and then erect the perpendicular to line l' from the same point A.

b) Let l be the given line, A the given point on it and BC the given segment. Let us draw through point O lines OD and OE parallel to lines l and BC, respectively (D and E are the intersection points of these lines with circle S). Let us draw through point C the line parallel to OB to its intersection with line OE at point F and through point F the line parallel to ED to its intersection with OD at point G and, finally, through point G the line parallel to OA to its intersection with l at point H. Then $AH = OG = OF = BC$, i.e., AH is the segment to be constructed.

c) Let us take two arbitrary lines that intersect at point P. Let us mark on one of them segment $PA = a$ and on the other one segments $PB = b$ and $PC = c$. Let D be the intersection point of line PA with the line that passes through B and is parallel to AC. Clearly, $PD = \frac{ab}{c}$.

d) Let H be the homothety (or the parallel translation) that sends the circle with center A and radius r to circle S (i.e., to the given circle with the marked center O). Since the radii of both circles are known, we can construct the image of any point X under the mapping H. For this we have to draw through point O the line parallel to line AX and mark on it a segment equal to $\frac{r_s \cdot AX}{r}$, where r_s is the radius of circle S.

We similarly construct the image of any point under the mapping H^{-1}. Hence, we can construct the line $l' = H(l)$ and find its intersection points with circle S and then construct the images of these points under the map H^{-1}.

e) Let A and B be the centers of the given circles, C one of the points to be constructed, CH the height of triangle ABC. From Pythagoras theorem for triangles ACH and BCH we deduce that $AH = \frac{b^2+c^2-a^2}{2c}$. The quantities a, b and c are known, hence, we can construct point H and the intersection points of line CH with one of the given circles.

8.80. a) Let us draw lines parallel to lines OA and OB, whose distance from the latter lines is equal to a and which intersect the legs of the angles. The intersection point of these lines lies on the bisector to be constructed.

b) Let us draw the line parallel to OB, whose distance from OB is equal to a and which intersects ray OA at a point M. Let us draw through points O and M another pair of parallel lines the distance between which is equal to a; the line that passes through point O contains the leg of the angle to be found.

8.81. Let us draw through point A an arbitrary line and then draw lines l_1 and l_2 parallel to it and whose distance from this line is equal to a; these lines intersect line l at points M_1 and M_2. Let us draw through points A and M_1 one more pair of parallel lines, l_a and l_m, the distance between which is equal to a. The intersection point of lines l_2 and l_m belongs to the perpendicular to be found.

8.82. Let us draw a line parallel to the given one at a distance of a. Now, we can make use of the results of Problems 8.77 and 8.74.

8.83. Let us draw through point P lines PA_1 and PB_1 so that $PA_1 \parallel OA$ and $PB_1 \parallel OB$. Let line PM divide the angle between lines l and PA_1 in halves. The symmetry through line PM sends line PA_1 to line l and, therefore, line PB_1 turns under this symmetry into one of the lines to be constructed.

8.84. Let us complement triangle ABM to parallelogram $ABMN$. Through point N draw lines parallel to the bisectors of the angles between lines l and MN. The intersection points of these lines with line l are the ones to be found.

8.85. Let us draw line l_1 parallel to line OA at a distance of a. On l, take an arbitrary point B. Let B_1 be the intersection point of lines OB and l_1. Through point B_1 draw the line parallel to AB; this line intersects line OA at point A_1. Now, let us draw through points O and A_1 a pair of parallel lines the distance between which is equal to a.

There could be two pairs of such lines. Let X and X_1 be the intersection points of the line that passes through point O with lines l and l_1. Since $OA_1 = OX_1$ and $\triangle OA_1X_1 \sim \triangle OAX$, point X is the one to be found.

8.86. Let us erect perpendiculars to line O_1O_2 at points O_1 and O_2 and on the perpendiculars mark segments $O_1B_1 = O_2A_2$ and $O_2B_2 = O_1A_1$. Let us construct the midpoint M of segment B_1B_2 and erect the perpendicular to B_1B_2 at point M. This perpendicular intersects line O_1O_2 at point N. Then $O_1N^2 + O_1B_1^2 = O_2N^2 + O_2B_2^2$ and, therefore, $O_1N^2 - O_1A_1^2 = O_2N^2 - O_2A_2^2$, i.e., point N lies on the radical axis. It remains to erect the perpendicular to O_1O_2 at point N.

8.87. First, let us construct an arbitrary line l_1 perpendicular to line l and then draw through point A the line perpendicular to l_1.

8.88. a) Let us draw through points A and B lines AB and BQ perpendicular to line AB and then draw an arbitrary perpendicular to line AP. As a result we get a rectangle. It remains to drop from the intersection point of its diagonals the perpendicular to line AB.

b) Let us raise from point B perpendicular l to line AB and draw through point A two perpendicular lines; they intersect line l at points M and N. Let us complement right triangle MAN to rectangle $MANR$. The base of the perpendicular dropped from point R to line AB is point C to be found.

8.89. a) Let us drop perpendicular AP from point A to line OB and construct segment AC whose midpoint is points P. Then angle $\angle AOC$ is the one to be found.

b) On line OB, take points B and B_1 such that $OB = OB_1$. Let us place the right angle so that its sides would pass through points B and B_1 and the vertex would lie on ray OA. If A is the vertex of the right angle, then angle $\angle AB_1B$ is the one to be found.

8.90. Let us draw through point O line l' parallel to line l. Let us drop perpendiculars BP and BQ from point B to lines l' and OA, respectively, and then drop perpendicular OX from point O to line PQ. Then line XO is the desired one (cf. Problem 2.3); if point Y is symmetric to point X through line l', then line YO is also the one to be found.

8.91. Let us complement triangle OAB to parallelogram $OABC$ and then construct segment CC_1 whose midpoint is point O. Let us place the right angle so that its legs pass through points C and C_1 and the vertex lies on line l. Then the vertex of the right angle coincides with point X to be found.

8.92. Let us construct segment AB whose midpoint is point O and place the right angle so that its legs passes through points A and B and the vertex lies on line l. Then the vertex of the right angle coincides with the point to be found.

Chapter 9. GEOMETRIC INEQUALITIES

Background

1) For elements of a triangle the following notations are used:

a, b, c are the lengths of sides BC, CA, AB, respectively;

α, β, γ the values of the angles at vertices A, B, C, respectively;

m_a, m_b, m_c are the lengths of the medians drawn from vertices A, B, C, respectively;

h_a, h_b, h_c are the lengths of the heights dropped from vertices A, B, C, respectively;

l_a, l_b, l_c are the lengths of the bisectors drawn from vertices A, B, C, respectively;

r and R are the radii of the inscribed and circumscribed circles, respectively.

2) If A, B, C are arbitrary points, then $AB \leq AC + CB$ and the equality takes place only if point C lies on segment AB (*the triangle inequality*).

3) The median of a triangle is shorter than a half sum of the sides that confine it: $m_a < \frac{1}{2(b+c)}$ (Problem 9.1).

4) If one convex polygon lies inside another one, then the perimeter of the outer polygon is greater than the perimeter of the inner one (Problem 9.27 b).

5) The sum of the lengths of the diagonals of a convex quadrilateral is greater than the sum of the length of any pair of the opposite sides of the quadrilateral (Problem 9.14).

6) The longer side of a triangle subtends the greater angle (Problem 10.59).

7) The length of the segment that lies inside a convex polygon does not exceed either that of its longest side or that of its longest diagonal (Problem 10.64).

Remark. While solving certain problems of this chapter we have to know various algebraic inequalities. The data on these inequalities and their proof are given in an appendix to this chapter; one should acquaint oneself with them but it should be taken into account that these inequalities are only needed in the solution of comparatively complicated problems; in order to solve simple problems we will only need the inequality $\sqrt{ab} \leq \frac{1}{2}a + b$ and its corollaries.

Introductory problems

1. Prove that $S_{ABC} \leq \frac{1}{2}AB \cdot BC$.

2. Prove that $S_{ABCD} \leq \frac{1}{2}(AB \cdot BC + AD \cdot DC)$.

3. Prove that $\angle ABC > 90^\circ$ if and only if point B lies inside the circle with diameter AC.

4. The radii of two circles are equal to R and r and the distance between the centers of the circles is equal to d. Prove that these circles intersect if and only if $|R - r| < d < R + r$.

5. Prove that any diagonal of a quadrilateral is shorter than the quadrilateral's semiperimeter.

§1. A median of a triangle

9.1. Prove that $\frac{1}{2}(a + b - c) < m_c < \frac{1}{2}(a + b)$.

9.2. Prove that in any triangle the sum of the medians is greater than $\frac{3}{4}$ of the perimeter but less than the perimeter.

9.3. Given n points $A_1, \dots, A_n$ and a unit circle, prove that it is possible to find a point M on the circle so that $MA_1 + \dots + MA_n \geq n$.

9.4. Points $A_1, \dots, A_n$ do not lie on one line. Let two distinct points P and Q have the following property

$$A_1P + \dots + A_nP = A_1Q + \dots + A_nQ = s.$$

Prove that $A_1K + \dots + A_nK < s$ for a point K.

9.5. On a table lies 50 working watches (old style, with hands); all work correctly. Prove that at a certain moment the sum of the distances from the center of the table to the endpoints of the minute's hands becomes greater than the sum of the distances from the center of the table to the centers of watches. (We assume that each watch is of the form of a disk.)

§2. Algebraic problems on the triangle inequality

In problems of this section a, b and c are the lengths of the sides of an arbitrary triangle.

9.6. Prove that $a = y + z$, $b = x + z$ and $c = x + y$, where x, y and z are positive numbers.

9.7. Prove that $a^2 + b^2 + c^2 < 2(ab + bc + ca)$.

9.8. For any positive integer n, a triangle can be composed of segments whose lengths are a^n, b^n and c^n. Prove that among numbers a, b and c two are equal.

9.9. Prove that

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 4abc > a^3 + b^3 + c^3.$$

9.10. Let $p = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ and $q = \frac{a}{c} + \frac{c}{b} + \frac{b}{a}$. Prove that $|p - q| < 1$.

9.11. Five segments are such that from any three of them a triangle can be constructed. Prove that at least one of these triangles is an acute one.

9.12. Prove that

$$(a + b - c)(a - b + c)(-a + b + c) \leq abc.$$

9.13. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0.$$

§3. The sum of the lengths of quadrilateral's diagonals

9.14. Let $ABCD$ be a convex quadrilateral. Prove that $AB + CD < AC + BD$.

9.15. Let $ABCD$ be a convex quadrilateral and $AB + BD \leq AC + CD$. Prove that $AB < AC$.

9.16. Inside a convex quadrilateral the sum of lengths of whose diagonals is equal to d, a convex quadrilateral the sum of lengths of whose diagonals is equal to d' is placed. Prove that $d' < 2d$.

9.17. Given closed broken line has the property that any other closed broken line with the same vertices (?) is longer. Prove that the given broken line is not a self-intersecting one.

9.18. How many sides can a convex polygon have if all its diagonals are of equal length?

9.19. In plane, there are n red and n blue dots no three of which lie on one line. Prove that it is possible to draw n segments with the endpoints of distinct colours without common points.

9.20. Prove that the mean arithmetic of the lengths of sides of an arbitrary convex polygon is less than the mean arithmetic of the lengths of all its diagonals.

9.21. A convex $(2n+1)$-gon $A_1A_3A_5\dots A_{2n+1}A_2\dots A_{2n}$ is given. Prove that among all the closed broken lines with the vertices in the vertices of the given $(2n+1)$-gon the broken line $A_1A_2A_3\dots A_{2n+1}A_1$ is the longest.

§4. Miscellaneous problems on the triangle inequality

9.22. In a triangle, the lengths of two sides are equal to 3.14 and 0.67. Find the length of the third side if it is known that it is an integer.

9.23. Prove that the sum of lengths of diagonals of convex pentagon $ABCDE$ is greater than its perimeter but less than the doubled perimeter.

9.24. Prove that if the lengths of a triangle's sides satisfy the inequality $a^2+b^2>5c^2$, then c is the length of the shortest side.

9.25. The lengths of two heights of a triangle are equal to 12 and 20. Prove that the third height is shorter than 30.

9.26. On sides AB, BC, CA of triangle ABC, points C_1, A_1, B_1, respectively, are taken so that $BA_1=\lambda\cdot BC$, $CB_1=\lambda\cdot CA$ and $AC_1=\lambda\cdot AB$, where $\frac{1}{2}<\lambda<1$. Prove that the perimeter P of triangle ABC and the perimeter P_1 of triangle $A_1B_1C_1$ satisfy the following inequality: $(2\lambda-1)P<P_1<\lambda P$.

* * *

9.27. a) Prove that under the passage from a nonconvex polygon to its convex hull the perimeter diminishes. (The *convex hull* of a polygon is the smallest convex polygon that contains the given one.)

b) Inside a convex polygon there lies another convex polygon. Prove that the perimeter of the outer polygon is not less than the perimeter of the inner one.

9.28. Inside triangle ABC of perimeter P, a point O is taken. Prove that $\frac{1}{2}P<AO+BO+CO<P$.

9.29. On base AD of trapezoid $ABCD$, a point E is taken such that the perimeters of triangles ABE, BCE and CDE are equal. Prove that $BC=\frac{1}{2}AD$.

See also Problems 13.40, 20.11.

§5. The area of a triangle does not exceed a half product of two sides

9.30. Given a triangle of area 1 the lengths of whose sides satisfy $a\le b\le c$. Prove that $b\ge\sqrt{2}$.

9.31. Let E, F, G and H be the midpoints of sides AB, BC, CD and DA of quadrilateral $ABCD$. Prove that

$$S_{ABCD}\le EG\cdot HF\le\frac{(AB+CD)(AD+BC)}{4}.$$

9.32. The perimeter of a convex quadrilateral is equal to 4. Prove that its area does not exceed 1.

9.33. Inside triangle ABC a point M is taken. Prove that

$$4S\le AM\cdot BC+BM\cdot AC+CM\cdot AB,$$

where S is the area of triangle ABC.

9.34. In a circle of radius R a polygon of area S is inscribed; the polygon contains the center of the circle and on each of its sides a point is chosen. Prove that the perimeter of the convex polygon with vertices in the chosen points is not less than $\frac{2S}{R}$.

9.35. Inside a convex quadrilateral $ABCD$ of area S point O is taken such that $AO^2+BO^2+CO^2+DO^2=2S$. Prove that $ABCD$ is a square and O is its center.

§6. Inequalities of areas

9.36. Points M and N lie on sides AB and AC, respectively, of triangle ABC, where $AM = CN$ and $AN = BM$. Prove that the area of quadrilateral $BMNC$ is at least three times that of triangle AMN.

9.37. Areas of triangles $ABC, A_1B_1C_1, A_2B_2C_2$ are equal to S, S_1, S_2, respectively, and $AB = A_1B_1 + A_2B_2, AC = A_1C_1 + A_2B_2, BC = B_1C_1 + B_2C_2$. Prove that $S \leq 4\sqrt{S_1S_2}$.

9.38. Let $ABCD$ be a convex quadrilateral of area S. The angle between lines AB and CD is equal to α and the angle between AD and BC is equal to β. Prove that

$$AB \cdot CD \sin\alpha + AD \cdot BC \sin\beta \leq 2S \leq AB \cdot CD + AD \cdot BC.$$

9.39. Through a point inside a triangle three lines parallel to the triangle's sides are drawn.

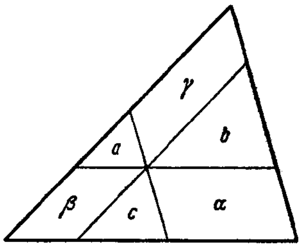


Figure 94 (9.39)

Denote the areas of the parts into which these lines divide the triangle as plotted on Fig. 94. Prove that $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} \geq \frac{3}{2}$.

9.40. The areas of triangles ABC and $A_1B_1C_1$ are equal to S and S_1, respectively, and we know that triangle ABC is not an obtuse one. The greatest of the ratios $\frac{a_1}{a}$, $\frac{b_1}{b}$ and $\frac{c_1}{c}$ is equal to k. Prove that $S_1 \leq k^2S$.

9.41. a) Points B, C and D divide the (smaller) arc $\smile AE$ of a circle into four equal parts. Prove that $S_{ACE} < 8S_{BCD}$.

b) From point A tangents AB and AC to a circle are drawn. Through the midpoint D of the (lesser) arc $\smile BC$ the tangent that intersects segments AB and AC at points M and N, respectively is drawn. Prove that $S_{BCD} < 2S_{MAN}$.

9.42. All sides of a convex polygon are moved outwards at distance h and extended to form a new polygon. Prove that the difference of areas of the polygons is more than $Ph + \pi h^2$, where P is the perimeter.

9.43. A square is cut into rectangles. Prove that the sum of areas of the disks circumscribed about all these rectangles is not less than the area of the disk circumscribed about the initial square.

9.44. Prove that the sum of areas of five triangles formed by the pairs of neighbouring sides and the corresponding diagonals of a convex pentagon is greater than the area of the pentagon itself.

9.45. a) Prove that in any convex hexagon of area S there exists a diagonal that cuts off the hexagon a triangle whose area does not exceed $\frac{1}{6}S$.

b) Prove that in any convex 8-gon of area S there exists a diagonal that cuts off it a triangle of area not greater than $\frac{1}{8}S$.

See also Problem 17.19.

§7. Area. One figure lies inside another

9.46. A convex polygon whose area is greater than 0.5 is placed in a unit square. Prove that inside the polygon one can place a segment of length 0.5 parallel to a side of the square.

9.47. Inside a unit square n points are given. Prove that:

a) the area of one of the triangles some of whose vertices are in these points and some in vertices of the square does not exceed $\frac{1}{2(n+1)}$;

b) the area of one of the triangles with the vertices in these points does not exceed $\frac{1}{n-2}$.

9.48. a) In a disk of area S a regular n-gon of area S_1 is inscribed and a regular n-gon of area S_2 is circumscribed about the disk. Prove that $S^2 > S_1S_2$.

b) In a circle of length L a regular n-gon of perimeter P_1 is inscribed and another regular n-gon of perimeter P_2 is circumscribed about the circle. Prove that $L^2 < P_1P_2$.

9.49. A polygon of area B is inscribed in a circle of area A and circumscribed about a circle of area C. Prove that $2B \leq A + C$.

9.50. In a unit disk two triangles the area of each of which is greater than 1 are placed. Prove that these triangles intersect.

9.51. a) Prove that inside a convex polygon of area S and perimeter P one can place a disk of radius $\frac{S}{P}$.

b) Inside a convex polygon of area S_1 and perimeter P_1 a convex polygon of area S_2 and perimeter P_2 is placed. Prove that $\frac{2S_1}{P_1} > \frac{S_2}{P_2}$.

9.52. Prove that the area of a parallelogram that lies inside a triangle does not exceed a half area of the triangle.

9.53. Prove that the area of a triangle whose vertices lie on sides of a parallelogram does not exceed a half area of the parallelogram.

* * *

9.54. Prove that any acute triangle of area 1 can be placed in a right triangle of area $\sqrt{3}$.

9.55. a) Prove that a convex polygon of area S can be placed in a rectangle of area not greater than $2S$.

b) Prove that in a convex polygon of area S a parallelogram of area not less than $\frac{1}{2}S$ can be inscribed.

9.56. Prove that in any convex polygon of area 1 a triangle whose area is not less than a) $\frac{1}{4}$; b) $\frac{3}{8}$ can be placed.

9.57. A convex n-gon is placed in a unit square. Prove that there are three vertices A, B and C of this n-gon, such that the area of triangle ABC does not exceed a) $\frac{8}{n^2}$; b) $\frac{16\pi}{n^3}$.

See also Problem 15.6.

§8. Broken lines inside a square

9.58. Inside a unit square a non-self-intersecting broken line of length 1000 is placed. Prove that there exists a line parallel to one of the sides of the square that intersects this broken line in at least 500 points.

9.59. In a unit square a broken line of length L is placed. It is known that each point of the square is distant from a point of this broken line less than by ε. Prove that $L \geq \frac{1}{2\varepsilon} - \frac{1}{2}\pi\varepsilon$.

9.60. Inside a unit square n^2 points are placed. Prove that there exists a broken line that passes through all these points and whose length does not exceed $2n$.

9.61. Inside a square of side 100 a broken line L is placed. This broken line has the following property: the distance from any point of the square to L does not exceed 0.5.

Prove that on L there are two points the distance between which does not exceed 1 and the distance between which along L is not less than 198.

§9. The quadrilateral

9.62. In quadrilateral $ABCD$ angles $\angle A$ and $\angle B$ are equal and $\angle D > \angle C$. Prove that $AD < BC$.

9.63. In trapezoid $ABCD$, the angles at base AD satisfy inequalities $\angle A < \angle D < 90^\circ$. Prove that $AC > BD$.

9.64. Prove that if two opposite angles of a quadrilateral are obtuse ones, then the diagonal that connects the vertices of these angles is shorter than the other diagonal.

9.65. Prove that the sum of distances from an arbitrary point to three vertices of an isosceles trapezoid is greater than the distance from this point to the fourth vertex.

9.66. Angle $\angle A$ of quadrilateral $ABCD$ is an obtuse one; F is the midpoint of side BC. Prove that $2FA < BD + CD$.

9.67. Quadrilateral $ABCD$ is given. Prove that $AC \cdot BD \leq AB \cdot CD + BC \cdot AD$. (*Ptolemy's inequality.*)

9.68. Let M and N be the midpoints of sides BC and CD, respectively, of a convex quadrilateral $ABCD$. Prove that $S_{ABCD} < 4S_{AMN}$.

9.69. Point P lies inside convex quadrilateral $ABCD$. Prove that the sum of distances from point P to the vertices of the quadrilateral is less than the sum of pairwise distances between the vertices of the quadrilateral.

9.70. The diagonals divide a convex quadrilateral $ABCD$ into four triangles. Let P be the perimeter of $ABCD$ and Q the perimeter of the quadrilateral formed by the centers of the inscribed circles of the obtained triangles. Prove that $PQ > 4S_{ABCD}$.

9.71. Prove that the distance from one of the vertices of a convex quadrilateral to the opposite diagonal does not exceed a half length of this diagonal.

9.72. Segment KL passes through the intersection point of diagonals of quadrilateral $ABCD$ and the endpoints of KL lie on sides AB and CD of the quadrilateral. Prove that the length of segment KL does not exceed the length of one of the diagonals of the quadrilateral.

9.73. Parallelogram P_2 is inscribed in parallelogram P_1 and parallelogram P_3 whose sides are parallel to the corresponding sides of P_1 is inscribed in parallelogram P_2. Prove that the length of at least one of the sides of P_1 does not exceed the doubled length of a parallel to it side of P_3.

See also Problems 13.19, 15.3 a).

§10. Polygons

9.74. Prove that if the angles of a convex pentagon form an arithmetic progression, then each of them is greater than 36°.

9.75. Let $ABCDE$ is a convex pentagon inscribed in a circle of radius 1 so that $AB = A$, $BC = b$, $CD = c$, $DE = d$, $AE = 2$. Prove that

$$a^2 + b^2 + c^2 + d^2 + abc + bcd < 4.$$

9.76. Inside a regular hexagon with side 1 point P is taken. Prove that the sum of the distances from point P to certain three vertices of the hexagon is not less than 1.

9.77. Prove that if the sides of convex hexagon $ABCDEF$ are equal to 1, then the radius of the circumscribed circle of one of triangles ACE and BDF does not exceed 1.

9.78. Each side of convex hexagon $ABCDEF$ is shorter than 1. Prove that one of the diagonals AD, BE, CF is shorter than 2.

9.79. Heptagon $A_1 \dots A_7$ is inscribed in a circle. Prove that if the center of this circle lies inside it, then the value of any angle at vertices A_1, A_3, A_5 is less than 450°.

* * *

9.80. a) Prove that if the lengths of the projections of a segment to two perpendicular lines are equal to a and b, then the segment's length is not less than $\frac{a+b}{\sqrt{2}}$.

b) The lengths of the projections of a polygon to coordinate axes are equal to a and b. Prove that its perimeter is not less than $\sqrt{2}(a+b)$.

9.81. Prove that from the sides of a convex polygon of perimeter P two segments whose lengths differ not more than by $\frac{1}{3}P$ can be constructed.

9.82. Inside a convex polygon $A_1 \dots A_n$ a point O is taken. Let α_k be the value of the angle at vertex $A_k, x_k = OA_k$ and d_k the distance from point O to line A_kA_{k+1}. Prove that $\sum x_k \sin\frac{\alpha_k}{2} \geq \sum d_k$ and $\sum x_k \cos\frac{\alpha_k}{2} \geq p$, where p is the semiperimeter of the polygon.

9.83. Regular $2n$-gon M_1 with side a lies inside regular $2n$-gon M_2 with side $2a$. Prove that M_1 contains the center of M_2.

9.84. Inside regular polygon $A_1 \dots A_n$ point O is taken.

Prove that at least one of the angles $\angle A_iOA_j$ satisfies the inequalities $\pi\left(1-\frac{1}{n}\right) \leq \angle A_iOA_j \leq \pi$.

9.85. Prove that for $n \geq 7$ inside a convex n-gon there is a point the sum of distances from which to the vertices is greater than the semiperimeter of the n-gon.

9.86. a) Convex polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ are such that all their corresponding sides except for A_1A_n and B_1B_n are equal and $\angle A_2 \geq \angle B_2, \dots, \angle A_{n-1} \geq \angle B_{n-1}$, where at least one of the inequalities is a strict one. Prove that $A_1A_n > B_1B_n$.

b) The corresponding sides of nonequal polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ are equal.

Let us write beside each vertex of polygon $A_1 \dots A_n$ the sign of the difference $\angle A_i - \angle B_i$. Prove that for $n \geq 4$ there are at least four pairs of neighbouring vertices with distinct signs. (The vertices with the zero difference are disregarded: two vertices between which there only stand vertices with the zero difference are considered to be neighbouring ones.)

See also Problems 4.37, 4.53, 13.42.

§11. Miscellaneous problems

9.87. On a segment of length 1 there are given n points. Prove that the sum of distances from a point of the segment to these points is not less than $\frac{1}{2}n$.

9.88. In a forest, trees of cylindrical form grow. A communication service person has to connect a line from point A to point B through this forest the distance between the points being equal to l. Prove that to acheave the goal a piece of wire of length $1.6l$ will be sufficient.

9.89. In a forest, the distance between any two trees does not exceed the difference of their heights. Any tree is shorter than 100 m. Prove that this forest can be fenced by a fence of length 200 m.

9.90. A (not necessarily convex) paper polygon is folded along a line and both halves are glued together. Can the perimeter of the obtained lamina be greater than the perimeter of the initial polygon?

* * *

9.91. Prove that a closed broken line of length 1 can be placed in a disk of radius 0.25.

9.92. An acute triangle is placed inside a circumscribed circle. Prove that the radius of the circle is not less than the radius of the circumscribed circle of the triangle.

Is a similar statement true for an obtuse triangle?

9.93. Prove that the perimeter of an acute triangle is not less than $4R$.

See also problems 14.23, 20.4.

Problems for independent study

9.94. Two circles divide rectangle $ABCD$ into four rectangles. Prove that the area of one of the rectangles, the one adjacent to vertices A and C, does not exceed a quarter of the area of $ABCD$.

9.95. Prove that if $AB + BD = AC + CD$, then the midperpendicular to side BC of quadrilateral $ABCD$ intersects segment AD.

9.96. Prove that if diagonal BD of convex quadrilateral $ABCD$ divides diagonal AC in halves and $AB > BC$, then $AD < DC$.

9.97. The lengths of bases of a circumscribed trapezoid are equal to 2 and 11. Prove that the angle between the extensions of its lateral sides is an acute one.

9.98. The bases of a trapezoid are equal to a and b and its height is equal to h. Prove that the length of one of its diagonals is not less than $\sqrt{\frac{h^2+(b+a)^2}{4}}$.

9.99. The vertices of an n-gon M_1 are the midpoints of sides of a convex n-gon M. Prove that for $n \geq 3$ the perimeter of M_1 is not less than the semiperimeter of M and for $n \geq 4$ the area of M_1 is not less than a half area of M.

9.100. In a unit circle a polygon the lengths of whose sides are confined between 1 and $\sqrt{2}$ is inscribed. Find how many sides does the polygon have.

Supplement. Certain inequalities

1. *The inequality between the mean arithmetic and the mean geometric of two numbers* $\sqrt{ab} \leq \frac{1}{2}(a+b)$, where a and b are positive numbers, is often encountered. This inequality follows from the fact that $a - 2\sqrt{ab} + b = (\sqrt{a} - \sqrt{b})^2 \geq 0$, where the equality takes place only if $a = b$.

This inequality implies several useful inequalities, for example:

$$\begin{aligned} x(a-x) &\leq \left(\tfrac{x+a-x}{2}\right)^2 = \tfrac{a^2}{4}; \\ a + \tfrac{1}{a} &\geq 2\sqrt{a \cdot \tfrac{1}{a}} = 2 \ \text{for} a > 0. \end{aligned}$$

2. *The inequality between the mean arithmetic and the mean geometric of n positive numbers* $(a_1a_2\dots a_n)^{\frac{1}{n}} \leq \frac{a_1+\dots+a_n}{n}$ is sometimes used. In this inequality the equality takes place only if $a_1 = \dots = a_n$.

First, let us prove this inequality for the numbers of the form $n = 2^m$ by induction on m. For $m = 1$ the equality was proved above.

Suppose that it is proved for m and let us prove it for $m+1$. Clearly, $a_ka_{k+2^m} \leq \left(\frac{a_k+a_{k+2^m}}{2}\right)^2$. Therefore,

$$(a_1a_2\dots a_{2^{m+1}})^{\frac{1}{2^{m+1}}} \leq (b_1b_2\dots b_{2^m})^{\frac{1}{2^m}},$$

where $b_k = \frac{1}{2}(a_k + a_{k+2^m})$ and by the inductive hypothesis

$$(b_1\dots b_{2^m})^{\frac{1}{2^m}} \leq \frac{1}{2^m}(b_1 + \dots + b_{2^m}) = \frac{1}{2^{m+1}}(a_1 + \dots + a_{2^{m+1}}).$$

Now, let n be an arbitrary number. Then $n < 2^m$ for some m. Suppose $a_{n+1} = \dots = a_{2^m} = \frac{a_1+\dots+a_n}{n} = A$. Clearly,

$$(a_1 + \dots + a_n) + (a_{n+1} + \dots + a_{2^m}) = nA + (2^m - n)A = 2^mA$$

and $a_1 \dots a_{2^m} = a_1 \dots a_n \cdot A^{2^m - n}$. Hence,

$$a_1 \dots a_n \cdot A^{2^m-n} \leq \left(\frac{2^m A}{2^m}\right)^{2^m} = A^{2^m}, \text{ i.e. } a_1 \dots a_n \leq A^n.$$

The equality is attained only for $a_1 = \dots = a_n$.

3. For arbitrary numbers $a_1, \dots, a_n$ we have

$$(a + \dots + a_n)^2 \leq n(a_1^2 + \dots + a_n^2).$$

Indeed,

$$(a_1 + \dots + a_n)^2 = \sum a_i^2 + 2\sum_{i<j} a_i a_j \leq \sum a_i^2 + \sum_{i<j}(a_i^2 + a_j^2) = n\sum a_i^2.$$

4. Since $\int_0^\alpha \cos t\, dt = \sin\alpha$ and $\int_0^\alpha \sin t\, dt = 1 - \cos\alpha$, it follows that starting from the inequality $\cos t \leq 1$ we get: first, $\sin\alpha \leq \alpha$, then $1 - \cos\alpha \leq \frac{\alpha^2}{2}$ (i.e. $\cos\alpha \geq 1 - \frac{\alpha^2}{2}$), next, $\sin\alpha \geq \alpha - \frac{\alpha^3}{6}$, $\cos\alpha \leq 1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{24}$, etc. (the inequalities are true for all $\alpha \geq 0$).

5. Let us prove that $\tan\alpha \geq \alpha$ for $0 \leq \alpha < \frac{\pi}{2}$. Let AB be the tangent to the unit circle centered at O; let B be the tangent point, C the intersection point of ray OA with the circle and S the area of the disk sector BOC. Then $\alpha = 2S < 2S_{AOB} = \tan\alpha$.

6. On the segment $[0, \frac{\pi}{2}]$ the function $f(x) = \frac{x}{\sin x}$ monotonously grows because $f'(x) = \frac{\tan x - x}{\cos x \sin^2 x} > 0$. In particular, $f(\alpha) \leq f\left(\frac{\pi}{2}\right)$, i.e.,

$$\frac{\alpha}{\sin\alpha} \leq \frac{\pi}{2} \text{ for } 0 < \alpha < \frac{\pi}{2}.$$

7. If $f(x) = a\cos x + b\sin x$, then $f(x) \leq \sqrt{a^2+b^2}$. Indeed, there exists an angle φ such that $\cos\varphi = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$; hence,

$$f(x) = \sqrt{a^2+b^2}\cos(\varphi - x) \leq \sqrt{a^2+b^2}.$$

The equality takes place only if $\varphi = x + 2k\pi$, i.e., $\cos x = \frac{a}{\sqrt{a^2+b^2}}$ and $\sin x = \frac{b}{\sqrt{a^2+b^2}}$.

Solutions

9.1. Let C_1 be the midpoint of side AB. Then $CC_1 + C_1A > CA$ and $BC_1 + C_1C > BC$. Therefore, $2CC_1 + BA > CA + BC$, i.e., $m_c > \frac{1}{2}(a + b - c)$.

Let point C' be symmetric to C through point C_1. Then $CC_1 = C_1C'$ and $BC' = CA$. Hence, $2m_c = CC' < CB + BC' = CB + CA$, i.e., $m_c < \frac{1}{2}(a + b)$.

9.2. The preceding problem implies that $m_a < \frac{1}{2}(b+c)$, $m_b < \frac{1}{2}(a+c)$ and $m_c < \frac{1}{2}(a+b)$ and, therefore, the sum of the lengths of medians does not exceed the perimeter.

Let O be the intersection point of medians of triangle ABC. Then $BO + OA > BA$, $AO + OC > AC$ and $CO + OB > CB$. Adding these inequalities and taking into account that $AO = \frac{2}{3}m_a$, $BO = \frac{2}{3}m_b$, $CO = \frac{2}{3}m_c$ we get $m_a + m_b + m_c > \frac{3}{4}(a + b + c)$.

9.3. Let M_1 and M_2 be diametrically opposite points on a circle. Then $M_1A_k + M_2A_k \geq M_1M_2 = 2$. Adding up these inequalities for $k = 1, \dots, n$ we get

$$(M_1A_1 + \dots + M_1A_n) + (M_2A_1 + \dots + M_2A_n) \geq 2n.$$

Therefore, either $M_1A_1 + \dots + M_1A_n \geq n$ and then we set $M = M_1$ or $M_2A_1 + \dots + M_2A_n \geq n$ and then we set $M = M_2$.

9.4. For K we can take the midpoint of segment PQ. Indeed, then $A_iK \leq \frac{1}{2}(A_iP + A_iQ)$ (cf. Problem 9.1), where at least one of the inequalities is a strict one because points A_i cannot all lie on line PQ.

9.5. Let A_i and B_i be the positions of the minute hands of the i-th watch at times t and $t + 30$ min, let O_i be the center of the i-th watch and O the center of the table. Then $OO_i \leq \frac{1}{2}(OA_i + OB_i)$ for any i, cf. Problem 9.1. Clearly, at a certain moment points A_i and B_i do not lie on line O_iO, i.e., at least one of n inequalities becomes a strict one. Then either $OO_1 + \dots + OO_n < OA_1 + \dots + OA_n$ or $OO_1 + \dots + OO_n < OB_1 + \dots + OB_n$.

9.6. Solving the system of equations

$$x + y = c, \quad x + z = b, \quad y + z = a$$

we get

$$x = \frac{-a + b + c}{2}, \quad y = \frac{a - b + c}{2}, \quad z = \frac{a + b - c}{2}.$$

The positivity of numbers x, y and z follows from the triangle inequality.

9.7. Thanks to the triangle inequality we have

$$a^2 > (b - c)^2 = b^2 - 2bc + c^2, \quad b^2 > a^2 - 2ac + c^2, \quad c^2 > a^2 - 2ab + b^2.$$

Adding these inequalities we get the desired statement.

9.8. We may assume that $a \geq b \geq c$. Let us prove that $a = b$. Indeed, if $b < a$, then $b \leq \lambda a$ and $c \leq \lambda a$, where $\lambda < 1$. Hence, $b^n + c^n \leq 2\lambda^n a^n$. For sufficiently large n we have $2\lambda^n < 1$ which contradicts the triangle inequality.

9.9. Since $c(a - b)^2 + 4abc = c(a + b)^2$, it follows that

$$\begin{gathered} a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 4abc - a^3 - b^3 - c^3 = \\ a((b - c)^2 - a^2) + +b((c - a)^2 - b^2) + c((a + b)^2 - c^2) = \\ (a + b - c)(a - b + c)(-a + b + c). \end{gathered}$$

The latter equality is subject to a direct verification. All three factors of the latter expression are positive thanks to the triangle inequality.

9.10. It is easy to verify that

$$abc|p - q| = |(b - c)(c - a)(a - b)|.$$

Since $|b - c| < a$, $|c - a| < b$ and $|a - b| < c$, we have $|(b - c)(c - a)(a - b)| < abc$.

9.11. Let us index the lengths of the segments so that $a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$. If all the triangles that can be composed of these segments are not acute ones, then $a_3^2 \geq a_1^2 + a_2^2$, $a_4^2 \geq a_2^2 + a_3^2$ and $a_5^2 \geq a_3^2 + a_4^2$. Hence,

$$a_5^2 \geq a_3^2 + a_4^2 \geq (a_1^2 + a_2^2) + (a_2^2 + a_3^2) \geq 2a_1^2 + 3a_2^2.$$

Since $a_1^2 + a_2^2 \geq 2a_1a_2$, it follows that

$$2a_1^2 + 3a_2^2 > a_1^2 + 2a_1a_2 + a_2^2 = (a_1 + a_2)^2.$$

We get the inequality $a_5^2 > (a_1 + a_2)^2$ which contradicts the triangle inequality.

9.12. First solution. Let us introduce new variables

$$x = -a + b + c, \quad y = a - b + c, \quad z = a + b - c.$$

Then $a = \frac{1}{2}(y + z)$, $b = \frac{1}{2}(x + z)$, $c = \frac{1}{2}(x + y)$, i.e., we have to prove that either

$$xyz \leq \frac{1}{8}(x + y)(y + z)(x + z)$$

or

$$6xyz \leq x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2).$$

The latter inequality follows from the fact that $2xyz \leq x(y^2 + z^2)$, $2xyz \leq y(x^2 + z^2)$ and $2xyz \leq z(x^2 + y^2)$, because x, y, z are positive numbers.

Second solution. Since $2S = ab\sin\gamma$ and $\sin\gamma = \frac{c}{2R}$, it follows that $abc = 2SR$. By Heron's formula

$$(a+b-c)(a-b+c)(-a+b+c) = \frac{8S^2}{p}.$$

Therefore, we have to prove that $\frac{8S^2}{p} \leq 4SR$, i.e., $2S \leq pR$. Since $S = pr$, we infer that $2r \leq R$, cf. Problem 10.26.

9.13. Let us introduce new variables

$$x = \frac{-a+b+c}{2}, \quad y = \frac{a-b+c}{2}, \quad z = \frac{a+b-c}{2}.$$

Then numbers x, y, z are positive and

$$a = y+z, \quad b = x+z, \quad c = x+y.$$

Simple but somewhat cumbersome calculations show that

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) = 2(x^3z + y^3x + z^3y - xyz(x+y+z)) =$$
$$2xyz\left(\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} - x - y - z\right).$$

Since $2 \leq \frac{x}{y} + \frac{y}{x}$, it follows that

$$2x \leq x\left(\frac{x}{y} + \frac{y}{x}\right) = \frac{x^2}{y} + y.$$

Similarly,

$$2y \leq y\left(\frac{y}{z} + \frac{z}{y}\right) = \frac{y^2}{z} + z; \quad 2z \leq z\left(\frac{z}{x} + \frac{x}{z}\right) = \frac{z^2}{x} + x.$$

Adding these inequalities we get

$$\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq x+y+z.$$

9.14. Let O be the intersection point of the diagonals of quadrilateral $ABCD$. Then

$$AC + BD = (AO + OC) + (BO + OD) = (AO + OB) + (OC + OD) > AB + CD.$$

9.15. By the above problem $AB + CD < AC + BD$. Adding this inequality to the inequality $AB + BD \leq AC + CD$ we get $2AB < 2AC$.

9.16. First, let us prove that if P is the perimeter of convex quadrilateral $ABCD$ and d_1 and d_2 are the lengths of its diagonals, then $P > d_1 + d_2 > \frac{1}{2}P$. Clearly, $AC < AB + BC$ and $AC < AD + DC$; hence,

$$AC < \frac{AB + BC + CD + AD}{2} = \frac{P}{2}.$$

Similarly, $BD < \frac{1}{2}P$. Therefore, $AC + BD < P$. On the other hand, adding the inequalities

$$AB + CD < AC + BD \quad \text{and} \quad BC + AD < AC + BD$$

(cf. Problem 9.14) we get $P < 2(AC + BD)$.

Let P be the perimeter of the outer quadrilateral, P' the perimeter of the inner one. Then $d > \frac{1}{2}P$ and since $P' < P$ (by Problem 9.27 b)), we have $d' < P' < P < 2d$.

9.17. Let the broken line of the shortest length be a self-intersecting one. Let us consider two intersecting links. The vertices of these links can be connected in one of the following three ways: Fig. 95. Let us consider a new broken line all the links of which are the same as of the initial one except that the two solid intersecting links are replaced by the dotted links (see Fig. 95).

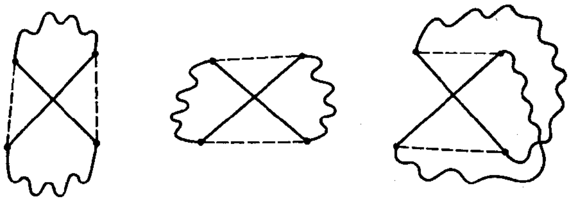


Figure 95 (Sol. 9.17)

Then we get again a broken line but its length is less than that of the initial one because the sum of the lengths of the opposite sides of a convex quadrilateral is less than the sum of the length of its diagonals. We have obtained a contradiction and, therefore, the closed broken line of the least length cannot have intersecting links.

9.18. Let us prove that the number of sides of such a polygon does not exceed 5. Suppose that all the diagonals of polygon $A_1 \dots A_n$ are of the same length and $n \geq 6$. Then segments A_1A_4, A_1A_5, A_2A_4 and A_2A_5 are of equal length since they are the diagonals of this polygon. But in convex quadrilateral $A_1A_2A_4A_5$ segments A_1A_5 and A_2A_4 are opposite sides whereas A_1A_4 and A_2A_5 are diagonals. Therefore, $A_1A_5 + A_2A_4 < A_1A_4 + A_2A_5$. Contradiction.

It is also clear that a regular pentagon and a square satisfy the required condition.

9.19. Consider all the partitions of the given points into pairs of points of distinct colours. There are finitely many such partitions and, therefore, there exists a partition for which the sum of lengths of segments given by pairs of points of the partition is the least one. Let us show that in this case these segments will not intersect. Indeed, if two segments would have intersected, then we could have selected a partition with the lesser sum of lengths of segments by replacing the diagonals of the convex quadrilateral by its opposite sides as shown on Fig. 96.

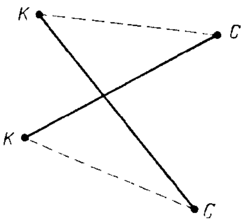


Figure 96 (Sol. 9.19)

9.20. Let A_pA_{p+1} and A_qA_{q+1} be nonadjacent sides of n-gon $A_1 \dots A_n$ (i.e., $|p-q| \geq 2$). Then

$$A_pA_{p+1} + A_qA_{q+1} < A_pA_q + A_{p+1}A_{q+1}.$$

Let us write all such inequalities and add them. For each side there exist precisely $n-3$ sides nonadjacent to it and, therefore, any side enters $n-3$ inequality, i.e., in the left-hand side of the obtained sum there stands $(n-3)p$, where p is the sum of lengths of the n-gon's sides. Diagonal A_mA_n enters two inequalities for $p = n$, $q = m$ and for $p = n-1$, $q = m-1$; hence, in the right-hand side stands $2d$, where d is the sum of lengths of diagonals. Thus, $(n-3)p < 2d$. Therefore, $\frac{p}{n} < \frac{d}{n(n-3)/2}$, as required.

9.21. Let us consider an arbitrary closed broken line with the vertices in vertices of the given polygon. If we have two nonintersecting links then by replacing these links by the diagonals of the quadrilateral determined by them we enlarge the sum of the lengths of the links. In this process, however, one broken line can get split into two nonintersecting ones. Let us prove that if the number of links is odd then after several such operations we will still get in the end a closed broken line (since the sum of lengths of the links increases each time, there can be only a finite number of such operations). One of the obtained closed broken lines should have an odd number of links but then any of the remaining links does not intersect at least one of the links of this broken line (cf. Problem 23.1 a)); therefore, in the end we get just one broken line.

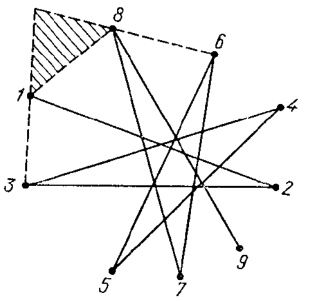


Figure 97 (Sol. 9.21)

Now, let us successively construct a broken line with pairwise intersecting links (Fig. 97). For instance, the 10-th vertex should lie inside the shaded triangle and therefore, the position of vertices is precisely as plotted on Fig. 97. Therefore, to convex polygon $A_1A_3A_5\dots A_{2n+1}A_2\dots A_{2n}$ the broken line $A_1A_2A_3\dots A_{2n+1}A_1$ corresponds.

9.22. Let the length of the third side be equal to n. From the triangle inequality we get $3.14 - 0.67 < n < 3.14 + 0.67$. Since n is an integer, $n = 3$.

9.23. Clearly, $AB + BC > AC$, $BC + CD > BD$, $CD + DE > CE$, $DE + EA > DA$, $EA + AB > EB$. Adding these inequalities we see that the sum of the lengths of the pentagon's diagonals is shorter than the doubled perimeter.

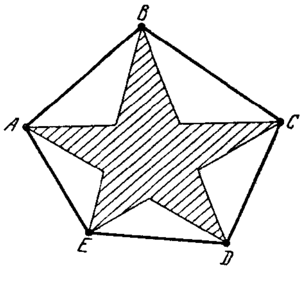


Figure 98 (Sol. 9.23)

The sum of the the diagonals' lengths is longer than the sum of lengths of the sides of the "rays of the star" and it, in turn, is greater than the perimeter of the pentagon (Fig. 98).

9.24. Suppose that c is the length of not the shortest side, for instance, $a \leq c$. Then $a^2 \leq c^2$ and $b^2 < (a+c)^2 \leq 4c^2$. Hence, $a^2 + b^2 < 5c^2$. Contradiction.

9.25. Since $c > |b-a|$ and $a = \frac{2S}{h_a}$, $c = \frac{2S}{h_c}$, it follows that $\frac{1}{h_c} > \left|\frac{1}{h_a} - \frac{1}{h_b}\right|$. Therefore, in our case $h_c < \frac{20\cdot 12}{8} = 30$.

9.26. On sides AB, BC, CA take points C_2, A_2, B_2, respectively, so that $A_1B_2 \parallel AB$, $B_1C_2 \parallel BC$, $CA_2 \parallel CA$ (Fig. 99). Then

$$A_1B_1 < A_1B_2 + B_2B_1 = (1-\lambda)AB + (2\lambda - 1)CA.$$

Similarly,

$$BC_1 < (1-\lambda)BC + (2\lambda-1)AB \text{ and } C_1A_1 < (1-\lambda)CA + (2\lambda-1)BC.$$

Adding these inequalities we get $P_1 < \lambda P$.

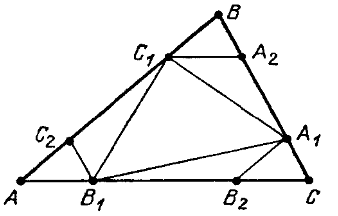


Figure 99 (Sol. 9.26)

Clearly, $A_1B_1 + AC > B_1C$, i.e.,

$$A_1B_1 + (1-\lambda)BC > \lambda \cdot CA.$$

Similarly,

$$B_1C_1 + (1-\lambda)CA > \lambda \cdot AB \quad \text{and} \quad C_1A_1 + (1-\lambda)AB > \lambda \cdot BC.$$

Adding these inequalities we get $P_1 > (2\lambda - 1)P$.

9.27. a) Passing from a nonconvex polygon to its convex hull we replace certain broken lines formed by sides with segments of straight lines (Fig. 100). It remains to take into account that any brokenline is longer than the line segment with the same endpoints.

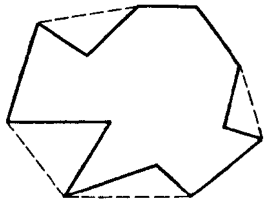


Figure 100 (Sol. 9.27 a))

b) On the sides of the inner polygon construct half bands directed outwards; let the parallel sides of half bands be perpendicular to the corresponding side of the polygon (Fig. 101).

Denote by P the part of the perimeter of the outer polygon corresponding to the boundary of the polygon contained inside these half bands. Then the perimeter of the inner polygon does not exceed P whereas the perimeter of the outer polygon is greater than P.

9.28. Since $AO + BO > AB$, $BO + OC > BC$ and $CO + OA > AC$, it follows that

$$AO + BO + CO > \frac{AB + BC + CA}{2}.$$

Since triangle ABC contains triangle ABO, it follows that $AB + BO + OA < AB + BC + CA$ (cf. Problem 9.27 b)), i.e., $BO + OA < BC + CA$. Similarly,

$$AO + OC < AB + BC \text{ and } CO + OB < CA + AB.$$

Adding these inequalities we get $AO + BO + CO < AB + BC + CA$.

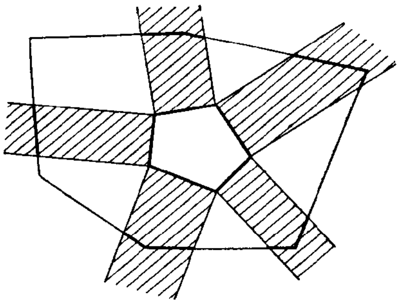


Figure 101 (Sol. 9.27 b))

9.29. It suffices to prove that $ABCE$ and $BCDE$ are parallelograms. Let us complement triangle ABE to parallelogram ABC_1E. Then perimeters of triangles BC_1E and ABE are equal and, therefore, perimeters of triangles BC_1E and BCE are equal. Hence, $C_1 = C$ because otherwise one of the triangles BC_1E and BCE would have lied inside the other one and their perimeters could not be equal. Hence, $ABCE$ is a parallelogram. We similarly prove that $BCDE$ is a parallelogram.

9.30. Clearly, $2 = 2S = ab\sin\gamma \leq ab \leq b^2$, i.e., $b \geq \sqrt{2}$.

9.31. Since EH is the midline of triangle ABD, it follows that $S_{AEH} = \frac{1}{4}S_{ABD}$. Similarly, $S_{CFG} = \frac{1}{4}S_{CBD}$. Therefore, $S_{AEH} + S_{CFG} = \frac{1}{4}S_{ABCD}$. Similarly, $S_{BFE} + S_{DGH} = \frac{1}{4}S_{ABCD}$. It follows that

$$S_{ABCD} = 2S_{EFGH} = EG \cdot HF \sin\alpha,$$

where α is the angle between lines EG and HF. Since $\sin\alpha \leq 1$, then $S_{ABCD} \leq EG \cdot HF$.

Adding equalities

$$\overrightarrow{EG} = \overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CG} \text{ and } \overrightarrow{EG} = \overrightarrow{EA} + \overrightarrow{AD} + \overrightarrow{DG}$$

we obtain

$$2\overrightarrow{EG} = (\overrightarrow{EB} + \overrightarrow{EA}) + (\overrightarrow{BC} + \overrightarrow{AD}) + (\overrightarrow{DG} + \overrightarrow{CG}) = \overrightarrow{BC} + \overrightarrow{AD}.$$

Therefore, $EG \leq \frac{1}{2}(BC + AD)$. Similarly, $HF \leq \frac{1}{2}(AB + CD)$. It follows that

$$S_{ABCD} \leq EG \cdot HF \leq \frac{(AB + CD)(BC + AD)}{4}.$$

9.32. By Problem 9.31 $S_{ABCD} \leq \frac{1}{4}(AB + CD)(BC + AD)$. Since $ab \leq \frac{1}{4}(a + b)^2$, it follows that $S_{ABCD} \leq \frac{1}{16}(AB + CD + AD + BC)^2 = 1$.

9.33. From points B and C drop perpendiculars BB_1 and CC_1 to line AM. Then

$$2S_{AMB} + 2S_{AMC} = AM \cdot BB_1 + AM \cdot CC_1 \leq AM \cdot BC$$

because $BB_1 + CC_1 \leq BC$. Similarly,

$$2S_{BMC} + 2S_{BMA} \leq BM \cdot AC \text{ and } 2S_{CMA} + 2S_{CMB} \leq CM \cdot AB.$$

Adding these inequalities we get the desired statement.

9.34. Let on sides A_1A_2, A_2A_3, ... , A_nA_1 points B_1, ... , B_n, respectively, be selected; let O be the center of the circle. Further, let

$$S_k = S_{OB_kA_{k+1}B_{k+1}} = \frac{OA_{k+1} \cdot B_kB_{k+1} \sin\varphi}{2},$$

where φ is the angle between OA_{k+1} and B_kB_{k+1}. Since $OA_{k+1} = R$ and $\sin\varphi \le 1$, it follows that $S_k \le \frac{1}{2}R \cdot B_kB_{k+1}$. Hence,

$$S = S_1 + \cdots + S_n \le \frac{R(B_1B_2 + \cdots + B_nB_1)}{2},$$

i.e., the perimeter of polygon $B_1B_2\dots B_n$ is not less than $\frac{2S}{R}$.

9.35. We have $2S_{AOB} \le AO \cdot OB \le \frac{1}{2}(AO^2 + BO^2)$, where the equality is only possible if $\angle AOB = 90^\circ$ and $AO = BO$. Similarly,

$$2S_{BOC} \le \frac{BO^2 + CO^2}{2},\quad 2S_{COD} \le \frac{CO^2 + DO^2}{2} \text{ and } 2S_{DOA} \le \frac{DO^2 + AO^2}{2}.$$

Adding these inequalities we get

$$2S = 2(S_{AOB} + S_{BOC} + S_{COD} + S_{DOA}) \le AO^2 + BO^2 + CO^2 + DO^2,$$

where the equality is only possible if $AO = BO = CO = DO$ and $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90^\circ$, i.e., $ABCD$ is a square and O is its center.

9.36. We have to prove that $\frac{S_{ABC}}{S_{AMN}} \ge 4$. Since $AB = AM + MB = AM + AN = AN + NC = AC$, it follows that

$$\frac{S_{ABC}}{S_{AMN}} = \frac{AB \cdot AC}{AM \cdot AN} = \frac{(AM + AN)^2}{AM \cdot AN} \ge 4.$$

9.37. Let us apply Heron's formula

$$S^2 = p(p-a)(p-b)(p-c).$$

Since $p-a = (p_1-a_1)+(p_2-a_2)$ and $(x+y)^2 \ge 4xy$, it follows that $(p-a)^2 \ge 4(p_1-a_1)(p_2-a_2)$. Similarly,

$$(p-b)^2 \ge 4(p_1-b_1)(p_2-b_2),\quad (p-c)^2 \ge 4(p_1-c_1)(p_2-c_2) \text{ and } p^2 \ge 4p_1p_2.$$

Multiplying these inequalities we get the desired statement.

9.38. For definiteness, we may assume that rays BA and CD, BC and AD intersect (Fig. 102). If we complement triangle ADC to parallelogram $ADCK$, then point K occurs inside quadrilateral $ABCD$. Therefore,

$$\begin{aligned}2S \ge 2S_{ABK} + 2S_{BCK} = AB \cdot AK \sin\alpha + BC \cdot CK \sin\beta = \\ AB \cdot CD \sin\alpha + BC \cdot AD \sin\beta.\end{aligned}$$

The equality is obtained if point D lies on segment AC.

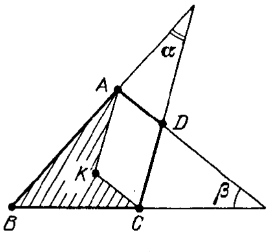


Figure 102 (Sol. 9.38)

Let point D' be symmetric to point D through the midperpendicular to segment AC. Then

$$2S = 2S_{ABCD'} = 2S_{ABD'} + 2S_{BCD'} \leq \\ AB \cdot AD' + BC \cdot CD' = AB \cdot CD + BC \cdot AD.$$

9.39. Thanks to the inequality between the mean geometric and the mean arithmetic, we have $\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} \geq 3\sqrt[3]{\frac{abc}{\alpha\beta\gamma}} = \frac{3}{2}$ because $\alpha = 2\sqrt{bc}$, $\beta = 2\sqrt{ca}$ and $\gamma = 2\sqrt{ab}$, cf. Problem 1.33.

9.40. The inequalities $\alpha < \alpha_1$, $\beta < \beta_1$ and $\gamma < \gamma_1$ cannot hold simultaneously. Therefore, for instance, $\alpha_1 \leq \alpha \leq 90^\circ$; hence, $\sin\alpha_1 \leq \sin\alpha$. It follows that $2S_1 = a_1b_1\sin\alpha_1 \leq k^2ab\sin\alpha = 2k^2S$.

9.41. a) Let chords AE and BD intersect diameter CM at points K and L, respectively. Then $AC^2 = CK \cdot CM$ and $BC^2 = CL \cdot CM$. It follows that $\frac{CK}{CL} = \frac{AC^2}{BC^2} < 4$. Moreover, $\frac{AE}{BD} = \frac{AE}{AC} < 2$. Therefore, $\frac{S_{ACE}}{S_{BCD}} = \frac{AE \cdot CK}{BD \cdot CL} < 8$.

b) Let H be the midpoint of segment BC. Since $\angle CBD = \angle BCD = \angle ABD$, it follows that D is the intersection point of the bisectors of triangle ABC. Hence, $\frac{AD}{DH} = \frac{AB}{BH} > 1$. Therefore, $S_{MAN} > \frac{1}{4}S_{ABC}$ and

$$S_{BCD} = \frac{BC \cdot DH}{2} < \frac{BC \cdot AH}{4} = \frac{S_{ABC}}{4}.$$

9.42. Let us cut off the obtained polygon rectangles with side h constructed outwards on the sides of the initial polygon (Fig. 103). Then beside the initial polygon there will be left several quadrilaterals from which one can compose a polygon circumscribed about a circle of radius h. The sum of the areas of these quadrilaterals is greater than the area of the circle of radius h, i.e., greater than πh^2. It is also clear that the sum of areas of the cut off rectangles is equal to Ph.

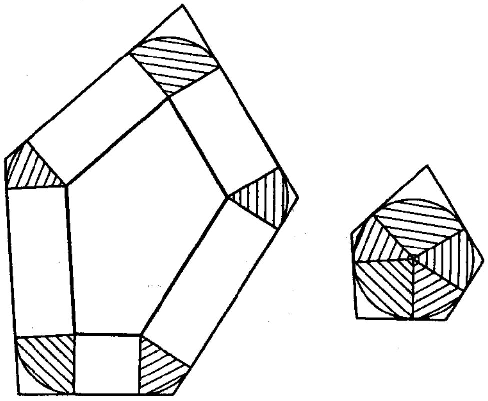


Figure 103 (Sol. 9.42)

9.43. Let $s, s_1, \dots, s_n$ be the areas of the square and the rectangles that constitute it, respectively; S, $S_1, \dots, S_n$ the areas of the disks circumscribed about the square and the rectangles, respectively. Let us prove that $s_k \leq \frac{2S_k}{\pi}$. Indeed, if the sides of the rectangle are equal to a and b, then $s_k = ab$ and $S_k = \pi R^2$, where $R^2 = \frac{a^2}{4} + \frac{b^2}{4}$. Therefore, $s_k = ab \leq \frac{a^2+b^2}{2} = \frac{2\pi R^2}{\pi} = \frac{2S_k}{\pi}$. It follows that

$$\frac{2S}{\pi} = s = s_1 + \cdots + s_n \leq \frac{2(S_1 + \cdots + S_n)}{\pi}.$$

9.44. Let, for definiteness, ABC be the triangle of the least area. Denote the intersection point of diagonals AD and EC by F. Then $S_{ABCDE} < S_{AED} + S_{EDC} + S_{ABCF}$. Since point F lies on segment EC and $S_{EAB} \geq S_{CAB}$, it follows that $S_{EAB} \geq S_{FAB}$. Similarly, $S_{DCB} \geq S_{FCB}$. Therefore, $S_{ABCF} = S_{FAB}+S_{FCB} \leq S_{EAB}+S_{DCB}$. It follows that $S_{ABCDE} < S_{AED} + S_{EDC} + S_{EAB} + S_{DCB}$ and this is even a stronger inequality than the one required.

9.45. a) Denote the intersection points of diagonals AD and CF, CF and BE, BE and AD by P, Q, R, respectively (Fig. 104). Quadrilaterals $ABCP$ and $CDEQ$ have no common inner points since sides CP and QC lie on line CF and segments AB and DE lie on distinct sides of it. Similarly, quadrilaterals $ABCP$, $CDEQ$ and $EFAR$ have no pairwise common inner points. Therefore, the sum of their areas does not exceed S.

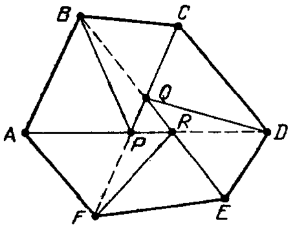


Figure 104 (Sol. 9.45 a))

It follows that the sum of the areas of triangles ABP, BCP, CDQ, DEQ, EFR, FAR does not exceed S, i.e., the area of one of them, say ABP, does not exceed $\frac{1}{6}S$. Point P lies on segment CF and, therefore, one of the points, C or F, is distant from line AB not further than point P. Therefore, either $S_{ABC} \leq S_{ABP} \leq \frac{1}{6}S$ or $S_{ABF} \leq S_{ABP} \leq \frac{1}{6}S$.

b) Let $ABCDEFGH$ be a convex octagon. First, let us prove that quadrilaterals $ABEF$, $BCFG$, $CDGH$ and $DEHA$ have a common point. Clearly, a convex quadrilateral $KLMN$ (Fig. 105) is the intersection of $ABEF$ and $CDGH$. Segments AF and HC lie inside angles $\angle DAH$ and $\angle AHE$, respectively; hence, point K lies inside quadrilateral $DEHA$. We similarly prove that point M lies inside quadrilateral $DEHA$, i.e., the whole segment KM lies inside it. Similarly, segment LN lies inside quadrilateral $BCFG$. The intersection point of diagonals KM and LN belongs to all our quadrilaterals; denote it by O.

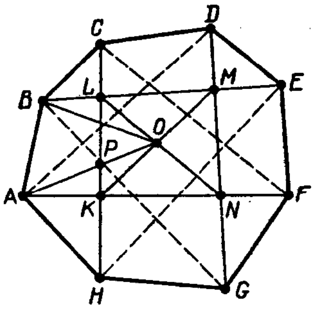


Figure 105 (Sol. 9.45 b))

Let us divide the 8-gon into triangles by connecting point O with the vertices. The area of one of these triangles, say ABO, does not exceed $\frac{1}{8}S$. Segment AO intersects side KL at a point P, therefore, $S_{ABP} < S_{ABO} \leq \frac{1}{8}S$. Since point P lies on diagonal CH, it follows that either $S_{ABC} \leq S_{ABP} \leq \frac{1}{8}S$ or $S_{ABH} \leq S_{ABP} \leq \frac{1}{8}S$.

9.46. Let us draw through all the vertices of the polygon lines parallel to one pair of sides of the square thus dividing the square into strips. Each such strip cuts off the polygon either a trapezoid or a triangle. It suffices to prove that the length of one of the bases of these trapezoids is greater than 0.5. Suppose that the length of each base of all the trapezoids does not exceed 0.5. Then the area of each trapezoid does not exceed a half height of the strip that confines it. Therefore, the area of the polygon, equal to the sum of areas of trapezoids and triangles into which it is cut, does not exceed a half sum of heights of the strips, i.e., does not exceed 0.5. Contradiction.

9.47. a) Let $P_1, \dots, P_n$ be the given points. Let us connect point P_1 with the vertices of the square. We will thus get four triangles. Next, for $k = 2, \dots, n$ let us perform the following operation. If point P_k lies strictly inside one of the triangles obtained earlier, then connect it with the vertices of this triangle.

If point P_k lies on the common side of two triangles, then connect it with the vertices of these triangles opposite to the common side. Each such operation increases the total number of triangles by 2. As a result we get $2(n+1)$ triangles. The sum of the areas of these triangles is equal to 1, therefore, the area of any of them does not exceed $\frac{1}{2(n+1)}$.

b) Let us consider the least convex polygon that contains the given points. Let is have k vertices. If $k = n$ then this k-gon can be divided into $n-2$ triangles by the diagonals that go out of one of its vertices. If $k < n$, then inside the k-gon there are $n-k$ points and it can be divided into triangles by the method indicated in heading a). We will thus get $k + 2(n-k-1) = 2n-k-2$ triangles. Since $k < n$, it follows that $2n-k-2 > n-2$.

The sum of the areas of the triangles of the partition is less than 1 and there are not less than $n-2$ of them; therefore, the area of at least one of them does not exceed $\frac{1}{n-2}$.

9.48. a) We may assume that the circumscribed n-gon $A_1 \dots A_n$ and the inscribed n-gon $B_1 \dots B_n$ are placed so that lines A_iB_i intersect at the center O of the given circle. Let C_i and D_i be the midpoints of sides A_iA_{i+1} and B_iB_{i+1}, respectively. Then

$$S_{OB_iC_i} = p \cdot OB_i \cdot OC_i, \quad S_{OB_iD_i} = p \cdot OB_i \cdot OD_i \text{ and } S_{OA_iC_i} = p \cdot OA_i \cdot OC_i,$$

where $p = \frac{1}{2}\sin\angle A_iOC_i$. Since $OA_i : OC_i = OB_i : OD_i$, it follows that $S^2_{OB_iC_i} = S_{OB_iD_i}S_{OA_iC_i}$. It remains to notice that the area of the part of the disk confined inside angle $\angle A_iOC_i$ is greater than $S_{OB_iC_i}$ and the areas of the parts of the inscribed and circumscribed n-gons confined inside this angle are equal to $S_{OB_iD_i}$ and $S_{OA_iC_i}$, respectively.

b) Let the radius of the circle be equal to R. Then $P_1 = 2nR\sin\frac{\pi}{n}$, $P_2 = 2nR\tan\frac{\pi}{n}$ and $L = 2\pi R$. We have to prove that $\sin x \tan x > x^2$ for $0 < x \le \frac{1}{3}\pi$. Since

$$\left(\frac{\sin x}{x}\right)^2 \ge \left(1 - \frac{x^2}{6}\right)^2 = 1 - \frac{x^2}{3} + \frac{x^4}{36}$$

and $0 < \cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{24}$ (see Supplement to this chapter), it remains to verify that $1 - \frac{x^2}{3} + \frac{x^4}{36} \ge 1 - \frac{x^2}{2} + \frac{x^4}{24}$, i.e., $12x^2 > x^4$. For $x \le \frac{1}{3}\pi$ this inequality is satisfied.

9.49. Let O be the center of homothety that sends the inscribed circle into the circumscribed one. Let us divide the plane by rays that exit from point O and pass through the vertices of the polygon and the tangent points of its sides with the inscribed circle (Fig. 106).

It suffices to prove the required inequality for the parts of disks and the polygon confined inside each of the angles formed by these rays. Let the legs of the angle intersect the inscribed circle at points P, Q and the circumscribed circle at points R, S so that P is the tangency point and S is a vertex of the polygon. The areas of the parts of disks are greater than the areas of triangles OPQ and ORS and, therefore, it suffices to prove that $2S_{OPS} \le S_{OPQ} + S_{ORS}$. Since $2S_{OPS} = 2S_{OPQ} + 2S_{PQS}$ and $S_{ORS} = S_{OPQ} + S_{PQS} + S_{PRS}$,

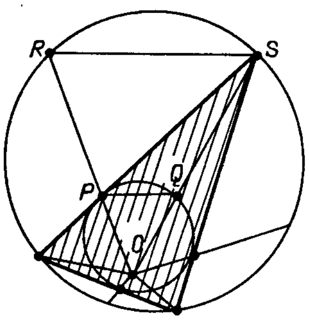


Figure 106 (Sol. 9.49)

it remains to prove that $S_{PQS} \leq S_{PRS}$. This inequality is obvious, because the heights of triangles PQS and PRS dropped to bases PQ and RS, respectively, are equal and $PQ < RS$.

9.50. It suffices to prove that both triangles contain the center O of the disk. Let us prove that if triangle ABC placed in the disk of radius 1 does not contain the center of the disk, then its area is less than 1. Indeed, for any point outside the triangle there exists a line that passes through two vertices and separating this point from the third vertex. Let, for definiteness, line AB separate points C and O. Then $h_c < 1$ and $AB < 2$, hence, $S = \frac{1}{2}h_c \cdot AB < 1$.

9.51. a) On the sides of the polygon, construct inwards rectangles whose other side is equal to $R = \frac{S}{P}$. The rectangles will not cover the whole polygon (these rectangles overlap and can stick out beyond the limits of the polygon whereas the sum of their areas is equal to the area of the polygon). An uncovered point is distant from every side of the polygon further than by R, consequently, the disk of radius R centered at this point entirely lies inside the polygon.

b) Heading a) implies that in the inner polygon a disk of radius $\frac{S_2}{P_2}$ can be placed. Clearly, this disk lies inside the outer polygon. It remains to prove that if inside a polygon a disk of radius R lies, then $R \leq \frac{2S}{P}$. For this let us connect (with lines) the center O of the disk with the vertices of the polygon. These lines split the polygon into triangles whose respective areas are equal to $\frac{1}{2}h_i a_i$, where h_i is the distance from point O to the i-th side and a_i is the length of the i-th side. Since $h_i \geq R$, we deduce that $2S = \sum h_i a_i \geq \sum R a_i = RP$.

9.52. First, let us consider the case when two sides of a parallelogram lie on lines AB and AC and the fourth vertex X lies on side BC. If $BX : CX = x : (1 - x)$, then the ratio of the area of the parallelogram to the area of the triangle is equal to $2x(1 - x) \leq \frac{1}{2}$.

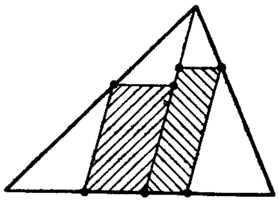


Figure 107 (Sol. 9.52)

In the general case let us draw parallel lines that contain a pair of sides of the given parallelogram (Fig. 107). The area of the given parallelogram does not exceed the sum of areas of the shaded parallelograms which fall in the case considered above. If lines that contain a pair of sides of the given parallelogram only intersect two sides of the triangle, then we can restrict ourselves to one shaded parallelogram only.

9.53. First, let us consider the following case: two vertices A and B of triangle ABC lie on one side PQ of the parallelogram. Then $AB \leq PQ$ and the height dropped to side AB is not longer than the height of the parallelogram. Therefore, the area of triangle ABC does not exceed a half area of the parallelogram.

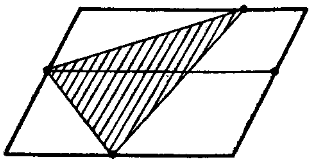


Figure 108 (Sol. 9.53)

If the vertices of the triangle lie on distinct sides of the parallelogram, then two of them lie on opposite sides. Let us draw through the third vertex of the triangle a line parallel to these sides (Fig. 108). This line cuts the parallelogram into two parallelograms and it cuts the triangle into two triangles so that two vertices of each of these triangles lie on sides of the parallelogram. We get the case already considered.

9.54. Let M be the midpoint of the longest side BC of the given acute triangle ABC. The circle of radius MA centered at M intersects rays MB and MC at points B_1 and C_1, respectively. Since $\angle BAC < 90^\circ$, it follows that $MB < MB_1$. Let, for definiteness, $\angle AMB \leq \angle AMC$, i.e., $\angle AMB \leq 90^\circ$. Then $AM^2 + MB^2 \leq AB^2 \leq BC^2 = 4MB^2$, i.e., $AM \leq \sqrt{3}BM$. If AH is a height of triangle ABC, then $AH \cdot BC = 2$ and, therefore,

$$S_{AB_1C_1} = \frac{B_1C_1 \cdot AH}{2} = AM \cdot AH \leq \sqrt{3}BM \cdot AH = \sqrt{3}.$$

9.55. a) Let AB be the longest of the diagonals and sides of the given polygon M. Polygon M is confined inside the strip formed by the perpendiculars to segment AB passing through points A and B. Let us draw two baselines to M parallel to AB. Let them intersect polygon M at points C and D. As a result we have confined M into a rectangle whose area is equal to $2S_{ABC} + 2S_{ABD} \leq 2S$.

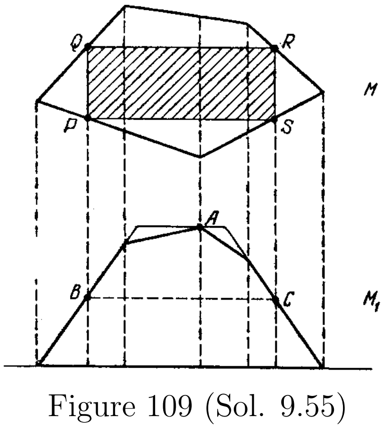


Figure 109 (Sol. 9.55)

b) Let M be the initial polygon, l an arbitrary line. Let us consider the polygon M_1 one of whose sides is the projection of M to l and the lengths of the sections of polygons M and M_1 by any line perpendicular to l are equal (Fig. 109). It is easy to verify that M_1 is also a convex polygon and its area is equal to S. Let A be the most distant from l point of M_1. The line equidistant from point A and line l intersects the sides of M_1 at points B and C.

Let us draw base lines through points B and C. As a result we will circumscribe a trapezoid about M_1 (through point A a base line can also be drawn); the area of this trapezoid is no less than S. If the height of the trapezoid, i.e., the distance from A to l is equal to h then its area is equal to $h \cdot BC$ and, therefore, $h \cdot BC \geq S$. Let us consider sections PQ and RS of polygon M by lines perpendicular to l and passing through B and C. The lengths of these sections are equal to $\frac{1}{2}h$ and, therefore, $PQRS$ is a parallelogram whose area is equal to $\frac{1}{2}BC \cdot h \geq \frac{1}{2}S$.

9.56. a) Let us confine the polygon in the strip formed by parallel lines. Let us shift these lines parallelly until some vertices A and B of the polygon lie on them. Then let us perform the same for the strip formed by lines parallel to AB. Let the vertices that lie on these new lines be C and D (Fig. 110). The initial polygon is confined in a parallelogram and, therefore, the area of this parallelogram is not less than 1. On the other hand, the sum of areas of triangles ABC and ADB is equal to a half area of the parallelogram and, therefore, the area of one of these triangles is not less than $\frac{1}{4}$.

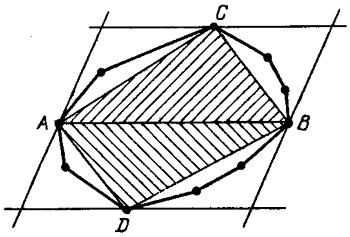


Figure 110 (Sol. 9.56 a))

b) As in heading a) let us confine the polygon in a strip formed by parallel lines so that some vertices, A and B, lie on these lines. Let d be the width of this strip. Let us draw three lines that divide this strip into equal strips of width $\frac{1}{4}d$. Let the first and the third lines intersect sides of the polygon at points K, L and M, N, respectively (Fig. 111).

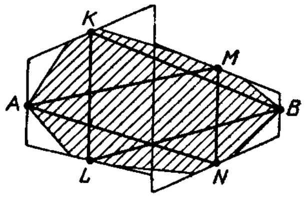


Figure 111 (Sol. 9.56 b))

Let us extend the sides on which points K, L, M and N lie to the intersection with the sides of the initial strip and with the line that divides it in halves. In this way we form two trapezoids with the midlines KL and MN and heights of length $\frac{1}{2}d$ each.

Since these trapezoids cover the whole polygon, the sum of their areas is not less than its area, i.e., $\frac{1}{2}(d \cdot KL + d \cdot MN) \geq 1$. The sum of areas of triangles AMN and BKL contained in the initial polygon is equal to $\frac{1}{8}(3d \cdot MN + 3d \cdot KL) \geq \frac{3}{4}$. Therefore, the area of one of these triangles is not less than $\frac{3}{8}$.

9.57. Let us prove that there exists even three last vertices satisfying the required condition. Let α_i be the angle between the i-th and $(i+1)$-th sides $\beta_i = \pi - \alpha_i$; let a_i be the length of the i-th side.

a) The area of the triangle formed by the i-th and $(i+1)$-th sides is equal to $S_i = \frac{a_i a_{i+1} \sin \alpha_i}{2}$. Let S be the least of these areas. Then $2S \leq a_i a_{i+1} \sin \alpha_i$; hence,

$$(2S)^n \leq (a_1^2 \dots a_n^2)(\sin \alpha_1 \dots \sin \alpha_n) \leq a_1^2 \dots a_n^2.$$

By the inequality between the mean arithmetic and the mean geometric we have $(a_1 \dots a_n)^{\frac{1}{n}} \leq \frac{a_1 + \dots + a_n}{n}$ and, therefore,

$$2S \leq (a_1 \dots a_n)^{\frac{2}{n}} \leq \frac{(a_1 + \dots + a_n)^2}{n^2}.$$

Since $a_i \leq p_i + q_i$, where p_i and q_i are projections of the i-th side to a vertical and a horizontal sides of the square, it follows that

$$a_1 + \dots + a_n \leq (p_1 + \dots + p_n) + (q_1 + \dots + q_n) \leq 4.$$

Hence, $2S \leq 16n^2$, i.e., $S \leq \frac{8}{n^2}$.

b) Let us make use of the inequality

$$2S \leq (a_1 \dots a_n)^{\frac{2}{n}} (\sin \alpha_1 \dots \sin \alpha_n)^{\frac{1}{n}} \leq \frac{16}{n^2} (\sin \alpha_1 \dots \sin \alpha_n)$$

proved above. Since $\sin \alpha_i = \sin \beta_i$ and $\beta_1 + \dots + \beta_n = 2\pi$, it follows that

$$(\sin \alpha_1 \dots \sin \alpha_n)^{\frac{1}{n}} = (\sin \beta_1 \dots \sin \beta_n)^{\frac{1}{n}} \leq \frac{\beta_1 + \dots + \beta_n}{n} = \frac{2\pi}{n}.$$

Hence, $2S \leq \frac{32\pi}{n^3}$, i.e., $S \leq \frac{16\pi}{n^3}$.

9.58. Let l_i be the length of the i-th link of the broken line; a_i and b_i the lengths of its projections to the sides of the square. Then $l_i \leq a_i + b_i$. It follows that

$$1000 = l_1 + \dots + l_n \leq (a_1 + \dots + a_n) + (b_1 + \dots + b_n),$$

i.e., either $a_1 + \dots + a_n \geq 500$ or $b_1 + \dots + b_n \geq 500$. If the sum of the lengths of the links' projections on a side of length 1 is not less than 500, then not fewer than 500 distinct lengths of the broken line are projected into one of the points of this side, i.e., the perpendicular to the side that passes through this point intersects the broken line at least at 500 points.

9.59. The locus of points distant from the given segment not further than by ε is depicted on Fig. 112. The area of this figure is equal to $\pi\varepsilon^2 + 2\varepsilon l$, where l is the length of the segment.

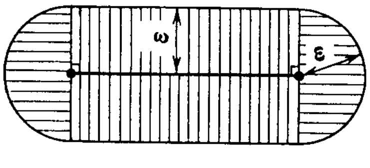


Figure 112 (Sol. 9.59)

Let us construct such figures for all N links of the given broken lines. Since neighbouring figures have $N-1$ common disks of radius ε centered at vertices of the broken line which are not its endpoints, it follows that the area covered by these figures does not exceed

$$N\pi\varepsilon^2 + 2\varepsilon(l_1 + \dots + l_n) - (N-1)\pi\varepsilon^2 = 2\varepsilon L + \pi\varepsilon^2.$$

This figure covers the whole square since any point of the square is distant from a point of the broken line by less than ε. Hence, $1 \leq 2\varepsilon L + \pi\varepsilon^2$, i.e., $L \geq \frac{1}{2\varepsilon} - \frac{\pi\varepsilon}{2}$.

9.60. Let us divide the square into n vertical strips that contain n points each. Inside each strip let us connect points downwards thus getting n broken lines. These broken lines can be connected into one broken line in two ways: Fig. 113 a) and b).

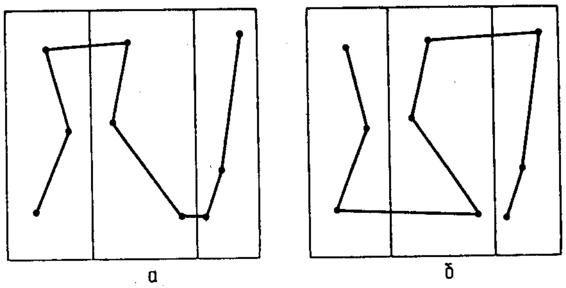


Figure 113 (Sol. 9.60)

Let us consider the segments that connect distinct bands. The union of all such segments obtained in both ways is a pair of broken lines such that the sum of the lengths of the horizontal projections of each of them does not exceed 1. Therefore, the sum of the lengths of horizontal projections of the connecting segments for one of these ways does not exceed 1.

Let us consider such a connection. The sum of the lengths of the horizontal projections for connecting links does not exceed 1 and for all the other links it does not exceed $(n-1)(h_1 + \dots + h_n)$, where h_i is the width of the i-th strip. Clearly, $h_1 + \dots + h_n = 1$. The sum of the vertical projections of all links of the broken line does not exceed n. As a result we deduce that the sum of the vertical and horizontal projections of all the links does not exceed $1 + (n-1) + n = 2n$ and, therefore, the length of the broken line does not exceed $2n$.

9.61. Let M and N be the endpoints of the broken line. Let us traverse along the broken line from M to N. Let A_1 be the first of points of the broken line that we meet whose distance from a vertex of the square is equal to 0.5. Let us consider the vertices of the square neighboring to this vertex. Let B_1 be the first after A_1 point of the broken line distant from one of these vertices by 0.5. Denote the vertices of the square nearest to points A_1 and B_1 by A and B, respectively (Fig. 114).

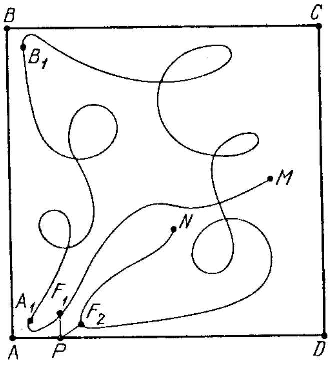


Figure 114 (Sol. 9.61)

Denote the part of the broken line from M to A_1 by L_1 and the part from A_1 to N by L_2. Let X and Y be the sets of points that lie on AD and distant not further than by 0.5 from L_1 and L_2, respectively. By hypothesis, X and Y cover the whole side AD. Clearly, $A \in X$ and $D \notin X$; hence, $D \in Y$, i.e., both sets, X and Y, are nonempty. But each of these sets consists of several segments and, therefore, they should have a common point P. Therefore, on L_1 and L_2, there are points F_1 and F_2 for which $PF_1 \le 0.5$ and $PF_2 \le 0.5$.

Let us prove that F_1 and F_2 are the points to be found. Indeed, $F_1F_2 \le F_1P + PF_2 \le 1$. On the other hand, while traversing from F_1 to F_2 we should pass through point B; and we have $F_1B_1 \geq 99$ and $F_2B_1 \geq 99$ because point B_1 is distant from side BC no further than by 0.5 while F_1 and F_2 are distant from side AD not further than by 0.5.

9.62. Let $\angle A = \angle B$. It suffices to prove that if $AD < BC$; then $\angle D > \angle C$. On side BC, take point D_1 such that $BD_1 = AD$. Then ABD_1D is an isosceles trapezoid. Hence, $\angle D > \angle D_1DA = \angle DD_1B \geq \angle C$.

9.63. Let B_1 and C_1 be the projections of points B and C on base AD. Since $\angle BAB_1 < \angle CDC_1$ and $BB_1 = CC_1$, it follows that $AB_1 > DC_1$ and, therefore, $B_1D < AC_1$. It follows that

$$BD^2 = B_1D^2 + B_1B^2 < AC_1^2 + CC_1^2 = AC^2.$$

9.64. Let angles $\angle B$ and $\angle D$ of quadrilateral $ABCD$ be obtuse ones. Then points B and D lie inside the circle with diameter AC. Since the distance between any two points that lie inside the circle is less than its diameter, $BD < AC$.

9.65. In an isosceles trapezoid $ABCD$ diagonals AC and BD are equal. Therefore,

$$BM + (AM + CM) \geq BM + AC = BM + BD \geq DM.$$

9.66. Let O be the midpoint of segment BD. Point A lies inside the circle with diameter BD, hence, $OA < \frac{1}{2}BD$. Moreover, $FO = \frac{1}{2}CD$. Therefore, $2FA \leq 2FO + 2OA < CD + BD$.

9.67. On rays AB, AC and AD mark segments AB', AC' and AD' of length $\frac{1}{AB}$, $\frac{1}{AC}$ and $\frac{1}{AD}$. Then $AB : AC = AC' : AB'$, i.e., $\triangle ABC \sim \triangle AC'B'$. The similarity coefficient of these triangles is equal to $\frac{1}{AB \cdot AC}$ and therefore, $B'C' = \frac{BC}{AB \cdot AC}$. Analogously, $C'D' = \frac{CD}{AC \cdot AD}$ and $B'D' = \frac{BD}{AB \cdot AD}$. Substituting these expressions in the inequality $B'D' \leq B'C' + C'D'$ and multiplying both sides by $AB \cdot AC \cdot AD$, we get the desired statement.

9.68. Clearly,

$$S_{ABCD} = S_{ABC} + S_{ACD} = 2S_{AMC} + 2S_{ANC} = 2(S_{AMN} + S_{CMN}).$$

If segment AM intersects diagonal BD at point A_1, then $S_{CMN} = S_{A_1MN} < S_{AMN}$. Therefore, $S_{ABCD} < 4S_{AMN}$.

9.69. Diagonals AC and BD intersect at point O. Let, for definiteness, point P lie in side of AOB. Then $AP + BP \leq AO + BO < AC + BD$ (cf. the solution of Problem 9.28) and $CP + DP < CB + BA + AD$.

9.70. Let r_i, S_i and p_i be the radii of the inscribed circles, the areas and semiperimeters of the obtained triangles, respectively. Then

$$Q \geq 2\sum r_i = 2\sum\left(\frac{S_i}{p_i}\right) > 4\sum\left(\frac{S_i}{P}\right) = \frac{4S}{P}.$$

9.71. Let $AC \leq BD$. Let us drop from vertices A and C perpendiculars AA_1 and CC_1 to diagonal BD. Then $AA_1 + CC_1 \leq AC \leq BD$ and, therefore, either $AA_1 \leq \frac{1}{2}BD$ or $CC_1 \leq \frac{1}{2}BD$.

9.72. Let us draw through the endpoints of segment KL lines perpendicular to it and consider projections to these lines of the vertices of the quadrilateral. Consider also the intersection points of lines AC and BD with these lines, cf. Fig. 115.

Let, for definiteness, point A lie inside the strip determined by these lines and point B outside it. Then we may assume that D lies inside the strip, because otherwise $BD > KL$ and the proof is completed. Since

$$\frac{AA'}{BB'} \leq \frac{A_1K}{B_1K} = \frac{C_1L}{D_1L} \leq \frac{CC'}{DD'},$$

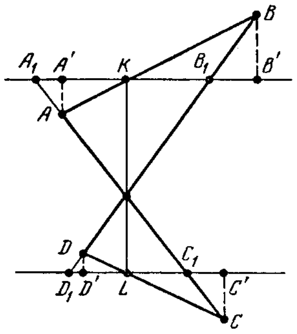


Figure 115 (Sol. 9.72)

then either $AA' \leq CC'$ (and, therefore, $AC > KL$) or $BB' \geq DD'$ (and, therefore, $BD > KL$).

9.73. Let us introduce the notations as plotted on Fig. 116. All the parallelograms considered have a common center (thanks to Problem 1.7). The lengths of the sides of parallelogram P_3 are equal to $a + a_1$ and $b + b_1$ and the lengths of the sides of parallelogram P_1 are equal to $a + a_1 + 2x$ and $b + b_1 + 2y$, consequently, we have to verify that either $a + a_1 + 2x \leq 2(a + a_1)$ or $b + b_1 + 2y \leq 2(b + b_1)$, i.e., either $2x \leq a + a_1$ or $2y \leq b + b_1$.

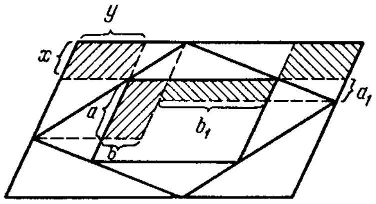


Figure 116 (Sol. 9.73)

Suppose that $a + a_1 < 2x$ and $b + b_1 < 2y$. Then $\sqrt{aa_1} \leq \frac{1}{2}(a + a_1) < x$ and $\sqrt{bb_1} < y$. On the other hand, the equality of the areas of shaded parallelograms (cf. Problem 4.19) shows that $ab = xy = a_1b$ and, therefore, $\sqrt{aa_1}\sqrt{bb_1} = xy$. Contradiction.

9.74. Let the angles of the pentagon be equal to α, $\alpha + \gamma$, $\alpha + 2\gamma$, $\alpha + 3\gamma$, $\alpha + 4\gamma$, where α, $\gamma \geq 0$. Since the sum of the angles of the pentagon is equal to 3π, it follows that $5\alpha + 10\gamma = 3\pi$. Since the pentagon is a convex one, each of its angles is less than π, i.e., either $\alpha + 4\gamma < \pi$ or $-5\frac{1}{2}\alpha - 10\gamma > -\frac{1}{2}5\pi$. Taking the sum of the latter inequality with $5\alpha + 10\gamma = 3\pi$ we get $\frac{5\alpha}{2} > \frac{\pi}{2}$, i.e., $\alpha > \frac{\pi}{5} = 36^\circ$.

9.75. Clearly,

$$4 = AE^2 = |\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE}|^2 =$$
$$|\overrightarrow{AB} + \overrightarrow{BC}|^2 + 2(\overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{CD} + \overrightarrow{DE}) + |\overrightarrow{CD} + \overrightarrow{DE}|^2.$$

Since $\angle ACE = 90^\circ$, we have

$$(\overrightarrow{AB} + \overrightarrow{BC}, \overrightarrow{CD} + \overrightarrow{DE}) = (\overrightarrow{AC}, \overrightarrow{CE}) = 0.$$

Hence,

$$4 = |\overrightarrow{AB} + \overrightarrow{BC}|^2 + |\overrightarrow{CD} + \overrightarrow{DE}|^2 =$$
$$AB^2 + BC^2 + CD^2 + DE^2 + 2(\overrightarrow{AB}, \overrightarrow{BC}) + 2(\overrightarrow{CD}, \overrightarrow{DE}),$$

i.e., it suffices to prove that

$$abc < 2(\overrightarrow{AB}, \overrightarrow{BC}) \text{ and } bcd < 2(\overrightarrow{CD}, \overrightarrow{DE}).$$

Since

$$2(\overrightarrow{AB},\overrightarrow{BC}) = 2ab\cos(180^\circ - \angle ABC) = 2ab\cos AEC = ab\cdot CE \text{ and } c < CE,$$

it follows that $abc < 2(\overrightarrow{AB},\overrightarrow{BC})$.

The second inequality is similarly proved, because in notations $A_1 = E$, $B_1 = D$, $C_1 = C$, $a_1 = d$, $b_1 = c$, $c_1 = b$ the inequality $bcd < 2(\overrightarrow{CD},\overrightarrow{DE})$ takes the form $a_1b_1c_1 < 2(\overrightarrow{A_1B_1},\overrightarrow{B_1C_1})$.

9.76. Let B be the midpoint of side A_1A_2 of the given hexagon $A_1\dots A_6$ and O its center. We may assume that point P lies inside triangle A_1OB. Then $PA_3 \geq 1$ because the distance from point A_3 to line BO is equal to 1; since the distances from points A_4 and A_5 to line A_3A_6 are equal to 1, we deduce that $PA_4 \geq 1$ and $PA_5 \geq 1$.

9.77. Suppose that the radii of the circumscribed circles of triangles ACE and BDF are greater than 1. Let O be the center of the circumscribed circle of triangle ACE. Then $\angle ABC > \angle AOC$, $\angle CDE > \angle COE$ and $\angle EFA > \angle EOA$ and, therefore, $\angle B + \angle D + \angle F > 2\pi$. Similarly, $\angle A + \angle C + \angle E > 2\pi$, i.e., the sum of the angles of hexagon $ABCDEF$ is greater than 4π. Contradiction.

Remark. We can similarly prove that the radius of the circumscribed circle of one of triangles ACE and BDF is not less than 1.

9.78. We may assume that $AE \leq AC \leq CE$. By Problem 9.67

$$AD\cdot CE \leq AE\cdot CD + AC\cdot DE < AE + AC \leq 2CE,$$

i.e., $AD < 2$.

9.79. Since $\angle A_1 = 180^\circ - \frac{1}{2}\smile A_2A_7$, $\angle A_3 = 180^\circ - \frac{1}{2}\smile A_4A_2$ and $\angle A_5 = 180^\circ - \frac{1}{2}\smile A_6A_4$, it follows that

$$\begin{gathered}\angle A_1 + \angle A_3 + \angle A_5 = 2\cdot 180^\circ + \tfrac{360^\circ - \smile A_2A_7 - \smile A_4A_2 - \smile A_6A_4}{2} = \\ 2\cdot 180^\circ + \tfrac{\smile A_7A_6}{2}.\end{gathered}$$

Since the center of the circle lies inside the hexagon, it follows that $\smile A_7A_6 < 180^\circ$ and, therefore, $\angle A_1 + \angle A_3 + \angle A_5 < 360^\circ + 90^\circ = 450^\circ$.

9.80. a) We have to prove that if c is the hypothenuse of the right triangle and a and b are its legs, then $c \geq \frac{a+b}{\sqrt{2}}$, i.e., $(a+b)^2 \leq 2(a^2+b^2)$. Clearly,

$$(a+b)^2 = (a^2+b^2) + 2ab \leq (a^2+b^2) + (a^2+b^2) = 2(a^2+b^2).$$

b) Let d_i be the length of the i-th side of the polygon; x_i and y_i the lengths of its projections to coordinate axes. Then $x_1 + \dots + x_n \geq 2a$, $y_1 + \dots + y_n \geq 2b$. By heading a) $d_i \geq \frac{x_i+y_i}{\sqrt{2}}$. Therefore,

$$d_1 + \dots + d_n \geq \frac{x_1 + \dots + x_n + y_1 + \dots + y_n}{\sqrt{2}} \geq \sqrt{2}(a+b).$$

9.81. Let us take a segment of length P and place the sides of the polygon on the segment as follows: on one end of the segment place the greatest side, on the other end place the second long side; place all the other sides between them. Since any side of the polygon is shorter than $\frac{1}{2}P$, the midpoint O of the segment cannot lie on these two longest sides. The length of the side on which point O lies, does not exceed $\frac{1}{3}P$ (otherwise the first two sides would also have been longer than $\frac{1}{3}P$ and the sum of the three sides would have been greater than P) and, therefore, one of its vertices is distant from O not further than by $\frac{1}{6}P$. This vertex divides the segment into two segments to be found since the difference of their lengths does not exceed $\frac{2}{6}P = \frac{1}{3}P$.

9.82. Let $\beta_k = \angle OA_kA_{k+1}$. Then $x_k \sin\beta_k = d_k = x_{k+1}\sin(\alpha_{k+1} - \beta_{k+1})$. Hence,

$$\begin{gathered} 2\sum d_k = \sum x_k(\sin(\alpha_k - \beta_k) + \sin\beta_k) = \\ 2\sum x_k \sin\tfrac{\alpha_k}{2}\cos\left(\tfrac{\alpha_k}{2} - \beta_k\right) \le 2\sum x_k \sin\tfrac{\alpha_k}{2}. \end{gathered}$$

It is also clear that

$$A_kA_{k+1} = x_k\cos\beta_k + x_{k+1}\cos(\alpha_{k+1} - \beta_{k+1}).$$

Therefore,

$$\begin{gathered} 2p = \sum A_kA_{k+1} = \sum x_k(\cos(\alpha_k - \beta_k) + \cos\beta_k) = \\ 2\sum x_k \cos\tfrac{\alpha_k}{2}\cos\left(\tfrac{\alpha_k}{2} - \beta_k\right) \le 2\sum x_k \cos\tfrac{\alpha_k}{2}. \end{gathered}$$

In both cases the equality is only attained if $\alpha_k = 2\beta_k$, i.e., O is the center of the inscribed circle.

9.83. Suppose that the center O of polygon M_2 lies outside polygon M_1. Then there exists a side AB of polygon M_1 such that polygon M_1 and point O lie on distinct sides of line AB. Let CD be a side of M_1 parallel to AB. The distance between lines AB and CD is equal to the radius of the inscribed circle S of polygon M_2 and, therefore, line CD lies outside S. On the other hand, segment CD lies inside M_2. Therefore, segment CD is shorter than a half side of polygon M_2, cf. Problem 10.66. Contradiction.

9.84. Let A_1 be the nearest to O vertex of the polygon. Let us divide the polygon into triangles by the diagonals that pass through vertex A_1. Point O lies inside one of these triangles, say, in triangle $A_1A_kA_{k+1}$. If point O lies on side A_1A_k, then $\angle A_1OA_k = \pi$ and the problem is solved.

Therefore, let us assume that point O lies strictly inside triangle $A_1A_kA_{k+1}$. Since $A_1O \le A_kO$ and $A_1O \le A_{k+1}O$, it follows that $\angle A_1A_kO \le \angle A_kA_1O$ and $\angle A_1A_{k+1}O \le \angle A_{k+1}A_1O$. Hence,

$$\begin{gathered} \angle A_kOA_1 + \angle A_{k+1}OA_1 = \\ (\pi - \angle OA_1A_k - \angle OA_kA_1) + (\pi - \angle OA_1A_{k+1} - \angle OA_{k+1}A_1) \ge \\ 2\pi - 2\angle OA_1A_k - 2\angle OA_1A_{k+1} = 2\pi - 2\angle A_kA_1A_{k+1} = 2\pi - \tfrac{2\pi}{n}, \end{gathered}$$

i.e., one of the angles $\angle A_kOA_1$ and $\angle A_{k+1}OA_1$ is not less than $\pi\left(1 - \frac{1}{n}\right)$.

9.85. Let d be the length of the longest diagonal (or side) AB of the given n-gon. Then the perimeter of the n-gon does not exceed πd (Problem 13.42). Let A'_i be the projection of A_i to segment AB. Then either $\sum AA'_i \ge \frac{1}{2}nd$ or $\sum BA'_i \ge \frac{1}{2}nd$ (Problem 9.87); let, for definiteness, the first inequality hold. Then $\sum AA_i > \sum AA'_i \ge \frac{1}{2}nd > \pi d \ge P$ because $\frac{1}{2}n \ge 3.5 > \pi$. Any point of the n-gon sufficiently close to vertex A possesses the required property.

9.86. a) First, suppose that $\angle A_i > \angle B_i$ and for all the other considered pairs of angles an equality takes place. Let us arrange polygons so that vertices $A_1, \dots, A_i$ coincide with $B_1, \dots, B_i$. In triangles $A_1A_iA_n$ and $A_1A_iB_n$ sides A_iA_n and A_iB_n are equal and $\angle A_1A_iA_n > \angle A_1A_iB_n$; hence, $A_1A_n > A_1B_n$.

If several angles are distinct, then polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ can be included in a chain of polygons whose successive terms are such as in the example considered above.

b) As we completely traverse the polygon we encounter the changes of minus sign by plus sign as often as the opposite change. Therefore, the number of pairs of neighbouring vertices with equal signs is an even one. It remains to verify that the number of sign changes cannot be equal to 2 (the number of sign changes is not equal to zero because the sums of the angles of each polygon are equal).

Suppose the number of sign changes is equal to 2. Let P and Q, as well as P' and Q' be the midpoints of sides of polygons $A_1 \dots A_n$ and $B_1 \dots B_n$ on which a change of sign occurs. We can apply the statement of heading a) to pairs of polygons M_1 and M'_1, M_2 and M'_2

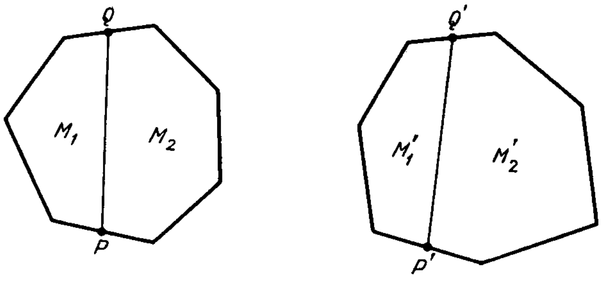


Figure 117 (Sol. 9.86)

(Fig. 117); we get $PQ > P'Q'$ in the one case, and $PQ < P'Q'$ in the other one, which is impossible.

9.87. Let A and B be the midpoints of the segment; $X_1, \dots, X_n$ the given points. Since $AX_i + BX_i = 1$, it follows that $\sum AX_i + \sum BX_i = n$. Therefore, either $\sum AX_i \geq \frac{1}{2}n$ or $\sum BX_i \geq \frac{1}{2}n$.

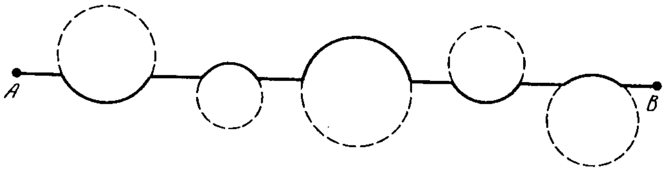


Figure 118 (Sol. 9.88)

9.88. Let us draw a wire along segment AB circumventing the encountered trees along the shortest arc as on Fig. 118. It suffices to prove that the way along an arc of the circle is not more than 1.6 times longer than the way along the line. The ratio of the length of an arc with the angle value 2φ to the chord it subtends is equal to $\frac{\varphi}{\sin\varphi}$. Since $0 < \varphi \leq \frac{\pi}{2}$, it follows that $\frac{\varphi}{\sin\varphi} \leq \frac{\pi}{2} < 1.6$.

9.89. Let the trees of height $a_1 > a_2 > \dots > a_n$ grow at points $A_1, \dots, A_n$. Then by the hypothesis

$$A_1A_2 \leq |a_1 - a_2| = a_1 - a_2, \dots, A_{n-1}A_n \leq a_{n-1} - a_n.$$

It follows that the length of the broken line $A_1A_2 \dots A_n$ does not exceed

$$(a_1 - a_2) + (a_2 - a_3) + \dots + (a_{n-1} - a_n) = a_1 - a_n < 100 \text{ m}.$$

This broken line can be fenced by a fence, whose length does not exceed 200 m (Fig. 119).

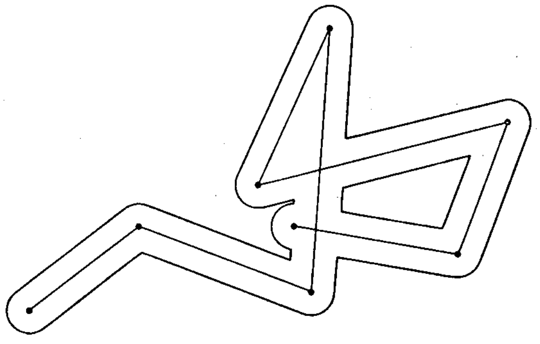


Figure 119 (Sol. 9.89)

9.90. In the obtained pentagon, distinguish the parts that were glued (on Fig. 120 these parts are shaded). All the sides that do not belong to the shaded polygons enter the perimeters of the initial and the obtained polygons. The sides of the shaded polygons that lie on the line along which the folding was performed enter the perimeter of the obtained polygon whereas all the other sides enter the perimeter of the initial polygon.

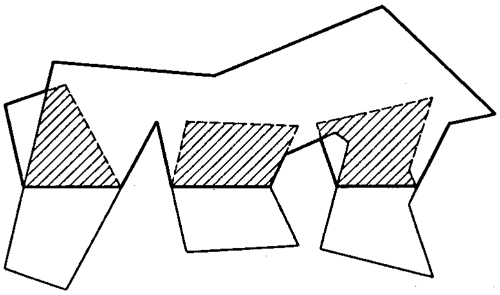


Figure 120 (Sol. 9.90)

Since for any polygon the sum of its sides that lie on a line is less than the sum of the other sides, the perimeter of the initial polygon is always longer than the perimeter of the obtained one.

9.91. On the broken line, take two points A and B, that divide its perimeter in halves. Then $AB \leq \frac{1}{2}$. Let us prove that all the points of the broken line lie inside the circle of radius $\frac{1}{4}$ centered at the midpoint O of segment AB. Let M be an arbitrary point of the broken line and point M_1 be symmetric to M through point O. Then

$$MO = \frac{M_1M}{2} \leq \frac{M_1A + AM}{2} = \frac{BM + AM}{2} \leq \frac{1}{4}$$

because $BM + AM$ does not exceed a half length of the broken line.

9.92. Let acute triangle ABC be placed inside circle S. Let us construct the circumscribed circle S_1 of triangle ABC. Since triangle ABC is an acute one, the angle value of the arc of circle S_1 that lies inside S is greater than 180°. Therefore, on this arc we can select diametrically opposite points, i.e., inside circle S a diameter of circle S_1 is contained. It follows that the radius of S is not shorter than the radius of S_1.

A similar statement for an acute triangle is false. An acute triangle lies inside a circle constructed on the longest side a as on diameter. The radius of this circle is equal to $\frac{1}{2}a$ and the radius of the circle circumscribed about the triangle is equal to $\frac{a}{2\sin\alpha}$. Clearly, $\frac{1}{2}a < \frac{a}{2\sin\alpha}$.

9.93. First solution. Any triangle of perimeter P can be placed in a disk of radius $\frac{1}{4}P$ and if an acute triangle is placed in a disk of radius R_1, then $R_1 \geq R$ (Problem 9.92). Hence, $\frac{1}{4}P = R_1 \geq R$.

Second solution. If $0 < x < \frac{\pi}{2}$, then $\sin x > \frac{2x}{\pi}$. Hence,

$$a + b + c = 2R(\sin\alpha + \sin\beta + \sin\gamma) > \frac{2R(2\alpha + 2\beta + 2\gamma)}{\pi} = 4R.$$

Chapter 10. INEQUALITIES BETWEEN THE ELEMENTS OF A TRIANGLE

This chapter is in close connection with the preceding one. For background see the preceding chapter.

§1. Medians

10.1. Prove that if $a > b$, then $m_a < m_b$.

10.2. Medians AA_1 and BB_1 of triangle ABC intersect at point M. Prove that if quadrilateral A_1MB_1C is a circumscribed one, then $AC = BC$.

10.3. Perimeters of triangles ABM, BCM and ACM, where M is the intersection point of medians of triangle ABC, are equal. Prove that triangle ABC is an equilateral one.

10.4. a) Prove that if a, b, c are the lengths of sides of an arbitrary triangle, then $a^2 + b^2 \geq \frac{1}{2}c^2$.

b) Prove that $m_a^2 + m_b^2 \geq \frac{2}{8}c^2$.

10.5. Prove that $m_a^2 + m_b^2 + m_c^2 \leq \frac{27}{4}R^2$.

b) Prove that $m_a + m_b + m_c \leq \frac{9}{2}R$.

10.6. Prove that $\frac{|a^2-b^2|}{2c} < m_c \leq \frac{a^2+b^2}{2c}$.

10.7. Let $x = ab + bc + ca$, $x_1 = m_am_b + m_bm_c = m_cm_a$. Prove that $\frac{9}{20} < \frac{x_1}{x} < \frac{5}{4}$.

See also Problems 9.1, 10.74, 10.76, 17.17.

§2. Heights

10.8. Prove that in any triangle the sum of the lengths of its heights is less than its semiperimeter.

10.9. Two heights of a triangle are longer than 1. Prove that its area is greater than $\frac{1}{2}$.

10.10. In triangle ABC, height AM is not shorter than BC and height BH is not shorter than AC. Find the angles of triangle ABC.

10.11. Prove that $\frac{1}{2r} < \frac{1}{h_a} + \frac{1}{h_b} < \frac{1}{r}$.

10.12. Prove that $h_a + h_b + h_c \geq 9r$.

10.13. Let $a < b$. Prove that $a + h_a \leq b + h_b$.

10.14. Prove that $h_a \leq \sqrt{r_br_c}$.

10.15. Prove that $h_a \leq \frac{a}{2}\cot\frac{\alpha}{2}$.

10.16. Let $a \leq b \leq c$. Prove that

$$h_a + h_b + h_c \leq \frac{3b(a^2 + ac + c^2)}{4pR}.$$

See also Problems 10.28, 10.55, 10.74, 10.79.

§3. The bisectors

10.17. Prove that $l_a \leq \sqrt{p(p-a)}$.

10.18. Prove that $\frac{h_a}{l_a} \geq \sqrt{\frac{2r}{R}}$.
10.19. Prove that a) $l_a^2 + l_b^2 + l_c^2 \leq p^2$; b) $l_a + l_b + l_c \leq \sqrt{3}p$.
10.20. Prove that $l_a + l_b + m_c \leq \sqrt{3}p$.

See also Problems 6.38, 10.75, 10.94.

§4. The lengths of sides

10.21. Prove that $\frac{9r}{2S} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9R}{4S}$.
10.22. Prove that $\frac{2bc\cos\alpha}{b+c} < b + c - a < \frac{2bc}{a}$.
10.23. Prove that if a, b, c are the lengths of sides of a triangle of perimeter 2, then $a^2 + b^2 + c^2 < 2(1 - abc)$.
10.24. Prove that $20Rr - 4r^2 \leq ab + bc + ca \leq 4(R + r)^2$.

§5. The radii of the circumscribed, inscribed and escribed circles

10.25. Prove that $rr_c \leq \frac{c^2}{4}$.
10.26. Prove that $\frac{r}{R} \leq 2\sin\frac{\alpha}{2}\left(1 - \sin\frac{\alpha}{2}\right)$.
10.27. Prove that $6r \leq a + b$.
10.28. Prove that $\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 3$.
10.29. Prove that $27Rr \leq 2p^2 \leq \frac{1}{2}27R^2$.
10.30. Let O be the centre of the inscribed circle of triangle ABC and $OA \geq OB \geq OC$. Prove that $OA \geq 2r$ and $OB \geq r\sqrt{2}$.
10.31. Prove that the sum of distances from any point inside of a triangle to its vertices is not less than $6r$.
10.32. Prove that $3\left(\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c}\right) \geq 4\left(\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c}\right)$.
10.33. Prove that:
a) $5R - r \geq \sqrt{3}p$;
b) $4R - r_a \geq (p - a)\left[\sqrt{3} + \frac{a^2+(b-c)^2}{2S}\right]$.
10.34. Prove that $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.
10.35. Prove that $r_a^2 + r_b^2 + r_c^2 \geq \frac{1}{4}27R^2$.

See also Problems 10.11, 10.12, 10.14, 10.18, 10.24, 10.55, 10.79, 10.82, 19.7.

§6. Symmetric inequalities between the angles of a triangle

Let α, β and γ be the angles of triangle ABC. In problems of this section you have to prove the inequalities indicated.

Remark. If α, β and γ are the angles of a triangle, then there exists a triangle with angles $\frac{\pi-\alpha}{2}$, $\frac{\pi-\beta}{2}$ and $\frac{\pi-\gamma}{2}$. Indeed, these numbers are positive and their sum is equal to π. It follows that if a symmetric inequality holds for sines, cosines, tangents and cotangents of the angles of any triangle then a similar inequality in which $\sin x$ is replaced with $\cos\frac{x}{2}$, $\cos x$ with $\sin\frac{x}{2}$, $\tan x$ with $\cot\frac{x}{2}$ and $\cot x$ with $\tan\frac{x}{2}$ is also true.

The converse passage from inequalities for halved angles to inequalities with whole angles is only possible for acute triangles. Indeed, if $\alpha' = \frac{1}{2}(\pi - \alpha)$, then $\alpha = \pi - 2\alpha'$. Therefore, for an acute triangle with angles α', β', γ' there exists a triangle with angles $\pi - 2\alpha'$, $\pi - 2\beta'$ and $\pi - 2\gamma'$. Under such a passage $\sin\frac{x}{2}$ turns into $\cos x$, etc., but the inequality obtained can only be true for acute triangles.

10.36. a) $1 < \cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$.
b) $1 < \sin\frac{\alpha}{2} + \sin\frac{\beta}{2} + \sin\frac{\gamma}{2} \leq \frac{3}{2}$.
10.37. a) $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3}{2}\sqrt{3}$.
b) $\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \leq \frac{3}{2}\sqrt{3}$.
10.38. a) $\cot\alpha + \cot\beta + \cot\gamma \geq \sqrt{3}$.
b) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} \geq \sqrt{3}$.
10.39. $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} + \cot\frac{\gamma}{2} \geq 3\sqrt{3}$.
b) For an acute triangle $\tan\alpha + \tan\beta + \tan\gamma \geq 3\sqrt{3}$.
10.40. a) $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leq \frac{1}{8}$.
b) $\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8}$.
10.41. a) $\sin\alpha\sin\beta\sin\gamma \leq \frac{3\sqrt{3}}{8}$;
b) $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leq \frac{3}{8}\sqrt{3}$.
10.42. a) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \geq \frac{3}{4}$.
b) For an obtuse triangle

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma > 1.$$

10.43. $\cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha \leq \frac{3}{4}$.
10.44. For an acute triangle

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma \leq \sin(\alpha+\beta) + \sin(\beta+\gamma) + \sin(\gamma+\alpha).$$

§7. Inequalities between the angles of a triangle

10.45. Prove that $1 - \sin\frac{\alpha}{2} \leq 2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.
10.46. Prove that $\sin\frac{\gamma}{2} \leq \frac{c}{a+b}$.
10.47. Prove that if $a + b < 3c$, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} < \frac{1}{2}$.
10.48. In an acute triangle, if $\alpha < \beta < \gamma$, then $\sin 2\alpha > \sin 2\beta > \sin 2\gamma$.
10.49. Prove that $\cos 2\alpha + \cos 2\beta - \cos 2\gamma \leq \frac{3}{2}$.

10.50. On median BM of triangle ABC, point X is taken. Prove that if $AB < BC$, then $\angle XAB < \angle XCB$.

10.51. The inscribed circle is tangent to sides of triangle ABC at points A_1, B_1 and C_1. Prove that triangle $A_1B_1C_1$ is an acute one.

10.52. From the medians of a triangle whose angles are α, β and γ a triangle whose angles are α_m, β_m and γ_m is constructed. (Angle α_m subtends median AA_1, etc.) Prove that if $\alpha > \beta > \gamma$, then $\alpha > \alpha_m$, $\alpha > \beta_m$, $\gamma_m > \beta > \alpha_m$, $\beta_m > \gamma$ and $\gamma_m > \gamma$.

See also Problems 10.90, 10.91, 10.93.

§8. Inequalities for the area of a triangle

10.53. Prove that: a) $3\sqrt{3}r^2 \leq S \leq \frac{p^2}{3\sqrt{3}}$; b) $S \leq \frac{a^2+b^2+c^2}{4\sqrt{3}}$.
10.54. Prove that

$$a^2 + b^2 + c^2 - (a-b)^2 - (b-c)^2 - (c-a)^2 \geq 4\sqrt{3}S.$$

10.55. Prove that: a) $S^3 \leq \left(\frac{\sqrt{3}}{4}\right)^3 (abc)^2$; b) $\sqrt{h_ah_bh_c} \leq \sqrt[4]{3}\sqrt{S} \leq \sqrt[3]{r_ar_br_c}$.

* * *

10.56. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1, respectively, are taken so that AA_1, BB_1 and CC_1 meett at one point. Prove that $\frac{S_{A_1B_1C_1}}{S_{ABC}} \leq \frac{1}{4}$.

10.57. On sides BC, CA and AB of triangle ABC arbitrary points A_1, B_1 and C_1 are taken. Let $a = S_{AB_1C_1}$, $b = S_{A_1BC_1}$, $c = S_{A_1B_1C}$ and $u = S_{A_1B_1C_1}$. Prove that

$$u^3 + (a+b+c)u^2 \geq 4abc.$$

10.58. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1 are taken. Prove that the area of one of the triangles AB_1C_1, A_1BC_1, A_1B_1C does not exceed: a) $\frac{1}{4}S_{ABC}$; b) $S_{A_1B_1C_1}$.

See also Problems 9.33, 9.37, 9.40, 10.9, 20.1, 20.7.

§9. The greater angle subtends the longer side

10.59. In a triangle ABC, prove that $\angle ABC < \angle BAC$ if and only if $AC < BC$, i.e., the longer side subtends the greater angle and the greater angle subtends the longer side.

10.60. Prove that in a triangle ABC angle $\angle A$ is an acute one if and only if $m_b > \frac{1}{2}a$.

10.61. Let $ABCD$ and $A_1B_1C_1D_1$ be two convex quadrilaterals with equal corresponding sides. Prove that if $\angle A > \angle A_1$, then $\angle B < \angle B_1$, $\angle C < \angle C_1$, $\angle D < \angle D_1$.

10.62. In an acute triangle ABC the longest height AH is equal to median BM. Prove that $\angle B \leq 60^\circ$.

10.63. Prove that a convex pentagon $ABCDE$ with equal sides whose angles satisfy inequalities $\angle A \geq \angle B \geq \angle C \geq \angle D \geq \angle E$ is a regular one.

§10. Any segment inside a triangle is shorter than the longest side

10.64. a) Segment MN is placed inside triangle ABC. Prove that the length of MN does not exceed the length of the longest side of the triangle.

b) Segment MN is placed inside a convex polygon. Prove that the length of MN does not exceed that of the longest side or of the greatest diagonal of this polygon.

10.65. Segment MN lies inside sector AOB of a disk of radius $R = AO = BO$. Prove that either $MN \leq R$ or $MN \leq AB$ (we assume that $\angle AOB < 180^\circ$).

10.66. In an angle with vertex A, a circle tangent to the legs at points B and C is inscribed. In the domain bounded by segments AB, AC and the shorter arc $\smile BC$ a segment is placed. Prove that the length of the segment does not exceed that of AB.

10.67. A convex pentagon lies inside a circle. Prove that at least one of the sides of the pentagon is not longer than a side of the regular pentagon inscribed in the circle.

10.68. Given triangle ABC the lengths of whose sides satisfy inequalities $a > b > c$ and an arbitrary point O inside the triangle. Let lines AO, BO, CO intersect the sides of the triangle at points P, Q, R, respectively. Prove that $OP + OQ + OR < a$.

§11. Inequalities for right triangles

In all problems of this section ABC is a right triangle with right angle $\angle C$.

10.69. Prove that $c^n > a^n + b^n$ for $n > 2$.

10.70. Prove that $a + b < c + h_c$.

10.71. Prove that for a right triangle $0.4 < \frac{r}{h} < 0.5$, where h is the height dropped from the vertex of the right angle.

10.72. Prove that $\frac{c}{r} \geq 2(1 + \sqrt{2})$.

10.73. Prove that $m_a^2 + m_b^2 > 29r^2$.

§12. Inequalities for acute triangles

10.74. Prove that for an acute triangle

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\leq 1+\frac{R}{r}.$$

10.75. Prove that for an acute triangle

$$\frac{1}{l_a}+\frac{1}{l_b}+\frac{1}{l_c}\leq \sqrt{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

10.76. Prove that if a triangle is not an obtuse one, then $m_a+m_b+m_c\geq 4R$.

10.77. Prove that if in an acute triangle $h_a=l_b=m_c$, then this triangle is an equilateral one.

10.78. In an acute triangle ABC heights AA_1, BB_1 and CC_1 are drawn. Prove that the perimeter of triangle $A_1B_1C_1$ does not exceed a semiperimeter of triangle ABC.

10.79. Let h be the longest height of a non-obtuse triangle. Prove that $r+R\leq h$.

10.80. On sides BC, CA and AB of an acute triangle ABC, points A_1, B_1 and C_1, respectively, are taken. Prove that

$$2(B_1C_1\cos\alpha+C_1A_1\cos\beta+A_1B_1\cos\gamma)\geq a\cos\alpha+b\cos\beta+c\cos\gamma).$$

10.81. Prove that a triangle is an acute one if and only if $a^2+b^2+c^2>8R^2$.

10.82. Prove that a triangle is an acute one if and only if $p>2R+r$.

10.83. Prove that triangle ABC is an acute one if and only if on its sides BC, CA and AB interior points A_1, B_1 and C_1, respectively, can be selected so that $AA_1=BB_1=CC_1$.

10.84. Prove that triangle ABC is an acute one if and only if the lengths of its projections onto three distinct directions are equal.

See also Problems 9.93, 10.39, 10.44, 10.48, 10.62.

§13. Inequalities in triangles

10.85. A line is drawn through the intersection point O of the medians of triangle ABC. The line intersects the triangle at points M and N. Prove that $NO\leq 2MO$.

10.86. Prove that if triangle ABC lies inside triangle $A'B'C'$, then $r_{ABC}<r_{A'B'C'}$.

10.87. In triangle ABC side c is the longest and a is the shortest. Prove that $l_c\leq h_a$.

10.88. Medians AA_1 and BB_1 of triangle ABC are perpendicular. Prove that $\cot\angle A+\cot\angle B\geq\frac{2}{3}$.

10.89. Through vertex A of an isosceles triangle ABC with base AC a circle tangent to side BC at point M and intersecting side AB at point N is drawn. Prove that $AN>CM$.

10.90. In an acute triangle ABC bisector AB, median BM and height CH intersect at one point. What are the limits inside which the value of angle A can vary?

10.91. In triangle ABC, prove that $\frac{1}{3}\pi\leq\pi a\alpha+b\beta+c\gamma a+b+c<\frac{1}{2}\pi$.

10.92. Inside triangle ABC point O is taken. Prove that

$$AO\sin\angle BOC+BO\sin\angle AOC+CO\sin\angle AOB\leq p.$$

10.93. On the extension of the longest side AC of triangle ABC beyond point C, point D is taken so that $CD=CB$. Prove that angle $\angle ABD$ is not an acute one.

10.94. In triangle ABC bisectors AK and CM are drawn. Prove that if $AB>BC$, then $AM>MK>KC$.

10.95. On sides BC, CA, AB of triangle ABC points X, Y, Z are taken so that lines AX, BY, CZ meet at one point O. Prove that of ratios $OA:OX$, $OB:OY$, $OC:OZ$ at least one is not greater than 2 and one is not less than 2.

10.96. Circle S_1 is tangent to sides AC and AB of triangle ABC, circle S_2 is tangent to sides BC and AB and, moreover, S_1 and S_2 are tangent to each other from the outside. Prove that the sum of radii of these circles is greater than the radius of the inscribed circle S.

See also Problems 14.24, 17.16, 17.18.

Problems for independent study

10.97. In a triangle ABC, let $P = a+b+c$, $Q = ab+bc+ca$. Prove that $3Q < P^2 < 4Q$.

10.98. Prove that the product of any two sides of a triangle is greater than $4Rr$.

10.99. In triangle ABC bisector AA_1 is drawn. Prove that $A_1C < AC$.

10.100. Prove that if $a > b$ and $a + h_a \leq b + h_b$, then $\angle C = 90^\circ$.

10.101. Let O be the centre of the inscribed circle of triangle ABC. Prove that $ab + bc + ca \geq (AO + BO + CO)^2$.

10.102. On sides of triangle ABC equilateral triangles with centers at D, E and F are constructed outwards. Prove that $S_{DEF} \geq S_{ABC}$.

10.103. In plane, triangles ABC and MNK are given so that line MN passes through the midpoints of sides AB and AC and the intersection of these triangles is a hexagon of area S with pairwise parallel opposite sides. Prove that $3S < S_{ABC} + S_{MNK}$.

Solutions

10.1. Let medians AA_1 and BB_1 meet at point M. Since $BC > AC$, points A and C lie on one side of the midperpendicular to segment AB and therefore, both median CC_1 and its point M lie on the same side. It follows that $AM < BM$, i.e., $m_a < m_b$.

10.2. Suppose that, for instance, $a > b$. Then $m < m_b$ (Problem 10.1). Since quadrilateral A_1MB_1C is a circumscribed one, it follows that $\frac{1}{2}a + \frac{1}{3}m_b = \frac{1}{2}b + \frac{1}{3}m_a$, i.e., $\frac{1}{2}(a-b) = \frac{1}{3}(m_a - m_b)$. Contradiction.

10.3. Let, for instance, $BC > AC$. Then $MA < MB$ (cf. Problem 10.1); hence, $BC + MB + MC > AC + MA + MC$.

10.4. a) Since $c \leq a + b$, it follows that $c^2 \leq (a+b)^2 = a^2 + b^2 + 2ab \leq 2(a^2 + b^2)$.

b) Let M be the intersection point of medians of triangle ABC. By heading a) $MA^2 + MB^2 \geq \frac{1}{2}AB^2$, i.e., $\frac{4}{9}m_a^2 + \frac{4}{9}m_b^2 \geq \frac{1}{2}c^2$.

10.5. a) Let M be the intersection point of medians, O the center of the circumscribed circle of triangle ABC. Then

$$\begin{gathered} AO^2 + BO^2 + CO^2 = \\ (\overrightarrow{AM} + \overrightarrow{MO})^2 + (\overrightarrow{BM} + \overrightarrow{MO})^2 + (\overrightarrow{CM} + \overrightarrow{MO})^2 = \\ AM^2 + BM^2 + CM^2 + 2(\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM}, \overrightarrow{MO}) + 3MO^2. \end{gathered}$$

Since $\overrightarrow{AM} + \overrightarrow{BM} + \overrightarrow{CM} = \overrightarrow{0}$, it follows that

$$AO^2 + BO^2 + CO^2 = AM^2 + BM^2 + CM^2 + 3MO^2 \geq AM^2 + BM^2 + CM^2,$$

i.e., $3R^2 \geq \frac{4}{9}(m_a^2 + m_b^2 + m_c^2)$.

b) It suffices to notice that $(m_a + m_b + m_c)^2 \leq 3(m_a^2 + m_b^2 + m_c^2)$, cf. Supplement to Ch. 9.

10.6. Heron's formula can be rewritten as

$$16S^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4.$$

Since $m_c^2 = \frac{1}{4}(2a^2 + 2b^2 - c^2)$ (Problem 12.11 a)), it follows that the inequalities

$$m_c^2 \le \left(\frac{a^2 + b^2}{2c}\right)^2; \quad m_c^2 \ge \left(\frac{a^2 - b^2}{2c}\right)^2$$

are equivalent to the inequalities $16S^2 \le 4a^2b^2$ and $16S^2 > 0$, respectively.

10.7. Let $y = a^2 + b^2 + c^2$ and $y_1 = m_a^2 + m_b^2 + m_c^2$. Then $3y = 4y_1$ (Problem 12.11, b), $y < 2x$ (Problem 9.7) and $2x_1 + y_1 < 2x + y$ because $(m_a + m_b + m_c)^2 < (a + b + c)^2$ (cf. Problem 9.2). By adding $8x_1 + 4y_1 < 8x + 4y$ to $3y = 4y_1$ we get $8x_1 < y + 8x < 10x$, i.e., $\frac{x_1}{x} < \frac{5}{4}$.

Let M be the intersection point of the medians of triangle ABC. Let us complement triangle AMB to parallelogram $AMBN$. Applying the above-proved statement to triangle AMN we get $\frac{(x/4)}{(4x_1/9)} < \frac{5}{4}$, i.e., $\frac{x}{x_1} < \frac{20}{9}$.

10.8. Clearly, $h_a \le b$, $h_b \le c$, $h_c \le a$, where at least one of these inequalities is a strict one. Hence, $h_a + h_b + h_c < a + b + c$.

10.9. Let $h_a > 1$ and $h_b > 1$. Then $a \ge h_b > 1$. Hence, $S = \frac{1}{2}ah_a > \frac{1}{2}$.

10.10. By the hypothesis $BH \ge AC$ and since the perpendicular is shorter than a slanted line, $BH \ge AC \ge AM$. Similarly, $AM \ge BC \ge BH$. Hence, $BH = AM = AC = BC$. Since $AC = AM$, segments AC and AM coincide, i.e., $\angle C = 90°$; since $AC = BC$, the angles of triangle ABC are equal to 45°, 45°, 90°.

10.11. Clearly, $\frac{1}{h_a} + \frac{1}{h_b} = \frac{a+b}{2S} = \frac{a+b}{(a+b+c)r}$ and $a + b + c < 2(a + b) < 2(a + b + c)$.

10.12. Since $ah_a = r(a + b + c)$, it follows that $h_a = r\left(1 + \frac{b}{a} + \frac{c}{a}\right)$. Adding these equalities for h_a, h_b and h_c and taking into account that $\frac{x}{y} + \frac{y}{x} \ge 2$ we get the desired statement.

10.13. Since $h_a - h_b = 2S\left(\frac{1}{a} - \frac{1}{b}\right) = 2S\frac{b-a}{ab}$ and $2S \le ab$, it follows that $h_a - h_b \le b - a$.

10.14. By Problem 12.21 $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$. Moreover, $\frac{1}{r_b} + \frac{1}{r_c} \ge \frac{2}{\sqrt{r_b r_c}}$.

10.15. Since

$$2\sin\beta\sin\gamma = \cos(\beta - \gamma) - \cos(\beta + \gamma) \le 1 + \cos\alpha,$$

we have

$$\frac{h_a}{a} = \frac{\sin\beta\sin\gamma}{\sin\alpha} \le \frac{1 + \cos\alpha}{2\sin\alpha} = \frac{1}{2}\cot\frac{\alpha}{2}.$$

10.16. Since $\frac{b}{2R} = \sin\beta$, then multiplying by $2p$ we get

$$(a + b + c)(h_a + h_b + h_c) \le 3\sin\beta(a^2 + ac + c^2).$$

Subtracting $6S$ from both sides we get

$$a(h_b + h_c) + b(h_a + h_c) + c(h_a + h_b) \le 3\sin\beta(a^2 + c^2).$$

Since, for instance, $ah_b = a^2\sin\gamma = \frac{a^2c}{2R}$, we obtain $a(b^2 + c^2) - 2b(a^2 + c^2) + c(a^2 + b^2) \le 0$. To prove the latter inequality let us consider the quadratic exoression

$$f(x) = x^2(a + c) - 2x(a^2 + c^2) + ac(a + c).$$

It is easy to verify that $f(a) = -a(a - c)^2 \le 0$ and $f(c) = -c(a - c)^2 \le 0$. Since the coefficient of x is positive and $a \le b \le c$, it follows that $f(b) \le 0$.

10.17. By Problem 12.35 a) $l_a^2 = \frac{4bcp(p-a)}{(b+c)^2}$. Moreover, $4bc \le (b + c)^2$.

10.18. Clearly, $\frac{h_a}{l_a} = \cos\frac{1}{2}(\beta - \gamma)$. By Problem 12.36 a)

$$\frac{2r}{R} = 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = 4\sin\frac{\alpha}{2}\left[\,\cos\frac{\beta-\gamma}{2} - \cos\frac{\beta+\gamma}{2}\,\right] = 4x(q-x),$$

$$\text{where } x = \sin\frac{\alpha}{2} \text{ and } q = \cos\frac{\beta-\gamma}{2}.$$

It remains to notice that $4x(q-x) \le q^2$.

10.19. a) By Problem 10.17 $l_a^2 \le p(p-a)$. Adding three similar inequalities we get the desired statement.

b) For any numbers l_a, l_b and l_c we have $(l_a + l_b + l_c)^2 \le 3(l_a^2 + l_b^2 + l_c^2)$.

10.20. It suffices to prove that $\sqrt{p(p-a)} + \sqrt{p(p-b)} + m_c \le \sqrt{3p}$. We may assume that $p = 1$; let $x = 1 - a$ and $y = 1 - b$. Then

$$m_c^2 = \frac{2a^2 + 2b^2 - c^2}{4} = 1 - (x+y) + \frac{(x-y)^2}{4} = m(x,y).$$

Let us consider the function

$$f(x,y) = \sqrt{x} + \sqrt{y} + \sqrt{m(x,y)}.$$

We have to prove that $f(x,y) \le \sqrt{3}$ for x, $y \ge 0$ and $x + y \le 1$. Let

$$g(x) = f(x,x) = 2\sqrt{x} + \sqrt{1-2x}.$$

Since $g'(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{1-2x}}$, it follows that as x grows from 0 to $\frac{1}{3}$ and $g(x)$ grows from 1 to $\sqrt{3}$ and as x grows from $\frac{1}{3}$ to $\frac{1}{2}$; we also see that $g(x)$ diminishes from $\sqrt{3}$ to $\sqrt{2}$. Introduce new variables: $d = x - y$ and $q = \sqrt{x} + \sqrt{y}$. It is easy to verify that $(x-y)^2 - 2q^2(x+y) + q^4 = 0$, i.e., $x + y = \frac{d^2+q^4}{2q^2}$. Hence,

$$f(x,y) = q + \sqrt{1 - \frac{q^2}{2} - \frac{d^2(2-q^2)}{4q^2}}.$$

Now, observe that $q^2 = (\sqrt{x} + \sqrt{y})^2 \le 2(x+y) \le 2$, i.e., $\frac{d^2(2-q^2)}{4q^2} \ge 0$. It follows that for a fixed q the value of function $f(x,y)$ is the maximal one for $d = 0$, i.e., $x = y$; the case $x = y(?)$ is the one considered above.

10.21. Clearly, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{h_a+h_b+h_c}{2S}$. Moreover, $9r \le h_a + h_b + h_c$ (Problem 10.12) and $h_a + h_b + h_c \le m_a + m_b + m_c \le \frac{9}{2}R$ (Problem 10.5 b)).

10.22. First, let us prove that $b + c - a < \frac{2bc}{a}$. Let $2x = b + c - a$, $2y = a + c - b$ and $2z = a + b - c$. We have to prove that

$$2x < \frac{2(x+y)(x+z)}{y+z}, \quad \text{i.e.,} \quad xy + xz < xy + xz + x^2 + yz.$$

The latter inequality is obvious.

Since

$$2bc\cos\alpha = b^2 + c^2 - a^2 = (b+c-a)(b+c+a) - 2bc,$$

it follows that

$$\frac{2bc\cos\alpha}{b+c} = b + c - a + \left[\frac{(b+c-a)a}{b+c} - \frac{2bc}{b+c}\right].$$

The expression in square brackets is negative because $b + c - a < \frac{2bc}{a}$.

10.23. By Problem 12.30 we have

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac) = 4p^2 - 2r^2 - 2p^2 - 8rR = 2p^2 - 2r^2 - 8rR$$

and $abc = 4prR$. Thus, we have to prove that

$$2p^2 - 2r^2 - 8rR < 2(1 - 4prR), \quad \text{where } p = 1.$$

This inequality is obvious.

10.24. By Problem 12.30, $ab + bc + ca = r^2 + p^2 + 4Rr$. Moreover, $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$ (Problem 10.34).

10.25. Since

$$r(\cot\alpha + \cot\beta) = c = r_c(\tan\alpha + \tan\beta),$$

it follows that

$$c^2 = rr_c\left(2 + \frac{\tan\alpha}{\tan\beta} + \frac{\tan\beta}{\tan\beta}\right) \geq 4rr_c.$$

10.26. It suffices to apply the results of Problems 12.36 a) and 10.45. Notice also that $x(1-x) \leq \frac{1}{4}$, i.e., $\frac{r}{R} \leq \frac{1}{2}$.

10.27. Since $h_c \leq a$ and $h_c \leq b$, it follows that $4S = 2ch_c \leq c(a+b)$. Hence,

$$6r(a+b+c) = 12S \leq 4ab + 4S \leq (a+b)^2 + c(a+b) = (a+b)(a+b+c).$$

10.28. Since $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$ (Problem 12.21), it follows that $\frac{r_a}{h_a} = \frac{1}{2}\left(\frac{r_a}{r_b} + \frac{r_a}{r_c}\right)$. Let us write similar equalities for $\frac{r_b}{h_b}$ and $\frac{r_c}{h_c}$ and add them. Taking into account that $\frac{x}{y} + \frac{y}{x} \geq 2$ we get the desired statement.

10.29. Since $Rr = \frac{PS}{p} = \frac{abc}{4p}$ (cf. Problem 12.1), we obtain $27abc \leq 8p^3 = (a+b+c)^3$.

Since $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$ for any numbers a, b and c, we have

$$p^2 \leq \frac{3}{4}(a^2+b^2+c^2) = m_a^2 + m_b^2 + m_c^2$$

(cf. Problem 12.11 b)). It remains to notice that $m_a^2 + m_b^2 + m_c^2 \leq \frac{27}{4}R^2$ (Problem 10.5 a)).

10.30. Since $OA = \frac{r}{\sin\frac{A}{2}}$, $OB = \frac{r}{\sin\frac{\angle B}{2}}$ and $OC = \frac{r}{\sin\frac{\angle X}{2}}$ and since angles $\frac{1}{2}\angle A$, $\frac{1}{2}\angle B$ and $\frac{1}{2}\angle C$ are acute ones, it follows that $\angle A \leq \angle B \leq \angle C$. Hence, $\angle A \leq 60^\circ$ and $\angle B \leq 90^\circ$ and, therefore, $\sin\frac{\angle A}{2} \leq \frac{1}{2}$ and $\sin\frac{\angle B}{2} \leq \frac{1}{\sqrt{2}}$.

10.31. If $\angle C \geq 120^\circ$, then the sum of distances from any point inside the triangle to its vertices is not less than $a+b$ (Problem 11.21); moreover, $a+b \geq 6r$ (Problem 10.27).

If each angle of the triangle is less than 120°, then at a point the sum of whose distances from the vertices of the triangle is the least one the square of this sum is equal to $\frac{1}{2}(a^2+b^2+c^2) + 2\sqrt{3}S$ (Problem 18.21 b)). Further, $\frac{1}{2}(a^2+b^2+c^2) \geq 2\sqrt{3}S$ (Problem 10.53 b)) and $4\sqrt{3}S \geq 36r^2$ (Problem 10.53 a)).

10.32. Let $\alpha = \cos\frac{\angle A}{2}$, $\beta = \cos\frac{\angle B}{2}$ and $\gamma = \cos\frac{\angle C}{2}$. By Problem 12.17 b) $\frac{a}{r_a} = \frac{\alpha}{\beta\gamma}$, $\frac{b}{r_b} = \frac{\beta}{\gamma\alpha}$ and $\frac{c}{r_c} = \frac{\gamma}{\alpha\beta}$. Therefore, multiplying by $\alpha\beta\gamma$ we express the inequality to be proved in the form

$$3(\alpha^2+\beta^2+\gamma^2) \geq 4(\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2).$$

Since $\alpha^2 = \frac{1+\cos\angle A}{2}$, $\beta^2 = \frac{1+\cos\angle B}{2}$ and $\gamma^2 = \frac{1+\cos\angle C}{2}$, we obtain the inequality

$$\cos\angle A + \cos\angle B + \cos\angle C + 2(\cos\angle A\cos\angle B + \cos\angle B\cos\angle C + \cos\angle C\cos\angle A) \leq 3.$$

It remains to make use of results of Problems 10.36 and 10.43.

10.33. a) Adding equality $4R + r = r_a + r_b + r_c$ (Problem 12.24) with inequality $R - 2r \geq 0$ (Problem 10.26) we get

$$5R - r \geq r_a + r_b + r_c = pr((p-a)^{-1} + (p-b)^{-1} + (p-c)^{-1}) =$$
$$\frac{p(ab+bc+ca-p^2)}{S} == \frac{p(2(ab+bc+ca)-a^2-b^2-c^2)}{4S}.$$

It remains to observe that

$$2(ab+bc+ca)-a^2-b^2-c^2\geq 4\sqrt{3}S$$

(Problem 10.54).

b) It is easy to verify that

$$4R-r_a=r_b+r_c-r=\frac{pr}{p-b}+\frac{pr}{p-c}-\frac{pr}{p}=\frac{(p-a)(p^2-bc)}{S}.$$

It remains to observe that

$$\begin{gathered}4(p^2-bc)=a^2+b^2+c^2+2(ab-bc+ca)=\\=2(ab+bc+ca)=\\-a^2-b^2-c^2+2(a^2+b^2+c^2-2bc)\geq 4\sqrt{3}S+2(a^2+(b-c)^2).\end{gathered}$$

10.34. Let a, b and c be the lengths of the sides of the triangle, $F=(a-b)(b-c)(c-a)=A-B$, where $A=ab^2+bc^2+ca^2$ and $B=a^2b+b^2c+c^2a$. Let us prove that the required inequalities can be obtained by a transformation of an obvious inequality $F^2\geq 0$. Let $\sigma_1=a+b+c=2p$, $\sigma_2=ab+bc+ca=r^2+p^2+4rR$ and $\sigma_3=abc=4prR$, cf. Problem 12.30. It is easy to verify that

$$F^2=\sigma_1^2\sigma_2^2-4\sigma_2^3-4\sigma_1^3\sigma_3+18\sigma_1\sigma_2\sigma_3-27\sigma_3^2.$$

Indeed,

$$\begin{gathered}(\sigma_1\sigma_2)^2-F^2=(A+B+3abc)^2-(A-B)^2=4AB+6(A+B)\sigma_3+9\sigma_3^2=\\4(a^3b^3+\dots)+4(a^4bc+\dots)+6(A+B)\sigma_3+21\sigma_3^2.\end{gathered}$$

It is also clear that

$$\begin{gathered}4\sigma_2^3=4(a^3b^3+\dots)+12(A+B)\sigma_3+24\sigma_3^2,\\4\sigma_1^3\sigma_3=4(a^4bc+\dots)+12(A+B)\sigma_3+24\sigma_3^2,\\18\sigma_1\sigma_2\sigma_3=18(A+B)\sigma_3+54\sigma_3^2.\end{gathered}$$

Expressing σ_1, σ_2 and σ_3 via p, r and R, we obtain

$$F^2=-4r^2[(p^2-2R^2-10Rr+r^2)^2-4R(R-2r)^3]\geq 0.$$

Thus, we obtain

$$\begin{gathered}p^2\geq 2R^2+10Rr-r^2-2(R-2r)\sqrt{R(R-2r)}=\\{}[(R-2r)-\sqrt{R(R-2r)}]^2+16Rr-5r^2\geq 16Rr-5r^2\\p^2\leq 2R^2+10Rr+r^2+2(R-2r)\sqrt{R(R-2r)}=\\4R^2+4Rr+3r^2-[(R-2r)-\sqrt{R(R-2r)}]^2\leq\\4R^2+4Rr+3r^2.\end{gathered}$$

10.35. Since $r_a+r_b+r_c=4R+r$ and $r_ar_b+r_br_c+r_cr_a=p^2$ (Problems 12.24 and 12.25), it follows that $r_a^2+r_b^2+r_c^2=(4R+r)^2-2p^2$. By Problem 10.34 $p^2\leq 4R^2+4Rr+3r^2$; hence, $r_a^2+r_b^2+r_c^2=8R^2-5r^2$. It remains to notice that $r\leq\frac{1}{2}R$ (Problem 10.26).

10.36. a) By Problem 12.38 $\cos\alpha+\cos\beta+\cos\gamma=\frac{R+r}{R}$. Moreover, $r\leq\frac{1}{2}R$ (Problem 10.26).

b) Follows from heading a), cf. Remark.

10.37. a) Clearly, $\sin\alpha+\sin\beta+\sin\gamma=\frac{p}{R}$. Moreover, $p\leq\frac{3}{2}\sqrt{3}R$ (Problem 10.29).

b) Follows from heading a), cf. Remark.

10.38. a) By Problem 12.44 a)

$$\cot\alpha+\cot\beta+\cot\gamma=\frac{a^2+b^2+c^2}{4S}.$$

Moreover, $a^2+b^2+c^2 \geq 4\sqrt{3}S$ (Problem 10.53 b)).

b) Follows from heading a), cf. Remark.

10.39. a) By Problem 12.45 a)

$$\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\frac{p}{r}.$$

Moreover, $p \geq 3\sqrt{3}r$ (Problem 10.53 a)).

b) Follows from heading a), cf. Remark. For an acute triangle $\tan\alpha+\tan\beta+\tan\gamma<0$; cf., for instance, Problem 12.46.

10.40. a) By Problem 12.36 a)

$$\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}=\frac{r}{4R}.$$

Moreover, $r \leq \frac{1}{2}R$ (Problem 10.26).

b) For an acute triangle it follows from heading a), cf. Remark. For an obtuse triangle $\cos\alpha\cos\beta\cos\gamma<0$.

10.41. a) Since $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$, we see that making use of results of Problems 12.36 a) and 12.36 c) we obtain $\sin\alpha\sin\beta\sin\gamma=\frac{pr}{2R^2}$. Moreover, $p \leq \frac{3}{2}\sqrt{3}R$ (Problem 10.29) and $r \leq \frac{1}{2}R$ (Problem 10.26).

b) Follows from heading a), cf. Remark.

10.42. By Problem 12.39 b)

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1-2\cos\alpha\cos\beta\cos\gamma.$$

It remains to notice that $\cos\alpha\cos\beta\cos\gamma \leq \frac{1}{8}$ (Problem 10.40 b)) and for an obtuse triangle $\cos\alpha\cos\beta\cos\gamma<0$.

10.43. Clearly,

$$2(\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha)= (\cos\alpha+\cos\beta+\cos\gamma)^2-\cos^2\alpha-\cos^2\beta-\cos^2\gamma.$$

It remains to notice that $\cos\alpha+\cos\beta+\cos\gamma \leq \frac{3}{2}$ (Problem 10.36 a)) and $\cos^2\alpha+\cos^2\beta+\cos^2\gamma \geq \frac{3}{4}$ (Problem 10.42).

10.44. Let the extensions of bisectors of acute triangle ABC with angles α, β and γ intersect the circumscribed circle at points A_1, B_1 and C_1, respectively. Then

$$S_{ABC}=\frac{R^2(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)}{2};\quad S_{A_1B_1C_1}=\frac{R^2(\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha))}{2}.$$

It remains to make use of results of Problems 12.72 and 10.26.

10.45. Clearly,

$$2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}=\cos\frac{\beta-\gamma}{2}-\cos\frac{\beta+\gamma}{2} \leq 1-\sin\frac{\alpha}{2}.$$

10.46. Let us drop perpendiculars AA_1 and BB_1 from vertices A and B to the bisector of angle $\angle ACB$. Then $AB \geq AA_1+BB_1 = b\sin\frac{\gamma}{2}+a\sin\frac{\gamma}{2}$.

10.47. By Problem 12.32 $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{a+b-c}{a+b+c}$. Since $a+b<3c$, it follows that $a+b-c<\frac{1}{2}(a+b+c)$.

10.48. Since $\pi-2\alpha>0$, $\pi-2\beta>0$, $\pi-2\gamma>0$ and $(\pi-2\alpha)+(\pi-2\beta)+(\pi-2\gamma)=\pi$, it follows that there exists a triangle whose angles are $\pi-2\alpha$, $\pi-2\beta$, $\pi-2\gamma$. The lengths of sides opposite to angles $\pi-2\alpha$, $\pi-2\beta$, $\pi-2\gamma$ are proportional to $\sin(\pi-2\alpha)=\sin 2\alpha$, $\sin 2\beta$, $\sin 2\gamma$, respectively. Since $\pi-2\alpha>\pi-2\beta>\pi-2\gamma$ and the greater angle subtends the longer side, $\sin 2\alpha>\sin 2\beta>\sin 2\gamma$.

10.49. First, notice that

$$\cos 2\gamma=\cos 2(\pi-\alpha-\beta)=\cos 2\alpha\cos 2\beta-\sin 2\alpha\sin 2\beta.$$

Hence,

$$\cos 2\alpha+\cos 2\beta-\cos 2\gamma=\cos 2\alpha+\cos 2\beta-\cos 2\alpha\cos 2\beta+\sin 2\alpha\sin 2\beta.$$

Since $a\cos\varphi+b\sin\varphi\leq\sqrt{a^2+b^2}$ (cf. Supplement to Ch. 9), it follows that

$$(1-\cos 2\beta)\cos 2\alpha+\sin 2\beta\sin 2\alpha+\cos 2\beta\leq \\ \sqrt{(1-\cos 2\beta)^2+\sin^2 2\beta}+\cos 2\beta=2|\sin\beta|+1-2\sin^2\beta.$$

It remains to notice that the greatest value of the quadratic $2t+1-2t^2$ is attained at point $t=\frac{1}{2}$ and this value is equal to $\frac{3}{2}$. The maximal value corresponds to angles $\alpha=\beta=30^\circ$ and $\gamma=120^\circ$.

10.50. Since $AB<CB$, $AX<CX=S_{ABX}=S_{BCX}$, it follows that $\sin\angle XAB>\sin\angle XCB$. Taking into account that angle $\angle XCB$ is an acute one, we get the desired statement.

10.51. If the angles of triangle ABC are equal to α, β and γ, then the angles of triangle $A_1B_1C_1$ are equal to $\frac{1}{2}(\beta+\gamma)$, $\frac{1}{2}(\gamma+\alpha)$ and $\frac{1}{2}(\alpha+\beta)$.

10.52. Let M be the intersection point of medians AA_1, BB_1 and CC_1. Complementing triangle AMB to parallelogram $AMBN$ we get $\angle BMC_1=\alpha_m$ and $\angle AMC_1=\beta_m$. It is easy to verify that $\angle C_1CB<\frac{1}{2}\gamma$ and $\angle B_1BC<\frac{1}{2}\beta$. It follows that $\alpha_m=\angle C_1CB+\angle B_1BC<\frac{1}{2}(\beta+\gamma)<\beta$. Similarly, $\gamma_m=\angle A_1AB+\angle B_1BA>\frac{1}{2}(\alpha+\beta)>\beta$.

First, suppose that triangle ABC is an acute one. Then the heights' intersection point H lies inside triangle AMC_1. Hence, $\angle AMB<\angle AHB$, i.e., $\pi-\gamma_m<\pi-\gamma$ and $\angle CMB<\angle CHB$, i.e., $\pi-\alpha_m>\pi-\alpha$. Now, suppose that angle α is an obtuse one. Then angle CC_1B is also an obtuse one and therefore, angle α_m is an acute one, i.e., $\alpha_m<\alpha$. Let us drop perpendicular MX from point M to BC. Then $\gamma_m>\angle XMB>180^\circ-\angle HAB>\gamma$.

Since $\alpha>\alpha_m$, it follows that $\alpha+(\pi-\alpha_m)>\pi$, i.e., point M lies inside the circumscribed circle of triangle AB_1C_1. Therefore, $\gamma=\angle AB_1C_1<\angle AMC_1=\beta_m$. Similarly, $\alpha=\angle CB_1A_1>\angle CMA_1=\beta_m$ because $\gamma+(\pi-\gamma_m)<\pi$.

10.53 a) Clearly,

$$\frac{S^2}{p}=(p-a)(p-b)(p-c)\leq\left(\frac{p-a+p-b+p-c}{3}\right)^2=\frac{p^3}{27}.$$

Hence, $pr=S\leq\frac{p^2}{3\sqrt{3}}$, i.e., $r\leq\frac{p}{3\sqrt{3}}$. By multiplying the latter inequality by r we get the desired statement.

b) Since $(a+b+c)^2\leq 3(a^2+b^2+c^2)$, it follows that

$$S\leq\frac{p^2}{3\sqrt{3}}=\frac{(a+b+c)^2}{12\sqrt{3}}\leq\frac{a^2+b^2+c^2}{4\sqrt{3}}.$$

10.54. Let $x = p - a$, $y = p - b$, $z = p - c$. Then

$$\begin{gathered}(a^2 - (b-c)^2) + (b^2 - (a-c)^2) + (c^2 - (a-b)^2) = \\ 4(p-b)(p-c) + 4(p-a)(p-c) + 4(p-a)(p-b) = \\ 4(yz + zx + xy)\end{gathered}$$

and

$$4\sqrt{3}S = 4\sqrt{3p(p-a)(p-b)(p-c)} = 4\sqrt{3(x+y+z)xyz}.$$

Thus, we have to prove that $xy + yz + zx \geq \sqrt{3(x+y+z)xyz}$. After squaring and simplification we obtain

$$x^2y^2 + y^2z^2 + z^2x^2 \geq x^2yz + y^2xz + z^2xy.$$

Adding inequalities

$$x^2yz \leq \frac{x^2(y^2+z^2)}{2}, \quad y^2xz \leq \frac{y^2(x^2+z^2)}{2} \quad \text{and} \quad z^2xy \leq \frac{z^2(x^2+y^2)}{2}$$

we get the desired statement.

10.55. a) By multiplying three equalities of the form $S = \frac{1}{2}ab\sin\gamma$ we get $S^3 = \frac{1}{8}(abc)^2 \sin\gamma \sin\beta \sin\alpha$. It remains to make use of a result of Problem 10.41.

b) Since $(h_ah_bh_c)^2 = \frac{(2S)^6}{(abc)^2}$ and $(abc)^2 \geq \left(\frac{4}{\sqrt{3}}\right)^3 S^3$, it follows that $(h_ah_bh_c)^2 \leq \frac{(2S)^6(\sqrt{3}/4)^3}{S^3} = (\sqrt{3}S)^3$.

Since $(r_ar_br_c)^2 = \frac{S^4}{r^2}$ (Problem 12.18, c) and $r^2(\sqrt{3})^3 \leq S$ (Problem 10.53 a), it follows that $(r_ar_br_c)^2 \geq (\sqrt{3}S)^3$.

10.56. Let $p = \frac{BA}{BC}$, $q = \frac{CB_1}{CA}$ and $r = \frac{AC_1}{AC}$. Then

$$\frac{S_{A_1B_1C_1}}{S_{ABC}} = 1 - p(1-r) - q(1-p) - r(1-q) = 1 - (p+q+r) + (pq+qr+rp).$$

By Cheva's theorem (Problem 5.70) $pqr = (1-p)(1-q)(1-r)$, i.e., $2pqr = 1 - (p+q+r) + (pq+qr+rp)$. Moreover,

$$(pqr)^2 = p(1-p)(1-q)r(1-r) \leq \left(\frac{1}{4}\right)^3.$$

Therefore, $\frac{S_{A_1B_1C_1}}{S_{ABC}} = 2pqr \leq \frac{1}{4}$.

10.57. We can assume that the area of triangle ABC is equal to 1. Then $a + b + c = 1$ and, therefore, the given inequality takes the form $u^2 \geq 4abc$. Let $x = \frac{BA_1}{BC}$, $y = \frac{CB_1}{CA}$ and $z = \frac{AC_1}{AB}$. Then

$$u = 1 - (x+y+z) + xy + yz + zx \quad \text{and} \quad abc = xyz(1-x)(1-y)(1-z) = v(u-v),$$

where $v = xyz$. Therefore, we pass to inequality $u^2 \geq 4v(u-v)$, i.e., $(u-2v)^2 \geq 0$ which is obvious.

10.58. a) Let $x = \frac{BA_1}{BC}$, $y = \frac{BC_1}{BA}$ and $z = \frac{AC_1}{AB}$. We may assume that the area of triangle ABC is equal to 1. Then $S_{AB_1C_1} = z(1-y)$, $S_{A_1BC_1} = x(1-z)$ and $S_{A_1B_1C} = y(1-x)$. Since $x(1-x) \leq \frac{1}{4}$, $y(1-y) \leq \frac{1}{4}$ and $z(1-z) \leq \frac{1}{4}$, it follows that the product of numbers $S_{AB_1C_1}$, $S_{A_1BC_1}$ and $S_{A_1B_1C}$ does not exceed $\left(\frac{1}{4}\right)^3$; hence, one of them does not exceed $\frac{1}{4}$.

b) Let, for definiteness, $x \geq \frac{1}{2}$. If $y \leq \frac{1}{2}$, then the homothety with center C and coefficient 2 sends points A_1 and B_1 to inner points on sides BC and AC, consequently, $S_{A_1B_1C} \leq S_{A_1B_1C_1}$. Hence, we can assume that $y \geq \frac{1}{2}$ and, similarly, $z \geq \frac{1}{2}$. Let $x = \frac{1}{2}(1+\alpha)$, $y = \frac{1}{2}(1+\beta)$ and $z = \frac{1}{2}(1+\gamma)$. Then $S_{AB_1C_1} = \frac{1}{4}(1 + \gamma - \beta - \beta\gamma)$, $S_{A_1B_1C_1} = \frac{1}{4}(1 + \alpha - \gamma - \alpha\gamma)$ and $S_{A_1B_1C} = \frac{1}{4}(1 + \beta - \alpha - \alpha\beta)$; hence, $S_{A_1B_1C_1} = \frac{1}{4}(1 + \alpha\beta + \beta\gamma + \alpha\gamma) \geq \frac{1}{4}$ and $S_{AB_1C_1} + S_{A_1BC_1} + S_{A_1B_1C} \leq \frac{3}{4}$.

10.59. It suffices to prove that if $AC < BC$, then $\angle ABC < \angle BAC$. Since $AC < BC$, on side BC point A_1 can be selected so that $A_1C = AC$. Then $\angle BAC < \angle AAC = \angle AA_1C > \angle ABC$.

10.60. Let A_1 be the midpoint of side BC. If $AA_1 < \frac{1}{2}BC = BA_1 = A_1C$, then $\angle BAA_1 > \angle ABA_1$ and $\angle CAA_1 > \angle ACA_1$; hence, $\angle A = \angle BAA_1 + \angle CAA_1 < \angle B + \angle C$, i.e., $\angle A > 90^\circ$. Similarly, if $AA_1 > \frac{1}{2}BC$ then $\angle A > 90^\circ$.

10.61. If we fix two sides of the triangle, then the greater the angle between these sides the longer the third side. Therefore, inequality $\angle A > \angle A_1$ implies that $BD > B_1D_1$, i.e., $\angle C < \angle C_1$. Now, suppose that $\angle B \geq \angle B_1$. Then $AC \geq A_1C_1$, i.e., $\angle D \geq \angle D_1$. Hence,

$$360^\circ = \angle A + \angle B + \angle C + \angle D > \angle A_1 + \angle B_1 + \angle C_1 + \angle D_1 = 360^\circ.$$

Contradiction; therefore, $\angle B < \angle B_1$ and $\angle D < \angle D_1$.

10.62. Let point B_1 be symmetric to B through point M. Since the height dropped from point M to side BC is equal to a half of AH, i.e., to a half of BM, it follows that $\angle MBC = 30^\circ$. Since AH is the longest of heights, BC is the shortest of sides. Hence, $AB_1 = BC \leq AB$, i.e., $\angle ABB_1 \leq \angle AB_1B = \angle MBC = 30^\circ$. Therefore, $\angle ABC = \angle ABB_1 + \angle MBC \leq 30^\circ + 30^\circ = 60^\circ$.

10.63. First, let us suppose that $\angle A > \angle D$. Then $BE > EC$ and $\angle EBA < \angle ECD$. Since in triangle EBC side BE is longer than side EC, it follows that $\angle EBC < \angle ECB$. Therefore,

$$\angle B = \angle ABE + \angle EBC < \angle ECD + \angle ECB = \angle C$$

which contradicts the hypothesis. Thus, $\angle A = \angle B = \angle C = \angle D$. Similarly, the assumption $\angle B > \angle E$ leads to inequality $\angle C < \angle D$. Hence, $\angle B = \angle C = \angle D = \angle E$.

10.64. Let us carry out the proof for the general case. Let line MN intersect the sides of the polygon at points M_1 and N_1. Clearly, $MN \leq M_1N_1$. Let point M_1 lie on side AB and point N_1 lie on PQ. Since $\angle AM_1N_1 + \angle BM_1N_1 = 180^\circ$, one of these angles is not less than 90°. Let, for definiteness, $\angle AM_1N_1 \geq 90^\circ$. Then $AN_1 \geq M_1N_1$ because the longer side subtends the greater angle.

We similarly prove that either $AN_1 \leq AP$ or $AN_1 \leq AQ$. Therefore, the length of segment MN does not exceed the length of a segment with the endpoints at vertices of the polygon.

10.65. The segment can be extended to its intersection with the boundary of the sector because this will only increase its length. Therefore, we may assume that points M and N lie on the boundary of the disk sector. The following three cases are possible:

1) Points M and N lie on an arc of the circle. Then

$$MN = 2R\sin\frac{\angle MON}{2} \leq 2R\sin\frac{\angle AOB}{2} = AB$$

because $\frac{1}{2}\angle MON \leq \frac{1}{2}\angle AOB \leq 90^\circ$.

2) Points M and N lie on segments AO and BO, respectively. Then MN is not longer than the longest side of triangle AOB.

3) One of points M and N lies on an arc of the circle, the other one on one of segments AO or BO. Let, for definiteness, M lie on AO and N on an arc of the circle. Then the length of MN does not exceed that of the longest side of triangle ANO. It remains to notice that $AO = NO = R$ and $AN \leq AB$.

10.66. If the given segment has no common points with the circle, then a homothety with center A (and coefficient greater than 1) sends it into a segment that has a common point X with arc AB and lies in our domain. Let us draw through point X tangent DE to the circle (points D and E lie on segments AB and AC). Then segments AD and AE are shorter than AB and $DE < \frac{1}{2}(DE + AD + AE) = AB$, i.e., each side of triangle ADE is shorter than AB. Since our segment lies inside triangle ADE (or on its side DE), its length does not exceed that of AB.

10.67. First, suppose that the center O of the circle lies inside the given pentagon $A_1A_2A_3A_4A_5$. Consider angles $\angle A_1OA_2$, $\angle A_2OA_3$, $\dots$, $\angle A_5OA_1$. The sum of these five angles is equal to 2π; hence, one of them, say, $\angle A_1OA_2$, does not exceed $\frac{2}{5}\pi$. Then segment A_1A_2 can be placed in disk sector OBC, where $\angle BOC = \frac{2}{5}\pi$ and points B and C lie on the circle. In triangle OBC, side BC is the longest one; hence, $A_1A_2 \leq BC$.

If point O does not belong to the given pentagon, then the union of angles $\angle A_1OA_2$, $\dots$, $\angle A_5OA_1$ is less than π and each point of the angle — the union — is covered twice by these angles. Therefore, the sum of these five angles is less than 2π, i.e., one of them is less than $\frac{2}{5}\pi$. The continuation of the proof is similar to the preceding case.

If point O lies on a side of the pentagon, then one of the considered angles is not greater than $\frac{1}{4}\pi$ and if it is its vertex, then one of them is not greater than $\frac{1}{3}\pi$. Clearly, $\frac{1}{4}\pi < \frac{1}{3}\pi < \frac{2}{5}\pi$.

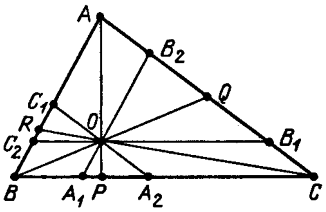


Figure 121 (Sol. 10.68)

10.68. On sides BC, CA, AB take points A_1 and A_2, B_1 and B_2, C_1 and C_2, respectively, so that $B_1C_2 \parallel BC$, $C_1A_2 \parallel CA$, $A_1B_2 \parallel AB$ (Fig. 121). In triangles A_1A_2O, B_1B_2O, C_1C_2O sides A_1A_2, B_1O, C_2O, respectively, are the longest ones. Hence, $OP < A_1A_2$, $OQ < BO$, $OR \leq C_2O$, i.e.,

$$OP + OQ + OR < A_1A_2 + B_1O + C_2O = A_1A_2 + CA_2 + BA_1 = BC.$$

10.69. Since $c^2 = a^2 + b^2$, it follows that

$$c^n = (a^2 + b^2)c^{n-2} = a^2c^{n-2} + b^2c^{n-2} > a^n + b^n.$$

10.70. The height of any of the triangles considered is longer than $2r$. Moreover, in a right triangle $2r = a + b - c$ (Problem 5.15).

10.71. Since $ch = 2S = r(a + b + c)$ and $c = \sqrt{a^2 + b^2}$, it follows that $\frac{r}{h} = \frac{\sqrt{a^2+b^2}}{a+b+\sqrt{a^2+b^2}} = \frac{1}{x+1}$, where $x = \frac{a+b}{\sqrt{a^2+b^2}} = \sqrt{1 + \frac{2ab}{a^2+b^2}}$. Since $0 < \frac{2ab}{a^2+b^2} \leq 1$, it follows that $1 < x \leq \sqrt{2}$. Hence, $\frac{2}{5} < \frac{1}{1+\sqrt{2}} \leq \frac{r}{h} < \frac{1}{2}$.

10.72. Clearly, $a + b \geq 2\sqrt{ab}$ and $c^2 + a^2 + b^2 \geq 2ab$. Hence,

$$\frac{c^2}{r^2} = \frac{(a + b + c)^2c^2}{a^2b^2} \geq \frac{(2\sqrt{ab} + \sqrt{2ab})^2 \cdot 2ab}{a^2b^2} = 4(1 + \sqrt{2})^2.$$

10.73. By Problem 12.11 a) $m_a^2 + m_b^2 = \frac{1}{4}(4c^2 + a^2 + b^2) = \frac{5}{4}c^2$. Moreover,

$$\frac{5c^2}{4} \geq 5(1 + \sqrt{2})^2r^2 = (15 + 10\sqrt{2})r^2 > 29r^2,$$

cf. Problem 10.72.

10.74. Let O be the center of the circumscribed circle, A_1, B_1, C_1 the midpoints of sides BC, CA, AB, respectively. Then $m_a = AA_1 \leq AO + OA_1 = R + OA_1$. Similarly,

$m_b \leq R + OB_1$ and $m_c \leq R + OC_1$. Hence,

$$\frac{m_a}{h_a} + \frac{m_b}{h_b} + \frac{m_c}{h_c} \leq R\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) + \frac{OA_1}{h_a} + \frac{OB_1}{h_b} + \frac{OC_1}{h_c}.$$

It remains to make use of the result of Problem 12.22 and the solution of Problem 4.46.

10.75. By Problem 4.47 $\frac{1}{b} + \frac{1}{c} = \frac{2\cos(\alpha/2)}{l_a} \geq \frac{\sqrt{2}}{l_a}$. Adding three analogous inequalities we get the required statement.

10.76. Denote the intersection point of medians by M and the center of the circumscribed circle by O. If triangle ABC is not an obtuse one, then point O lies inside it (or on its side); let us assume, for definiteness, that it lies inside triangle AMB. Then $AO+BO \leq AM+BM$, i.e., $2R \leq \frac{2}{3}m_a + \frac{2}{3}m_b$ or, which is the same, $m_a + m_b \geq 3R$. It remains to notice that since angle $\angle COC_1$ (where C_1 is the midpoint of AB) is obtuse, it follows that $CC_1 \geq CO$, i.e., $m_c \geq R$.

The equality is attained only for a degenerate triangle.

10.77. In any triangle $h_b \leq l_b \leq m_b$ (cf. Problem 2.67); hence, $h_a = l_b \geq h_b$ and $m_c = l_b \leq m_b$. Therefore, $a \leq b$ and $b \leq c$ (cf. Problem 10.1), i.e., c is the length of the longest side and γ is the greatest angle.

The equality $h_a = m_c$ yields $\gamma \leq 60^\circ$ (cf. Problem 10.62). Since the greatest angle γ of triangle ABC does not exceed 60°, all the angles of the triangle are equal to 60°.

10.78. By Problem 1.59 the ratio of the perimeters of triangles $A_1B_1C_1$ and ABC is equal to $\frac{r}{R}$. Moreover, $r \leq \frac{R}{2}$ (Problem 10.26).

Remark. Making use of the result of Problem 12.72 it is easy to verify that $\frac{S_{A_1B_1C_1}}{S_{ABC}} = \frac{r_1}{2R_1} \leq \frac{1}{4}$.

10.79. Let $90^\circ \geq \alpha \geq \beta \geq \gamma$, then CH is the longest height. Denote the centers of the inscribed and circumscribed circles by I and O, the tangent points of the inscribed circle with sides BC, CA, AB by K, L, M, respectively (Fig. 122).

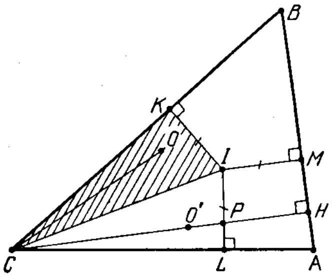


Figure 122 (Sol. 10.79)

First, let us prove that point O lies inside triangle KCI. For this it suffices to prove that $CK \geq KB$ and $\angle BCO \leq \angle BCI$. Clearly, $CK = r\cot\frac{\gamma}{2} \geq r\cot\frac{\beta}{2} = KB$ and

$$2\angle BCO = 180^\circ - \angle BOC = 180^\circ - 2\alpha \leq 180^\circ - \alpha - \beta = \gamma = 2\angle BCI.$$

Since $\angle BCO = 90^\circ - \alpha = \angle ACH$, the symmetry through CI sends line CO to line CH. Let O' be the image of O under this symmetry and P the intersection point of CH and IL. Then $CP \geq CO' = CO = R$. It remains to prove that $PH \geq IM = r$. It follows from the fact that $\angle MIL = 180^\circ - \alpha \geq 90^\circ$.

10.80. Let B_2C_2 be the projection of segment B_1C_1 on side BC. Then

$$BC_1 \geq B_2C_2 = BC - BC_1\cos\beta - CB_1\cos\gamma.$$

Similarly,

$$A_1C_1 \geq AC - AC_1\cos\alpha - CA_1\cos\gamma;$$
$$A_1B_1 \geq AB - AB_1\cos\alpha - BA_1\cos\beta.$$

Let us multiply these inequalities by $\cos\alpha$, $\cos\beta$ and $\cos\gamma$, respectively, and add them; we get

$$B_1C_1\cos\alpha + C_1A_1\cos\beta + AB_1\cos\gamma \geq a\cos\alpha + b\cos\beta + c\cos\gamma - \\ -(a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma + c\cos\alpha\cos\beta).$$

Since $c = a\cos\beta + b\cos\alpha$, it follows that $c\cos\gamma = a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma$. Write three analogous inequalities and add them; we get

$$a\cos\beta\cos\gamma + b\cos\alpha\cos\gamma + c\cos\alpha\cos\beta = \frac{a\cos\alpha + b\cos\beta + c\cos\gamma}{2}.$$

10.81. Since

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$$

(Problem 12.39 b)), it follows that triangle ABC is an acute one if and only if $\cos^2\alpha + \cos^2\beta + \cos^2\gamma < 1$, i.e., $\sin^2\alpha + \sin^2\beta + \sin^2\gamma > 2$. Multiplying both sides of the latter inequality by $4R^2$ we get the desired statement.

10.82. It suffices to notice that

$$p^2 - (2R + r)^2 = 4R^2\cos\alpha\cos\beta\cos\gamma$$

(cf. Problem 12.41 b).

10.83. Let $\angle A \leq \angle B \leq \angle C$. If triangle ABC is not an acute one, then $CC_1 < AC < AA_1$ for any points A_1 and C_1 on sides BC and AB, respectively. Now, let us prove that for an acute triangle we can select points A_1, B_1 and C_1 with the required property. For this it suffices to verify that there exists a number x satisfying the following inequalities:

$$h_a \leq x < \max(b, c) = c, \quad h_b \leq x < \max(a, c) = c \text{ and } h_c \leq x < \max(a, b) = b.$$

It remains to notice that $\max(h_a, h_b, h_c) = h_a$, $\min(b, c) = b$ and $h_a < h$.

10.84. Let $\angle A \leq \angle B \leq \angle C$. First, suppose that triangle ABC is an acute one. As line l that in its initial position is parallel to AB rotates, the length of the triangle's projection on l first varies monotonously from c to h_b, then from h_b to a, then from a to h_c, next from h_c to b, then from b to h_a and, finally, from h_a to c. Since $h_b < a$, there exists a number x such that $h_b < x < a$. It is easy to verify that a segment of length x is encountered on any of the first four intervals of monotonity.

Now, suppose that triangle ABC is not an acute one. As line l that in its initial position is parallel to AB rotates, the length of the triangle's projection on l monotonously decreases first from c to h_b, then from h_b to h_c; after that it monotonously increases, first, from h_c to h_a, then from h_a to c. Altogether we have two intervals of monotonity.

10.85. Let points M and N lie on sides AB and AC, respectively. Let us draw through vertex C the line parallel to side AB. Let N_1 be the intersection point of this line with MN. Then $N_1O : MO = 2$ but $NO \leq N_1O$; hence, $NO : MO \leq 2$.

10.86. Sircle S inscribed in triangle ABC lies inside triangle $A'B'C'$. Draw the tangents to this circle parallel to sides of triangle $A'B'C'$; we get triangle $A''B''C''$ similar to triangle $A'B'C'$ and S is the inscribed circle of triangle $A''B''C''$. Hence, $r_{ABC} = r_{A''B''C''} < r_{A'B'C'}$.

10.87. The bisector l_c divides triangle ABC into two triangles whose doubled areas are equal to $al_c\sin\frac{\gamma}{2}$ and $bl_c\sin\frac{\gamma}{2}$. Hence, $ah_a = 2S = l_c(a + b)\sin\frac{\gamma}{2}$. The conditions of the problem imply that $\frac{a}{a+b} \leq \frac{1}{2} \leq \sin\frac{\gamma}{2}$.

10.88. Clearly, $\cot\angle A + \cot\angle B = \frac{c}{h_c} \geq \frac{c}{m_c}$. Let M be the intersection point of the medians, N the midpoint of segment AB. Since triangle AMB is a right one, $MN = \frac{1}{2}AB$. Therefore, $c = 2MN = \frac{2}{3}m_c$.

10.89. Since $BN\cdot BA = BM^2$ and $BM < BA$, it follows that $BN < BM$ and, therefore, $AN > CN$.

10.90. Let us draw through point B the perpendicular to side AB. Let F be the intersection point of this perpendicular with the extension of side AC (Fig. 123). Let us prove that bisector AD, median BM and height CH intersect at one point if and only if $AB = CF$. Indeed, let L be the intersection point of BM and CH. Bisector AD passes through point L if and only if $BA : AM = BL : LM$ but $BL : LM = FC : CM = FC : AM$.

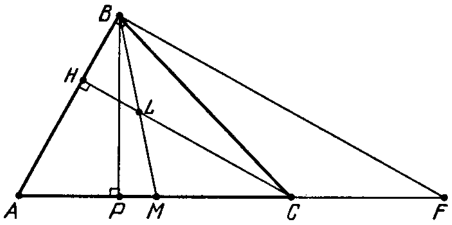


Figure 123 (Sol. 10.90)

If on side AF of right triangle ABF ($\angle ABF = 90^\circ$) segment CF equal to AB is marked, then angles $\angle BAC$ and $\angle ABC$ are acute ones. It remains to find out when angle $\angle ACB$ is acute.

Let us drop perpendicular BP from point B to side AF. Angle ACB is an acute one if $FP > FC = AB$, i.e., $BF\sin\angle A > BF\cot\angle A$. Therefore, $1-\cos^2\angle A = \sin^2\angle A > \cos\angle A$, i.e., $\cos A < \frac{1}{2}(\sqrt{5}-1)$. Finally, we see that

$$90^\circ > \angle A > \arccos\frac{\sqrt{5}-1}{2} \approx 51^\circ 50'.$$

10.91. Since the greater angle subtends the longer side,

$$(a-b)(\alpha-\beta) \geq 0,\ (b-c)(\beta-\gamma) \geq 0 \text{ and } (a-c)(\alpha-\gamma) \geq 0.$$

Adding these inequalities we get

$$\begin{gathered} 2(a\alpha + b\beta + c\gamma) \geq a(\beta+\gamma) + b(\alpha+\gamma) + c(\alpha+\beta) = \\ (a+b+c)\pi - a\alpha - b\beta - c\gamma, \end{gathered}$$

i.e., $\frac{1}{3}\pi \leq \frac{a\alpha+b\beta+c\gamma}{a+b+c}$. The triangle inequality implies that

$$\alpha(b+c-a) + \beta(a+c-b) + \gamma(a+b-c) > 0,$$

i.e.,

$$a(\beta+\gamma-\alpha) + b(\alpha+\gamma-\beta) + c(\alpha+\beta-\gamma) > 0.$$

Since $\alpha+\beta+\gamma = \pi$, it follows that $a(\pi-2\alpha)+b(\pi-2\beta) = c(\pi-2\gamma) > 0$, i.e., $\frac{a\alpha+b\beta+c\gamma}{a+b+c} < \frac{1}{2}\pi$.

10.92. On rays OB and OC, take points C_1 and B_1, respectively, such that $OC_1 = OC$ and $OB_1 = OB$. Let B_2 and C_2 be the projections of points B_1 and C_1, respectively, on a line perpendicular to AO. Then

$$BO\sin\angle AOC + CO\sin\angle AOB = B_2C_2 \leq BC.$$

Adding three analogous inequalities we get the desired statement. It is also easy to verify that the conditions $B_1C_1 \perp AO$, $C_1A_1 \perp BO$ and $A_1B_1 \perp CO$ are equivalent to the fact that O is the intersection point of the bisectors.

10.93. Since $\angle CBD = \frac{1}{2}\angle C$ and $\angle B \leq \angle A$, it follows that $\angle ABD = \angle B + \angle CBD \geq \frac{1}{2}(\angle A + \angle B + \angle C) = 90^\circ$.

10.94. By the bisector's property, $BM : MA = BC : CA$ and $BK : KC = BA : AC$. Hence, $BM : MA < BK : KC$, i.e.,

$$\frac{AB}{AM} = 1 + \frac{BM}{MA} < 1 + \frac{BK}{KC} = \frac{CB}{CK}.$$

Therefore, point M is more distant from line AC than point K, i.e., $\angle AKM < \angle KAC = \angle KAM$ and $\angle KMC < \angle MCA = \angle MCK$. Hence, $AM > MK$ and $MK > KC$, cf. Problem 10.59.

10.95. Suppose that all the given ratios are less than 2. Then

$$S_{ABO} + S_{AOC} < 2S_{XBO} + 2S_{XOC} = 2S_{OBC},$$
$$S_{ABO} + S_{OBC} < 2S_{AOC},\ S_{AOC} + S_{OBC} < 2S_{ABO}.$$

Adding these inequalities we come to a contradiction. We similarly prove that one of the given ratios is not greater than 2.

10.96. Denote the radii of the circles S, S_1 and S_2 by r, r_1 and r_2, respectively. Let triangles AB_1C_1 and A_2BC_2 be similar to triangle ABC with similarity coefficients $\frac{r_1}{r}$ and $\frac{r_2}{r}$, respectively. Circles S_1 and S_2 are the inscribed circles of triangles AB_1C_1 and A_2BC_2, respectively. Therefore, these triangles intersect because otherwise circles S_1 and S_2 would not have had common points. Hence, $AB_1 + A_2B > AB$, i.e., $r_1 + r_2 > r$.

Chapter 11. PROBLEMS ON MAXIMUM AND MINIMUM

Background

1) Geometric problems on maximum and minimum are in close connection with geometric inequalities because in order to solve these problems we always have to prove a corresponding geometric inequality and, moreover, to prove that sometimes it turns into an equality. Therefore, before solving problems on maximum and minimum we have to skim through Supplement to Ch. 9 once again with the special emphasis on the conditions under which strict inequalities become equalities.

2) For elements of a triangle we use the standard notations.

3) Problems on maximum and minimum are sometimes called *extremal problems* (from Latin *extremum*).

Introductory problems

1. Among all triangles ABC with given sides AB and AC find the one with the greatest area.

2. Inside triangle ABC find the vertex of the smallest angle that subtends side AB.

3. Prove that among all triangles with given side a and height h_a. an isosceles triangle is the one with the greatest value of angle α.

4. Among all triangles with given sides AB and AC ($AB < AC$), find the one for which the radius of the circumscribed circle is maximal.

5. The iagonals of a convex quadrilateral are equal to d_1 and d_2. What the greatest value the quadrilateral's area can attain?

§1. The triangle

11.1. Prove that among all the triangles with fixed angle α and area S, an isosceles triangle with base BC has the shortest length of side BC.

11.2. Prove that among all triangles with fixed angle α and semiperimeter p, an isosceles triangle with base BC is of the greatest area.

11.3. Prove that among all the triangles with fixed semiperimeter p, an equilateral triangle has the greatest area.

11.4. Consider all the acute triangles with given side a and angle α. What is the maximum of $b^2 + c^2$?

11.5. Among all the triangles inscribed in a given circle find the one with the maximal sum of squared lengths of the sides.

11.6. The perimeter of triangle ABC is equal to $2p$. On sides AB and AC points M and N, respectively, are taken so that $MN \parallel BC$ and MN is tangent to the inscribed circle of triangle ABC. Find the greatest value of the length of segment MN.

11.7. Into a given triangle place a centrally symmetric polygon of greatest area.

11.8. The area of triangle ABC is equal to 1. Let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. On segments AB_1, CA_1, BC_1, points K, L, M, respectively, are taken. What is the least area of the common part of triangles KLM and $A_1B_1C_1$?

11.9. What least width From an infinite strip of paper any triangle of area 1 can be cut. What is the least width of such a strip?

* * *

11.10. Prove that triangles with the lengths of sides a, b, c and a_1, b_1, c_1, respectively, are similar if and only if

$$\sqrt{aa_1}+\sqrt{bb_1}+\sqrt{cc_1}=\sqrt{(a+b+c)(a_1+b_1+c_1)}.$$

11.11. Prove that if α, β, γ and α_1, β_1, γ_1 are the respective angles of two triangles, then

$$\frac{\cos\alpha_1}{\sin\alpha}+\frac{\cos\beta_1}{\sin\beta}+\frac{\cos\gamma_1}{\sin\gamma}\leq\cot\alpha+\cot\beta+\cot\gamma.$$

11.12. Let a, b and c be the lengths of the sides of a triangle of area S; let α_1, β_1 and γ_1 be the angles of another triangle. Prove that

$$a^2\cot\alpha_1+b^2\cot\beta_1+c^2\cot\gamma_1\geq 4S,$$

where the equality is attained only if the considered triangles are similar.

11.13. In a triangle $a\geq b\geq c$; let x, y and z be the angles of another triangle. Prove that

$$bc+ca-ab<bc\cos x+ca\cos y+ab\cos z\leq\frac{a^2+b^2+c^2}{2}.$$

See also Problem 17.21.

§2. Extremal points of a triangle

11.14. On hypothenuse AB of right triangle ABC point X is taken; M and N are the projections of X on legs AC and BC, respectively.

a) What is the position of X for which the length of segment MN is the smallest one?

b) What is the position of point X for which the area of quadrilateral $CMXN$ is the greatest one?

11.15. From point M on side AB of an acute triangle ABC perpendiculars MP and MQ are dropped to sides BC and AC, respectively. What is the position of point M for which the length of segment PQ is the minimal one?

11.16. Triangle ABC is given. On line AB find point M for which the sum of the radii of the circumscribed circles of triangles ACM and ACN is the least possible one.

11.17. From point M of the circumscribed circle of triangle ABC perpendiculars MP and MQ are dropped on lines AB and AC, respectively. What is the position of point M for which the length of segment PQ is the maximal one?

11.18. Inside triangle ABC, point O is taken. Let d_a, d_b, d_c be distances from it to lines BC, CA, AB, respectively. What is the position of point O for which the product $d_ad_bd_c$ is the greatest one?

11.19. Points A_1, B_1 and C_1 are taken on sides BC, CA and AB, respectively, of triangle ABC so that segments AA_1, BB_1 and CC_1 meet at one point M. For what position of point M the value of $\frac{MA_1}{AA_1}\cdot\frac{MB_1}{BB_1}\cdot\frac{MC_1}{CC_1}$ is the maximal one?

11.20. From point M inside given triangle ABC perpendiculars MA_1, MB_1, MC_1 are dropped to lines BC, CA, AB, respectively. What are points M inside the given triangle ABC for which the quantity $\frac{a}{MA_1}+\frac{b}{MB_1}+\frac{c}{MC_1}$ takes the least possible value?

11.21. Triangle ABC is given. Find a point O inside of it for which the sum of lengths of segments OA, OB, OC is the minimal one. (Take a special heed to the case when one of the angles of the triangle is greater than 120°.)

11.22. Inside triangle ABC find a point O for which the sum of squares of distances from it to the sides of the triangle is the minimal one.

See also Problem 18.21 a).

§3. The angle

11.23. On a leg of an acute angle points A and B are given. On the other leg construct point C the vertex of the greatest angle that subtends segment AB.

11.24. Angle $\angle XAY$ and point O inside it are given. Through point O draw a line that cuts off the given angle a triangle of the least area.

11.25. Through given point P inside angle $\angle AOB$ draw line MN so that the value $OM + ON$ is minimal (points M and N lie on legs OA and OB, respectively).

11.26. Angle $\angle XAY$ and a circle inside it are given. On the circle construct a point the sum of the distances from which to lines AX and AY is the least.

11.27. A point M inside acute angle $\angle BAC$ is given. On legs BA and AC construct points X and Y, respectively, such that the perimeter of triangle XYM is the least.

11.28. Angle $\angle XAY$ is given. The endpoints B and C of unit segments BO and CO move along rays AX and AY, respectively. Construct quadrilateral $ABOC$ of the greatest area.

§4. The quadrilateral

11.29. Inside a convex quadrilateral find a point the sum of distances from which to the vertices were the least one.

11.30. The diagonals of convex quadrilateral $ABCD$ intersect at point O. What least area can this quadrilateral have if the area of triangle AOB is equal to 4 and the area of triangle COD is equal to 9?

11.31. Trapezoid $ABCD$ with base AD is cut by diagonal AC into two triangles. Line l parallel to the base cuts these triangles into two triangles and two quadrilaterals. What is the position of line l for which the sum of areas of the obtained triangles is the minimal one?

11.32. The area of a trapezoid is equal to 1. What is the least value the length of the longest diagonal of this trapezoid can attain?

11.33. On base AD of trapezoid $ABCD$ point K is given. On base BC find point M for which the area of the common part of triangles AMD and BKC is maximal.

11.34. Prove that among all quadrilaterals with fixed lengths of sides an inscribed quadrilateral has the greatest area.

See also Problems 9.35, 15.3 b).

§5. Polygons

11.35. A polygon has a center of symmetry, O. Prove that the sum of the distances from a point to the vertices attains its minimum at point O.

11.36. Among all the polygons inscribed in a given circle find the one for which the sum of squared lengths of its sides is minimal.

11.37. A convex polygon $A_1 \dots A_n$ is given. Prove that a point of the polygon for which the sum of distances from it to all the vertices is maximal is a vertex.

See also Problem 6.69.

§6. Miscellaneous problems

11.38. Inside a circle centered at O a point A is given. Find point M on the circle for which angle $\angle OMA$ is maximal.

11.39. In plane, line l and points A and B on distinct sides of l are given. Construct a circle that passes through points A and B so that line l intercepts on the circle a shortest chord.

11.40. Line l and points P and Q lying on one side of l are given. On line l, take point M and in triangle PQM draw heights PP' and QQ'. What is the position of point M for which segment $P'Q'$ is the shortest?

11.41. Points A, B and O do not lie on one line. Through point O draw line l so that the sum of distances from it to points A and B were: a) maximal; b) minimal.

11.42. If five points in plane are given, then considering all possible triples of these points we can form 30 angles. Denote the least of these angles by α. Find the greatest value of α.

11.43. In a town there are 10 streets parallel to each other and 10 streets that intersect them at right angles. A closed bus route passes all the road intersections. What is the least number of turns such a bus route can have?

11.44. What is the greatest number of cells on a 8×8 chessboard that one straight line can intersect? (An intersection should have a common *inner* point.)

11.45. What is the greatest number of points that can be placed on a segment of length 1 so that on any segment of length d contained in this segment not more than $1 + 1000d^2$ points lie?

See also Problems 15.1, 17.20.

§7. The extremal properties of regular polygons

11.46. a) Prove that among all n-gons circumscribed about a given circle a regular n-gon has the least area.

b) Prove that among all the n-gons circumscribed about a given circle a regular n-gon has the least perimeter.

11.47. Triangles ABC_1 and ABC_2 have common base AB and $\angle AC_1B = \angle AC_2B$. Prove that if $|AC_1 - C_1B| < |AC_2 - C_2B|$, then:

a) the area of triangle ABC_1 is greater than the area of triangle ABC_2;

b) the perimeter of triangle ABC_1 is greater than the perimeter of triangle ABC_2.

11.48. a) Prove that among all the n-gons inscribed in a given circle a regular n-gon has the greatest area.

b) Prove that among all n-gons inscribed in a given circle a regular n-gon has the greatest perimeter.

Problems for independent study

11.49. On a leg of an acute angle with vertex A point B is given. On the other leg construct point X such that the radius of the circumscribed circle of triangle ABX is the least possible.

11.50. Through a given point inside a (given?) circle draw a chord of the least length.

11.51. Among all triangles with a given sum of lengths of their bisectors find a triangle with the greatest sum of lengths of its heights.

11.52. Inside a convex quadrilateral find a point the sum of squared distances from which to the vertices is the least possible.

11.53. Among all triangles inscribed in a given circle find the one for which the value $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is the least possible.

11.54. On a chessboard with the usual coloring draw a circle of the greatest radius so that it does not intersect any white field.

11.55. Inside a square, point O is given. Any line that passes through O cuts the square into two parts. Through point O draw a line so that the difference of areas of these parts were the greatest possible.

11.56. What is the greatest length that the shortest side of a triangle inscribed in a given square can have?

11.57. What greatest area can an equilateral triangle inscribed in a given square can have?

Solutions

11.1. By the law of cosines

$$a^2 = b^2 + c^2 - 2bc\cos\alpha = (b-c)^2 + 2bc(1-\cos\alpha) =$$
$$(b-c)^2 + \frac{4S(1-\cos\alpha)}{\sin\alpha}.$$

Since the last summand is constant, a is minimal if $b = c$.

11.2. Let an escribed circle be tangent to sides AB and AC at points K and L, respectively. Since $AK = AL = p$, the escribed circle S_a is fixed. The radius r of the inscribed circle is maximal if it is tangent to circle S_a, i.e., triangle ABC is an isosceles one. It is also clear that $S = pr$.

11.3. By Problem 10.53 a) we have $S \leq \frac{p^2}{3\sqrt{3}}$, where the equality is only attained for an equilateral triangle.

11.4. By the law of cosines $b^2 + c^2 = a^2 + 2bc\cos\alpha$. Since $2bc \leq b^2 + c^2$ and $\cos\alpha > 0$, it follows that $b^2 + c^2 \leq a^2 + (b^2 + c^2)\cos\alpha$, i.e., $b^2 + c^2 \leq \frac{a^2}{1-\cos\alpha}$. The equality is attained if $b = c$.

11.5. Let R be the radius of the given circle, O its center; let A, B and C be the vertices of the triangle; $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$. Then

$$AB^2 + BC^2 + CA^2 = |\mathbf{a}-\mathbf{b}|^2 + |\mathbf{b}-\mathbf{c}|^2 + |\mathbf{c}-\mathbf{a}|^2 =$$
$$2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) - -2(\mathbf{a},\mathbf{b}) - 2(\mathbf{b},\mathbf{c}) - 2(\mathbf{c},\mathbf{a}).$$

Since

$$|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a},\mathbf{b}) + 2(\mathbf{b},\mathbf{c}) + 2(\mathbf{c},\mathbf{a}),$$

it follows that

$$AB^2 + BC^2 + CA^2 = 3(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) - |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 \leq$$
$$3(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2) = 9R^2,$$

where the equality is only attained if $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$. This equality means that triangle ABC is an equilateral one.

11.6. Denote the length of the height dropped on side BC by h. Since $\triangle AMN \sim \triangle ABC$, it follows that $\frac{MN}{BC} = \frac{h-2r}{h}$, i.e. $MN = a\left(1 - \frac{2r}{h}\right)$. Since $r = \frac{S}{p} = \frac{ah}{2p}$, we deduce that $MN = a\left(1 - \frac{a}{p}\right)$. The maximum of the quadratic expression $a\left(1 - \frac{a}{p}\right) = \frac{a(p-a)}{p}$ in a

is attained for $a = \frac{1}{2}p$. This maximum is equal to $\frac{p}{4}$. It remains to notice that there exists a triangle of perimeter $2p$ with side $a = \frac{1}{2}p$ (set $b = c = \frac{3}{4}p$).

11.7. Let O be the center of symmetry of polygon M lying inside triangle T, let $S(T)$ be the image of triangle T under the symmetry through point O. Then M lies both in T and in $S(T)$. Therefore, among all centrally symmetric polygons with the given center of symmetry lying in T the one with the greatest area is the intersection of T and $S(T)$. Point O lies inside triangle T because the intersection of T and $S(T)$ is a convex polygon and a convex polygon always contains its center of symmetry.

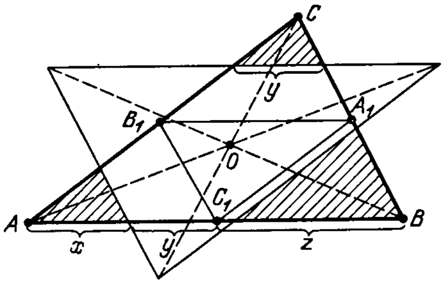


Figure 124 (Sol. 11.7)

Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively, of triangle $T = ABC$. First, let us suppose that point O lies inside triangle $A_1B_1C_1$. Then the intersection of T and $S(T)$ is a hexagon (Fig. 124). Let side AB be divided by the sides of triangle $S(T)$ in the ratio of $x : y : z$, where $x + y + z = 1$. Then the ratio of the sum of areas of the shaded triangles to the area of triangle ABC is equal to $x^2 + y^2 + z^2$ and we have to minimize this expression. Since

$$1 = (x + y + z)^2 = 3(x^2 + y^2 + z^2) - (x - y)^2 - (y - z)^2 - (z - x)^2,$$

it follows that $x^2 + y^2 + z^2 \geq \frac{1}{3}$, where the equality is only attained for $x = y = z$; the latter equality means that O is the intersection point of the medians of triangle ABC.

Now, consider another case: point O lies inside one of the triangles AB_1C_1, A_1BC_1, A_1B_1C; for instance, inside AB_1C_1. In this case the intersection of T and $S(T)$ is a parallelogram and if we replace point O with the intersection point of lines AO and B_1C_1, then the area of this parallelogram can only increase. If point O lies on side B_1C_1, then this is actually the case that we have already considered (set $x = 0$).

The polygon to be found is a hexagon with vertices at the points that divide the sides of the triangles into three equal parts. Its area is equal to $\frac{2}{3}$ of the area of the triangle.

11.8. Denote the intersection point of lines KM and BC by T and the intersection points of the sides of triangles $A_1B_1C_1$ and KLM as shown on Fig. 125.

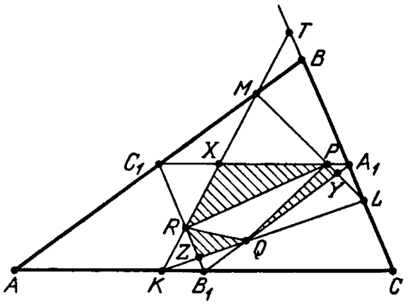


Figure 125 (Sol. 11.8)

Then $TL : RZ = KL : KZ = LC : ZB_1$. Since $TL \geq BA_1 = A_1C \geq LC$, it follows that $RZ \geq ZB_1$, i.e., $S_{RZQ} \geq S_{ZB_1Q}$. Similarly, $S_{QYP} \geq S_{YA_1P}$ and $S_{PXR} \geq S_{XC_1R}$. Adding all these inequalities and the inequality $S_{PQR} > 0$ we see that the area of hexagon $PXRZQY$ is not less than the area of the remaining part of triangle $A_1B_1C_1$, i.e., its area is not less than $\frac{S_{A_1B_1C_1}}{2} = \frac{1}{8}$. The equality is attained, for instance, if point K coincides with B_1 and point M with B.

11.9. Since the area of an equilateral triangle with side a is equal to $\frac{a^2\sqrt{3}}{4}$, the side of an equilateral triangle of area 1 is equal to $\frac{2}{\sqrt[4]{3}}$ and its height is equal to $\sqrt[4]{3}$. Let us prove that it is impossible to cut an equilateral triangle of area 1 off a strip of width less than $\sqrt[4]{3}$.

Let equilateral triangle ABC lie inside a strip of width less than $\sqrt[4]{3}$. Let, for definiteness, the projection of vertex B on the boundary of the strip lie between the projections of vertices A and C. Then the line drawn through point B perpendicularly to the boundary of the strip intersects segment AC at a point M. The length of a height of triangle ABC does not exceed BM and BM is not greater than the width of the strip and, therefore, a height of triangle ABC is shorter than $\sqrt[4]{3}$, i.e., its area is less than 1.

It remains to prove that any triangle of area 1 can be cut off a strip of width $\sqrt[4]{3}$. Let us prove that any triangle of area 1 has a height that does not exceed $\sqrt[4]{3}$. For this it suffices to prove that it has a side not shorter than $\frac{2}{\sqrt[4]{3}}$. Suppose that all sides of triangle ABC are shorter than $2\sqrt[4]{3}$. Let α be the smallest angle of this triangle. Then $\alpha \leq 60°$ and

$$S_{ABC} = \frac{AB \cdot AC \sin\alpha}{2} < \left(\frac{2}{\sqrt[4]{3}}\right)^2 \left(\frac{\sqrt{3}}{4}\right) = 1.$$

We have obtained a contradiction. A triangle that has a height not exceeding $\sqrt[4]{3}$ can be placed in a strip of width $\sqrt[4]{3}$: place the side to which this height is dropped on a boundary of the strip.

11.10. Squaring both sides of the given equality we easily reduce the equality to the form

$$(\sqrt{ab_1} - \sqrt{a_1b})^2 + (\sqrt{ca_1} - \sqrt{c_1a})^2 + (\sqrt{bc_1} - \sqrt{cb_1})^2 = 0,$$

i.e., $\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1}$.

11.11. Fix angles α, β and γ. Let $A_1B_1C_1$ be a triangle with angles α_1, β_1 and γ_1. Consider vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ codirected with vectors $\overrightarrow{B_1C_1}$, $\overrightarrow{C_1A_1}$ and $\overrightarrow{A_1B_1}$ and of length $\sin\alpha$, $\sin\beta$ and $\sin\gamma$, respectively. Then

$$\frac{\cos\alpha_1}{\sin\alpha} + \frac{\cos\beta_1}{\sin\beta} + \frac{\cos\gamma_1}{\sin\gamma} = -\frac{[(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})]}{\sin\alpha\sin\beta\sin\gamma}.$$

Since

$$2[(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})] = |\mathbf{a}+\mathbf{b}+\mathbf{c}|^2 - |\mathbf{a}|^2 - |\mathbf{b}|^2 - |\mathbf{c}|^2,$$

the quantity $(\mathbf{a},\mathbf{b}) + (\mathbf{b},\mathbf{c}) + (\mathbf{c},\mathbf{a})$ attains its minimum when $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$, i.e., $\alpha_1 = \alpha$, $\beta_1 = \beta$ and $\gamma_1 = \gamma$.

11.12. Let $x = \cot\alpha_1$ and $y = \cot\beta_1$. Then $x + y > 0$ (since $\alpha_1 + \beta_1 < \pi$) and

$$\cot\gamma_1 = \frac{1 - xy}{x + y} = \frac{x^2 + 1}{x + y} - x.$$

Therefore,

$$a^2\cot\alpha_1 + b^2\cot\beta_1 + c^2\cot\gamma_1 = (a^2 - b^2 - c^2)x + b^2(x + y) + c^2\frac{x^2 + 1}{x + y}.$$

For a fixed x this expression is minimal for a y such that $b^2(x+y) = c^2\frac{x^2+1}{x+y}$, i.e.,

$$\frac{c}{b} = \frac{x+y}{\sqrt{1+x^2}} = \sin\alpha_1(\cot\alpha_1 + \cot\beta_1) = \frac{\sin\gamma_1}{\sin\beta_1}.$$

Similar arguments show that if $a : b : c = \sin\alpha_1 : \sin\beta_1 : \sin\gamma_1$, then the considered expression is minimal. In this case triangles are similar and $a^2\cot\alpha + b^2\cot\beta + c^2\cot\gamma = 4S$, cf. Problem 12.44 b).

11.13. Let $f = bc\cos x + ca\cos y + ab\cos z$. Since $\cos x = -\cos y\cos z + \sin y\sin z$, it follows that

$$f = c(a - b\cos z)\cos y + bc\sin y\sin z + ab\cos z.$$

Consider a triangle the lengths of whose two sides are equal to a and b and the angle between them is equal to z; let ξ and η be the angles subtending sides a and b; let t be the length of the side that subtends angle z. Then

$$\cos z = \frac{a^2+b^2-t^2}{2ab} \text{ and } \cos\eta = \frac{t^2+a^2-b^2}{2at};$$

hence, $\frac{a-b\cos z}{t} = \cos\eta$. Moreover, $\frac{b}{t} = \frac{\sin\eta}{\sin z}$. Therefore, $f = ct\cos(\eta - y) + \frac{1}{2}(a^2+b^2-t^2)$.

Since $\cos(\eta - y) \leq 1$, it follows that $f \leq \frac{1}{2}(a^2+b^2+c^2) - \frac{1}{2}((c-t)^2) \leq \frac{1}{2}(a^2+b^2+c^2)$. Since $a \geq b$, it follows that $\xi \geq \eta$, consequently, $-\xi \leq -\eta < y - \eta < \pi - z - \eta = \xi$, i.e., $\cos(y - \psi) > \cos\xi$. Hence,

$$f > ct\cos\xi + \frac{a^2+b^2-t^2}{2} = \frac{c-b}{2b}t^2 + \frac{c(b^2-a^2)}{2b} + \frac{a^2+b^2}{2} = g(t).$$

The coefficient of t^2 is either negative or equal to zero; moreover, $t < a+b$. Hence, $g(t) \geq g(a+b) = bc + ca - ab$.

11.14. a) Since $CMXN$ is a rectangle, $MN = CX$. Therefore, the length of segment MN is the least possible if CX is a height.

b) Let $S_{ABC} = S$. Then $S_{AMX} = \frac{AX^2\cdot S}{AB^2}$ and $S_{BNX} = \frac{BX^2\cdot S}{AB^2}$. Since $AX^2 + BX^2 \geq \frac{1}{2}AB^2$ (where the equality is only attained if X is the midpoint of segment AB), it follows that $S_{CMXN} = S - S_{AMX} - S_{BNX} \leq \frac{1}{2}S$. The area of quadrilateral $CMXN$ is the greatest if X is the midpoint of side AB.

11.15. Points P and Q lie on the circle constructed on segment CM as on the diameter. In this circle the constant angle C intercepts chord PQ, therefore, the length of chord PQ is minimal if the diameter CM of the circle is minimal, i.e., if CM is a height of triangle ABC.

11.16. By the law of sines the radii of the circumscribed circles of triangles ACM and BCM are equal to $\frac{AC}{2\sin AMC}$ and $\frac{BC}{2\sin BMC}$, respectively. It is easy to verify that $\sin AMC = \sin BMC$. Therefore,

$$\frac{AC}{2\sin AMC} + \frac{BC}{2\sin BMC} = \frac{AC+BC}{2\sin BMC}.$$

The latter expression is minimal if $\sin BMC = 1$, i.e., $CM \perp AB$.

11.17. Points P and Q lie on the circle with diameter AM, hence, $PQ = AM\sin PAQ = AM\sin A$. It follows that the length of segment PQ is maximal if AM is a diameter of the circumscribed circle.

11.18. Clearly, $2S_{ABC} = ad_a + bd_b + cd_c$. Therefore, the product $(ad_a)(bd_b)(cd_c)$ takes its greatest value if $ad_a = bd_b = cd_c$ (cf. Supplement to Ch. 9, the inequality between the mean arithmetic and the mean geometric). Since the value abc is a constant, the product $(ad_a)(bd_b)(cd_c)$ attains its greatest value if and only if the product $d_ad_bd_c$ takes its greatest value.

Let us show that equality $ad_a = bd_b = cd_c$ means that O is the intersection point of the medians of triangle ABC. Denote the intersection point of lines AO and BC by A_1. Then

$$BA_1 : A_1C = S_{ABA_1} : S_{ACA_1} = S_{ABO} : S_{ACO} = (cd_c) : (bd_b) = 1,$$

i.e., AA_1 is a median. We similarly prove that point O lies on medians BB_1 and CC_1.

11.19. Let $\alpha = \frac{MA_1}{AA_1}$, $\beta = \frac{MB_1}{BB_1}$ and $\gamma = \frac{MC_1}{CC_1}$. Since $\alpha+\beta+\gamma = 1$ (cf. Problem 4.48 a)), we have $\sqrt{\alpha\beta\gamma} \leq \frac{1}{3}(\alpha+\beta+\gamma) = \frac{1}{3}$, where the equality is attained when $\alpha = \beta = \gamma = \frac{1}{3}$, i.e., M is the intersection point of the medians.

11.20. Let $x = MA_1$, $y = MB_1$ and $z = MC_1$. Then

$$ax + by + cz = 2S_{BMC} + 2S_{AMC} + 2S_{AMB} = 2S_{ABC}.$$

Hence,

$$\left(\tfrac{a}{x} + \tfrac{b}{y} + \tfrac{c}{z}\right) \cdot 2S_{ABC} =$$
$$\left(\tfrac{a}{x} + \tfrac{b}{y} + \tfrac{c}{z}\right)(ax+by+cz) = a^2+b^2+c^2 + ab\left(\tfrac{x}{y} + \tfrac{y}{x}\right) + bc\left(\tfrac{y}{z} + \tfrac{z}{y}\right) + ac\left(\tfrac{z}{x} + \tfrac{x}{z}\right) \geq$$
$$a^2+b^2+c^2+2ab+2bc+2ac,$$

where the equality is only attained if $x = y = z$, i.e., M is the center of the inscribed circle of triangle ABC.

11.21. First, suppose that all the angles of triangle ABC are less than 120°. Then inside triangle ABC there exists a point O — the vertex of angles of 120° that subtend each side. Let us draw through vertices A, B and C lines perpendicular to segments OA, OB and OC, respectively. These lines form an equilateral triangle $A_1B_1C_1$ (Fig. 126).

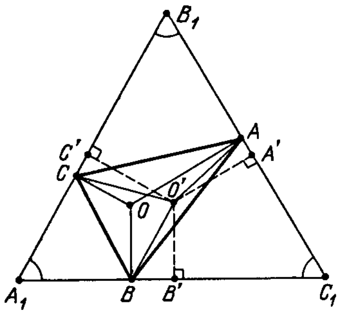


Figure 126 (Sol. 11.21)

Let O' be any point that lies inside triangle ABC and is distinct from O. Let us prove then that $O'A + O'B + O'C > OA + OB + OC$, i.e., O is the desired point. Let A', B' and C' be the bases of the perpendiculars dropped from point O' on sides B_1C_1, C_1A_1 and A_1B_1, respectively, a the length of the side of equilateral triangle $A_1B_1C_1$. Then

$$O'A' + O'B' + O'C' = \frac{2(S_{O'B_1C_1} + S_{O'A_1C_1})}{a} =$$
$$\frac{2S_{A_1B_1C_1}}{a} = OA + OB + OC.$$

Since a slanted line is longer than the perpendicular,

$$O'A + O'B + O'C > O'A' + O'B' + O'C' = OA + OB + OC.$$

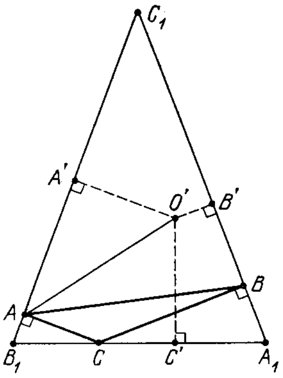


Figure 127 (Sol. 11.21)

Now, let one of the angles of triangle ABC, say $\angle C$, be greater than 120°. Let us draw through points A and B perpendiculars B_1C_1 and C_1A_1 to segments CA and CB and through point C line A_1B_1 perpendicular to the bisector of angle $\angle ACB$ (Fig. 127).

Since $\angle AC_1B = 180^\circ - \angle ACB < 60^\circ$, it follows that $B_1C_1 > A_1B_1$. Let O' be any point that lies inside triangle $A_1B_1C_1$. Since

$$B_1C_1 \cdot O'A' + C_1A_1 \cdot O'B' + A_1C_1 \cdot O'C' = 2S_{A_1B_1C_1},$$

it follows that

$$(O'A' + O'B' + O'C') \cdot B_1C_1 = 2S_{A_1B_1C_1} + (B_1C_1 - A_1B_1) \cdot O'C'.$$

Since $B_1C_1 > A_1B_1$, the sum $O'A' + O'B' + O'C'$ is minimal for points that lie on side B_1A_1. It is also clear that

$$O'A + O'B + O'C \geq O'A' + O'B' + O'C'.$$

Therefore, vertex C is the point to be found.

11.22. Let the distances from point O to sides BC, CA and AB be equal to x, y and z, respectively. Then

$$ax + by + cz = 2(S_{BOC} + S_{COA} + S_{AOB}) = 2S_{ABC}.$$

It is also clear that

$$x : y : z = \left(\frac{S_{BOC}}{a}\right) : \left(\frac{S_{COA}}{b}\right) : \left(\frac{S_{AOB}}{c}\right).$$

Equation $ax + by + cz = 2S$ determines a plane in 3-dimensional space with coordinates x, y, z; vector (a, b, c) is perpendicular to this plane because if $ax_1 + by_1 + cz_1 = 2S$ and $ax_2 + by_2 + cz_2 = 2S$, then $a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) = 0$.

We have to find a point (x_0, y_0, z_0) on this plane at which the minimum of expression $x^2 + y^2 + z^2$ is attained and verify that an inner point of the triangle corresponds to this point. Since $x^2 + y^2 + z^2$ is the squared distance from the origin to point (x, y, z), it follows that the base of the perpendicular dropped from the origin to the plane is the desired point, i.e., $x : y : z = a : b : c$. It remains to verify that inside the triangle there exists point O for which $x : y : z = a : b : c$. This equality is equivalent to the condition

$$\left(\frac{S_{BOC}}{a}\right) : \left(\frac{S_{COA}}{b}\right) : \left(\frac{S_{AOB}}{c}\right) = a : b : c,$$

i.e., $S_{BOC} : S_{COA} : S_{AOB} = a^2 : b^2 : c^2$. Since the equality $S_{BOC} : S_{AOB} = a^2 : c^2$ follows from equalities $S_{BOC} : S_{COA} = a^2 : b^2$ and $S_{COA} : S_{AOB} = b^2 : c^2$, the desired point is the intersection point of lines CC_1 and AA_1 that divide sides AB and BC, respectively, in the ratios of $BC_1 : C_1A = a^2 : b^2$ and $CA_1 : A_1B = b^2 : c^2$, respectively.

11.23. Let O be the vertex of the given angle. Point C is the tangent point of a leg with the circle that passes through points A and B, i.e., $OC^2 = OA \cdot OB$. To find the length of segment OC, it suffices to draw the tangent to any circle that passes through points A and B.

11.24. Let us consider angle $\angle X'A'Y'$ symmetric to angle $\angle XAY$ through point O. Let B and C be the intersection points of the legs of these angles. Denote the intersection points of the line that passes through point O with the legs of angles $\angle XAY$ and $\angle X'A'Y'$ by B_1, C_1 and B_1', C_1', respectively (Fig. 128).

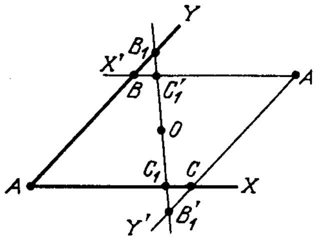


Figure 128 (Sol. 11.24)

Since $S_{AB_1C_1} = S_{A'B_1'C_1'}$, it follows that $S_{AB_1C_1} = \frac{1}{2}(S_{ABA'C} + S_{BB_1C_1'} + S_{CC_1B_1'})$. The area of triangle AB_1C_1 is the least if $B_1 = B$ and $C_1 = C$, i.e., line BC is the one to be found.

11.25. On legs OA and OB, take points K and L so that $KP \parallel OB$ and $LP \parallel OA$. Then $KM : KP = PL : LN$ and, therefore,

$$KM + LN \geq 2\sqrt{KM \cdot LN} = 2\sqrt{KP \cdot PL} = 2\sqrt{OK \cdot OL}$$

where the equality is attained when $KM = LN = \sqrt{OK \cdot OL}$. It is also clear that $OM + ON = (OK + OL) + (KM + LN)$.

11.26. On rays AX and AY, mark equal segments AB an AC. If point M lies on segment BC, then the sum of distances from it to lines AB and AC is equal to $\frac{2(S_{ABM} + S_{ACM})}{AB} = \frac{2S_{ABC}}{AB}$. Therefore, the sum of distances from a point to lines AX and AY is the lesser, the lesser is the distance between point A and the point's projection on the bisector of angle $\angle XAY$.

11.27. Let points M_1 and M_2 be symmetric to M through lines AB and AC, respectively. Since $\angle BAM_1 = \angle BAM$ and $\angle CAM_2 = \angle CAM$, it follows that $\angle M_1AM_2 = 2\angle BAC < 180°$. Hence, segment M_1M_2 intersects rays AB and AC at certain points X and Y (Fig. 129). Let us prove that X and Y are the points to be found.

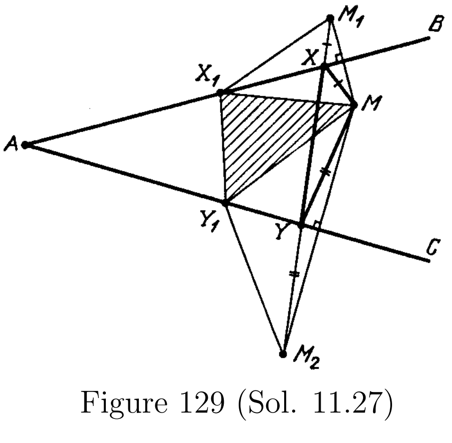


Figure 129 (Sol. 11.27)

Indeed, if points X_1 and Y_1 lie on rays AB and AC, respectively, then $MX_1 = M_1X_1$ and $MY_1 = M_2Y_1$, i.e., the perimeter of triangle MX_1Y_1 is equal to the length of the broken line $MX_1Y_1M_2$. Of all the broken lines with the endpoints at M_1 and M_2 segment M_1M_2 is the shortest one.

11.28. Quadrilateral $ABOC$ of the greatest area is a convex one. Among all the triangles ABC with the fixed angle $\angle A$ and side BC an isosceles triangle with base BC has the greatest area. Therefore, among all the considered quadrilaterals $ABOC$ with fixed diagonal BC the quadrilateral with $AB = AC$, i.e., for which point O lies on the bisector of angle $\angle A$, is of greatest area.

Further, let us consider triangle ABO in which angle $\angle BAO$ equal to $\frac{1}{2}\angle A$ and side BO are fixed. The area of this triangle is maximal when $AB = AO$.

11.29. Let O be the intersection point of the diagonals of convex quadrilateral $ABCD$ and O_1 any other point. Then $AO_1 + CO_1 \geq AC = AO + CO$ and $BO_1 + DO_1 \geq BD = BO + DO$, where at least one of the inequalities is a strict one. Therefore, O is the point to be found.

11.30. Since $S_{AOB} : S_{BOC} = AO : OC = S_{AOD} : S_{DOC}$, it follows that $S_{BOC} \cdot S_{AOD} = S_{AOB} \cdot S_{DOC} = 36$. Therefore, $S_{BOC} + S_{AOD} \geq 2\sqrt{S_{BOC} \cdot S_{AOD}} = 12$, where the equality takes place if $S_{BOC} = S_{AOD}$, i.e., $S_{ABC} = S_{ABD}$. This implies that $AB \parallel CD$. In this case the area of the triangle is equal to 4+9+12=25.

11.31. Let S_0 and S be the considered sums of areas of triangles for line l_0 that passes through the intersection point of the diagonals of the trapezoid and for another line l. It is easy to verify that $S = S_0 + s$, where s is the area of the triangle formed by diagonals AC and BD and line l. Hence, l_0 is the line to be found.

11.32. Denote the lengths of the diagonals of the trapezoid by d_1 and d_2 and the lengths of their projections on the bottom base by p_1 and p_2, respectively; denote the lengths of the bases by a and b and that of the height by h. Let, for definiteness, $d_1 \geq d_2$. Then $p_1 \geq p_2$. Clearly, $p_1 + p_2 \geq a + b$. Hence, $p_1 \geq \frac{a+b}{2} = \frac{S}{h} = \frac{1}{h}$. Therefore, $d_1^2 = p_1^2 + h^2 \geq \frac{1}{h^2} + h^2 \geq 2$, where the equality is attained only if $p_1 = p_2 = h = 1$. In this case $d_1 = \sqrt{2}$.

11.33. Let us prove that point M that divides side BC in the ratio of $BM : NC = AK : KD$ is the desired one. Denote the intersection points of segments AM and BK, DM and CK by P and Q, respectively. Then $KQ : QC = KD : MC = KA : MB = KP : PB$, i.e., line PQ is parallel to the basis of the trapezoid.

Let M_1 be any other point on side BC. For definiteness, we may assume that M_1 lies on segment BM. Denote the intersection points of AM_1 and BK, DM_1 and CK, AM_1 and PQ, DM_1 and PQ, AM and DM_1 by P_1, Q_1, P_2, Q_2, O, respectively (Fig. 130).

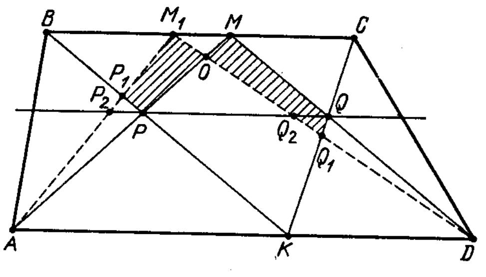


Figure 130 (Sol. 11.33)

We have to prove that $S_{MPKQ} > S_{M_1P_1KQ_1}$, i.e., $S_{MOQ_1Q} > S_{M_1OPP_1}$. Clearly, $S_{MOQ_1Q} > S_{MOQ_2Q} = S_{M_1OPP_2} > S_{M_1OPP_1}$.

11.34. By Problem 4.45 a) we have

$$S^2 = (p-a)(p-b)(p-c)(p-d) - abcd\cos^2\frac{\angle B + \angle D}{2}.$$

This quantity takes its maximal value when $\cos\frac{\angle B+\angle D}{2}=0$, i.e., $\angle B+\angle D=180^\circ$.

11.35. If A and A' are vertices of the polygon symmetric through point O, then the sum of distances from any point of segment AA' to points A and A' is the same whereas for any other point it is greater. Point O belongs to all such segments.

11.36. If in triangle ABC, angle $\angle B$ is either obtuse or right, then by the law of sines $AC^2 \geq AB^2+BC^2$. Therefore, if in a polygon the angle at vertex B is not acute, then deleting vertex B we obtain a polygon with the sum of squared lengths of the sides not less than that of the initial polygon. Since for $n \geq 3$ any n-gon has a nonacute angle, it follows that by repeating such an operation we eventually get a triangle. Among all the triangles inscribed in the given circle an equilateral triangle has the greatest sum of squared lengths of the sides, cf. Problem 11.5.

11.37. If point X divides segment PQ in the ratio of $\lambda : (1-\lambda)$, then $\overrightarrow{A_iX}=(1-\lambda)\overrightarrow{A_iP}+\lambda\overrightarrow{A_iQ}$; hence, $A_iX \leq (1-\lambda)A_iP+\lambda A_iQ$. Therefore,

$$f(X)=\sum A_iX \leq (1-\lambda)\sum A_iP+\lambda\sum A_iQ=(1-\lambda)f(P)+\lambda f(Q).$$

Let, for instance, $f(P) \leq f(Q)$, then $f(X) \leq f(Q)$; hence, on segment PQ the function f attains its maximal value at one of the endpoints; more precisely, inside the segment there can be no point of strict maximum of f. Hence, if X is any point of the polygon, then $f(X) \leq f(Y)$, where Y is a point on a side of the polygon and $f(Y) \leq f(Z)$, where Z is a vertex.

11.38. The locus of points X for which angle $\angle OXA$ is a constant consists of two arcs of circles S_1 and S_2 symmetric through line OA.

Consider the case when the diameter of circles S_1 and S_2 is equal to the radius of the initial circle, i.e., when these circles are tangent to the initial circle at points M_1 and M_2 for which $\angle OAM_1=\angle OAM_2=90^\circ$. Points M_1 and M_2 are the desired ones because if $\angle OXA>\angle OM_1A=\angle OM_2A$, then point X lies strictly inside the figure formed by circles S_1 and S_2, i.e., cannot lie on the initial circle.

11.39. Let us denote the intersection point of line l and segment AB by O. Let us consider an arbitrary circle S that passes through points A and B. It intersects l at certain points M and N. Since $MO\cdot NO=AO\cdot BO$ is a constant,

$$MN=MO+NO \geq 2\sqrt{MO\cdot NO}=2\sqrt{AO\cdot BO},$$

where the equality is only attained if $MO=NO$. In the latter case the center of S is the intersection point of the midperpendicular to AB and the perpendicular to l that passes through point O.

11.40. Let us construct the circle with diameter PQ. If this circle intersects with l, then any of the intersection points is the desired one because in this case $P'=Q'$. If the circle does not intersect with l, then for any point M on l angle $\angle PMQ$ is an acute one and $\angle P'PQ'=90^\circ \pm \angle PMQ$. Now it is easy to establish that the length of chord $P'Q'$ is minimal if angle $\angle PMQ$ is maximal.

To find point M it remains to draw through points P and Q circles tangent to l (cf. Problem 8.56 a)) and select the needed point among the tangent points.

11.41. Let the sum of distances from points A and B to line l be equal to $2h$. If l intersects segment AB at point X, then $S_{AOB}=h\cdot OX$ and, therefore, the value of h is extremal when the value of OX is extremal, i.e., when line OX corresponds to a side or a height of triangle AOB.

If line l does not intersect segment AB, then the value of h is equal to the length of the midline of the trapezoid confined between the perpendiculars dropped from points A and B on line l. This quantity is an extremal one when l is either perpendicular to median OM of triangle AOB or corresponds to a side of triangle AOB. Now it only remains to select two of the obtained four straight lines.

11.42. First, suppose that the points are the vertices of a convex pentagon. The sum of angles of the pentagon is equal to 540°; hence, one of its angles does not exceed $\frac{540^\circ}{5} = 108^\circ$. The diagonals divide this angle into three angles, hence, one of them does not exceed $\frac{108^\circ}{3} = 36^\circ$. In this case $\alpha \leq 36^\circ$.

If the points are not the vertices of a convex pentagon, then one of them lies inside the triangle formed by some other three points. One of the angles of this triangle does not exceed 60°. The segment that connects the corresponding vertex with an inner point divides this angle into two angles, hence, one of them does not exceed 30°. In this case $\alpha \leq 30^\circ$. In all the cases $\alpha \leq 36^\circ$. Clearly, for a regular pentagon $\alpha = 36^\circ$.

11.43. A closed route that passes through all the road crossings can have 20 turns (Fig. 131). It remains to prove that such a route cannot have less than 20 turns. After each turn a passage from a horizontal street to a vertical one or from a vertical street to a horizontal one occurs.

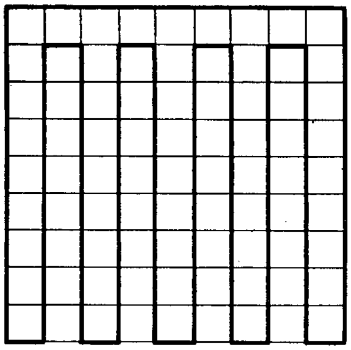


Figure 131 (Sol. 11.43)

Hence, the number of horizontal links of a closed route is equal to the number of vertical links and is equal to half the number of turns. Suppose that a closed route has less than 20 turns. Then there are streets directed horizontally, as well as streets directed vertically, along which the route does not pass. Therefore, the route does not pass through the intersection point of these streets.

11.44. A line can intersect 15 cells (Fig. 132). Let us prove now that a line cannot intersect more than 15 cells. The number of cells that the line intersects is by 1 less than the number of intersection points of the line with the segments that determine the sides of the cells. Inside a square there are 14 such segments.

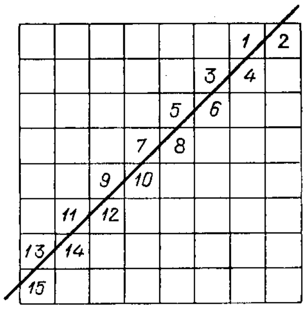


Figure 132 (Sol. 11.44)

Hence, inside a square there are not more than 14 intersection points of the line with sides of cells. No line can intersect the boundary of the chessboard at more than 2 points; hence, the number of intersection points of the line with the segments does not exceed 16. Hence, the maximal number of cells on the chessboard of size 8 × 8 that can be intersected by one line is equal to 15.

11.45. First, let us prove that 33 points are impossible to place in the required way. Indeed, if on a segment of length 1 there are 33 points, then the distance between some two of them does not exceed $\frac{1}{32}$. The segment with the endpoints at these points contains two points and it should contain not more than $1 + \frac{1000}{32^2}$ points, i.e., not less than two points.

Now, let us prove that it is possible to place 32 points. Let us take 32 points that divide the segment into equal parts (the endpoints of the given segment should be among these 32 points). Then a segment of length d contains either $[31d]$ or $[31d] + 1$ points. (Recall that $[x]$ denotes the integer part of the number x, i.e., the greatest integer that does not exceed x.) We have to prove that $[31d] \le 1000d^2$. If $31d < 1$, then $[31d] = 0 < 1000d^2$. If $31d \ge 1$, then $[31d] \le 31d \le (31d)^2 = 961d^2 < 1000d^2$.

11.46. a) Let a non-regular n-gon be circumscribed about circle S. Let us circumscribe a regular n-gon about this circle and let us circumscribe circle S_1 about this regular n-gon (Fig. 133). Let us prove that the area of the part of the non-regular n-gon confined inside S_1 is greater than the area of the regular n-gon.

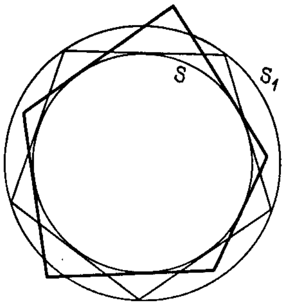


Figure 133 (Sol. 11.46)

All the tangents to S cut off S_1 equal segments. Hence, the sum of areas of the segments cut off S_1 by the sides of the regular n-gon is equal to the sum of segments cut off S_1 by the sides of the non-regular n-gon or by their extensions.

But for the regular n-gon these segments do not intersect (more exactly, they do not have common interior points) and for the non-regular n-gon some of them must overlap, hence, the area of the union of these segments for a regular-gon is greater than for a non-regular one. Therefore, the area of the part of the non-regular n-gon confined inside S_1 is greater than the area of the regular n-gon and the area of the whole non-regular n-gon is still greater than the area of the regular one.

b) This heading follows from heading a) because the perimeter of the polygon circumscribed about a circle of radius R is equal to $\frac{2S}{R}$, where S is the area of the polygon.

11.47. The sides of triangle ABC are proportional to $\sin\alpha$, $\sin\beta$ and $\sin\gamma$. If angle γ is fixed, then the value of

$$|\sin\alpha - \sin\beta| = 2\left|\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma}{2}\right|$$

is the greater the greater is $\varphi = |\alpha - \beta|$. It remains to observe that quantities

$$S = 2R^2 \sin\alpha \sin\beta \sin\gamma = R^2 \sin\gamma(\cos\alpha - \beta + \cos\gamma) =$$
$$R^2 \sin\gamma(\cos\varphi + \cos\gamma)$$
$$\text{and} \quad \sin\alpha + \sin\beta = 2\cos\tfrac{\gamma}{2}\cos\tfrac{\varphi}{2}$$

monotonously decrease as φ increases.

11.48. a) Denote the length of the side of a regular n-gon inscribed in the given circle by a_n. Consider an arbitrary non-regular n-gon inscribed in the same circle. It will necessarily have a side shorter than a_n.

On the other hand, it can have no side longer than a_n and in such a case such a polygon can be confined in a segment cut off a side of the regular n-gon. Since the symmetry through a side of a regular n-gon sends the segment cut off this side inside the n-gon, the area of the n-gon is greater than the area of the segment(??). Therefore, we may assume that the considered n-gon has a side shorter than a_n and a side longer than a_n.

We can replace neighbouring sides of the n-gon, i.e., replace $A_1A_2A_3\dots A_n$ with polygon $A_1A_2'A_3\dots A_n$, where point A_2' is symmetric to A_2 through the midperpendicular to segment A_1A_3 (Fig. 134). Clearly, both polygons are inscribed in the same circle and their areas are equal. It is also clear that with the help of this operation we can make any two sides of the polygon neighbouring ones. Therefore, let us assume that for the n-gon considered, $A_1A_2 > a_n$ and $A_2A_3 < a_n$.

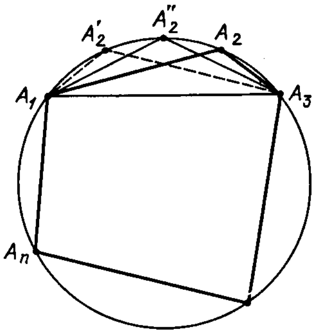


Figure 134 (Sol. 11.48)

Let A_2' be the point symmetric to A_2 through the midperpendicular to segment A_1A_3. If point A_2'' lies on arc $\smile A_2A_2'$, then the difference of the angles at the base A_1A_3 of triangle $A_1A_2''A_3$ is less than that of triangle $A_1A_2A_3$ because the values of angles $\angle A_1A_2A_2''$ and $\angle A_3A_1A_2''$ are confined between the values of angles $\angle A_1A_3A_2$ and $\angle A_3A_1A_2$.

Since $A_1A_2' < a_n$ and $A_1A_2 > a_n$, on arc $\smile A_2A_2'$ there exists a point A_2'' for which $A_1A_2'' = a_n$. The area of triangle $A_1A_2''A_3$ is greater than the area of triangle $A_1A_2A_3$, cf. Problem 11.47 a). The area of polygon $A_1A_2''A_3\dots A_n$ is greater than the area of the initial polygon and it has at least by 1 more sides equal to a_n.

After finitely many steps we get a regular n-gon and at each step the area increases. Therefore, the area of any non-regular n-gon inscribed in a circle is less than the area of a regular n-gon inscribed in the same circle.

b) Proof is similar to the proof of heading a); one only has to make use of the result of Problem 11.47 b) instead of that of Problem 11.47 a).

Chapter 12. CALCULATIONS AND METRIC RELATIONS

Introductory problems

1. Prove the law of cosines:

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos \angle A.$$

2. Prove the law of sines:

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R.$$

3. Prove that the area of a triangle is equal to $\sqrt{p(p-a)(p-b)(p-c)}$, where p is semiperimeter (*Heron's formula.*)

4. The sides of a parallelogram are equal to a and b and its diagonals are equal to d and e. Prove that $2(a^2+b^2) = d^2+e^2$.

5. Prove that for convex quadrilateral $ABCD$ with the angle φ between the diagonals we have $S_{ABCD} = \frac{1}{2}AC \cdot BD \sin\varphi$.

§1. The law of sines

12.1. Prove that the area S of triangle ABC is equal to $\frac{abc}{4R}$.

12.2. Point D lies on base AC of equilateral triangle ABC. Prove that the radii of the circumscribed circles of triangles ABD and CBD are equal.

12.3. Express the area of triangle ABC in terms of the length of side BC and the value of angles $\angle B$ and $\angle C$.

12.4. Prove that $\frac{a+b}{c} = \frac{\cos\frac{\alpha-\beta}{2}}{\sin\frac{\gamma}{2}}$ and $\frac{a-b}{c} = \frac{\sin\frac{\alpha-\beta}{2}}{\cos\frac{\gamma}{2}}$.

12.5. In an acute triangle ABC heights AA_1 and CC_1 are drawn. Points A_2 and C_2 are symmetric to A_1 and C_1 through the midpoints of sides BC and AB, respectively. Prove that the line that connects vertex B with the center O of the circumscribed circle divides segment A_2C_2 in halves.

12.6. Through point S lines a, b, c and d are drawn; line l intersects them at points A, B, C and D. Prove that the quantity $\frac{AC \cdot BD}{BC \cdot AD}$ does not depend on the choice of line l.

12.7. Given lines a and b that intersect at point O and an arbitrary point P. Line l that passes through point P intersects lines a and b at points A and B. Prove that the value of $\frac{\frac{OA}{OB}}{\frac{PA}{PB}}$ does not depend on the choice of line l.

12.8. Denote the vertices and the intersection points of links of a (non-regular) five-angled star as shown on Fig. 135. Prove that

$$A_1C \cdot B_1D \cdot C_1E \cdot D_1A \cdot E_1B = A_1D \cdot B_1E \cdot C_1A \cdot D_1B \cdot E_1C.$$

12.9. Two similar isosceles triangles have a common vertex. Prove that the projections of their bases on the line that connects the midpoints of the bases are equal.

12.10. On the circle with diameter AB, points C and D are taken. Line CD and the tangent to the circle at point B intersect at point X. Express BX in terms of the radius R of the circle and angles $\varphi = \angle BAC$ and $\psi = \angle BAD$.

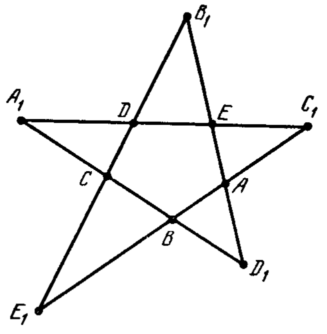


Figure 135 (12.8)

§2. The law of cosines

12.11. Prove that:
a) $m_a^2 = \frac{2b^2+2c^2-a^2}{4}$;
b) $m_a^2 + m_b^2 + m_c^2 = \frac{3(a^2+b^2+c^2)}{4}$.

12.12. Prove that $4S = (a^2 - (b-c)^2)\cot\frac{\alpha}{2}$.

12.13. Prove that

$$\cos^2\frac{\alpha}{2} = \frac{p(p-a)}{bc} \text{ and } \sin^2\frac{\alpha}{2} = \frac{(p-b)(p-c)}{bc}.$$

12.14. The lengths of sides of a parallelogram are equal to a and b; the lengths of the diagonals are equal to m and n. Prove that $a^4 + b^4 = m^2n^2$ if and only if the acute angle of the parallelogram is equal to 45°.

12.15. Prove that medians AA_1 and BB_1 of triangle ABC are perpendicular if and only if $a^2 + b^2 = 5c^2$.

12.16. Let O be the center of the circumscribed circle of scalane triangle ABC, let M be the intersection point of the medians. Prove that line OM is perpendicular to median CC_1 if and only if $a^2 + b^2 = 2c^2$.

§3. The inscribed, the circumscribed and escribed circles; their radii

12.17. Prove that:
a) $a = r\left(\cot\frac{\beta}{2} + \cot\frac{\gamma}{2}\right) = \frac{r\cos\frac{\alpha}{2}}{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$;
b) $a = r_a\left(\tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right) = \frac{r_a\cos\frac{\alpha}{2}}{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$;
c) $p - b = r\cot\frac{\beta}{2} = r_a\tan\frac{\gamma}{2}$;
d) $p = r_a\cot\frac{\alpha}{2}$.

12.18. Prove that:
a) $rp = r_a(p-a)$, $rr_a = (p-b)(p-c)$ and $r_br_c = p(p-a)$;
b) $S^2 = p(p-a)(p-b)(p-c)$; (*Heron's formula.*)
c) $S^2 = rr_ar_br_c$.

12.19. Prove that $S = r_c^2\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\cot\frac{\gamma}{2}$.

12.20. Prove that $S = \frac{cr_ar_b}{r_a+r_b}$.

12.21. Prove that $\frac{2}{h_a} = \frac{1}{r_b} + \frac{1}{r_c}$.

12.22. Prove that $\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = 1r$.

12.23. Prove that

$$\frac{1}{(p-a)(p-b)} + \frac{1}{(p-b)(p-c)} + \frac{1}{(p-c)(p-a)} = \frac{1}{r^2}.$$

12.24. Prove that $r_a + r_b + r_c = 4R + r$.
12.25. Prove that $r_a r_b + r_b r_c + r_c r_a = p^2$.
12.26. Prove that $\frac{1}{r^3} - \frac{1}{r_a^3} - \frac{1}{r_b^3} - \frac{1}{r_c^2} = \frac{12R}{S^2}$.
12.27. Prove that

$$a(b+c) = (r + r_a)(4R + r - r_a) \text{ and } a(b-c) = (r_b - r_c)(4R - r_b - r_c).$$

12.28. Let O be the center of the inscribed circle of triangle ABC. Prove that $\frac{OA^2}{bc} + \frac{OB^2}{ac} + \frac{OC^2}{ab} = 1$.

12.29. a) Prove that if for a triangle we have $p = 2R + r$, then this triangle is a right one.

b) Prove that if $p = 2R\sin\varphi + r\cot\frac{\varphi}{2}$, then φ is one of the angles of the triangle (we assume here that $0 < \varphi < \pi$).

§4. The lengths of the sides, heights, bisectors

12.30. Prove that $abc = 4prR$ and $ab + bc + ca = r^2 + p^2 + 4rR$.
12.31. Prove that $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}$.
12.32. Prove that $\frac{a+b-c}{a+b+c} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$.
12.33. Prove that $h_a = \frac{bc}{2R}$.
12.34. Prove that

$$h_a = \frac{2(p-a)\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = \frac{2(p-b)\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}.$$

12.35. Prove that the length of bisector l_a can be computed from the following formulas:
a) $l_a = \sqrt{\frac{4p(p-a)bc}{(b+c)^2}}$;
b) $l_a = \frac{2bc\cos\frac{\alpha}{2}}{b+c}$;
c) $l_a = \frac{2R\sin\beta\sin\gamma}{\cos\frac{\beta-\gamma}{2}}$;
d) $l_a = \frac{4p\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\sin\beta+\sin\gamma}$.

§5. The sines and cosines of a triangle's angles

Let α, β and γ be the angles of triangle ABC. In the problems of this section one should prove the relations indicated.

12.36. a) $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{r}{4R}$;
b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = \frac{r}{p}$;
c) $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \frac{p}{4R}$.
12.37. a) $\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{p-a}{4R}$;
b) $\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \frac{r_a}{4R}$.
12.38. $\cos\alpha + \cos\beta + \cos\gamma = \frac{R+r}{R}$.
12.39. a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 4\cos\alpha\cos\beta\cos\gamma + 1 = 0$;
b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma = 1$.
12.40. $\sin 2\alpha + \cos 2\beta + \cos 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.
12.41. a) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{p^2-r^2-4rR}{2R^2}$.
b) $4R^2\cos\alpha\cos\beta\cos\gamma = p^2 - (2R+r)^2$.
12.42. $ab\cos\gamma + bc\cos\alpha + ca\cos\beta = \frac{a^2+b^2+c^2}{2}$.
12.43. $\frac{\cos^2\frac{\alpha}{2}}{a} + \frac{\cos^2\frac{\beta}{2}}{b} + \frac{\cos^2\frac{\gamma}{2}}{c} = \frac{p}{4Rr}$.

§6. The tangents and cotangents of a triangle's angles

In problems of this section one has to prove the relations indicated between the values α, β and γ of the angles of triangle ABC.

12.44. a) $\cot\alpha+\cot\beta+\cot\gamma=\frac{a^2+b^2+c^2}{4S}$;
b) $a^2\cot\alpha+b^2\cot\beta+c^2\cot\gamma=4S$.

12.45. a) $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\frac{p}{r}$;
b) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\frac{1}{2}\left(\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}\right)$.

12.46. $\tan\alpha+\tan\beta+\tan\gamma=\tan\sigma\tan\beta\tan\gamma$.

12.47. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$.

12.48. a) $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\alpha\cot\gamma=1$;
b) $\cot\alpha+\cot\beta+\cot\gamma-\cot\alpha\cot\beta\cot\gamma=\frac{1}{\sin\alpha\sin\beta\sin\gamma}$.

12.49. For a non-right triangle we have

$$\tan\sigma+\tan\beta+\tan\gamma=\frac{4S}{a^2+b^2+c^2-8R^2}.$$

§7. Calculation of angles

12.50. Two intersecting circles, each of radius R with the distance between their centers greater than R are given. Prove that $\beta=3\alpha$ (Fig. 136).

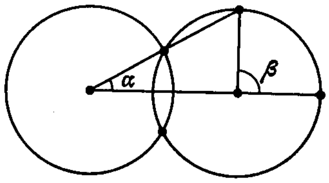


Figure 136 (12.50)

12.51. Prove that if $\frac{1}{b}+\frac{1}{c}=\frac{1}{l_a}$, then $\angle A=120^\circ$.

12.52. In triangle ABC height AH is equal to median BM. Find angle $\angle MBC$.

12.53. In triangle ABC bisectors AD and BE are drawn. Find the value of angle $\angle C$ if it is given that $AD\cdot BC=BE\cdot AC$ and $AC\neq BC$.

12.54. Find angle $\angle B$ of triangle ABC if the length of height CH is equal to a half length of side AB and $\angle BAC=75^\circ$.

12.55. In right triangle ABC with right angle $\angle A$ the circle is constructed with height AD of the triangle as a diameter; the circle intersects leg AB at point K and leg AC at point M. Segments AD and KM intersect at point L. Find the acute angles of triangle ABC if $AK:AL=AL:AM$.

12.56. In triangle ABC, angle $\angle C=2\angle A$ and $b=2a$. Find the angles of triangle ABC.

12.57. In triangle ABC bisector BE is drawn and on side BC point K is taken so that $\angle AKB=2\angle AEB$. Find the value of angle $\angle AKE$ if $\angle AEB=\alpha$.

* * *

12.58. In an isosceles triangle ABC with base BC angle at vertex A is equal to 80°. Inside triangle ABC point M is taken so that $\angle MBC=30^\circ$ and $\angle MCB=10^\circ$. Find the value of angle $\angle AMC$.

12.59. In an isosceles triangle ABC with base AC the angle at vertex B is equal to 20°. On sides BC and AB points D and E, respectively, are taken so that $\angle DAC = 60°$ and $\angle ECA = 50°$. Find angle $\angle ADE$.

12.60. In an acute triangle ABC segments BO and CO, where O is the center of the circumscribed circle, are extended to their intersection at points D and E with sides AC and AB, respectively. It turned out that $\angle BDE = 50°$ and $\angle CED = 30°$. Find the value of the angles of triangle ABC.

§8. The circles

12.61. Circle S with center O on base BC of isosceles triangle ABC is tangent to equal sides AB and AC. On sides AB and AC, points P and Q, respectively, are taken so that segment PQ is tangent to S. Prove that $4PB \cdot CQ = BC^2$.

12.62. Let E be the midpoint of side AB of square $ABCD$ and points F and G are taken on sides BC and CD, respectively, so that $AG \parallel EF$. Prove that segment FG is tangent to the circle inscribed in square $ABCD$.

12.63. A chord of a circle is distant from the center by h. A square is inscribed in each of the disk segments subtended by the chord so that two neighbouring vertices of the square lie on an arc and two other vertices lie either on the chord or on its extension (Fig. 137). What is the difference of lengths of sides of these squares?

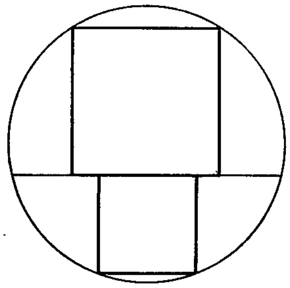


Figure 137 (12.63)

12.64. Find the ratio of sides of a triangle one of whose medians is divided by the inscribed circle into three equal parts.

* * *

12.65. In a circle, a square is inscribed; in the disk segment cut off the disk by one of the sides of this square another square is inscribed. Find the ratio of the lengths of the sides of these squares.

12.66. On segment AB, point C is taken and on segments AC, BC and AB as on diameters semicircles are constructed lying on one side of line AB. Through point C the line perpendicular to AB is drawn and in the obtained curvilinear triangles ACD and BCD circles S_1 and S_2 are inscribed (Fig. 138). Prove that the radii of these circles are equal.

12.67. The centers of circles with radii 1, 3 and 4 are positioned on sides AD and BC of rectangle $ABCD$. These circles are tangent to each other and lines AB and CD as shown on Fig. 139. Prove that there exists a circle tangent to all these circles and line AB.

§9. Miscellaneous problems

12.68. Find all the triangles whose angles form an arithmetic projection and sides form a) an arithmetic progression; b) a geometric progression.

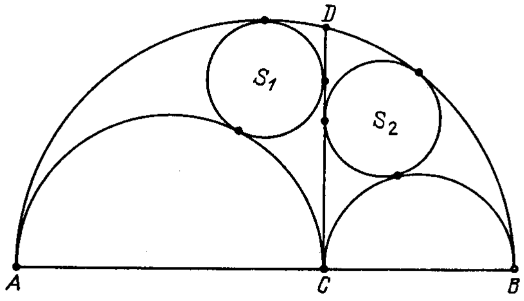


Figure 138 (12.66)

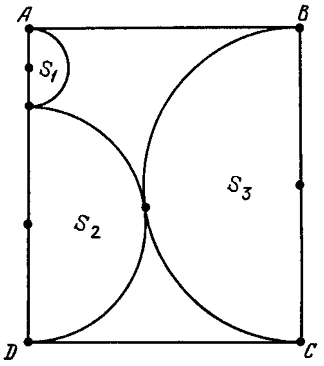


Figure 139 (12.67)

12.69. Find the height of a trapezoid the lengths of whose bases AB and CD are equal to a and b $(a < b)$, the angle between the diagonals is equal to 90°, and the angle between the extensions of the lateral sides is equal to 45°.

12.70. An inscribed circle is tangent to side BC of triangle ABC at point K. Prove that the area of the triangle is equal to $BK \cdot KC \cot \frac{\alpha}{2}$.

12.71. Prove that if $\cot \frac{\alpha}{2} = \frac{a+b}{a}$, then the triangle is a right one.

12.72. The extensions of the bisectors of triangle ABC intersect the circumscribed circle at points A_1, B_1 and C_1. Prove that $\frac{S_{ABC}}{S_{A_1B_1C_1}} = \frac{2r}{R}$, where r and R are the radii of the inscribed and circumscribed circles, respectively, of triangle ABC.

12.73. Prove that the sum of cotangents of the angles of triangle ABC is equal to the sum of cotangents of the angles of the triangle formed by the medians of triangle ABC.

12.74. Let A_4 be the orthocenter of triangle $A_1A_2A_3$. Prove that there exist numbers $\lambda_1, \dots, \lambda_4$ such that $A_iA_j^2 = \lambda_i + \lambda_j$ and if the triangle is not a right one, then $\sum \frac{1}{\lambda_i} = 0$.

§10. The method of coordinates

12.75. Coordinates of the vertices of a triangle are rational numbers. Prove that then the coordinates of the center of the circumscribed circle are also rational.

12.76. Diameters AB and CD of circle S are perpendicular. Chord EA intersects diameter CD at point K, chord EC intersects diameter AB at point L. Prove that if $CK : KD = 2 : 1$, then $AL : LB = 3 : 1$.

12.77. In triangle ABC angle $\angle C$ is a right one. Prove that under the homothety with center C and coefficient 2 the inscribed circle turns into a circle tangent to the circumscribed circle.

12.78. A line l is fixed. Square $ABCD$ is rotated about its center. Find the locus of the midpoints of segments PQ, where P is the base of the perpendicular dropped from point D on l and Q is the midpoint of side AB.

See also Problems 7.6, 7.14, 7.47, 22.15.

Problems for independent study

12.79. Each of two circles is tangent to both sides of the given right angle. Find the ratio of the circles' radii if it is known that one of the circles passes through the center of the other one.

12.80. Let the extensions of sides AB and CD, BC and AD of convex quadrilateral $ABCD$ intersect at points K and M, respectively. Prove that the radii of the circles circumscribed about triangles ACM, BDK, ACK, BDM are related by the formula $R_{ACM} \cdot R_{BDK} = R_{ACK} \cdot R_{BDM}$.

12.81. Three circles of radii 1, 2, 3 are tangent to each other from the outside. Find the radius of the circle that passes through the tangent points of these circles.

12.82. Let point K lie on side BC of triangle ABC. Prove that

$$AC^2 \cdot BK + AB^2 \cdot CK = BC(AK^2 + BK \cdot KC).$$

12.83. Prove that the length of the bisector of an outer angle $\angle A$ of triangle ABC is equal to $\frac{2bc \sin\frac{\alpha}{2}}{|b-c|}$.

12.84. Two circles of radii R and r are placed so that their common inner tangents are perpendicular. Find the area of the triangle formed by these tangents and their common outer tangent.

12.85. Prove that the sum of angles at rays of any (nonregular) five-angled star is equal to 180°.

12.86. Prove that in any triangle $S = (p-a)^2 \tan\frac{\alpha}{2} \cot\frac{\beta}{2} \cot\frac{\gamma}{2}$.

12.87. Let $a < b < c$ be the lengths of sides of a triangle; l_a, l_b, l_c and l'_a, l'_b, l'_c the lengths of its bisectors and the bisectors of its outer angles, respectively. Prove that $\frac{1}{al_al'_a} + \frac{1}{cl_cl'_c} = \frac{1}{bl_bl'_b}$.

12.88. In every angle of a triangle a circle tangent to the inscribed circle of the triangle is inscribed. Find the radius of the inscribed circle if the radii of these smaller circles are known.

12.89. The inscribed circle is tangent to sides AB, BC, CA at points K, L, M, respectively. Prove that:

a) $S = \frac{1}{2}\left(\frac{MK^2}{\sin\alpha} + \frac{KL^2}{\sin\beta} + \frac{LM^2}{\sin\gamma}\right)$;

b) $S^2 = \frac{1}{4}(bcMK^2 + caKL^2 + abLM^2)$;

c) $\frac{MK^2}{h_bh_c} + \frac{KL^2}{h_ch_a} + \frac{LM^2}{h_ah_b} = 1$.

Solutions

12.1. By the law of sines $\sin\gamma = c2R$; hence, $S = \frac{1}{2}ab\sin\gamma = \frac{abc}{4R}$.

12.2. The radii of the circumscribed circles of triangles ABD and CBD are equal to $\frac{AB}{2\sin\angle ADB}$ and $\frac{BC}{2\sin\angle BDC}$. It remains to notice that $AB = BC$ and $\sin\angle ADB = \sin\angle BDC$.

12.3. By the law of sines $b = \frac{a\sin\beta}{\sin\alpha} = \frac{a\sin\beta}{\sin(\beta+y)}$ and, therefore, $S = \frac{1}{2}ab\sin\gamma = \frac{a^2\sin\beta\sin\gamma}{2\sin(\beta+\gamma)}$.

12.4. By the law of sines $\frac{1}{2}(a+b) = \frac{\sin\alpha+\sin\beta}{\sin\gamma}$. Moreover,

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\cos\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}$$

and $\sin\gamma = 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$. The second equality is similarly proved.

12.5. In triangle A_2BC_2, the lengths of sides A_2B and BC_2 are equal to $b\cos\gamma$ and $b\cos\alpha$; line BO divides angle $\angle A_2BC_2$ into angles of $90^\circ-\gamma$ and $90^\circ-\alpha$. Let line BO intersect segment A_2C_2 at point M. By the law of sines

$$A_2M=\frac{A_2B\sin\angle A_2BM}{\sin\angle A_2MB}=\frac{b\cos\gamma\cos\alpha}{\sin\angle C_2MB}=C_2M.$$

12.6. Let $\alpha=\angle(a,c)$, $\beta=\angle(c,d)$ and $\gamma=\angle(d,b)$. Then

$$\frac{\frac{AC}{AS}}{\frac{BC}{BS}}=\frac{\sin\alpha}{\sin(\beta+\gamma)}\ \text{ and }\ \frac{\frac{BD}{BS}}{\frac{AD}{AS}}=\frac{\sin\gamma}{\sin(\alpha+\beta)}.$$

Hence

$$\frac{(AC\cdot BD)}{(BC\cdot AD)}=\frac{\sin\alpha\sin\gamma}{\sin(\alpha+\beta)\sin(\beta+\gamma)}.$$

12.7. Since $\frac{OA}{PA}=\frac{\sin\angle OPA}{\sin\angle POA}$ and $\frac{OB}{PB}=\frac{\sin\angle OPB}{\sin\angle POB}$, it follows that

$$\frac{(OA:OB)}{(PA:PB)}=\frac{\sin\angle POB}{\sin\angle POA}.$$

12.8. It suffices to multiply five equalities of the form $\frac{D_1A}{D_1B}=\frac{\sin\angle B}{\sin\angle A}$.

12.9. Let O be the common vertex of the given triangles, M and N the midpoints of the bases, k the ratio of the lengths of the bases to that of heights. The projections of the bases of given triangles on line MN are equal to $k\cdot OM\sin\angle OMN$ and $k\cdot ON\sin\angle ONM$. It remains to notice that $\frac{OM}{\sin\angle ONM}=\frac{ON}{\sin\angle OMN}$.

12.10. By the law of sines

$$\frac{BX}{\sin\angle BDX}=\frac{BD}{\sin\angle BXD}=\frac{2R\sin\psi}{\sin\angle BXD}.$$

Moreover, $\sin\angle BDX=\sin\angle BDC=\sin\varphi$ and the value of angle $\angle BXD$ is easy to calculate: if points C and D lie on one side of AB, then $\angle BXD=\pi-\varphi-\psi$ and if they lie on distinct sides, then $\angle BXD=|\varphi-\psi|$. Hence, $BX=\frac{2R\sin\varphi\sin\psi}{\sin|\varphi\pm\psi|}$.

12.11. a) Let A_1 be the midpoint of segment BC. Adding equalities

$$AB^2=AA_1^2+A_1B^2-2AA_1\cdot BA_1\cos\angle BA_1A$$

and

$$AC^2=AA_1^2+A_1C^2-2AA_1\cdot A_1C\cos\angle CA_1A$$

and taking into account that $\cos\angle BA_1A=-\cos\angle CA_1A$ we get the statement desired.

b) Follows in an obvious way from heading a).

12.12. By the law of cosines

$$a^2-(b-c)^2=2bc(1-\cos\alpha)=\frac{4S(1-\cos\alpha)}{\sin\alpha}=4S\tan\frac{\alpha}{2}.$$

12.13. By the law of cosines $\cos\alpha=\frac{b^2+c^2-a^2}{2bc}$. It remains to make use of the formulas $\cos^2\frac{\alpha}{2}=\frac{1}{2}(1+\cos\alpha)$ and $\sin^2\frac{\alpha}{2}=\frac{1}{2}(1-\cos\alpha)$.

12.14. Let α be the angle at a vertex of the parallelogram. By the law of cosines

$$m^2=a^2+b^2+2ab\cos\alpha\ \text{ and }\ n^2=a^2+b^2-2ab\cos\alpha.$$

Hence,

$$m^2n^2=(a^2+b^2)^2-(2ab\cos\alpha)^2=a^4+b^4+2a^2b^2(1-2\cos^2\alpha).$$

Therefore, $m^2n^2=a^4+b^4$ if and only if $\cos^2\alpha=\frac{1}{2}$.

12.15. Let M be the intersection point of medians AA_1 and BB_1. Angle $\angle AMB$ is a right one if and only if $AM^2 + BM^2 = AB^2$, i.e. $\frac{4}{9}(m_a^2 + m_b^2) = c^2$. By Problem 12.11 $m_a^2 + m_b^2 = \frac{4c^2+a^2+b^2}{4}$.

12.16. Let $m = C_1M$ and $\varphi = \angle C_1MO$. Then

$$OC_1^2 = C_1M^2 + OM^2 - 2OM \cdot C_1M \cos\varphi$$

and

$$BO^2 = CO^2 = OM^2 + MC^2 + 2OM \cdot CM \cos\varphi = OM^2 + 4C_1M^2 + 4OM \cdot C_1M \cos\varphi.$$

Hence,

$$BC_1^2 = BO^2 - OC_1^2 = 3C_1M^2 + 6OM \cdot C_1M \cos\varphi,$$

i.e.,

$$c^2 = 4BC_1^2 = 12m^2 + 24OM \cdot C_1M \cos\varphi.$$

It is also clear that $18m^2 = 2m_c^2 = a^2 + b^2 - \frac{c^2}{2}$, cf. Problem 12.11. Therefore, equality $a^2 + b^2 = 2c^2$ is equivalent to the fact that $18m^2 = \frac{3c^2}{2}$, i.e., $c^2 = 12m^2$. Since $c^2 = 12m^2 + 24OM \cdot C_1M \cos\varphi$, equality $a^2 + b^2 = 2c^2$ is equivalent to the fact that $\angle C_1MO = \varphi = 90^\circ$, i.e., $CC_1 \perp OM$.

12.17. Let the inscribed circle be tangent to side BC at point K and the escribed one at point L. Then

$$BC = BK + KC = t\cot\frac{\beta}{2} + r\cot\frac{\gamma}{2}$$

and

$$BC = BL + LC = r_a \cot LBO_a + r_a \cot LCO_a = r_a \tan\frac{\beta}{2} + r_a \tan\frac{\gamma}{2}.$$

Moreover, $\cos\frac{\alpha}{2} = \sin\left(\frac{\beta}{2} + \frac{\gamma}{2}\right)$.

By Problem 3.2, $p - b = BK = r\cot\frac{\beta}{2}$ and $p - b = CL = r_a \tan\frac{\gamma}{2}$.

If the inscribed circle is tangent to the extensions of sides AB and AC at points P and Q, respectively, then $p = AP = AQ = r_a \cot\frac{\alpha}{2}$.

12.18. a) By Problem 12.17,

$$\begin{gathered} p = r_a \cot\tfrac{\alpha}{2} \quad\text{and}\quad r\cot\tfrac{\alpha}{2} = p - a; \\ r\cot\tfrac{\beta}{2} = p - b \quad\text{and}\quad r_a \tan\tfrac{\beta}{2} = p - c; \\ r_c \tan\tfrac{\beta}{2} = p - a \quad\text{and}\quad r_b \cot\tfrac{\beta}{2} = p. \end{gathered}$$

By multiplying these pairs of equalities we get the desired statement.

b) By multiplying equalities $rp = r_a(p - a)$ and $rr_a = (p - b)(p - c)$ we get $r^2p = (p - a)(p - b)(p - c)$. It is also clear that $S^2 = p(r^2p)$.

c) It suffices to multiply $rr_a = (p - b)(p - c)$ and $r_br_c = p(p - a)$ and make use of Heron's formula.

12.19. By Problem 12.17, $r = r_c \tan\frac{\alpha}{2}\tan\frac{\beta}{2}$ and $p = r_c \cot\frac{\gamma}{2}$.

12.20. By Problem 12.18 a), $r_a = \frac{rp}{p-a}$ and $r_b = \frac{rp}{p-b}$. Hence,

$$cr_ar_b = \frac{cr^2p^2}{(p-a)(p-b)} \quad\text{and}\quad r_a + r_b = \frac{rpc}{(p-a)(p-b)}$$

and, therefore, $\frac{cr_ar_b}{r_a+r_b} = rp = S$.

12.21. By Problem 12.18 a), $\frac{1}{r_b} = \frac{p-b}{pr}$ and $\frac{1}{r_c} = \frac{p-c}{pr}$, hence, $\frac{1}{r_b} + \frac{1}{r_c} = \frac{a}{pr} = \frac{a}{S} = \frac{2}{h_a}$.

12.22. It is easy to verify that $\frac{1}{h_a} = \frac{a}{2pr}$ and $\frac{1}{r_a} = \frac{p-a}{pr}$. Adding similar equalities we get the desired statement.

12.23. By Problem 12.18 a) $\frac{1}{(p-b)(p-c)} = \frac{1}{rr_a}$. It remains to add similar equalities and make use of the result of Problem 12.22.

12.24. By Problem 12.1, $4SR = abc$. It is also clear that

$$abc = p(p-b)(p-c) + p(p-c)(p-a) + p(p-a)(p-b) - (p-a)(p-b)(p-c) = \\ \frac{S^2}{p-a} + \frac{S^2}{p-b} + \frac{S^2}{p-c} - S^2p = S(r_a + r_b + r_c - r).$$

12.25. By Problem 12.18 a)

$$r_ar_b = p(p-c),\ r_br_c = p(p-a) \text{ and } r_cr_a = p(p-b).$$

Adding these equalities we get the desired statement.

12.26. Since

$$S = rp = r_a(p-a) = r_b(p-b) = r_c(p-c),$$

the right-most expression is equal to

$$\frac{p^3 - (p-a)^3 - (p-b)^3 - (p-c)^3}{S^3} = \frac{3abc}{S^3}.$$

It remains to observe that $\frac{abc}{S} = 4R$ (Problem 12.1).

12.27. Let the angles of triangle ABC be equal to 2α, 2β and 2γ. Thanks to Problems 12.36 a) and 12.37 b) we have $r = 4R\sin\alpha\sin\beta\sin\gamma$ and $r_a = 4R\sin\alpha\cos\beta\cos\gamma$. Therefore,

$$(r + r_a)(4R + r - r_a) = \\ 16R^2\sin\alpha\cdot(\sin\beta\sin\gamma + \cos\beta\cos\gamma)(1 + \sin\alpha(\sin\beta\sin\gamma - \cos\beta\cos\gamma)) = \\ 16R^2\sin\alpha\cos(\beta-\gamma)(1 - \sin\alpha\cos(\beta+\gamma)) = \\ 16R^2\sin\alpha\cos(\beta-\gamma)\cos^2\alpha.$$

It remains to notice that $4R\sin\alpha\cos\alpha = a$ and

$$4R\sin(\beta+\gamma)\cos(\beta-\gamma) = 2R(\sin 2\beta + \sin 2\gamma) = b + c.$$

The second equality is similarly proved.

12.28. Since $OA = \frac{r}{\sin\frac{\alpha}{2}}$ and $bc = \frac{2S}{\sin\alpha}$, it follows that

$$\frac{OA^2}{bc} = \frac{r^2\cot\frac{\alpha}{2}}{S} = \frac{r(p-a)}{S},$$

cf. Problem 12.17 c). It remains to notice that $r(p-a+p-b+p-c) = rp = S$.

12.29. Let us solve heading b); heading a) is its particular case. Since $\cot\frac{\varphi}{2} = \frac{\sin\varphi}{1-\cos\varphi}$, it follows that

$$p^2(1-x)^2 = (1-x^2)(2R(1-x)+r)^2, \text{ where } x = \cos\varphi.$$

The root $x_0 = 1$ of this equation is of no interest to us because in this case $\cot\frac{\varphi}{2}$ is undefined; therefore, by dividing both parts of the equation by $1-x$ we get a cubic equation. Making use of results of Problems 12.38, 12.41 b) and 12.39 b) we can verify that this equation coincides with the equation

$$(x - \cos\alpha)(x - \cos\beta)(x - \cos\gamma) = 0,$$

where α, β and γ are the angles of the triangle. Therefore the cosine of φ is equal to the cosine of one of the angles of the triangle; moreover, the cosine is monotonous on the interval $[0, \pi]$.

12.30. It is clear that $2pr = 2S = ab\sin\gamma = \frac{abc}{2R}$, i.e., $4prR = abc$. To prove the second equality make use of Heron's formula: $S^2 = p(p-a)(p-b)(p-c)$, i.e.,

$$\begin{aligned} pr^2 &= (p-a)(p-b)(p-c) = p^3 - p^2(a+b+c) + p(ab+bc+ca) - abc = \\ &= -p^3 + p(ab+bc+ca) - 4prR. \end{aligned}$$

By dividing by p we get the desired equality.

12.31. Since $abc = 4RS$ (Problem 12.1), the expression in the left-hand side is equal to $\frac{c+a+b}{4RS} = \frac{2p}{4Rpr} = \frac{1}{2Rr}$.

12.32. It suffices to observe that $\frac{p-c}{p} = \frac{r}{r_c}$ (Problem 12.18 a)),

$$r = \frac{c \sin\frac{\alpha}{2}\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}} \text{ and } r_c = \frac{c\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}}$$

(Problem 12.17).

12.33. By Problem 12.1, $S = \frac{abc}{4R}$. On the other hand, $S = \frac{ah_a}{2}$. Hence, $h_a = \frac{bc}{2R}$.

12.34. Since $ah_a = 2S = 2(p-a)r_a$ and $\frac{r_a}{a} = \frac{\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$ (Problem 12.17 b)), we have

$$h_a = \frac{2(p-a)\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}.$$

Taking into account that $(p-a)\cot\frac{\beta}{2} = r_c = (p-b)\cot\frac{\alpha}{2}$ (Problem 12.17 c)), we get $h_a = \frac{2(p-b)\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}}$.

12.35. a) Let the extension of bisector AD intersect the circumscribed circle of triangle ABC at point M. Then $AD \cdot DM = BD \cdot DC$ and since $\triangle ABC \sim \triangle AMC$, it follows that

$$AB \cdot AC = AD \cdot AM = AD(AD + DM) = AD^2 + BD \cdot DC.$$

Moreover, $BD = \frac{ac}{b+c}$ and $DC = \frac{ab}{b+c}$. Hence,

$$AD^2 = bc - \frac{bca^2}{(b+c)^2} = \frac{4p(p-a)bc}{(b+c)^2}.$$

b) See the solution of Problem 4.47.

c) Let AD be a bisector, AH a height of triangle ABC. Then $AH = c\sin\beta = 2R\sin\beta\sin\gamma$. On the other hand,

$$AH = AD\sin\angle ADH = l_a \sin\left(\beta + \frac{\alpha}{2}\right) = l_a \sin\frac{\pi+\beta-\gamma}{2} = l_a\cos\frac{\beta-\gamma}{2}.$$

d) Taking into account that $p = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ (Problem 12.36 c)) and

$$\sin\beta + \sin\gamma = 2\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2} = 2\cos\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}$$

we arrive at the formula of heading c).

12.36. a) Let O be the center of the inscribed circle, K the tangent point of the inscribed circle with side AB. Then

$$2R\sin\gamma = AB = AK + KB = r\left(\cot\frac{\alpha}{2} + \cot\frac{\beta}{2}\right) = r\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Taking into account that $\sin\gamma = 2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}$ and $\sin\frac{\alpha+\beta}{2} = \cos\frac{\gamma}{2}$ we get the desired statement.

b) By Problem 3.2, $p-a = AK = r\cot\frac{\alpha}{2}$. Similarly, $p-b = r\cot\frac{\beta}{2}$ and $p-c = r\cot\frac{\gamma}{2}$. By multiplying these equalities and taking into account that $p(p-a)(p-b)(p-c) = S^2 = (pr)^2$ we get the desired statement.

c) Obviously follows from headings a) and b).

12.37. a) By multiplying equalities $r\cos\frac{\alpha}{2}\sin\frac{\alpha}{2} = p - a$ and

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{r}{4R}$$

(cf. Problems 12.17 c) and 12.36 a)) we get the desired statement.

b) By Problem 12.17 c), $r_a \tan\frac{\gamma}{2} = p - b = r\cot\frac{\beta}{2}$. By multiplying this equality by $\frac{r}{4R} = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ we get the desired statement.

12.38. By adding equalities

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\gamma = -\cos(\alpha+\beta) = -2\cos^2\frac{\alpha+\beta}{2} + 1$$

and taking into account that

$$\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2} = 2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}$$

we get

$$\cos\alpha + \cos\beta + \cos\gamma = 4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} + 1 = \frac{r}{R} + 1,$$

cf. Problem 12.36 a).

12.39. a) Adding equalities

$$\cos 2\alpha + \cos 2\beta = 2\cos(\alpha+\beta)\cos(\alpha-\beta) = -2\cos\gamma\cos(\alpha-\beta);$$
$$\cos 2\gamma = 2\cos^2\gamma - 1 = -2\cos\gamma\cos(\alpha+\beta) - 1$$

and taking into account that

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

we get the desired statement.

b) It suffices to substitute expressions of the form $\cos 2\alpha = 2\cos^2\alpha - 1$ in the equality obtained in heading a).

12.40. Adding equalities

$$\sin 2\alpha + \sin 2\beta = 2\sin(\alpha+\beta)\cos(\alpha-\beta) = 2\sin\gamma\cos(\alpha-\beta);$$
$$\sin 2\gamma = 2\sin\gamma\cos\gamma = -2\sin\gamma\cos(\alpha+\beta)$$

and taking into account that

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

we get the desired statement.

12.41. a) Clearly,

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{a^2+b^2+c^2}{4R}$$

and

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 4p^2 - 2(r^2+p^2+4rR),$$

cf. Problem 12.30.

b) By Problem 12.39 b)

$$2\cos\alpha\cos\beta\cos\gamma = \sin^2\alpha + \sin^2\beta + \sin^2\gamma - 2.$$

It remains to make use of a result of heading a).

12.42. The law of cosines can be expressed as $ab\cos\gamma = \frac{a^2+b^2-c^2}{2}$. By adding three similar equalities we get the desired statement.

12.43. By Problem 12.13 $\frac{\cos^2\frac{\alpha}{2}}{a} = \frac{p(p-a)}{abc}$. It remains to notice that $p(p-a) + p(p-b) + p(p-c) = p^2$ and $abc = 4SR = 4prR$.

12.44. a) Since $bc\cos\alpha = 2S\cot\alpha$, it follows that $a^2 = b^2 + c^2 - 4S\cot\alpha$. By adding three similar equalities we get the desired statement.

b) For an acute triangle $a^2\cot\alpha = 2R^2\sin 2\alpha = 4S_{BOC}$, where O is the center of the circumscribed circle. It remains to add three analogous equalities. For a triangle with an obtuse angle α the quality S_{BOC} should be taken with the minus sign.

12.45. By Problem 12.17 $\cot\frac{\alpha}{2} + \cot\frac{\beta}{2} = \frac{c}{r}$ and $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} = \frac{c}{r_c}$. It remains to add such equalities for all pairs of angles of the triangle.

12.46. Clearly,

$$\tan\gamma = -\tan(\alpha+\beta) = -\frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

By multiplying both sides of equality by $1 - \tan\alpha\tan\beta$ we get the desired statement.

12.47.

$$\tan\frac{\gamma}{2} = \cot\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \left[1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right]\left[\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}\right].$$

It remains to multiply both sides of the equality by $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}$.

12.48. a) Let us multiply both sides of the equality by $\sin\alpha\sin\beta\sin\gamma$. Further on:

$$\begin{gathered}\cos\gamma(\sin\alpha\cos\beta + \sin\beta\cos\alpha) + \sin\gamma(\cos\alpha\cos\beta - \sin\alpha\sin\beta) = \\ \cos\gamma\sin(\alpha+\beta) + \sin\gamma\cos(\alpha+\beta) = \\ \cos\gamma\sin\gamma - \sin\gamma\cos\gamma = 0.\end{gathered}$$

b) Let us multiply both sides of the equality by $\sin\alpha\sin\beta\sin\gamma$. Further on:

$$\cos\alpha(\sin\beta\sin\gamma - \cos\beta\cos\gamma) + \sin\alpha(\cos\beta\sin\gamma + \cos\gamma\sin\beta) = \cos^2\alpha + \sin^2\alpha = 1.$$

12.49. Since

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma - 2 = 2\cos\alpha\cos\beta\cos\gamma$$

(see Problem 12.39 b) and $S = 2R^2\sin\alpha\sin\beta\sin\gamma$, it remains to verify that

$$(\tan\alpha + \tan\beta + \tan\gamma)\cos\alpha\cos\beta\cos\gamma = \sin\gamma\sin\beta\sin\alpha.$$

The latter equality is proved in the solution of Problem 12.48 a).

12.50. Let A and B be the vertices of angles α and β, let P be the intersection point of non-coinciding legs of these angles, Q the common point of the given circles that lies on segment PA. Triangle AQB is an isosceles one, hence, $\angle PQB = 2\alpha$. Since $\angle PQB + \angle QPB = \beta + \angle QBA$, it follows that $\beta = 3\alpha$.

12.51. By Problem 4.47, $\frac{1}{b} + \frac{1}{c} = \frac{2\cos\frac{\alpha}{2}}{l_a}$, hence, $\cos\frac{\alpha}{2} = \frac{1}{2}$, i.e., $\alpha = 120^\circ$.

12.52. Let us drop perpendicular MD from point M to line BC. Then $MD = \frac{1}{2}AH = \frac{1}{2}BM$. In right triangle BDM, leg MD is equal to a half hypothenuse BM. Hence, $\angle MBC = \angle MBD = 30^\circ$.

12.53. The quantities $AD\cdot BC\sin ADB$ and $BE\cdot AC\sin AEB$ are equal because each of them is equal to the doubled area of triangle ABC. Hence, $\sin ADB = \sin AEB$. Two cases are possible:

1) $\angle ADB = \angle AEB$. In this case points A, E, D, B lie on one circle; hence, $\angle EAD = \angle EBD$, i.e., $\angle A = \angle B$ which contradicts the hypothesis.

2) $\angle ADB + \angle AEB = 180^\circ$. In this case $\angle ECD + \angle EOD = 180^\circ$, where O is the intersection point of bisectors. Since $\angle EOD = 90^\circ + \frac{\angle C}{2}$ (Problem 5.3), it follows that $\angle C = 60^\circ$.

12.54. Let B' be the intersection point of the midperpendicular to segment AC with line AB. Then $AB' = CB'$ and $\angle AB'C = 180^\circ - 2\cdot 75^\circ = 30^\circ$. Hence, $AB' = CB' = 2CH = AB$, i.e., $B' = B$ and $\angle B = 30^\circ$.

12.55. Clearly, $AKDM$ is a rectangle and L the intersection point of its diagonals. Since $AD \perp BC$ and $AM \perp BA$, it follows that $\angle DAM = \angle ABC$. Similarly, $\angle KAD = \angle ACB$. Let us drop perpendicular AP from point A to line KM. Let, for definiteness, $\angle B < \angle C$. Then point P lies on segment KL. Since $\triangle AKP \sim \triangle MKA$, it follows that $AK : AP = MK : MA$. Hence, $AK \cdot AM = AP \cdot MK = AP \cdot AD = 2AP \cdot AL$. By the hypothesis $AL^2 = AK \cdot AM$; hence, $AL = 2AP$, i.e., $\angle ALP = 30^\circ$. Clearly, $\angle KMA = \frac{\angle ALP}{2} = 15^\circ$. Therefore, the acute angles of triangle ABC are equal to 15° and 75°.

12.56. Let CD be a bisector. Then $BD = \frac{ac}{a+b}$. On the other hand, $\triangle BDC \sim \triangle BCA$, consequently, $BD : BC = BC : BA$, i.e., $BD = \frac{a^2}{c}$. Hence $c^2 = a(a+b) = 3a^2$. The lengths of the sides of triangle ABC are equal to a, $2a$ and $\sqrt{3}a$; hence, its angles are equal to 30°, 90° and 60°, respectively.

12.57. Let $\angle ABC = 2x$. Then the outer angle $\angle A$ of triangle ABE is equal to $\angle ABE + \angle AEB = x + \alpha$. Further,

$$\angle KAE = \angle BAE - \angle BAK = (180^\circ - x - \alpha) - (180^\circ - 2x - 2\alpha) = x + \alpha.$$

Therefore, AE is the bisector of the outer angle $\angle A$ of triangle ABK. Since BE is the bisector of the inner angle $\angle B$ of triangle ABK, it follows that E is the center of its escribed circle tangent to side AK. Hence, $\angle AKE = \frac{1}{2}\angle AKC = 90^\circ - \alpha$.

12.58. Let $A_1 \dots A_{18}$ be a regular 18-gon. For triangle ABC we can take triangle $A_{14}A_1A_9$. By Problem 6.59 b) the diagonals A_1A_{12}, A_2A_{14} and A_9A_{18} meet at one point, hence, $\angle AMC = \frac{1}{2}(\smile A_{18}A_2 + \smile A_9A_{14}) = 70^\circ$.

12.59. Let $A_1 \dots A_{18}$ be a regular 18-gon, O its center. For triangle ABC we can take triangle A_1OA_{18}. The diagonals A_2A_{14} and $A_{18}A_6$ are symmetric through diameter A_1A_{10}; diagonal A_2A_{14} passes through the intersection point of diagonals A_1A_{12} and A_9A_{18} (cf. the solution of Problem 12.58), therefore, $\angle ADE = \frac{1}{2}(\smile A_1A_2 + \smile A_{12}A_{14}) = 30^\circ$.

12.60. Since $\angle BDE = 50^\circ$ and $\angle CDE = 30^\circ$, it follows that $\angle BOC = \angle EOD = 180^\circ - 50^\circ - 30^\circ = 100^\circ$. Let us assume that diameters BB' and CC' of the circle are fixed, $\angle BOC = 100^\circ$ and point A moves along arc $\smile B'C'$. Let D be the intersection point of BB' and AC, E the intersection point of CC' and AB (Fig. 140). As point A moves from B' to C', segment OE increases while OD decreases, consequently, angle $\angle OED$ decreases and angle $\angle ODE$ increases. Therefore, there exists a unique position of point A for which $\angle CED = \angle OED = 30^\circ$ and $\angle BDE = \angle ODE = 50^\circ$.

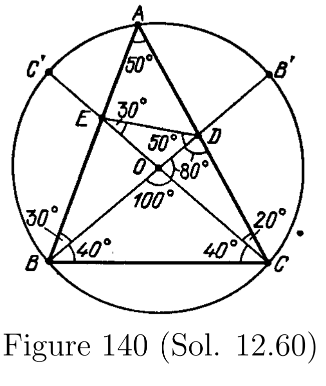


Figure 140 (Sol. 12.60)

Now, let us prove that triangle ABC with angles $\angle A = 50^\circ$, $\angle B = 70^\circ$, $\angle C = 60^\circ$ possesses the required property. Let $A_1 \dots A_{18}$ be a regular 18-gon. For triangle ABC we can take triangle $A_2A_{14}A_9$. Diagonal A_1A_{12} passes through point E (cf. solution of Problem 12.58). Let F be the intersection point of lines A_1A_{12} and A_5A_{14}; line A_9A_{16} is symmetric to line A_1A_{12} through line A_5A_{14} and, therefore, it passes through point F. In triangle CDF, ray CE is the bisector of angle $\angle C$ and line FE is the bisector of the outer angle at vertex F. Hence, DE is the bisector of angle $\angle ADB$, i.e., $\angle ODE = \frac{1}{4}(\smile A_2A_{14} + \smile A_5A_9) = 50^\circ$.

12.61. Let D, E and F be the tangent points of the circle with BP, PQ and QC, respectively; $\angle BOD = 90^\circ - \angle B = 90^\circ - \angle C = \angle COF = \alpha$, $\angle DOP = \angle POE = \beta$ and $\angle EOQ = \angle QOF = \gamma$. Then $180^\circ = \angle BOC = 2\alpha + 2\beta + 2\gamma$, i.e., $\alpha + \beta + \gamma = 90^\circ$. Since $\angle BPO = \frac{1}{2}\angle DPE = \frac{1}{2}(180^\circ - \angle DOE) = 90^\circ - \beta$ and $\angle QOC = \gamma + \alpha = 90^\circ - \beta$, it follows that $\angle BPO = \angle COQ$. It is also clear that $\angle PBO = \angle OCQ$. Hence, $\triangle BPO \sim \triangle COQ$, i.e., $PB \cdot CQ = BO \cdot CO = \frac{1}{4}BC^2$.

12.62. Let P and Q be the midpoints of sides BC and CD, respectively. Points P and Q are the tangent points of the inscribed circle with sides BC and CD. Therefore, it suffices to verify that $PF + GQ = FG$. Indeed, if $F'G'$ is the segment parallel to FG and tangent to the inscribed circle, then $PF' + G'Q = F'G'$; hence, $F' = F$ and $G' = G$.

We may assume that the side of the square is equal to 2. Let $GD = x$. Since $BF : EB = AD : GD$, then $BF = \frac{2}{x}$. Therefore, $CG = 2 - x$, $GQ = x - 1$, $CF = 2 - \frac{2}{x}$, $FP = \frac{2}{x} - 1$, i.e., $PF + GQ = x + \frac{2}{x} - 2$ and

$$FG^2 = CG^2 + CF^2 = (2-x)^2 + \left(2 - \frac{2}{x}\right)^2 = \\ 4 - 4x + x^2 + 4 - \frac{8}{x} + \frac{4}{x^2} = \left(x + \frac{2}{x} - 2\right)^2 = (PF + GQ)^2.$$

12.63. Denote the vertices of the squares as shown on Fig. 141. Let O be the center of the circle, H the midpoint of the given chord, K the midpoint of segment AA_1.

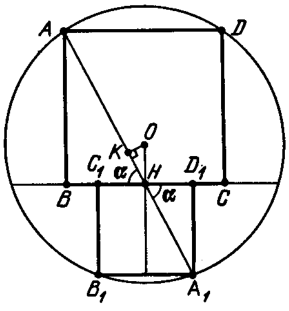


Figure 141 (Sol. 12.63)

Since $\tan AHB = 2 = \tan A_1HD_1$, point H lies on line AA_1. Let $\alpha = \angle AHB = \angle A_1HD_1$, then

$$AB - A_1D_1 = (AH - A_1H) \cdot \sin\alpha = 2KH \sin\alpha = 2OH \sin^2\alpha.$$

Since $\tan\alpha = 2$ and $1 + \cot^2\alpha = \frac{1}{\sin^2\alpha}$, it follows that $\sin^2\alpha = \frac{4}{5}$. Therefore, the difference of the lengths of the squares' sides is equal to $\frac{8}{5}h$.

12.64. Let median BM of triangle ABC intersect the inscribed circle at points K and L, where $BK = KL = LM = x$. Let, for definiteness, the tangent point of the inscribed circle with side AC lie on segment MC. Then since the symmetry through the midperpendicular to segment BM interchanges points B and M and fixes the inscribed circle, tangent MC turns into tangent BC. Therefore, $BC = MC = \frac{1}{2}AC$, i.e., $b = 2a$.

Since $BM^2 = \frac{2a^2+2c^2-b^2}{4}$ by Problem 12.11 a), we have $9x^2 = \frac{2a^2+2c^2-4a^2}{4} = \frac{c^2-a^2}{2}$. Let P be the tangent point of the inscribed circle with side BC. Then $BP = \frac{a+c-b}{2} = \frac{c-a}{2}$. On the other hand, by a property of the tangent, $BP^2 = BK \cdot BL$, i.e., $BP^2 = 2x^2$. Hence,

$2x^2 = \left(\frac{c-a}{2}\right)^2$. Multiplying inequalities $9x^2 = \frac{c^2-a^2}{2}$ and $\left(\frac{c-a}{2}\right)^2 = 2x^2$ we get $\frac{c+a}{c-a} = \frac{9}{4}$, i.e., $c : a = 13 : 5$. As a result we get $a : b : c = 5 : 10 : 13$.

12.65. Let $2a$ and $2b$ be the length of the side of the first and second squares, respectively. Then the distance from the center of the circle to any of the vertices of the second square that lie on the circle is equal to $\sqrt{(a+2b)^2+b^2}$. On the other hand, this distance is equal to $\sqrt{2}a$. Therefore, $(a+2b)^2+b^2 = 2a^2$, i.e., $a = 2b \pm \sqrt{4b^2+5b^2} = (2 \pm 3)b$. Only the solution $a = 5b$ is positive.

12.66. Let P and Q be the midpoints of segments AC and AB, respectively, R the center of circle S_1; $a = \frac{1}{2}AC$, $b = \frac{1}{2}BC$, x the radius of circle S_1. It is easy to verify that $PR = a+x$, $QR = a+b-x$ and $PQ = b$. In triangle PQR, draw height RH. The distance from point R to line CD is equal to x, hence, $PH = a-x$, consequently, $QH = |b-a+x|$. It follows that

$$(a+x)^2 - (a-x)^2 = RH^2 = (a+b-x)^2 - (b-a+x)^2,$$

i.e., $ax = b(a-x)$. As a result we get $x = \frac{ab}{a+b}$.

For the radius of circle S_2 we get precisely the same expression.

12.67. Let x be the radius of circle S tangent to circles S_1 and S_2 and ray AB, let y be the radius of circle S' tangent to circles S_2 and S_3 and ray BA. The position of the circle tangent to circle S_1 and ray AB (resp. S_3 and BA) is uniquely determined by its radius, consequently, it suffices to verify that $x = y$.

By equating two expressions for the squared distance from the center of circle S to line AD we get

$$(x+1)^2 - (x-1)^2 = (3+x)^2 - (5-x)^2, \text{ i.e., } x = \frac{4}{3}.$$

Considering circles S_2 and S_3 it is easy to verify that $AB^2 = (3+4)^2 - 1^2 = 48$. On the other hand, the squared distances from the center of circle S' to lines AD and BC are equal to $(y+3)^2 - (5-y)^2 = 16(y-1)$ and $(4+y)^2 - (4-y)^2 = 16y$, respectively. Therefore, $4\sqrt{y-1} + 4\sqrt{y} = \sqrt{48}$, i.e., $y = \frac{4}{3}$.

12.68. If the angles of a triangle form an arithmetic progression, then they are equal to $\alpha-\gamma$, α, $\alpha+\gamma$, where $\gamma \geq 0$. Since the sum of the angles of a triangle is equal to 180°, we deduce that $\alpha = 60^\circ$. The sides of this triangle are equal to $2R\sin(\alpha-\gamma)$, $2R\sin\alpha$, $2R\sin(\alpha+\gamma)$. Since the greater side subtends the greater angle, $\sin(\alpha-\gamma) \leq \sin\alpha \leq \sin(\alpha+\gamma)$.

a) If the numbers $\sin(\alpha-\gamma) \leq \sin\alpha \leq \sin(\alpha+\gamma)$ form an arithmetic progression, then $\sin\alpha = \frac{1}{2}(\sin(\alpha+\gamma)+\sin(\alpha-\gamma)) = \sin\alpha\cos\gamma$, i.e., either $\cos\gamma = 1$ or $\gamma = 0$. Therefore, each of the triangle's angles is equal to 60°.

b) If the numbers $\sin(\alpha-\gamma) \leq \sin\alpha \leq \sin(\alpha+\gamma)$ form a geometric progression, then

$$\sin^2\alpha = \sin(\alpha-\gamma)\sin(\alpha+\gamma) = \sin^2\alpha\cos^2\gamma - \sin^2\gamma\cos^2\alpha \leq \sin^2\alpha\cos^2\gamma.$$

Hence, $\cos\gamma = 1$, i.e., each of the triangle's angles is equal to 60°.

12.69. Let us complement triangle ABC to parallelogram $ABCE$ (Fig. 142). Let $BC = x$ and $AD = y$. Then $(b-a)h = 2S_{AED} = xy\sin 45^\circ$ and

$$(b-a)^2 = x^2 + y^2 - 2xy\cos 45^\circ = x^2 + y^2 - 2xy\sin 45^\circ.$$

By Pythagoras theorem

$$a^2 + b^2 = (AO^2 + BO^2) + (CO^2 + DO^2) = (BO^2 + CO^2) + (DO^2 + AO^2) = x^2 + y^2.$$

Therefore,

$$(b-a)^2 = x^2 + y^2 - 2xy\sin 45^\circ = a^2 + b^2 - 2(b-a)h,$$

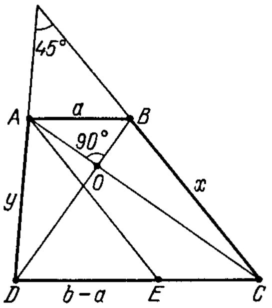


Figure 142 (Sol. 12.69)

i.e., $h = \frac{ab}{b-a}$.

12.70. Since $BK = \frac{1}{2}(a+c-b)$ and $KC = \frac{1}{2}(a+b-c)$ (cf. Problem 3.2), it follows that $BK \cdot KC = \frac{a^2-(b-c)^2}{4} = S\tan\frac{\alpha}{2}$, cf. Problem 12.12.

12.71. Since $\frac{b+c}{a} = \frac{\cos\frac{\beta-\gamma}{2}}{\sin\frac{\alpha}{2}}$ (Problem 12.4), it follows that $\cos\frac{\beta-\gamma}{2} = \cos\frac{\alpha}{2}$, i.e., $\beta-\gamma = \pm\alpha$. If $\beta = \gamma+\alpha$, then $\beta = 90^\circ$ and if $\beta+\alpha = \gamma$, then $\gamma = 90^\circ$.

12.72. It is easy to verify that $S_{ABC} = 2R^2\sin\alpha\sin\beta\sin\gamma$. Analogously,

$$S_{A_1B_1C_1} = 2R^2\sin\frac{\beta+\gamma}{2}\sin\frac{\alpha+\gamma}{2}\sin\frac{\alpha+\beta}{2} = 2R^2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

Hence,

$$\frac{S_{ABC}}{S_{A_1B_1C_1}} = 8\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} = \frac{2r}{R},$$

cf. Problem 12.36 a).

12.73. The sum of cotangents of the angles of a triangle is equal to $\frac{a^2+b^2+c^2}{4S}$, cf. Problem 12.44 a). Moreover, $m_a^2+m_b^2+m_c^2 = \frac{3(a^2+b^2+c^2)}{4}$ (by Problem 12.11 b)) and the area of the triangle formed by the medians of triangle ABC is equal to $\frac{3}{4}S_{ABC}$ (Problem 1.36).

12.74. One of the points A_i lies inside the triangle formed by the other three points; hence, we can assume that triangle $A_1A_2A_3$ is an acute one (or a right one). Numbers λ_1, λ_2 and λ_3 are easy to obtain from the corresponding system of equations; as a result we get

$$\lambda_1 = \frac{b^2+c^2-a^2}{2},\ \lambda_2 = \frac{a^2+c^2-b^2}{2} \text{ and } \lambda_3 = \frac{a^2+b^2-c^2}{2},$$

where $a = A_2A_3, b = A_1A_3$ and $c = A_1A_2$. By Problem 5.45 b) $A_1A_4^2 = 4R^2-a^2$, where R is the radius of the circumscribed circle of triangle $A_1A_2A_3$. Hence,

$$\lambda_4 = A_1A_4^2-\lambda_1 = 4R^2-\frac{a^2+b^2+c^2}{2} = A_2A_4^2-\lambda_2 = A_3A_4^2-\lambda_3.$$

Now, let us verify that $\sum\frac{1}{\lambda_i} = 0$. Since $\frac{b^2+c^2-a^2}{2} = bc\cos\alpha = 2S\cot\alpha$, it follows that $\frac{1}{\lambda_1} = \tan\frac{\alpha}{2S}$. It remains to observe that

$$\frac{2}{a^2+b^2+c^2-8R^2} = \frac{\tan\alpha+\tan\beta+\tan\gamma}{2S}$$

?Problem 12.49.

12.75. Let (a_1,b_1), (a_2,b_2) and (a_3,b_3) be the coordinates of the triangle's vertices. The coordinates of the center of the circumscribed circle of the triangle are given by the system of equations

$$(x-a_1)^2+(y-b_1)^2 = (x-a_2)^2+(y-b_2)^2,$$
$$(x-a_1)^2+(y-b_1)^2 = (x-a_3)^2+(y-b_3)^2.$$

It is easy to verify that these equations are actually linear ones and, therefore, the solution of the considered system is a rational one.

12.76. On segments AB and CD, take points K and L that divide the segments in the ratios indicated. It suffices to prove that the intersection point of lines AK and CL lies on circle S. Let us take the coordinate system with the origin at the center O of circle S and axes Ox and Oy directed along rays OB and OD. The radius of circle S can be assumed to be equal to 1. Lines AK and CL are given by equations $y = \frac{x+1}{3}$ and $y = 2x - 1$, respectively. Therefore, the coordinates of their intersection point are $x_0 = \frac{4}{5}$ and $y_0 = \frac{3}{5}$. Clearly, $x_0^2 + y_0^2 = 1$.

12.77. Let d be the distance between the center of the circumscribed circle and the image of the center of the inscribed circle under the considered homothety. It suffices to verify that $R = d + 2r$. Let $(0, 0)$, $(2a, 0)$ and $(0, 2b)$ be the coordinates of the vertices of the given triangle. Then (a, b) are the coordinates of the center of the circumscribed circle, (r, r) the coordinates of the center of the inscribed circle, where $r = a + b - R$. Therefore,

$$d^2 = (2r - a)^2 + (2r - b)^2 = a^2 + b^2 - 4r(a + b - r) + 4r^2 = (R - 2r)^2$$

because $a^2 + b^2 = R^2$.

12.78. Let us consider the coordinate system with the origin at the center of the square and the Ox-axis parallel to line l. Let the coordinates of the vertices of the square be $(A(x, y)$, $B(y, -x)$, $C(-x, -y)$ and $D(-y, x)$; let line l be given by the equation $y = a$. Then the coordinates of point Q are $\left(\frac{x+y}{2}, \frac{y-x}{2}\right)$ and those of P are $(-y, a)$. Therefore, the locus to be found consists of points $\left(t, -t + \frac{1}{2}a\right)$, where $t = \frac{x-y}{4}$. It remains to observe that the quantity $x - y$ varies from $-\sqrt{2(x^2 + y^2)} = -AB$ to AB.

Chapter 13. VECTORS

Background

1. We will make use of the following notations:

a) $\overrightarrow{AB}$ and $\mathbf{a}$ denote vectors;

b) AB and $|\mathbf{a}|$ denote the lengths of these vectors; sometimes the length of vector $\mathbf{a}$ will be denoted by a; a *unit* vector is a vector of unit length;

c) $(\overrightarrow{AB}, \overrightarrow{CD})$, $(\mathbf{a}, \mathbf{b})$ and $(\overrightarrow{AB}, \mathbf{a})$ denote the inner products of the vectors;

d) (x, y) is the vector with coordinates x, y;

e) $\overrightarrow{0}$ or $\mathbf{0}$ denotes the zero vector.

2. The *oriented angle between the nonzero vectors* $\mathbf{a}$ and $\mathbf{b}$ (notation $\angle(\mathbf{a}, \mathbf{b})$) is the angle through which one should rotate the vector $\mathbf{a}$ counterclockwise to make it directed as $\mathbf{b}$ is. The angles that differ by 360 degrees are assumed to be equal. It is easy to verify the following properties of oriented angles between vectors:

a) $\angle(\mathbf{a}, \mathbf{b}) = -\angle(\mathbf{b}, \mathbf{a})$;

b) $\angle(\mathbf{a}, \mathbf{b}) + \angle(\mathbf{b}, \mathbf{c}) = \angle(\mathbf{a}, \mathbf{c})$;

c) $\angle(-\mathbf{a}, \mathbf{b}) = \angle(\mathbf{a}, \mathbf{b}) + 180^\circ$.

3. The *inner product of vectors* $\mathbf{a}$ and $\mathbf{b}$ is the number

$$(\mathbf{a}, \mathbf{b}) = |\mathbf{a}| \cdot |\mathbf{b}| \cos \angle(\mathbf{a}, \mathbf{b})$$

(if one of these vectors is the zero one, then by definition $(\mathbf{a}, \mathbf{b}) = 0$). The following properties of the inner product are easily verified:

a) $(\mathbf{a}, \mathbf{b}) = (\mathbf{b}, \mathbf{a})$;

b) $|(\mathbf{a}, \mathbf{b})| \leq |\mathbf{a}| \cdot |\mathbf{b}|$;

c) $(\lambda\mathbf{a} + \mu\mathbf{b}, \mathbf{c}) = \lambda(\mathbf{a}, \mathbf{c}) + \mu(\mathbf{b}, \mathbf{c})$;

d) if $\mathbf{a}, \mathbf{b} \neq 0$ then $(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{a} \perp \mathbf{b}$.

4. Many of vector inequalities can be proved with the help of the following fact.

Given two sets of vectors such that the sum of lengths of projections of the vectors of the first set to any straight line does not exceed the sum of the lengths of projections of the vectors from the second set to the same line, the sum of the lengths of the vectors from the first set does not exceed the sum of the lengths of the vectors of the second set, cf. Problem 13.39.

In this way a problem on a plane reduces to a problem on a straight line which is usually easier.

Introductory problems

1. Let AA_1 be the median of triangle ABC. Prove that $\overrightarrow{AA_1} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$.

2. Prove that $|\mathbf{a} + \mathbf{b}|^2 + |\mathbf{a} - \mathbf{b}|^2 = 2(|\mathbf{a}|^2 + |\mathbf{b}|^2)$.

3. Prove that if vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are perpendicular, then $|\mathbf{a}| = |\mathbf{b}|$.

4. Let $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{0}$ and $OA = OB = OC$. Prove that ABC is an equilateral triangle.

5. Let M and N be the midpoints of segments AB and CD, respcetively. Prove that $\overrightarrow{MN}=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{BD})$.

§1. Vectors formed by polygons' (?) sides

13.1. a) Prove that from the medians of a triangle one can construct a triangle.

b) From the medians of triangle ABC one constructed triangle $A_1B_1C_1$ and from the medians of triangle $A_1B_1C_1$ one constructed triangle $A_2B_2C_2$. Prove that triangles ABC and $A_2B_2C_2$ are similar with simlarity coefficient $\frac{3}{4}$.

13.2. The sides of triangle T are parallel to the respective medians of triangle T_1. Prove that the medians of T are parallel to the corresponding sides of T_1.

13.3. Let M_1, $M_2, \dots, M_6$ be the midpoints of a convex hexagon $A_1A_2\dots A_6$. Prove that there exists a triangle whose sides are equal and parallel to the segments M_1M_2, M_3M_4, M_5M_6.

13.4. From a point inside a convex n-gon, the rays are drawn perpendicular to the sides and intersecting the sides (or their continuations). On these rays the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ whose lengths are equal to the lengths of the corresponding sides are drawn. Prove that $\mathbf{a}_1+\dots+\mathbf{a}_n=\mathbf{0}$.

13.5. The sum of four unit vectors is equal to zero. Prove that the vectors can be divided into two pairs of opposite vectors.

13.6. Let E and F be the midpoints of sides AB and CD of quadrilateral $ABCD$ and K, L, M and N are the midpoints of segments AF, CE, BF and DE, respectively. Prove that $KLMN$ is a parallelogram.

13.7. Consider n pairwise noncodirected vectors ($n\geq 3$) whose sum is equal to zero. Prove that there exists a convex n-gon such that the set of vectors formed by its sides coincides with the given set of vectors.

13.8. Given four pairwise nonparallel vectors whose sum is equal to zero. Prove that we can construct from them:

a) a nonconvex quadrilateral;

b) a self-intersecting broken line of four links.

13.9. Given four pairwise nonparallel vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ whose sum is equal to zero, prove that

$$|\mathbf{a}|+|\mathbf{b}|+|\mathbf{c}|+|\mathbf{d}|>|\mathbf{a}+\mathbf{b}|+|\mathbf{a}+\mathbf{c}|+|\mathbf{a}+\mathbf{d}|.$$

13.10. In a convex pentagon $ABCDE$ side BC is parallel to diagonal AD, in addition we have $CD\parallel BE$, $DE\parallel AC$ and $AE\parallel BD$. Prove that $AB\parallel CE$.

§2. Inner product. Relations

13.11. Prove that if the diagonals of quadrilateral $ABCD$ are perpendicular to each other, then the diagonals of any other quadrilateral with the same lengths of its sides are perpendicular to each other.

13.12. a) Let A, B, C and D be arbitrary points on a plane. Prove that

$$(\overrightarrow{AB},\overrightarrow{CD})+(\overrightarrow{BC},\overrightarrow{AD})+(\overrightarrow{CA},\overrightarrow{BD})=0.$$

b) Prove that the hights of a triangle intersect at one point.

13.13. Let O be the center of the circle inscribed in triangle ABC and let point H satisfy $OH=OA+OB+OC$. Prove that H is the intersection point of heights of triangle ABC.

13.14. Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be vectors formed by the sides of an n-gon, $\varphi_{ij} = \angle(\mathbf{a}_i, \mathbf{a}_j)$. Prove that

$$a_1^2 = a_2^2 + \dots + a_n^2 + 2\sum_{i>j>1} a_i a_j \cos\varphi_{ij}, \text{ where } a_i = |\mathbf{a}_i|.$$

13.15. Given quadrilateral $ABCD$ and the numbers

$$u = AD^2,\ \ v = BD^2,\ \ w = CD^2,\ \ U = BD^2 + CD^2 - BC^2,$$
$$V = AD^2 + CD^2 - AC^2,\ \ W = AD^2 + BD^2 - AB^2.$$

Prove that

((Gauss).) $$uU^2 + vV^2 + wW^2 = UVW + 4uvw.$$

13.16. Points A, B, C and D are such that for any point M the numbers $(\overrightarrow{MA}, \overrightarrow{MB})$ and $(\overrightarrow{MC}, \overrightarrow{MD})$ are distinct. Prove that $\overrightarrow{AC} = \overrightarrow{DB}$.

13.17. Prove that in a convex k-gon the sum of distances from any inner point to the sides of the k-gon is constant if and only if the sum of vectors of unit exterior normals to the sides is equal to zero.

13.18. In a convex quadrilateral the sum of distances from a vertex to the sides is the same for all vertices. Prove that this quadrilateral is a parallelogram.

§3. Inequalities

13.19. Given points A, B, C and D. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 \geq AC^2 + BD^2,$$

where the equality is attained only if $ADCD$ is a parallelogram.

13.20. Prove that from any five vectors one can always select two so that the length of their sum does not exceed the length of the sum of the remaining three vectors.

13.21. Ten vectors are such that the length of the sum of any nine of them is smaller than the length of the sum of all the ten vectors. Prove that there exists an axis such that the projection of every of the ten vectors to the axis is positive.

13.22. Points $A_1, \dots, A_n$ lie on a circle with center O and $\overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \overrightarrow{0}$. Prove that for any point X we have

$$XA_1 + \dots + XA_n \geq nR,$$

where R is the radius of the circle.

13.23. Given eight real numbers a, b, c, d, e, f, g, h. Prove that at least one of the six numbers

$$ac + bd,\ \ ae + bf,\ \ ag + bh,\ \ ce + df,\ \ cg + dh,\ \ eg + fh$$

is nonnegative.

13.24. On the circle of radius 1 with center O there are given $2n+1$ points $P_1, \dots, P_{2n+1}$ which lie on one side of a diameter. Prove that

$$|\overrightarrow{OP_1} + \dots + \overrightarrow{OP_{2n+1}}| \geq 1.$$

13.25. Let $\mathbf{a}_1$, $\mathbf{a}_2$, … , $\mathbf{a}_n$ be vectors whose length does not exceed 1. Prove that in the sum

$$\mathbf{c} = \pm\mathbf{a}_1 \pm \mathbf{a}_2 \pm \dots \pm \mathbf{a}_n$$

we can select signs so that $|\mathbf{c}| \leq \sqrt{2}$.

13.26. Point O is the beginning point of n unit vectors such that in any half plane bounded by a straight line through O there are contained not less than k vectors (we assume that the boundary line belongs to the half-plane). Prove that the length of the sum of these vectors does not exceed $n-2k$.

§4. Sums of vectors

13.27. Prove that point X belongs to line AB if and only if

$$\overrightarrow{OX} = t\overrightarrow{OA} + (1-t)\overrightarrow{OB}$$

for some t and any point O.

13.28. We are given several points and for several pairs (A, B) of these points the vectors AB are taken in such a way that as many vectors exit from every point as terminate in it. Prove that the sum of all the selected vectors is equal to $\mathbf{0}$.

13.29. Inside triangle ABC, point O is taken. Prove that

$$S_{BOC} \cdot \overrightarrow{OA} + S_{AOC} \cdot \overrightarrow{OB} + S_{AOB} \cdot \overrightarrow{OC} = \overrightarrow{0}.$$

13.30. Points A and B move along two fixed rays with common origin O so that $\frac{p}{OA} + \frac{q}{OB}$ is a constant. Prove that line AB passes through a fixed point.

13.31. Through the intersection point M of medians of triangle ABC a straight line is drawn intersecting BC, CA and AB at points A_1, B_1 and C_1, respectively. Prove that

$$\left(\frac{1}{\overline{MA_1}}\right) + \left(\frac{1}{\overline{MB_1}}\right) + \left(\frac{1}{\overline{MC_1}}\right) = 0.$$

(Segments MA_1, MB_1 and MC_1 are assumed to be oriented.)

13.32. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1, respectively, are taken. Segments BB_1 and CC_1, CC_1 and AA_1, AA_1 and BB_1 intersect at points A_1, B_2 and C_2, respectively. Prove that if $\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \overrightarrow{0}$, then

$$AB_1 : B_1C = CA_1 : A_1B = BC_1 : C_1A.$$

13.33. Quadrilateral $ABCD$ is an inscribed one. Let H_a be the orthocenter of BCD, let M_a be the midpoint of AH_a; let points M_b, M_c and M_d be similarly defined. Prove that points M_a, M_b, M_c and M_d coincide.

13.34. Quadrilateral $ABCD$ is inscribed in a circle of radius R.

a) Let S_a be the circle of radius R with center at the orthocenter of triangle BCD; let circles S_b, S_c and S_d be similarly defined. Prove that these four circles intersect at one point.

b) Prove that the circles of nine points of triangles ABC, BCD, CDA and DAB intersect at one point.

§5. Auxiliary projections

13.35. Point X belongs to the interior of triangle ABC; let $\alpha = S_{BXC}$, $\beta = S_{CXA}$ and $\gamma = S_{AXB}$. Let A_1, B_1 and C_1 be the projections of points A, B and C, respectively, on an arbitrary line l. Prove that the length of vector $\alpha\overrightarrow{AA_1} + \beta\overrightarrow{BB_1} + \gamma\overrightarrow{CC_1}$ is equal to $(\alpha+\beta+\gamma)d$, where d is the distance from X to l.

13.36. A convex $2n$-gon $A_1A_2\dots A_{2n}$ is inscribed into a unit circle. Prove that

$$|\overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \dots + \overrightarrow{A_{2n-1}A_{2n}}| \leq 2.$$

13.37. Let a, b and c be the lengths of the sides of triangle ABC; let $\mathbf{n}_a$, $\mathbf{n}_b$ and $\mathbf{n}_c$ be unit vectors perpendicular to the corresponding sides and directed outwards. Prove that

$$a^3\mathbf{n}_a + b^2\mathbf{n}_b + c^2\mathbf{n}_c = 12S \cdot \overrightarrow{MO},$$

where S is the area, M the intersection point of the medians, O the center of the circle inscribed into triangle ABC.

13.38. Let O and R be the center and the radius, respectively, of an escribed circle of triangle ABC; let Z and r be the center and the radius of the inscribed circle, K the intersection point of the medians of the triangle with vertices at the tangent points of the inscribed circle of triangle ABC with the sides of triangle ABC. Prove that Z belongs to segment OK and

$$OZ : ZK = 3R : r.$$

§6. The method of averaging

13.39. Given two sets of vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ and $\mathbf{b}_1, \dots, \mathbf{b}_m$ such that the sum of the lengths of the projections of the vectors from the first set to any straight line does not exceed the sum of the lengths of the projections of the vectors from the second set to the same straight line. Prove that the sum of the lengths of the vectors from the first set does not exceed the sum of the lengths of the vectors from the second set.

13.40. Prove that if one convex polygon lies inside another one, then the perimeter of the inner polygon does not exceed the perimeter of the outer one.

13.41. The sum of the length of several vectors on a plane is equal to L. Prove that from these vectors one can select several vectors (perhaps, just one) so that the length of their sum is not less than $\frac{L}{\pi}$.

13.42. Prove that if the lengths of any side and diagonal of a convex polygon are shorter than d, then its perimeter is shorter than πd.

13.43. On the plane, there are given four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ whose sum is equal to zero. Prove that

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq |\mathbf{a} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}|.$$

13.44. Inside a convex n-gon $A_1A_2 \dots A_n$ a point O is selected so that $\overrightarrow{OA_1} + \dots + \overrightarrow{OA_n} = \overrightarrow{0}$. Let $d = OA_1 + \dots + OA_n$. Prove that the perimeter of the polygon is not shorter than $\frac{4d}{n}$ for n even and not shorter than $\frac{4dn}{n^2-1}$ for n odd.

13.45. The length of the projection of a closed convex curve to any line is equal to 1. Prove that its length is equal to π.

13.46. Given several convex polygons so that it is impossible to draw a line which does not intersect any of the polygons and at least one polygon would lie on both sides of it. Prove that all the polygons are inside a polygon whose perimeter does not exceed the sum of the perimeters of the given polygons.

§7. Pseudoinner product

The *pseudoinner product* of nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$c = |\mathbf{a}| \cdot |\mathbf{b}| \sin \angle(\mathbf{a}, \mathbf{b});$$

the pseudoinner product is equal to 0 if at least one of the vectors $\mathbf{a}$ or $\mathbf{b}$ is zero. The pseudoinner product is denoted by $c = \mathbf{a} \vee \mathbf{b}$. Clearly, $\mathbf{a} \vee \mathbf{b} = -\mathbf{b} \vee \mathbf{a}$.

The absolute value of the pseudoinner product of $\mathbf{a}$ and $\mathbf{b}$ is equal to the area of the parallelogram spanned by these vectors. In this connection the *oriented area* of the triple of points A, B and C is the number

$$S(A, B, C) = \frac{1}{2}(\overrightarrow{AB} \vee \overrightarrow{AC}).$$

The absolute value of $S(A, B, C)$ is equal to the area of triangle ABC.

13.47. Prove that:

a) $(\lambda\mathbf{a}) \vee \mathbf{b} = \lambda(\mathbf{a}, \mathbf{b})$;
b) $\mathbf{a} \vee (\mathbf{b} + \mathbf{c}) = \mathbf{a} \vee \mathbf{b} + \mathbf{a} \vee \mathbf{c}$.

13.48. Let $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$. Prove that

$$\mathbf{a} \vee \mathbf{b} = a_1b_2 - a_2b_1.$$

13.49. a) Prove that

$$S(A, B, C) = -S(B, A, C) = S(B, C, A).$$

b) Prove that for any points A, B, C and D we have

$$S(A, B, C) = S(D, A, B) + S(D, B, C) + S(D, C, A).$$

13.50. Three runners A, B and C run along the parallel lanes with constant speeds. At the initial moment the area of triangle ABC is equal to 2 in 5 seconds it is equal to 3. What might be its value after 5 more seconds?

13.51. Three pedestrians walk at constant speeds along three straight roads. At the initial moment the pedestrians were not on one straight line. Prove that the pedestrians can occure on one straight line not more than twice.

13.52. Prove Problem 4.29 b) with the help of a pseudoinner product.

13.53. Points P_1, P_2 and P_3 not on one line are inside a convex $2n$-gon $A_1 \dots A_{2n}$. Prove that if the sum of the areas of triangles $A_1A_2P_i$, $A_3A_4P_i$, $\dots$, $A_{2n-1}A_{2n}P_i$ is equal to the same number c for $i = 1, 2, 3$, then for any inner point P the sum of the areas of these triangles is equal to c.

13.54. Given triangle ABC and point P. Let point Q be such that $CQ \parallel AP$ and point R be such that $AR \parallel BQ$ and $CR \parallel BP$. Prove that $S_{ABC} = S_{PQR}$.

13.55. Let H_1, H_2 and H_3 be the orthocenters of triangles $A_2A_3A_4$, $A_1A_3A_4$ and $A_1A_2A_4$. Prove that the areas of triangles $A_1A_2A_3$ and $H_1H_2H_3$ are equal.

13.56. In a convex 5-gon $ABCDE$ whose area is equal to S the areas of triangles ABC, BCD, CDE, DEA and EAB are equal to a, b, c, d and e, respectively. Prove that

$$S^2 - S(a + b + c + d + e) + ab + bc + cd + de + ea = 0.$$

Problems for independent study

13.57. Let M and N be the midpoints of segments AB and AC, respectively, P the midpoint of MN and O an arbitrary point. Prove that $2\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = 4\overrightarrow{OP}$.

13.58. Points A, B and C move uniformly with the same angle velocities along the three circles in the same direction. Prove that the intersection point of the medians of triangle ABC moves along a circle.

13.59. Let A, B, C, D and E be arbitrary points. Is there a point O such that $\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{OE}$? Find all such points, if any.

13.60. Let P and Q be the midpoints of the diagonals of a convex quadrilateral $ABCD$. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4PQ^2.$$

13.61. The midpoints of segments AB and CD are connected by a segment; so are the midpoints of segments BC and DE. The midpoints of the segments obtained are also connected by a segment. Prove that the last segment is parallel to segment AE and its length is equal to $\frac{1}{4}AE$.

13.62. The inscribed circle is tangent to sides BC, CA and AB of triangle ABC at points A_1, B_1 and C_1, respectively. Prove that if $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then triangle ABC is an equilateral one.

13.63. Quadrilaterals $ABCD$, $AEFG$, $ADFH$, $FIJE$ and $BIJC$ are parallelograms. Prove that quadrilateral $AFHG$ is also a parallelogram.

Solutions

13.1. a) Let $\mathbf{a} = \overrightarrow{BC}$, $\mathbf{b} = \overrightarrow{CA}$ and $\mathbf{c} = \overrightarrow{AB}$; let AA', BB' and CC' be medians of triangle ABC. Then $\overrightarrow{AA'} = \frac{1}{2}(\mathbf{c} - \mathbf{b})$, $\overrightarrow{BB'} = \frac{1}{2}(\mathbf{a} - \mathbf{c})$ and $\overrightarrow{CC'} = \frac{1}{2}(\mathbf{b} - \mathbf{c})$. Therefore, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}$.

b) Let $\mathbf{a}_1 = \overrightarrow{AA'}$, $\mathbf{b}_1 = \overrightarrow{BB'}$ and $\mathbf{c} = \overrightarrow{CC'}$. Then $\frac{1}{2}(\mathbf{c}_1 - \mathbf{b}_1) = \frac{1}{4}(\mathbf{b} - \mathbf{a} - \mathbf{a} + \mathbf{c}) = -\frac{3}{4}\mathbf{a}$ is the vector of one of the sides of triangle $A_2B_2C_2$.

13.2. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the vectors of the sides of T. Then $\frac{1}{2}(\mathbf{b} - \mathbf{a})$, $\frac{1}{2}(\mathbf{a} - \mathbf{c})$ and $\frac{1}{2}(\mathbf{c} - \mathbf{b})$ are the vectors of its medians. We may assume that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the vectors directed from the intersection point of the medians of triangle T_1 to its vertices. Then $\mathbf{b} - \mathbf{a}$, $\mathbf{a} - \mathbf{c}$ and $\mathbf{c} - \mathbf{a}$ are the vectors of its sides.

13.3. It is clear that $2\overrightarrow{M_1M_2} = \overrightarrow{A_1A_2} + \overrightarrow{A_2A_3} = \overrightarrow{A_1A_3}$, $2\overrightarrow{M_3M_4} = \overrightarrow{A_3A_5}$ and $2\overrightarrow{M_5M_6} = \overrightarrow{A_5A_1}$. Therefore, $\overrightarrow{M_1M_2} + \overrightarrow{M_3M_4} + \overrightarrow{M_5M_6} = \overrightarrow{0}$.

13.4. After rotation through 90° the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ turn into the vectors of sides of the n-gon.

13.5. From given vectors one can construct a convex quadrilateral. The lengths of all the sides of this quadrilateral are equal to 1, therefore, this quadrilateral is a rhombus; the pairs of its opposite sides provide us with the division desired.

13.6. Let $\mathbf{a} = \overrightarrow{AE}$, $\mathbf{b} = \overrightarrow{DF}$ and $\mathbf{v} = \overrightarrow{AD}$. Then $2\overrightarrow{AK} = \mathbf{b} + \mathbf{v}$ and $2\overrightarrow{AL} = \mathbf{a} + \mathbf{v} + 2\mathbf{b}$ and, therefore, $\overrightarrow{KL} = \overrightarrow{AL} - \overrightarrow{AK} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$. Similarly, $\overrightarrow{NM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

13.7. Let us draw the given vectors from one point and index them clockwise: $\mathbf{a}_1, \dots,$ $\mathbf{a}_n$. Consider a closed broken line $A_1 \dots A_n$, where $\overrightarrow{A_iA_{i+1}} = \mathbf{a}_i$. Let us prove that $A_1 \dots A_n$ is a convex polygon. Introduce a coordinate system and direct the Ox-axis along $\mathbf{a}_1$. Let the vectors $\mathbf{a}_2, \dots, \mathbf{a}_k$ lie on one side of Ox-axis and the vectors $\mathbf{a}_{k+1}, \dots, \mathbf{a}_n$ lie on the other side (if there is a vector directed opposite to $\mathbf{a}_1$ it can be referred to either of these two groups).

The projections of the vectors from the first group on the Oy-axis are of one sign and the projections of the vectors of the other group are of the opposite sign. Therefore, the second coordinate of the points $A_2, A_3, \dots, A_{k+1}$ and the points $A_{k+1}, \dots, A_n, A_1$ vary monotonously: for the first group from 0 to a quantity d, for the second group they decrease from d to 0. Since there are two intervals of monotonity, all the vertices of the polygon lie on one side of the line A_1A_2.

For the other lines passing through the sides of the polygon the proof is similar.

13.8. Thanks to Problem 13.7 the given vectors form a convex quadrilateral. The rest is clear from Fig. 143.

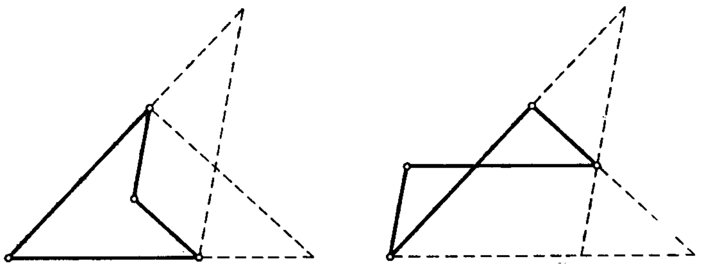


Figure 143 (Sol. 13.8)

13.9. By Problem 13.8 b) from the given vectors we can construct a self-intersecting broken line of four links; this broken line can be viewed as the two diagonals and two opposite sides of a convex quadrilateral. Two cases are possible: the vector **a** can be either a side or a diagonal of this quadrilateral.

But in both cases the sum in the left-hand side of the inequality is the sum of lengths of two opposite sides and two diagonals of the quadrilateral and the sum in the right-hand side is constituted by the length of the sum of vectors of the same opposite sides and the lengths of the two other opposite sides. It only remains to observe that the sum of lengths of two vectors is not shorter than the length of their sum and the sum of the length of diagonals of a convex quadrilateral is longer than the sum of lengths of the two opposite sides: cf. Problem 19.14.

13.10. Let diagonal BE intersect diagonals AD and AC at points F and G, respectively. The respective sides of triangles AFE and BCD are parallel; hence, the triangles are similar and $AF : FE = BC : CD$. Therefore,

$$AD : BE = (AF + BC) : (EF + CD) = BC : CD.$$

Similarly, $AE : BD = DE : AC$. From the similarity of BED and EGA we deduce that $AE : DB = EG : BE = CD : BE$. Thus,

$$\frac{BC}{AD} = \frac{CD}{BE} = \frac{AE}{BD} = \frac{DE}{AC} = \lambda.$$

Clearly,

$$\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA} + \overrightarrow{AB} = \overrightarrow{0}, \qquad \overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CA} + \overrightarrow{DB} + \overrightarrow{EC} = \overrightarrow{0}$$

and

$$\overrightarrow{BC} = \lambda\overrightarrow{AD},\ \ \overrightarrow{CD} = \lambda\overrightarrow{BE},\ \ \overrightarrow{DE} = \lambda\overrightarrow{CA},\ \ \overrightarrow{EA} = \lambda\overrightarrow{DB}.$$

It follows that

$$\overrightarrow{0} = \lambda(\overrightarrow{AD} + \overrightarrow{BE} + \overrightarrow{CA} + \overrightarrow{DB}) + \overrightarrow{AB} = -\lambda\overrightarrow{EC} + \overrightarrow{AB},$$

i.e., $\overrightarrow{AB} = \lambda\overrightarrow{EC}$. Hence, $AB \parallel EC$.

13.11. Let $\mathbf{a} = \overrightarrow{AB}$, $\mathbf{b} = \overrightarrow{BC}$, $\mathbf{c} = \overrightarrow{CD}$ and $\mathbf{d} = \overrightarrow{DA}$. It suffices to verify that $AC \perp BD$ if and only if $a^2 + c^2 = b^2 + d^2$. Clearly,

$$d^2 = |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = a^2 + b^2 + c^2 + 2[(\mathbf{a}, \mathbf{b}) + (\mathbf{b}, \mathbf{c}) + (\mathbf{c}, \mathbf{a})].$$

Therefore, the condition $AC \perp BD$, i.e.,

$$0 = (\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}) = b^2 + (\mathbf{b}, \mathbf{c}) + (\mathbf{a}, \mathbf{c}) + (\mathbf{a}, \mathbf{b})$$

is equivalent to the fact that

$$d^2 = a^2 + b^2 + c^2 - 2b^2.$$

13.12. a) Let us express all the vectors that enter the formula through $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$, i.e., let us write $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, $\overrightarrow{CA} = -\overrightarrow{AB} - \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$. After simplification we get the statement desired.

b) Let D be the intersection point of heights drawn from vertices A and C of triangle ABC. Then in the formula proved in heading a) the first two summands are zero and, therefore, the last summand is also zero, i.e., $BD \perp AC$.

13.13. Let us prove that $AH \perp BC$. Indeed, $\overrightarrow{AH} = \overrightarrow{AO} + \overrightarrow{OH} = \overrightarrow{AO} + \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{OC}$ and $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{OB} + \overrightarrow{OC}$ and, therefore,

$$(\overrightarrow{AH}, \overrightarrow{BC}) = OC^2 - OB^2 = R^2 - R^2 = 0$$

because O is the center of the circumscribed circle. We similarly prove that $BH \perp AC$ and $CH \perp AB$.

13.14. Let $\alpha_i = \angle(\mathbf{a}_i, \mathbf{a}_1)$. Considering the projections to the straight line parallel to $\mathbf{a}_1$ and the straight line perpendicular to $\mathbf{a}_1$ we get $a_1 = \sum_{i>1} a_i \cos\alpha_i$ and $0 = \sum_{i>1} a_i \sin\alpha_i$, respectively. Squaring these equalities and summing we get

$$\begin{aligned} a_1^2 &= \textstyle\sum_{i>1} a_i^2(\cos^2\alpha_i + \sin^2\alpha_i) + 2\sum_{i>j>1} a_i a_j(\cos\alpha_i\cos\alpha_j + \sin\alpha_i\sin\alpha_j) = \\ &\qquad a_2^2 + \dots + a_n^2 + 2\textstyle\sum_{i>j>1} a_i a_j \cos(\alpha_i - \alpha_j). \end{aligned}$$

It remains to notice that $\alpha_i - \alpha_j = \angle(\mathbf{a}_i, \mathbf{a}_1) - \angle(\mathbf{a}_j, \mathbf{a}_1) = \angle(\mathbf{a}_i, \mathbf{a}_j) = \varphi_{ij}$.

13.15. Let $\mathbf{a} = \overrightarrow{AD}$, $\mathbf{b} = \overrightarrow{BD}$ and $\mathbf{c} = \overrightarrow{CD}$. Since $BC^2 = |\mathbf{b}-\mathbf{c}|^2 = BD^2 + CD^2 - 2(\mathbf{b}, \mathbf{c})$, it follows that $U = 2(\mathbf{b}, \mathbf{c})$. Similarly, $V = 2(\mathbf{a}, \mathbf{c})$ and $W = 2(\mathbf{a}, \mathbf{b})$. Let $\alpha = \angle(\mathbf{a}, \mathbf{b})$ and $\beta = \angle(\mathbf{b}, \mathbf{c})$. Multiplying the equality

$$\cos^2\alpha + \cos^2\beta + \cos^2(\alpha+\beta) = 2\cos\alpha\cos\beta\cos(\alpha+\beta) + 1$$

(cf. Problem 12.39 b)) by $4uvw = 4|a|^2|b|^2|c|^2$ we get the statement desired.

13.16. Fix an arbitrary point O. Let $\mathbf{m} = \overrightarrow{OM}$, $\mathbf{a} = \overrightarrow{OA}, \dots, \mathbf{d} = \overrightarrow{OD}$. Then

$$\begin{aligned} (\overrightarrow{MA}, \overrightarrow{MB}) - (\overrightarrow{MC}, \overrightarrow{MD}) = (\mathbf{a}-\mathbf{m}, \mathbf{b}-\mathbf{m}) - (\mathbf{c}-\mathbf{m}, \mathbf{d}-\mathbf{m}) = \\ (\mathbf{c}+\mathbf{d}-\mathbf{a}-\mathbf{b}, \mathbf{m}) + (\mathbf{a}, \mathbf{b}) - (\mathbf{c}, \mathbf{d}). \end{aligned}$$

If $\mathbf{v} = \mathbf{c}+\mathbf{d}-\mathbf{a}-\mathbf{b} \neq \mathbf{0}$, then as the point M runs over the plane the value $(\mathbf{v}, \mathbf{m})$ attains all the real values, in particular, it takes the value $(\mathbf{c}, \mathbf{d}) - (\mathbf{a}, \mathbf{b})$. Hence, $\mathbf{v} = \mathbf{0}$, i.e., $\overrightarrow{OC} + \overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB}$ and, therefore, $\overrightarrow{AC} = \overrightarrow{DB}$.

13.17. Let $\mathbf{n}_1, \dots, \mathbf{n}_k$ be the unit exterior normals to the sides and let $M_1, \dots, M_k$ be arbitrary points on these sides. For any point X inside the polygon the distance from X to the i-th side is equal to $(\overrightarrow{XM_i}, \mathbf{n}_i)$. Therefore, the sums of distances from the inner points A and B to the sides of the polygon are equal if and only if

$$\sum_{i=1}^k(\overrightarrow{AM_i}, \mathbf{n}_i) = \sum_{i=1}^k(\overrightarrow{BM_i}, \mathbf{n}_i) = \\ \sum_{i=1}^k(\overrightarrow{BA}, \mathbf{n}_i) + \sum_{i=1}^k(\overrightarrow{AM_i}, \mathbf{n}_i),$$

i.e., $(\overrightarrow{BA}, \sum_{i=1}^k \mathbf{n}_i) = 0$. Hence, the sum of distances from any inner point of the polygon to the sides is constant if and only if $\sum \mathbf{n}_i = \mathbf{0}$.

13.18. Let l be an arbitrary line, $\mathbf{n}$ the unit vector perpendicular to l. If points A and B belong to the same half-plane given by the line l the vector $\mathbf{n}$ belongs to, then $\rho(B, l) - \rho(A, l) = (\overrightarrow{AB}, \mathbf{n})$, where $\rho(X, l)$ is the distance from X to l.

Let $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$ and $\mathbf{n}_4$ be unit vectors perpendicular to the consecutive sides of quadrilateral $ABCD$ and directed inwards. Denote the sum of distances from point X to the sides of quadrilateral $ABCD$ by $\sum(X)$. Then

$$0 = \sum(B) - \sum(A) = (\overrightarrow{AB}, \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4).$$

Similarly,

$$(\overrightarrow{BC}, \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 + \mathbf{n}_4) = 0.$$

Since points A, B and C do not belong to the same line, $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4=\mathbf{0}$. It remains to make use of the result of Problem 13.5.

13.19. Let $\mathbf{a}=\overrightarrow{AB}$, $\mathbf{b}=\overrightarrow{BC}$ and $\mathbf{c}=\overrightarrow{CD}$. Then $\overrightarrow{AD}=\mathbf{a}+\mathbf{b}+\mathbf{c}$, $\overrightarrow{AC}=\mathbf{a}+\mathbf{b}$ and $\overrightarrow{BD}=\mathbf{b}+\mathbf{c}$. It is also clear that

$$|\mathbf{a}|^2+|\mathbf{b}|^2+|\mathbf{c}|^2+|\mathbf{a}+\mathbf{b}+\mathbf{c}|^2-|\mathbf{a}+\mathbf{b}|^2-|\mathbf{b}+\mathbf{c}|^2=|\mathbf{a}|^2+2(\mathbf{a},\quad \mathbf{c})+|\mathbf{c}|^2=|\mathbf{a}+\mathbf{c}|^2\geq 0.$$

The equality is only attained if $\mathbf{a}=-\mathbf{c}$, i.e., $ABCD$ is a parallelogram.

13.20. Consider five vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$ and suppose that the length of the sum of any two of them is longer than the length of the sum of the three remaining ones. Since $|\mathbf{a}_1+\mathbf{a}_2|>|\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5|$, it follows that

$$|\mathbf{a}_1|^2+2(\mathbf{a}_1,\mathbf{a}_2)+|\mathbf{a}_2|^2>|\mathbf{a}_3|^2+|\mathbf{a}_4|^2+|\mathbf{a}_5|^2+2(\mathbf{a}_3,\mathbf{a}_4)+2(\mathbf{a}_4,\mathbf{a}_5)+2(\mathbf{a}_3,\mathbf{a}_5).$$

Adding such inequalities for all ten pairs of vectors we get

$$4(|\mathbf{a}_1|^2+\dots)+2((\mathbf{a}_1,\mathbf{a}_2)+\dots)>6(|\mathbf{a}_1|^2+\dots)+6((\mathbf{a}_1,\mathbf{a}_2)+\dots)$$

i.e., $|\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5|^2<0$. Contradiction.

13.21. Denote the given vectors by $\mathbf{e}_1,\dots,\mathbf{e}_{10}$. Let $\overrightarrow{AB}=\mathbf{e}_1+\dots+\mathbf{e}_{10}$. Let us prove that the ray AB determines the required axis. Clearly, $|\overrightarrow{AB}-\mathbf{e}_i|^2=AB^2-2(\overrightarrow{AB},\mathbf{e}_i)+|\mathbf{e}_i|^2$, i.e., $(\overrightarrow{AB},\mathbf{e}_i)=\frac{1}{2}(AB^2+|\mathbf{e}_i|^2-|\overrightarrow{AB}-\mathbf{e}_i|^2)$. By the hypothesis $AB>|\overrightarrow{AB}-\mathbf{e}_i|$ and, therefore, $(\overrightarrow{AB},\mathbf{e}_i)>0$, i.e., the projection of $\mathbf{e}_i$ to AB is positive.

13.22. Let $\mathbf{a}_i=\overrightarrow{OA_i}$ and $\mathbf{x}=\overrightarrow{OX}$. Then $|\mathbf{a}_i|=R$ and $\overrightarrow{XA_i}=\mathbf{a}_i-\mathbf{x}$. Therefore,

$$\sum XA_i=\sum|\mathbf{a}_i-\mathbf{x}|=\sum\frac{|\mathbf{a}_i-\mathbf{x}|\cdot|\mathbf{a}_i|}{R}\geq$$
$$\sum\frac{\mathbf{a}_i-\mathbf{x},\mathbf{a}_i}{R}=\sum\frac{\mathbf{a}_i,\mathbf{a}_i}{R}-\frac{(\mathbf{x},\sum\mathbf{a}_i)}{R}.$$

It remains to observe that $(\mathbf{a}_i,\mathbf{a}_i)=R^2$ and $\sum\mathbf{a}_i=\mathbf{0}$.

13.23. On the plane, consider four vectors (a,b), (c,d), (e,f) and (g,h). One of the angles between these vectors does not exceed $\frac{360^\circ}{4}=90^\circ$. If the angle between the vectors does not exceed 90°, then the inner product is nonnegative.

The given six numbers are inner products of all the pairs of our four vectors and, therefore, at least one of them is nonnegative.

13.24. Let us prove this statement by induction. For $n=0$ the statement is obviously true. Let us assume that the statement is proved for $2n+1$ vectors. In a system of $2n+3$ vectors consider two extreme vectors (i.e., the vectors the angle between which is maximal).

For definiteness sake, suppose that these are vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_{2n+3}}$. By the inductive hypothesis the length of $\overrightarrow{OR}=\overrightarrow{OP_2}+\dots+\overrightarrow{OP}_{2n+2}$ is not shorter than 1. The vector $\overrightarrow{OR}$ belongs to the interior of angle $\angle P_1OP_{2n+3}$ and, therefore, it forms an acute angle with the vector $\overrightarrow{OS}=\overrightarrow{OP}_1+\overrightarrow{OP}_{2n+3}$. Hence, $|\overrightarrow{OS}+\overrightarrow{OR}|\geq OR\geq 1$.

13.25. First, let us prove that if $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors whose length does not exceed 1, then at least one of the vectors $\mathbf{a}\pm\mathbf{b}$, $\mathbf{a}\pm\mathbf{c}$, $\mathbf{b}\pm\mathbf{c}$ is not longer than 1.

Indeed, two of the vectors $\pm\mathbf{a}$, $\pm\mathbf{b}$, $\pm\mathbf{c}$ form an angle not greater than 60° and, therefore, the difference of these two vectors is not longer than 1 (if (??) in triangle ABC we have $AB\leq 1$, $BC\leq 1$ and $\angle ABC\leq 60^\circ$, then AC is not the greatest side and $AC\leq 1$).

Thus, we can reduce the discussion to two vectors $\mathbf{a}$ and $\mathbf{b}$. Then either the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ or between vectors $\mathbf{a}$ and $-\mathbf{b}$ does not exceed 90°; hence, either $|\mathbf{a}-\mathbf{b}|\leq\sqrt{2}$ or $|\mathbf{a}+\mathbf{b}|\leq\sqrt{2}$.

13.26. We can assume that the sum **a** of the given vectors is nonzero because otherwise the statement of the problem is obvious.

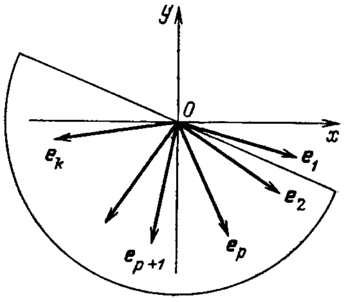


Figure 144 (Sol. 13.26)

Let us introduce a coordinate system directing Oy-axis along **a**. Let us enumerate the vectors of the lower half-plane clockwise: $\mathbf{e}_1, \mathbf{e}_2, \dots$ as on Fig. 144. By the hypothesis there are not less than k of these vectors. Let us prove that among the given vectors there are also vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ such that the second coordinate of the vector $\mathbf{v}_i + \mathbf{e}_i$ is nonpositive for any $i = 1, \dots, k$. This will prove the required statement.

Indeed, the length of the sum of the given vectors is equal to the sum of the second coordinates (the coordinate system was introduced just like this). The second coordinate of the sum of the vectors $\mathbf{e}_1$, $\mathbf{v}_1, \dots, \mathbf{e}_k$, $\mathbf{v}_k$ is nonpositive and the second coordinate of any of the remaining vectors does not exceed 1. Therefore, the second coordinate of the sum of all the given vectors does not exceed $n - 2k$.

Let vectors $\mathbf{e}_1, \dots, \mathbf{e}_p$ belong to the fourth quadrant. Let us start assigning to them the vectors $\mathbf{v}_1, \dots, \mathbf{v}_p$. Let us rotate the lower half plane that consists of points with nonpositive second coordinate by rotating the Ox-axis clockwise through an angle between 0° and 90°. If one of the two vectors that belongs to the half plane rotated this way lies in the fourth quadrant, then their sum has a nonpositive second coordinate. As the Ox-axis rotates beyond vector $\mathbf{e}_1$, at least one vector that belongs to the half plane should be added to the vectors $\mathbf{e}_2, \dots, \mathbf{e}_k$; hence, the vector which follows $\mathbf{e}_k$ should be taken for $\mathbf{v}_1$.

Similarly, while the Ox-axis is rotated beyond $\mathbf{e}_2$ we get vector $\mathbf{v}_2$, and so on. These arguments remain valid until the Ox-axis remains in the fourth quadrant. For the vectors $\mathbf{e}_{p+1}, \dots, \mathbf{e}_k$ which belong to the third quadrant the proof is given similarly (if the first coordinate of the vector $\mathbf{e}_{p+1}$ is zero, then we should first disregard it; then take any of the remaining vectors for its(whose?) partner).

13.27. Point X belongs to line AB if and only if $\overrightarrow{AX} = \lambda\overrightarrow{AB}$, i.e.,

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX} = (1-\lambda)\overrightarrow{OA} + \lambda\overrightarrow{OB}.$$

13.28. Let us take an arbitrary point O and express all the selected vectors in the form $\overrightarrow{A_iA_j} = \overrightarrow{OA_j} - \overrightarrow{OA_i}$. By the hypothesis every vector $\overrightarrow{OA_i}$ enters the sum of all the chosen vectors with the "plus" sign as many times as with the "minus" sign.

13.29. Let $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ be unit vectors codirected with vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, respectively; let $\alpha = \angle BOC$, $\beta = \angle COA$ and $\gamma = \angle AOB$. We have to prove that

$$\mathbf{e}_1 \sin\alpha + \mathbf{e}_2 \sin\beta + \mathbf{e}_3 \sin\gamma = \overrightarrow{0}.$$

Consider triangle $A_1B_1C_1$ whose sides are parallel to lines OC, OA and OB. Then

$$\overrightarrow{0} = \overrightarrow{A_1B_1} + \overrightarrow{B_1C_1} + \overrightarrow{C_1A_1} = \pm 2R(\mathbf{e}_1 \sin\alpha + \mathbf{e}_2 \sin\beta + \mathbf{e}_3 \sin\gamma),$$

where R is the radius of the circumscribed circle of triangle ABC.

13.30. Let $\mathbf{a}$ and $\mathbf{b}$ be unit vectors codirected with rays OA and OB, let $\lambda = OA$ and $\mu = OB$. Line AB consists of all points X such that

$$\overrightarrow{OX} = t\overrightarrow{OA} + (1-t)\overrightarrow{OB} = t\lambda\mathbf{a} + (1-t)\mu\mathbf{b}.$$

We have to find numbers x_0 and y_0 such that $\frac{x_0}{\lambda} = t = 1 - \frac{y_0}{\mu}$ for all the considered values of λ and μ. It remains to set $x_0 = \frac{p}{c}$ and $y_0 = \frac{q}{c}$. As a result we see that if $\frac{p}{OA} + \frac{q}{OB} = c$, then line AB passes through a point X such that $\overrightarrow{OX} = \frac{p\mathbf{a}+q\mathbf{b}}{c}$.

13.31. Let $\mathbf{a} = \overrightarrow{MA}$, $\mathbf{b} = \overrightarrow{MB}$ and $\mathbf{c} = \overrightarrow{MC}$. Then $\mathbf{e} = \overrightarrow{MC_1} = p\mathbf{a} + (1-p)\mathbf{b}$ and

$$\overrightarrow{MA_1} = q\mathbf{c} + (1-q)\mathbf{b} = -q\mathbf{a} + (1-2q)\mathbf{b}.$$

On the other hand, $\overrightarrow{MA_1} = \alpha\mathbf{e}$. Similarly,

$$\beta\mathbf{e} = \overrightarrow{MB_1} = -r\mathbf{b} + (1-2r)\mathbf{a}.$$

We have to show that $1 + \frac{1}{\alpha} + \frac{1}{\beta} = 0$. Since $\alpha p\mathbf{a} + \alpha(1-p)\mathbf{b} = \alpha\mathbf{e} = -q\mathbf{a} + (1-2q)\mathbf{b}$, it follows that $\alpha p = 1-2r$ and $\alpha(1-p) = 1-2q$ and, therefore, $\frac{1}{\alpha} = 1-3p$. Similarly, $\beta p = 1-2r$ and $\beta(1-p) = -r$ and, therefore, $\frac{1}{\beta} = 3p-2$.

13.32. Summing up the equalities $\overrightarrow{AA_2} + \overrightarrow{BB_2} + \overrightarrow{CC_2} = \overrightarrow{0}$ and $\overrightarrow{A_2B_2} + \overrightarrow{B_2C_2} + \overrightarrow{C_2A_2} = \overrightarrow{0}$ we get $\overrightarrow{AB_2} + \overrightarrow{BC_2} + \overrightarrow{CA_2} = \overrightarrow{0}$. It follows that $\overrightarrow{AB_2} = \lambda\overrightarrow{C_2B_2}, \overrightarrow{BC_2} = \lambda\overrightarrow{A_2C_2}$ and $\overrightarrow{CA_2} = \lambda\overrightarrow{B_2A_2}$. Let E be a point on line BC such that $A_2E \parallel AA_1$. Then $\overrightarrow{BA_1} = \lambda\overrightarrow{EA_1}$ and $\overrightarrow{EC} = \lambda\overrightarrow{EA_1}$; hence, $\overrightarrow{A_1C} = \overrightarrow{EC} - \overrightarrow{EA_1} = (\lambda-1)\overrightarrow{EA_1}$. Therefore, $\frac{\overline{A_1C}}{\overline{BA_1}} = \frac{\lambda-1}{\lambda}$. Similarly, $\frac{\overline{AB_1}}{\overline{B_1C}} = \frac{\overline{BC_1}}{\overline{C_1A}} = \frac{\lambda-1}{\lambda}$.

13.33. Let O be the center of the inscribed circle of the given quadrilateral, $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $\mathbf{d} = \overrightarrow{OD}$. If H_a is the orthocenter of triangle BCD, then $\overrightarrow{OH_a} = \mathbf{b} + \mathbf{c} + \mathbf{d}$ (cf. Problem 13.13). Therefore,

$$\overrightarrow{OM_a} = \frac{1}{2}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \overrightarrow{OM_b} = \overrightarrow{OM_c} = \overrightarrow{OM_d}.$$

13.34. Let O be the center of the circumscribed circle of the given quadrilateral; $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$, $\mathbf{c} = \overrightarrow{OC}$ and $\mathbf{d} = \overrightarrow{OD}$. If H_d is the orthocenter of triangle ABC, then $\overrightarrow{OH_d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ (Problem 13.13).

a) Take a point K such that $\overrightarrow{OK} = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$. Then

$$KH_d = |\overrightarrow{OK} - \overrightarrow{OH_d}| = |\mathbf{d}| = R,$$

i.e., K belongs to circle S_d. We similarly prove that K belongs to circles S_a, S_b and S_c.

b) Let O_d be the center of the circle of nine points of triangle ABC, i.e., the midpoint of OH_d. Then $\overrightarrow{OO_d} = \overrightarrow{OH_d}/2 = (\mathbf{a}+\mathbf{b}+\mathbf{c})/2$. Take point X such that $\overrightarrow{OX} = (\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})/2$. Then $XO_d = \frac{1}{2}|\mathbf{d}| = \frac{1}{2}R$, i.e., X belongs to the circle of nine points of triangle ABC. We similarly prove that X belongs to the circles of nine points of triangles BCD, CDA and DAB.

13.35. Let X_1 be the projection of X on l. Vector $\alpha\overrightarrow{AA_1} + \beta\overrightarrow{BB_1} + \gamma\overrightarrow{CC_1}$ is the projection of vector $\alpha\overrightarrow{AX_1} + \beta\overrightarrow{BX_1} + \gamma\overrightarrow{CX_1}$ to a line perpendicular to l. Since

$$\alpha\overrightarrow{AX_1} + \beta\overrightarrow{BX_1} + \gamma\overrightarrow{CX_1} = \alpha\overrightarrow{AX} + \beta\overrightarrow{BX} + \gamma\overrightarrow{CX} + (\alpha+\beta+\gamma)\overrightarrow{XX_1}$$

and $\alpha\overrightarrow{AX} + \beta\overrightarrow{BX} + \gamma\overrightarrow{CX} = \overrightarrow{0}$ (by Problem 13.29), we get the statement required.

(?)**13.36.** Let $\mathbf{a} = \overrightarrow{A_1A_2} + \overrightarrow{A_3A_4} + \dots + \overrightarrow{A_{2n-1}A_{2n}}$ and $\mathbf{a} \neq \mathbf{0}$. Introduce the coordinate system directing the Ox-axis along vector $\mathbf{a}$. Since the sum of projections of vectors $\overrightarrow{A_1A_2}$, $\overrightarrow{A_3A_4}$, ... , $\overrightarrow{A_{2n-1}A_{2n}}$ on Oy is zero, it follows that the length of $\mathbf{a}$ is equal to the absolute value of the difference between the sum of the lengths of positive projections of these vectors to the Ox-axis and the sum of lengths of their negative projections.

Therefore, the length of $\mathbf{a}$ does not exceed either the sum of the lengths of the positive projections or the sum of the lengths of the negative projections.

It is easy to verify that the sum of the lengths of positive projections as well as the sum of the lengths of negative projections of the given vectors on any axis does not exceed the diameter of the circle, i.e., does not exceed 2.

13.37. In the proof of the equality of vectors it suffices to verify the equality of their projections (minding the sign) on lines BC, CA and AB. Let us carry out the proof, for example, for the projections on line BC, where the direction of ray BC will be assumed to be the positive one. Let P be the projection of point A on line BC and N the midpoint of BC. Then

$$\overrightarrow{PN}=\overrightarrow{PC}+\overrightarrow{CN}=\frac{b^2+a^2-c^2}{2a}-\frac{a}{2}=\frac{b^2-c^2}{2a}$$

(PC is found from the equation $AB^2-BP^2=AC^2-CP^2$). Since $NM:NA=1:3$, the projection of $\overrightarrow{MO}$ on line BC is equal to $\frac{1}{3}\overrightarrow{PN}=\frac{b^2-c^2}{6a}$. It remains to notice that the projection of vector $a^3\mathbf{n}_a+b^3\mathbf{n}_b+c^3\mathbf{n}_c$ on BC is equal to

$$b^3\sin\gamma-c^3\sin\beta=\frac{b^3c-c^3b}{2R}=\frac{abc}{2R}\cdot\frac{b^2-c^2}{a}=2S\frac{b^2-c^2}{a}.$$

13.38. Let the inscribed circle be tangent to sides AB, BC and CA at points U, V and W, respectively. We have to prove that $\overrightarrow{OZ}=\frac{3R}{r}\overrightarrow{ZK}$, i.e., $\overrightarrow{OZ}=\frac{R}{r}(\overrightarrow{ZU}+\overrightarrow{ZV}+\overrightarrow{ZW})$. Let us prove, for example, that the (oriented) projections of these vectors on line BC are equal; the direction of ray BC will be assumed to be the positive one.

Let N be the projection of point O on line BC. Then the projection of vector OZ on line BC is equal to

$$\overrightarrow{NV}=\overrightarrow{NC}+\overrightarrow{CV}=(\frac{a}{2})-\frac{(a+b-c)}{2}=\frac{(c-b)}{2}.$$

The projection of vector $\overrightarrow{ZU}+\overrightarrow{ZV}+\overrightarrow{ZW}$ on this line is equal to the projection of vector $\overrightarrow{ZU}+\overrightarrow{ZW}$, i.e., it is equal to

$$-r\sin VZU+r\sin VZW=-r\sin B+r\sin C=\frac{r(c-b)}{2R}.$$

13.39. Introduce the coordinate system Oxy. Let l_φ be the straight line through O and constituting an angle of φ ($0<\varphi<\pi$) with the Ox-axis, i.e., if point A belongs to l_φ and the second coordinate of A is positive, then $\angle AOX=\varphi$; in particular, $l_0=l_\pi=Ox$.

If vector $\mathbf{a}$ forms an angle of α with the Ox-axis (the angle is counted counterclockwise from the Ox-axis to the vector $\mathbf{a}$), then the length of the projection of $\mathbf{a}$ on l_φ is equal to $|\mathbf{a}|\cdot|\cos(\varphi-\alpha)|$. The integral $\int_o^\pi|\mathbf{a}|\cdot|\cos(\varphi-\alpha)|d\varphi=2|\mathbf{a}|$ does not depend on α.

Let vectors $\mathbf{a}_1,\dots,\mathbf{a}_n$; $\mathbf{b}_1,\dots,\mathbf{b}_m$ constitute angles of $\alpha_1,\dots,\alpha_n$; $\beta_1,\dots,\beta_n$, respectively, with the Ox-axis. Then by the hypothesis

$$|\mathbf{a}_1|\cdot|\cos(\varphi-\alpha_1)|+\dots+|\mathbf{a}_n|\cdot|\cos(\varphi-\alpha_n)|\leq\\ |\mathbf{b}_1|\cdot|\cos(\varphi-\beta_1)|+\dots+|\mathbf{b}_m|\cdot|\cos(\varphi-\beta_m)|$$

for any φ. Integrating these inequalities over φ from 0 to π we get

$$|\mathbf{a}_1|+\dots+|\mathbf{a}_n|\leq|\mathbf{b}_1|+\dots+|\mathbf{b}_m|.$$

Remark. The value $\frac{1}{b-a}\int_a^b f(x)dx$ is called the *mean value* of the function f on the segment $[a, b]$. The equality

$$\int_0^\pi |\mathbf{a}| \cdot |\cos(\varphi - \alpha)| d\varphi = 2|\mathbf{a}|$$

means that the mean value of the length of the projection of vector $\mathbf{a}$ is equal to $\frac{2}{\pi}|\mathbf{a}|$; more precisely, the mean value of the function $f(\varphi)$ equal to the length of the projection of $\mathbf{a}$ to l_φ on the segment $[0, \pi]$ is equal to $\frac{2}{\pi}|\mathbf{a}|$.

13.40. The sum of the lengths of the projections of a convex polygon on any line is equal to twice the length of the projection of the polygon on this line. Therefore, the sum of the lengths of the projections of vectors formed by edges on any line is not longer for the inner polygon than for the outer one. Hence, by Problem 13.39 the sum of the lengths of vectors formed by the sides, i.e., the perimeter of the inner polygon, is not longer than that of the outer one.

13.41. If the sum of the lengths of vectors is equal to L, then by Remark to Problem 13.39 the mean value of the sum of the lengths of projections of these vectors is equal to $2L/\pi$.

The value of function f on segment $[a, b]$ cannot be always less than its mean value c because otherwise

$$c = \frac{1}{a-b}\int_a^b f(x)dx < \frac{(b-a)c}{b-a} = c.$$

Therefore, there exists a line l such that the sum of the lengths of the projections of the initial vectors on l is not shorter than $2L/\pi$.

On l, select a direction. Then either the sum of the lengths of the positive projections to this directed line or the sum of the lengths of the negative projections is not shorter than L/π. Therefore, either the length of the sum of vectors with positive projections or the length of the sum of vectors with negative porjections is not shorter than L/π.

13.42. Let AB denote the projection of the polygon on line l. Clearly, points A and B are projections of certain vertices A_1 and B_1 of the polygon. Therefore, $A_1B_1 \geq AB$, i.e., the length of the projection of the polygon is not longer than A_1B_1 and $A_1B_1 < d$ by the hypothesis. Since the sum of the lengths of the projections of the sides of the polygon on l is equal to $2AB$, it does not exceed $2d$.

The mean value of the sum of the lengths of the projections of sides is equal to $\frac{2}{\pi}P$, where P is a perimeter (see Problem 13.39). The mean value does not exceed the maximal one; hence, $\frac{2}{\pi}P < 2d$, i.e., $P < \pi d$.

13.43. By Problem 13.39 it suffices to prove the inequality

$$|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}| + |\mathbf{d}| \geq |\mathbf{a} + \mathbf{d}| + |\mathbf{b} + \mathbf{d}| + |\mathbf{c} + \mathbf{d}|$$

for the projections of the vectors on a line, i.e., we may assume that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are vectors parallel to one line, i.e., they are just numbers such that $a + b + c + d = 0$. Let us assume that $d \geq 0$ because otherwise we can change the sign of all the numbers.

We can assume that $a \leq b \leq c$. We have to consider three cases:

1) $a, b, c \leq 0$;
2) $a \leq 0$ and $b, c \geq 0$;
3) $a, b \leq 0, c \geq 0$.

All arising inequalities are quite easy to verify. In the third case we have to consider separately the subcases $|d| \leq |b|$, $|b| \leq |d| \leq |a|$ and $|a| \leq |d|$ (in the last subcase we have to take into account that $|d| = |a| + |b| - |c| \leq |a| + |b|$).

13.44. By Problem 13.39 it suffices to prove the inequality for the projections of vectors on any line. Let the projections of $\overrightarrow{OA}_1, \dots, \overrightarrow{OA}_n$ on a line l be equal (up to a sign) to $a_1, \dots, a_n$. Let us divide the numbers $a_1, \dots, a_n$ into two groups: $x_1 \geq x_2 \geq \dots \geq x_k > 0$ and $y'_1 \leq y'_2 \leq \dots \leq y'_{n-k} \leq 0$. Let $y_i = -y'_i$. Then $x_1 + \dots + x_k = y_1 + \dots + y_{n-k} = a$ and, therefore, $x_1 \geq \frac{a}{k}$ and $y_1 \geq \frac{a}{n-k}$. To the perimeter the number $2(x_1 + y_1)$ in the projection corresponds. To the sum of the vectors $\overrightarrow{OA}_i$ the number $x_1 + \dots + x_k + y_1 + \dots + y_{n-k} = 2a$ in the projection corresponds. And since

$$\frac{2(x_1 + y_1)}{x_1 + \dots + y_{n-k}} \geq \frac{2((a/k) + (a/(n-k)))}{2a} = \frac{n}{k(n-k)},$$

it remains to notice that the quantity $k(n-k)$ is maximal for $k = n/2$ if n is even and for $k = (n \pm 1)/2$ if n is odd.

13.45. By definition the *length of a curve* is the limit of perimeters of the polygons inscribed in it. [Vo vvedenie]

Consider an inscribed polygon with perimeter P and let the length of the projection on line l be equal to d_i. Let $1 - \varepsilon < d_i < 1$ for all lines l. The polygon can be selected so that ε is however small. Since the polygon is a convex one, the sum of the lengths of the projections of its sides on l is equal to $2d_i$.

By Problem 13.39 the mean value of the quantity $2d_i$ is equal to $\frac{2}{\pi}P$ (cf. Problem 13.39) and, therefore, $2 - 2\varepsilon < \frac{2}{\pi}P < 2$, i.e., $\pi - \pi\varepsilon < P < \pi$. Tending ε to zero we see that the length of the curve is equal to π.

13.46. Let us prove that the perimeter of the convex hull of all the vertices of given polygons does not exceed the sum of their perimeters. To this end it suffices to notice that by the hypothesis the projections of given polygons to any line cover the projection of the convex hull.

13.47. a) If $\lambda < 0$, then

$$(\lambda \mathbf{a}) \vee \mathbf{b} = -\lambda|\mathbf{a}| \cdot |\mathbf{b}| \sin\angle(-\mathbf{a}, \mathbf{b}) = \lambda|\mathbf{a}| \cdot |\mathbf{a}| \sin\angle(\mathbf{a}, \mathbf{b}) = \lambda(\mathbf{a} \vee \mathbf{b}).$$

For $\lambda > 0$ the proof is obvious.

b) Let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$ and $\mathbf{c} = \overrightarrow{OC}$. Introduce the coordinate system directing the Oy-axis along ray OA. Let $A = (0, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. Then

$$\mathbf{a} \vee \mathbf{b} = x_2 y_1, \mathbf{a} \vee \mathbf{c} = x_3 y_1; \quad \mathbf{a} \vee (\mathbf{b} + \mathbf{c}) = (x_2 + x_3)y_1 = \mathbf{a} \vee \mathbf{b} + \mathbf{a} \vee \mathbf{c}.$$

13.48. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be unit vectors directed along the axes Ox and Oy. Then $\mathbf{e}_1 \vee \mathbf{e}_2 = -\mathbf{e}_2 \vee \mathbf{e}_1 = 1$ and $\mathbf{e}_1 \vee \mathbf{e}_1 = \mathbf{e}_2 \vee \mathbf{e}_2 = 0$; hence,

$$\mathbf{a} \vee \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \vee (b_1\mathbf{e}_1 + b_2\mathbf{e}_2) = a_1b_2 - a_2b_1.$$

13.49. a) Clearly,

$$\overrightarrow{AB} \vee \overrightarrow{AC} = \overrightarrow{AB} \vee (\overrightarrow{AB} + \overrightarrow{BC}) = -\overrightarrow{BA} \vee \overrightarrow{BC} = \overrightarrow{BC} \vee \overrightarrow{BA}.$$

b) In the proof it suffices to make use of the chain of inequalities

$$\begin{gathered}\overrightarrow{AB} \vee \overrightarrow{AC} = (\overrightarrow{AD} + \overrightarrow{DB}) \vee (\overrightarrow{AD} + \overrightarrow{DC}) = \\ \overrightarrow{AD} \vee \overrightarrow{DC} + \overrightarrow{DB} \vee \overrightarrow{AD} + \overrightarrow{DB} \vee \overrightarrow{DC} = \\ = \overrightarrow{DC} \vee \overrightarrow{DA} + \overrightarrow{DA} \vee \overrightarrow{DB} + \overrightarrow{DB} \vee \overrightarrow{DC}.\end{gathered}$$

13.50. Let at the initial moment, i.e., at $t = 0$ we have $\overrightarrow{AB} = \mathbf{v}$ and $\overrightarrow{AC} = \mathbf{w}$. Then at the moment t we get $\overrightarrow{AB} = \mathbf{v} + t(\mathbf{a} - \mathbf{b})$ and $\overrightarrow{AC} = \mathbf{w} + t(\mathbf{c} - \mathbf{a})$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the velocity vectors of the runners A, B and C, respectively. Since vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are parallel, it follows that $(\mathbf{b}-\mathbf{a})\vee(\mathbf{c}-\mathbf{a})=0$ and, therefore, $|S(A,B,C)|=\frac{1}{2}|\overrightarrow{AB}\vee\overrightarrow{AC}|=|x+ty|$, where x and y are some constants.

Solving the system $|x|=2$, $|x+5y|=3$ we get two solutions with the help of which we express the dependence of the area of triangle ABC of time t as $|2+\frac{t}{5}|$ or $|2-t|$. Therefore, at $t=10$ the value of the area can be either 4 or 8.

13.51. Let $\mathbf{v}(t)$ and $\mathbf{w}(t)$ be the vectors directed from the first pedestrian to the second and the third ones, respectively, at time t. Clearly, $\mathbf{v}(t)=t\mathbf{a}+\mathbf{b}$ and $\mathbf{w}(t)=t\mathbf{c}+\mathbf{d}$. The pedestrians are on the same line if and only if $\mathbf{v}(t)\parallel\mathbf{w}(t)$, i.e., $\mathbf{v}(t)\vee\mathbf{w}(t)=0$. The function

$$f(t)=\mathbf{v}(t)\vee\mathbf{w}(t)=t^2\mathbf{a}\vee\mathbf{c}+t(\mathbf{a}\vee\mathbf{d}+\mathbf{b}\vee\mathbf{c})+\mathbf{b}\vee\mathbf{d}$$

is a quadratic and $f(0)\neq 0$. We know that a quadratic not identically equal to zero has not more than 2 roots.

13.52. Let $\overrightarrow{OC}=\mathbf{a}$, $\overrightarrow{OB}=\lambda\mathbf{a}$, $\overrightarrow{OD}=\mathbf{b}$ and $\overrightarrow{OA}=\mu\mathbf{b}$. Then

$$\pm 2S_{OPQ}=\overrightarrow{OP}\vee\overrightarrow{OQ}=\frac{\mathbf{a}+\mu\mathbf{b}}{2}\vee\frac{\lambda\mathbf{a}+\mathbf{b}}{2}=\frac{1-\lambda\mu}{4}(\mathbf{a}\vee\mathbf{b})$$

and

$$\pm S_{ABCD}=\pm 2(S_{COD}-S_{AOB})=\pm(\mathbf{a}\vee\mathbf{b}-\lambda\mathbf{a}\vee\mu\mathbf{b})=\pm(1-\lambda\mu)\mathbf{a}\vee\mathbf{b}.$$

13.53. Let $\mathbf{a}_j=\overrightarrow{P_1A_j}$. Then the doubled sum of the areas of the given triangles is equal for any inner point P to

$$(\mathbf{x}+\mathbf{a}_1)\vee(\mathbf{x}+\mathbf{a}_2)+(\mathbf{x}+\mathbf{a}_3)\vee(\mathbf{x}+\mathbf{a}_4)+\dots+(\mathbf{x}+\mathbf{a}_{2n-1})\vee(\mathbf{x}+\mathbf{a}_{2n}),$$

where $\mathbf{x}=\overrightarrow{PP_1}$ and it differs from the doubled sum of the areas of these triangles for point P_1 by

$$\mathbf{x}\vee(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3-\mathbf{a}_4+\dots+\mathbf{a}_{2n-1}-\mathbf{a}_{2n})=\mathbf{x}\vee\mathbf{a}.$$

By the hypothesis $\mathbf{x}\vee\mathbf{a}=0$ for $\mathbf{x}=\overrightarrow{P_1P_1}$ and $\mathbf{x}=\overrightarrow{P_3P_1}$ and these vectors are not parallel. Hence, $\mathbf{a}=\mathbf{0}$, i.e., $\mathbf{x}\vee\mathbf{a}=0$ for any $\mathbf{x}$.

13.54. Let $\mathbf{a}=\overrightarrow{AP}$, $\mathbf{b}=\overrightarrow{BQ}$ and $\mathbf{c}=\overrightarrow{CR}$. Then $\overrightarrow{QC}=\alpha\mathbf{a}$, $\overrightarrow{RA}=\beta\mathbf{b}$ and $\overrightarrow{PB}=\gamma\mathbf{c}$; we additionally have

$$(1+\alpha)\mathbf{a}+(1+\beta)\mathbf{b}+(1+\gamma)\mathbf{c}=\mathbf{0}.$$

It suffices to verify that $\overrightarrow{AB}\vee\overrightarrow{CA}=\overrightarrow{PQ}\vee\overrightarrow{RP}$. The difference between these quantities is equal to

$$\begin{aligned}(\mathbf{a}+\gamma\mathbf{c})\vee(\mathbf{c}+\beta\mathbf{b})-(\gamma\mathbf{c}+\mathbf{b})\vee(\mathbf{a}+\beta\mathbf{b})&=\mathbf{a}\vee\mathbf{c}+\beta\mathbf{a}\vee\mathbf{b}+\mathbf{a}\vee\mathbf{b}+\gamma\mathbf{a}\vee\mathbf{c}=\\&=\mathbf{a}\vee[(1+\gamma)\mathbf{c}+(1+\beta)\mathbf{b}]=-\mathbf{a}\vee(1+\alpha)\mathbf{a}=0.\end{aligned}$$

13.55. Let $\mathbf{a}_i=\overrightarrow{A_4A_i}$ and $\mathbf{w}_i=\overrightarrow{A_4H_i}$. By Problem 13.49 b) it suffices to verify that

$$\mathbf{a}_1\vee\mathbf{a}_2+\mathbf{a}_2\vee\mathbf{a}_3+\mathbf{a}_3\vee\mathbf{a}_1=\mathbf{w}_1\vee\mathbf{w}_2+\mathbf{w}_2\vee\mathbf{w}_3+\mathbf{w}_3\vee\mathbf{w}_1.$$

Vectors $\mathbf{a}_1-\mathbf{w}_2$ and $\mathbf{a}_2-\mathbf{w}_1$ are perpendicular to vector $\mathbf{a}_3$ and, therefore, they are parallel to each other, i.e., $(\mathbf{a}_1-\mathbf{w}_2)\vee(\mathbf{a}_2-\mathbf{w}_1)=0$. Adding this equality to the equalities $(\mathbf{a}_2-\mathbf{w}_3)\vee(\mathbf{a}_3-\mathbf{w}_2)=0$ and $(\mathbf{a}_3-\mathbf{w}_1)\vee(\mathbf{a}_1-\mathbf{w}_3)=0$ we get the statement required.

13.56. Let $\mathbf{x}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$. Then $\mathbf{e}_1\vee\mathbf{x}=x_2(\mathbf{e}_1\vee\mathbf{e}_2)$ and $\mathbf{x}\vee\mathbf{e}_2=x_1(\mathbf{e}_1\vee\mathbf{e}_2)$, i.e.,

$$\mathbf{x}=\frac{(\mathbf{x}\vee\mathbf{e}_2)\mathbf{e}_1+(\mathbf{e}_1\vee\mathbf{x})\mathbf{e}_2}{\mathbf{e}_1\vee\mathbf{e}_2}.$$

Multiplying this expression by $(\mathbf{e}_1\vee\mathbf{e}_2)\mathbf{y}$ from the right we get

$$(\mathbf{x}\vee\mathbf{e}_2)(\mathbf{e}_1\vee\mathbf{y})+(\mathbf{e}_1\vee\mathbf{x})(\mathbf{e}_2\vee\mathbf{y})+(\mathbf{e}_2\vee\mathbf{e}_1)(\mathbf{x}\vee\mathbf{y})=0. \tag{1}$$

Let $\mathbf{e}_1 = \overrightarrow{AB}$, $\mathbf{e}_2 = \overrightarrow{AC}$, $\mathbf{x} = \overrightarrow{AD}$ and $\mathbf{y} = \overrightarrow{AE}$. Then

$$S = a + \mathbf{x} \vee \mathbf{e}_2 + d = c + \mathbf{y} \vee \mathbf{e}_2 + a = d + \mathbf{x} \vee \mathbf{e}_1 + b,$$

i.e.,

$$\mathbf{x} \vee \mathbf{e}_2 = S - a - d, \mathbf{y} \vee \mathbf{e}_2 = S - c - a$$

and $\mathbf{x} \vee \mathbf{e}_1 = S - d - b$. Substituting these expressions into (1) we get the statement required.

Chapter 14. THE CENTER OF MASS

Background

1. Consider a system of mass points on a plane, i.e., there is a set of pairs (X_i, m_i), where X_i is a point on the plane and m_i a positive number. The *center of mass* of the system of points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$, respectively, is a point, O, which satisfies

$$m_1\overrightarrow{OX_1} + \dots + m_n\overrightarrow{OX_n} = \overrightarrow{0}.$$

The center of mass of any system of points exists and is unique (Problem 14.1).

2. A careful study of the solution of Problem 14.1 reveals that the positivity of the numbers m_i is not actually used; it is only important that their sum is nonzero. Sometimes it is convenient to consider systems of points for which certain masses are positive and certain are negative (but the sum of masses is nonzero).

3. The most important property of the center of mass which lies in the base of almost all its applications is the following

Theorem on mass regroupping. *The center of mass of a system of points does not change if part of the points are replaced by one point situated in their center of mass and whose mass is equal to the sum of their masses* (Problem 14.2).

4. The *moment of inertia* of a system of points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$ with respect to point M is the number

$$I_M = m_1MX_1^2 + \dots + m_nMX_n^2.$$

The applications of this notion in geometry are based on the relation $I_M = I_O + mOM^2$, where O is the center of mass of a system and $m = m_1 + \dots + m_n$ (Problem 14.17).

§1. Main properties of the center of mass

14.1. a) Prove that the center of mass exists and is unique for any system of points.

b) Prove that if X is an arbitrary point and O the center of mass of points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$, then

$$\overrightarrow{XO} = \frac{1}{m_1 + \dots + m_n}(m_1\overrightarrow{XX_1} + \dots + m_n\overrightarrow{XX_n}).$$

14.2. Prove that the center of mass of the system of points $X_1, \dots, X_n, Y_1, \dots, Y_m$ with masses $a_1, \dots, a_n, b_1, \dots, b_m$ coincides with the center of mass of two points — the center of mass X of the first system with mass $a_1 + \dots + a_n$ and the center of mass Y of the second system with mass $b_1 + \dots + b_m$.

14.3. Prove that the center of mass of points A and B with masses a and b belongs to segment AB and divides it in the ratio of $b : a$.

§2. A theorem on mass regroupping

14.4. Prove that the medians of triangle ABC intersect at one point and are divided by it in the ratio of $2:1$ counting from the vertices.

14.5. Let $ABCD$ be a convex quadrilateral; let K, L, M and N be the midpoints of sides AB, BC, CD and DA, respectively. Prove that the intersection point of segments KM and LN is the midpoint of these segments and also the midpoint of the segment that connects the midpoints of the diagonals.

14.6. Let $A_1, B_1, \dots, F_1$ be the midpoints of sides $AB, BC, \dots, FA$, respectively, of a hexagon. Prove that the intersection points of the medians of triangles $A_1C_1E_1$ and $B_1D_1F_1$ coincide.

14.7. Prove Ceva's theorem (Problem 4.48 b)) with the help of mass regrouping.

14.8. On sides AB, BC, CD and DA of convex quadrilateral $ABCD$ points K, L, M and N, respectively, are taken so that $AK : KB = DM : MC = \alpha$ and $BL : LC = AN : ND = \beta$. Let P be the intersection point of segments KL and LN. Prove that $NP : PL = \alpha$ and $KP : PM = \beta$.

14.9. Inside triangle ABC find point O such that for any straight line through O, intersecting AB at K and intersecting BC at L the equality $p\frac{AK}{KB} + q\frac{CL}{LB} = 1$ holds, where p and q are given positive numbers.

14.10. Three flies of equal mass crawl along the sides of triangle ABC so that the center of their mass is fixed. Prove that the center of their mass coincides with the intersection point of medians of ABC if it is known that one fly had crawled along the whole boundary of the triangle.

14.11. On sides AB, BC and CA of triangle ABC, points C_1, A_1 and B_1, respectively, are taken so that straight lines CC_1, AA_1 and BB_1 intersect at point O. Prove that

a) $\frac{CO}{OC_1} = \frac{CA_1}{A_1B} + \frac{CB_1}{B_1A}$;

b) $\frac{AO}{OA_1} \cdot \frac{BO}{OB_1} \cdot \frac{CO}{OC_1} = \frac{AO}{OA_1} + \frac{BO}{OB_1} + \frac{CO}{OC_1} + 2 \geq 8$.

14.12. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1, respectively, are taken so that $\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B}$. Prove that the centers of mass of triangles ABC and $A_1B_1C_1$ coincide.

14.13. On a circle, n points are given. Through the center of mass of $n-2$ points a straight line is drawn perpendicularly to the chord that connects the two remaining points. Prove that all such straight lines intersect at one point.

14.14. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1, respectively, are taken so that segments AA_1, BB_1 and CC_1 intersect at point P. Let l_a, l_b, l_c be the lines that connect the midpoints of segments BC and B_1C_1, CA and C_1A_1, AB and A_1B_1, respectively. Prove that lines l_a, l_b and l_c intersect at one point and this point belongs to segment PM, where M is the center of mass of triangle ABC.

14.15. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1, respectively, are taken; straight lines B_1C_1, BB_1 and CC_1 intersect straight line AA_1 at points M, P and Q, respectively. Prove that:

a) $\frac{A_1M}{MA} = \frac{A_1P}{PA} + \frac{A_1Q}{QA}$;

b) if $P = Q$, then $MC_1 : MB_1 = \frac{BC_1}{AB} : \frac{CB_1}{AC}$.

14.16. On line AB points P and P_1 are taken and on line AC points Q and Q_1 are taken. The line that connects point A with the intersection point of lines PQ and P_1Q_1

intersects line BC at point D. Prove that

$$\frac{\overline{BD}}{\overline{CD}} = \frac{\frac{\overline{BP}}{\overline{PA}} - \frac{\overline{BP_1}}{\overline{P_1A}}}{\frac{\overline{CQ}}{\overline{QA}} - \frac{\overline{CQ_1}}{\overline{Q_1A}}}.$$

§3. The moment of inertia

For point M and a system of mass points $X_1, \dots, X_n$ with masses $m_1, \dots, m_n$ the quantity $I_M = m_1MX_1^2 + \dots + m_nMX_n^2$ is called the *moment of inertia* with respect to M.

14.17. Let O be the center of mass of a system of points whose sum of masses is equal to m. Prove that the moments of inertia of this system with respect to O and with respect to an arbitrary point X are related as follows: $I_X = I_O + mXO^2$.

14.18. a) Prove that the moment of inertia with respect to the center of mass of a system of points of unit masses is equal to $\frac{1}{n}\sum_{i<j} a_{ij}^2$, where n is the number of points and a_{ij} the distance between points whose indices are i and j.

b) Prove that the moment of inertia with respect to the center of mass of a system of points whose masses are $m_1, \dots, m_n$ is equal to $\frac{1}{m}\sum_{i<j} m_im_ja_{ij}^2$, where $m = m_1 + \dots + m_n$ and a_{ij} is the distance between the points whose indices are i and j.

14.19. a) Triangle ABC is an equilateral one. Find the locus of points X such that $AX^2 = BX^2 + CX^2$.

b) Prove that for the points of the locus described in heading a) the pedal triangle with respect to the triangle ABC is a right one.

14.20. Let O be the center of the circumscribed circle of triangle ABC and H the intersection point of the heights of triangle ABC. Prove that $a^2 + b^2 + c^2 = 9R^2 - OH^2$.

14.21. Chords AA_1, BB_1 and CC_1 in a disc with center O intersect at point X. Prove that

$$\frac{AX}{XA_1} + \frac{BX}{XB_1} + \frac{CX}{XC_1} = 3$$

if and only if point X belongs to the circle with diameter OM, where M is the center of mass of triangle ABC.

14.22. On sides AB, BC, CA of triangle ABC pairs of points A_1 and B_2, B_1 and C_2, C_1 and A_2, respectively, are taken so that segments A_1A_2, B_1B_2 and C_1C_2 are parallel to the sides of triangle ABC and intersect at point P. Prove that

$$PA_1 \cdot PA_2 + PB_1 \cdot PB_2 + PC_1 \cdot PC_2 = R^2 - OP^2,$$

where O is the center of the circumscribed circle.

14.23. Inside a circle of radius R, consider n points. Prove that the sum of squares of the pairwise distances between the points does not exceed n^2R^2.

14.24. Inside triangle ABC point P is taken. Let d_a, d_b and d_c be the distances from P to the sides of the triangle; R_a, R_b and R_c the distances from P to the vertices. Prove that

$$3(d_a^2 + d_b^2 + d_c^2) \geq (R_a \sin A)^2 + (R_b \sin B)^2 + (R_c \sin C)^2.$$

14.25. Points $A_1, \dots, A_n$ belong to the same circle and M is their center of mass. Lines $MA_1, \dots, MA_n$ intersect this circle at points $B_1, \dots, B_n$ (distinct from $A_1, \dots, A_n$). Prove that

$$MA_1 + \dots + MA_n \leq MB_1 + \dots + MB_n.$$

§4. Miscellaneous problems

14.26. Prove that if a polygon has several axes of symmetry, then all of them intersect at one point.

14.27. A centrally symmetric figure on a graph paper consists of n "corners" and k rectangles of size 1 × 4 depicted on Fig. 145. Prove that n is even.

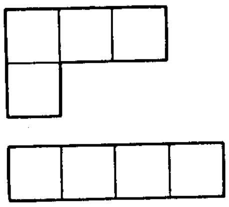


Figure 145 (14.27)

14.28. Solve Problem 13.44 making use the properties of the center of mass.

14.29. On sides BC and CD of parallelogram $ABCD$ points K and L, respectively, are taken so that $BK : KC = CL : LD$. Prove that the center of mass of triangle AKL belongs to diagonal BD.

§5. The barycentric coordinates

Consider triangle $A_1A_2A_3$ whose vertices are mass points with masses m_1, m_2 and m_3, respectively. If point X is the center of mass of the triangle's vertices, then the triple $(m_1 : m_2 : m_3)$ is called the *barycentric coordinates* of point X with respect to triangle $A_1A_2A_3$.

14.30. Consider triangle $A_1A_2A_3$. Prove that

a) any point X has some barycentric coordinates with respect to $\triangle A_1A_2A_3$;

b) provided $m_1 + m_2 + m_3 = 1$ the barycentric coordinates of X are uniquely defined.

14.31. Prove that the barycentric coordinates with respect to $\triangle ABC$ of point X which belongs to the interior of ABC are equal to $(S_{BCX} : S_{CAX} : S_{ABX})$.

14.32. Point X belongs to the interior of triangle ABC. The straight lines through X parallel to AC and BC intersect AB at points K and L, respectively. Prove that the barycentric coordinates of X with respect to $\triangle ABC$ are equal to $(BL : AK : LK)$.

14.33. Consider $\triangle ABC$. Find the barycentric coordinates with respect to $\triangle ABC$ of

a) the center of the circumscribed circle;

b) the center of the inscribed circle;

c) the orthocenter of the triangle.

14.34. The baricentric coordinates of point X with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$, where $\alpha + \beta + \gamma = 1$. Prove that $\overrightarrow{XA} = \beta\overrightarrow{BA} + \gamma\overrightarrow{CA}$.

14.35. Let $(\alpha : \beta : \gamma)$ be the barycentric coordinates of point X with respect to $\triangle ABC$ and $\alpha + \beta + \gamma = 1$ and let M be the center of mass of triangle ABC. Prove that

$$3\overrightarrow{XM} = (\alpha - \beta)\overrightarrow{AB} + (\beta - \gamma)\overrightarrow{BC} + (\gamma - \alpha)\overrightarrow{CA}.$$

14.36. Let M be the center of mass of triangle ABC and X an arbitrary point. On lines BC, CA and AB points A_1, B_1 and C_1, respectively, are taken so that $A_1X \parallel AM$, $B_1X \parallel BM$ and $C_1X \parallel CM$. Prove that the center of mass M_1 of triangle $A_1B_1C_1$ coincides with the midpoint of segment MX.

14.37. Find an equation of the circumscribed circle of triangle $A_1A_2A_3$ (kto sut' indexy? iz 14.36?) in the barycentric coordinates.

14.38. a) Prove that the points whose barycentric coordinates with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$ and $(\alpha^{-1} : \beta^{-1} : \gamma^{-1})$ are *isotomically conjugate* with respect to triangle ABC.

b) The lengths of the sides of triangle ABC are equal to a, b and c. Prove that the points whose barycentric coordinates with respect to $\triangle ABC$ are $(\alpha : \beta : \gamma)$ and $(\frac{a^2}{\alpha} : \frac{b^2}{\beta} : \frac{c^2}{\gamma})$ are *isogonally conjugate* with respect to ABC.

Solutions

14.1. Let X and O be arbitrary points. Then

$$m_1\overrightarrow{OX_1} + \dots + m_n\overrightarrow{OX_n} = (m_1 + \dots + m_n)\overrightarrow{OX} + m_1\overrightarrow{XX_1} + \dots + m_n\overrightarrow{XX_n}$$

and, therefore, O is the center of mass of the given system of points if and only if

$$(m_1 + \dots + m_n)\overrightarrow{OX} + m_1\overrightarrow{XX_1} + \dots + M_n\overrightarrow{XX_n} = \overrightarrow{0},$$

i.e., $\overrightarrow{OX} = \frac{1}{m_1+\dots+m_n}(m_1\overrightarrow{XX_1} + \dots + m_n\overrightarrow{XX_n})$.

This argument gives a solution to the problems of both headings.

14.2. Let Z be an arbitrary point; $a = a_1 + \dots + a_n$ and $b = b_1 + \dots + b_m$. Then $\overrightarrow{ZX} = \frac{a_1\overrightarrow{ZX_1}+\dots+a_n\overrightarrow{ZX_n}}{a}$ and $\overrightarrow{ZY} = \frac{b_1\overrightarrow{ZY_1}+\dots+b_m\overrightarrow{ZY_m}}{b}$. If O is the center of mass of point X whose mass is a and of point Y whose mass is b, then

$$\overrightarrow{ZO} = \frac{a\overrightarrow{ZX} + b\overrightarrow{ZY}}{a+b} = \frac{a_1\overrightarrow{ZX_1} + \dots + a_n\overrightarrow{ZX_n} + b_1\overrightarrow{ZY_1} + \dots + b_m\overrightarrow{ZY_m}}{a+b},$$

i.e., O is the center of mass of the system of points $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ with masses $a_1, \dots, a_n, b_1, \dots, b_m$.

14.3. Let O be the center of mass of the given system. Then $a\overrightarrow{OA} + b\overrightarrow{OB} = \overrightarrow{0}$ and, therefore, O belongs to segment AB and $aOA = bOB$, i.e., $AO : OB = b : a$.

14.4. Let us place unit masses at points A, B and C. Let O be the center of mass of this system of points. Point O is also the center of mass of points A of mass 1 and A_1 of mass 2, where A_1 is the center of mass of points B and C of unit mass, i.e., A_1 is the midpoint of segment BC. Therefore, O belongs to median AA_1 and divides it in the ratio $AO : OA_1 = 2 : 1$. We similarly prove that the remaining medians pass through O and are divided by it in the ratio of $2 : 1$.

14.5. Let us place unit masses in the vertices of quadrilateral $ABCD$. Let O be the center of mass of this system of points. It suffices to prove that O is the midpoint of segments KM and LN and the midpoint of the segment connecting the midpoints of the diagonals. Clearly, K is the center of mass of points A and B while M is the center of mass of points C and D. Therefore, O is the center of mass of points K and M of mass 2, i.e., O is the center of mass of segment KM.

Similarly, O is the midpoint of segment LN. Considering centers of mass of pairs of points (A, C) and (B, D) (i.e., the midpoints of diagonals) we see that O is the midpoint of the segment connecting the midpoints of diagonals.

14.6. Let us place unit masses in the vertices of the hexagon; let O be the center of mass of the obtained system of points. Since points A_1, C_1 and E_1 are the centers of mass of pairs of points (A, B), (C, D) and (E, F), respectively, point O is the center of mass of the system of points A_1, C_1 and E_1 of mass 2, i.e., O is the intersection point of the medians of triangle $A_1C_1E_1$ (cf. the solution of Problem 14.4).

We similarly prove that O is the intersection point of medians of triangle $B_1D_1F_1$.

14.7. Let lines AA_1 and CC_1 intersect at O and let $AC_1 : C_1B = p$ and $BA_1 : A_1C = q$. We have to prove that line BB_1 passes through O if and only if $CB_1 : B_1A = 1 : pq$.

Place masses 1, p and pq at points A, B and C, respectively. Then point C_1 is the center of mass of points A and B and point A_1 is the center of mass of points B and C. Therefore, the center of mass of points A, B and C with given masses is the intersection point O of lines CC_1 and AA_1.

On the other hand, O belongs to the segment which connects B with the center of mass of points A and C. If B_1 is the center of mass of points A and C of masses 1 and pq, respectively, then $AB_1 : B_1C = pq : 1$. It remains to notice that there is one point on segment AC which divides it in the given ratio $AB_1 : B_1C$.

14.8. Let us place masses 1, α, $\alpha\beta$ and β at points A, B, C and D, respectively. Then points K, L, M and N are the centers of mass of the pairs of points (A, B), (B, C), (C, D) and (D, A), respectively. Let O be the center of mass of points A, B, C and D of indicated mass. Then O belongs to segment NL and $NO : OL = (\alpha\beta + \alpha) : (1 + \beta) = \alpha$. Point O belongs to the segment KM and $KO : OM = (\beta + \alpha\beta) : (1 + \alpha) = \beta$. Therefore, O is the intersection point of segments KM and LN, i.e., $O = P$ and $NP : PL = NO : OL = \alpha$, $KP : PM = \beta$.

14.9. Let us place masses p, 1 and q in vertices A, B and C, respectively. Let O be the center of mass of this system of points. Let us consider a point of mass 1 as two coinciding points of mass x_a and x_c, where $x_a + x_c = 1$. Let K be the center of mass of points A and B of mass p and x_a and L the center of mass of points C and B of mass q and x_c, respectively. Then $AK : KB = x_a : p$ and $CL : LB = x_c : q$, whereas point O which is the center of mass of points K and L of mass $p + x_a$ and $q + x_c$, respectively, belongs to line KL. By varying x_a from 0 to 1 we get two straight lines passing through O and intersecting sides AB and BC. Therefore, for all these lines we have

$$\frac{pAK}{KB} + \frac{qCL}{LB} = x_a + x_c = 1.$$

14.10. Denote the center of mass of the flies by O. Let one fly be sited in vertex A and let A_1 be the center of mass of the two other flies. Clearly, point A_1 lies inside triangle ABC and point O belongs to segment AA_1 and divides it in the ratio of $AO : OA_1 = 2 : 1$. Therefore, point O belongs to the interior of the triangle obtained from triangle ABC by a homothety with coefficient $\frac{2}{3}$ and center A.

Considering such triangles for all the three vertices of triangle ABC we see that their unique common point is the intersection point of the medians of triangle ABC. Since one fly visited all the three vertices of the triangle ABC and point O was fixed during this, O should belong to all these three small triangles, i.e., O coincides with the intersection point of the medians of triangle ABC.

14.11. a) Let $AB_1 : B_1C = 1 : p$ and $BA_1 : A_1C = 1 : q$. Let us place masses p, q, 1 at points A, B, C, respectively. Then points A_1 and B_1 are the centers of mass of the pairs of points (B, C) and (A, C), respectively. Therefore, the center of mass of the system of points A, B and C belongs both to segment AA_1 and to segment BB_1, i.e., coincides with O. It follows that C_1 is the center of mass of points A and B. Therefore,

$$\frac{CO}{OC_1} = p + q = \frac{CB_1}{B_1A} + \frac{CA_1}{A_1B}.$$

b) By heading a) we have

$$\frac{AO}{OA_1}\cdot\frac{BO}{OB_1}\cdot\frac{CO}{OC_1}=\frac{1+q}{p}\cdot\frac{1+p}{q}\cdot\frac{p+q}{1}=$$
$$p+q+\frac{p}{q}+\frac{q}{p}+\frac{1}{p}+\frac{1}{q}+2=\frac{AO}{OA_1}+\frac{BO}{OB_1}+\frac{CO}{OC_1}+2.$$

It is also clear that

$$p+\frac{1}{p}\geq 2,\ q+\frac{1}{q}\geq 2 \text{ and } \frac{p}{q}+\frac{q}{p}\geq 2.$$

14.12. Let M be the center of mass of triangle ABC. Then

$$\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC}=\overrightarrow{0}.$$

Moreover,

$$\overrightarrow{AB_1}+\overrightarrow{BC_1}+\overrightarrow{CA_1}=k(\overrightarrow{AC}+\overrightarrow{BA}+\overrightarrow{CB})=\overrightarrow{0}.$$

Adding these identities we get $\overrightarrow{MB_1}+\overrightarrow{MC_1}+\overrightarrow{MA_1}=\overrightarrow{0}$, i.e., M is the center of mass of triangle $A_1B_1C_1$.

Remark. We similarly prove a similar statement for an arbitrary n-gon.

14.13. Let M_1 be the center of mass of $n-2$ points; K the midpoint of the chord connecting the two remaining points, O the center of the circle, and M the center of mass of all the given points. If line OM intersects a(?) line drawn through M_1 at point P, then

$$\frac{\overline{OM}}{\overline{MP}}=\frac{\overline{KM}}{\overline{MM_1}}=\frac{n-2}{2}$$

and, therefore, the position of point P is uniquely determined by the position of points O and M (if $M=O$, then $P=O$).

14.14. Let P be the center of mass of points A, B and C of masses a, b and c, respectively, M the center of mass of points A, B and C (the mass of M is $a+b+c$) and Q the center of mass of the union of these two systems of points. The midpoint of segment AB is the center of mass of points A, B and C of mass $a+b+c-\frac{ab}{c}$, $a+b+c-\frac{ab}{c}$ and 0, respectively, and the midpoint of segment A_1B_1 is the center of mass of points A, B and C of mass $\frac{a(b+c)}{c}$, $\frac{b(a+c)}{c}$ and $(b+c)+(a+c)$, respectively. Point O is the center of mass of the union of these systems of points.

14.15. a) Place masses β, γ and $b+c$ in points B, C and A so that $CA_1:BA_1=\beta:\gamma$, $BC_1:AC_1=b:\beta$ and $AB_1:CB_1=\gamma:c$. Then M is the center of mass of this system and, therefore, $\frac{A_1M}{AM}=\frac{b+c}{\beta+\gamma}$. Point P is the center of mass of points A, B and C of masses c, β and γ and, therefore, $\frac{A_1P}{PA}=\frac{c}{\beta+\gamma}$. Similarly, $\frac{A_1Q}{AQ}=\frac{b}{b+\gamma}$.

b) As in heading a), we get $\frac{MC_1}{MB_1}=\frac{c+\gamma}{b+\beta}$, $\frac{BC_1}{AB}=\frac{b}{b+\beta}$ and $\frac{AC}{CB_1}=\frac{c+\gamma}{c}$. Moreover, $b=c$ because straight lines AA_1, BB_1 and CC_1 intersect at one point (cf. Problem 14.7).

14.16. The intersection point of lines PQ and P_1Q_1 is the center of mass of points A, B and C of masses a, b and c and P is the center of mass of points A and B of masses $a-x$ and b while Q is the center of mass of points A and C of masses x and c. Let $p=\frac{BP}{PA}=\frac{a-x}{b}$ and $q=\frac{CQ}{QA}=\frac{x}{c}$. Then $pb+qc=a$. Similarly, $p_1b+q_1c=a$. It follows that

$$\frac{\overline{BD}}{\overline{CD}}=-\frac{c}{b}=\frac{(p-p_1)}{(q-q_1)}.$$

14.17. Let us enumerate the points of the given system. Let $\mathbf{x}_i$ be the vector with the beginning at O and the end at the point of index i and of mass m_i. Then $\sum m_i\mathbf{x}_i = \mathbf{0}$. Further, let $\mathbf{a} = \overrightarrow{OX}$. Then

$$I_O = \sum m_{ii}^2,$$
$$I_M = \sum m_i(\mathbf{x}_i + \mathbf{a})^2 = \sum m_i x_i^2 + 2(\sum m_i\mathbf{x}_i, \mathbf{a}) + \sum m_i a^2 = I_O + ma^2.$$

14.18. a) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass O and the end at the point of index i. Then

$$\sum_{i,j}(\mathbf{x}_i - \mathbf{x}_j)^2 = \sum_{i,j}(x_i^2 + x_j^2) - 2\sum_{i,j}(\mathbf{x}_i, \mathbf{x}_j),$$

where the sum runs over all the possible pairs of indices. Clearly,

$$\sum_{i,j}(x_i^2 + x_j^2) = 2n\sum_i x_i^2 = 2nI_O; \quad \sum_{i,j}(\mathbf{x}_i, \mathbf{x}_j) = \sum_i(\mathbf{x}_i, \sum_j \mathbf{x}_j) = 0.$$

Therefore, $2nI_O = \sum_{i,j}(\mathbf{x}_i - \mathbf{x}_j)^2 = 2\sum_{i<j} a_{ij}^2$.

b) Let $\mathbf{x}_i$ be the vector with the beginning at the center of mass O and the end at the point with index i. Then

$$\sum_{i,j} m_i m_j(\mathbf{x}_i - \mathbf{x}_j)^2 = \sum_{i,j} m_i m_j(x_i^2 + x_j^2) - 2\sum_{i,j} m_i m_j(\mathbf{x}_i, \mathbf{x}_j).$$

It is clear that

$$\sum_{i,j} m_i m_j(x_i^2 + x_j^2) = \sum_i m_i \sum_j(m_j x_i^2 + m_j x_j^2) = \sum_i m_i(m x_i^2 + I_O) = 2mI_O$$

and

$$\sum_{i,j} m_i m_j(\mathbf{x}_i, \mathbf{x}_j) = \sum_i m_i(\mathbf{x}_i, \sum_j m_j\mathbf{x}_j) = 0.$$

Therefore,

$$2mI_O = \sum_{i,j} m_i m_j(\mathbf{x}_i - \mathbf{x}_j)^2 = 2\sum_{i<j} m_i m_j a_{ij}^2.$$

14.19. a) Let M be the point symmetric to A through line BC. Then M is the center of mass of points A, B and C whose masses are -1, 1 and 1, respectively, and, therefore,

$$-AX^2 + BX^2 + CX^2 = I_X = I_M + (-1+1+1)MX^2 = (-3+1+1)a^2 + MX^2,$$

where a is the length of the side of triangle ABC. As a result we see that the locus to be found is the circle of radius a with the center at M.

b) Let A', B' and C' be the projections of point X to lines BC, CA and AB, respectively. Points B' and C' belong to the circle with diameter AX and, therefore, $B'C' = AX \sin B'AC' = \frac{\sqrt{3}}{2}AX$. Similarly, $C'A' = \frac{\sqrt{3}}{2}BX$ and $A'B' = \frac{\sqrt{3}}{2}CX$. Therefore, if $AX^2 = BX^2 + CX^2$, then $\angle B'A'C' = 90^\circ$.

14.20. Let M be the center of mass of the vertices of triangle ABC with unit masses in them. Then

$$I_O = I_M + 3MO^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3MO^2$$

(cf. Problems 14.17 and 14.18 a)). Since $OA = OB = OC = R$, it follows that $I_O = 3R^2$. It remains to notice that $OH = 3OM$ (Problem 5.105).

14.21. It is clear that

$$\frac{AX}{XA_1} = \frac{AX^2}{AX \cdot XA_1} = \frac{AX^2}{R^2 - OX^2}.$$

Therefore, we have to verify that $AX^2 + BX^2 + CX^2 = 3(R^2 - OX^2)$ if and only if $OM^2 = OX^2 + MX^2$. To this end it suffices to notice that

$$AX^2 + BX^2 + CX^2 = I_X = I_M + 3MX^2 = \\ I_O - 3MO^2 + 3MX^2 = 3(R^2 - MO^2 + MX^2).$$

14.22. Let P be the center of mass of points A, B and C whose masses are α, β and γ, respectively. We may assume that $\alpha + \beta + \gamma = 1$. If K is the intersection point of lines CP and AB, then

$$\frac{BC}{PA_1} = \frac{CK}{PK} = \frac{CP + PK}{PK} = 1 + \frac{CP}{PK} = 1 + \frac{\alpha + \beta}{\gamma} = \frac{1}{\gamma}.$$

Similar arguments show that the considered quantity is equal to $\beta\gamma a^2 + \gamma\alpha b^2 + \alpha\beta c^2 = I_P$ (cf. Problem 14.18 b)). Since $I_O = \alpha R^2 + \beta R^2 + \gamma R^2 = R^2$, we have $I_P = I_O - OP^2 = R^2 - OP^2$.

14.23. Let us place unit masses in the given points. As follows from the result of Problem 14.18 a) the sum of squared distances between the given points is equal to nI, where I is the moment of inertia of the system of points with respect to its center of mass. Now, consider the moment of inertia of the system with respect to the center O of the circle. On the one hand, $I \leq I_O$ (see Problem 14.17). On the other hand, since the distance from O to any of the given points does not exceed R, it follows that $I_O \leq nR^2$. Therefore, $nI \leq n^2R^2$ and the equality is attained only if $I = I_O$ (i.e., when the center of mass coincides with the center of the circle) and $I_O = nR^2$ (i.e., all the points lie on the given circle).

14.24. Let A_1, B_1 and C_1 be projections of point P to sides BC, CA and AB, respectively; let M be the center of mass of triangle $A_1B_1C_1$. Then

$$3(d_a^2 + d_b^2 + d_c^2) = 3I_P \geq \\ 3I_M = A_1B_1^2 + B_1C_1^2 + C_1A_1^2 = (R_c \sin C)^2 + (R_a \sin A)^2 + (R_b \sin B)^2$$

because, for example, segment A_1B_1 is a chord of the circle with diameter CP.

14.25. Let O be the center of the given circle. If chord AB passes through M, then $AM \cdot BM = R^2 - d^2$, where $d = MO$. Denote by I_X the moment of inertia of the system of points $A_1, \dots, A_n$ with respect to X. Then $I_O = I_M + nd^2$ (see Problem 14.17). On the other hand, since $OA_i = R$, we deduce that $I_O = nR^2$. Therefore,

$$A_iM \cdot B_iM = R^2 - d^2 = \frac{1}{n}(A_1M^2 + \dots + A_nM^2).$$

Set $a_i = A_iM$. Then the inequality to be proved takes the form

$$a_1 + \dots + a_n \leq \frac{1}{n}(a_1^2 + \dots + a_n^2)(\frac{1}{a_1} + \dots + \frac{1}{a_n}).$$

To prove this inequality we have to make use of the inequality

$$x + y \leq (\frac{x^2}{y}) + (\frac{y^2}{x})$$

which is obtained from the inequality $xy \leq x^2 - xy + y^2$ by multiplying both of its sides by $\frac{x+y}{xy}$.

14.26. Let us place unit masses in the vertices of the polygon. Under the symmetry through a line this system of points turns into itself and, therefore, its center of mass also turns into itself. It follows that all the axes of symmetry pass through the center of mass of the vertices.

14.27. Let us place unit masses in the centers of the cells which form "corners" and rectangles. Let us split each initial small cell of the graph paper into four smaller cells getting as a result a new graph paper. It is easy to verify that now the center of mass of a corner belongs to the center of a new small cell and the center of mass of a rectangle is a vertex of a new small cell, cf. Fig. 146.

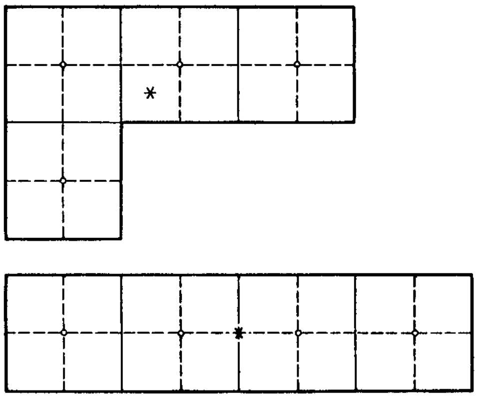


Figure 146 (Sol. 14.27)

It is clear that the center of mass of a figure coincides with its center of symmetry and the center of symmetry of the figure consisting of the initial cells can only be situated in a vertex of a new cell. Since the masses of corners and bars (rectangles) are equal, the sum of vectors with the source in the center of mass of a figure and the targets in the centers of mass of all the corners and bars is equal to zero. If the number of corners had been odd, then the sum of the vectors would have had half integer coordinates and would have been nonzero. Therefore, the number of corners is an even one.

14.28. Let us place unit masses in the vertices of the polygon $A_1 \dots A_n$. Then O is the center of mass of the given system of points. Therefore, $\overrightarrow{A_iO} = \frac{1}{n}(\overrightarrow{A_iA_1} + \dots + \overrightarrow{A_iA_n})$ and $A_iO \leq \frac{1}{n}(A_iA_1 + \dots + A_iA_n)$; it follows that

$$d = A_1O + \dots + A_nO \leq \frac{1}{n}\sum_{i,j=1}^{n} A_iA_j.$$

We can express the number n either in the form $n = 2m$ or in the form $n = 2m + 1$. Let P be the perimeter of the polygon. It is clear that

$$\begin{gathered} A_1A_2 + \dots + A_nA_1 = P, \\ A_1A_3 + A_2A_4 + \dots + A_nA_2 \leq 2P, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ A_1A_{m+1} + A_2A_{m+2} + \dots + A_nA_m \leq mP \end{gathered}$$

and in the left-hand sides of these inequalities all the sides and diagonals are encountered. Since they enter the sum $\sum_{i,j=1}^{n} A_iA_j$ twice, it is clear that

$$d \leq \frac{1}{n}\sum_{i,j=1}^{n} A_iA_j \leq \frac{2}{n}(P + 2P + \dots + mP) = \frac{m(m+1)}{n}P.$$

For n even this inequality can be strengthened due to the fact that in this case every diagonal occuring in the sum $A_1A_{m+1}+\dots+A_nA_{m+n}$ is counted twice, i.e., instead of mP we can take $\frac{m}{2}P$. This means that for n even we have

$$d\leq\frac{2}{n}(P+2P+\dots+(m-1)P+\frac{m}{2}P)=\frac{m^2}{n}P.$$

Thus, we have

$$d\leq\begin{cases}\frac{m^2}{n}P=\frac{n}{4}P & \text{if } n \text{ is even}\\ \frac{m(m+1)}{n}P=\frac{n^2-1}{4n}P & \text{if } n \text{ is odd.}\end{cases}$$

14.29. Let $k=\frac{BK}{BC}=1-\frac{DL}{DC}$. Under the projection to a line perpendicular to diagonal BD points A, B, K and L pass into points A', B', K' and L', respectively, such that

$$B'K'+B'L'=kA'B'+(1-k)A'B'=A'B'.$$

It follows that the center of mass of points A', K' and L' coincides with B'. It remains to notice that under the projection a center of mass turns into a center of mass.

14.30. Introduce the following notations: $\mathbf{e}_1=\overrightarrow{A_3A_1}$, $\mathbf{e}_2=\overrightarrow{A_3A_2}$ and $\mathbf{x}=\overrightarrow{XA_3}$. Point X is the center of mass of the vertices of triangle $A_1A_2A_3$ with masses m_1, m_2, m_3 attached to them if and only if

$$m_1(\mathbf{x}+\mathbf{e}_1)+m_2(\mathbf{x}+\mathbf{e}_2)+m_3\mathbf{x}=\mathbf{0},$$

i.e., $m\mathbf{x}=-(m_1\mathbf{e}_1+m_2\mathbf{e}_2)$, where $m=m_1+m_2+m_3$. Let us assume that $m=1$. Any vector $\mathbf{x}$ on the plane can be represented in the form $\mathbf{x}=-m_1\mathbf{e}_1-m_2\mathbf{e}_2$, where the numbers m_1 and m_2 are uniquely defined. The number m_3 is found from the relation $m_3=1-m_1-m_2$.

14.31. This problem is a reformulation of Problem 13.29.

Remark. If we assume that the areas of triangles BCX, CAX and ABX are *oriented*, then the statement of the problem remains true for all the points situated outside the triangle as well.

14.32. Under the projection to line AB parallel to line BC vector $\mathbf{u}=\overrightarrow{XA}\cdot BL+\overrightarrow{XB}\cdot AK+\overrightarrow{XC}\cdot LK$ turns into vector $\overrightarrow{LA}\cdot BL+\overrightarrow{LB}\cdot AK+\overrightarrow{LB}\cdot LK$. The latter vector is the zero one since $\overrightarrow{LA}=\overrightarrow{LK}+\overrightarrow{KA}$. Considering the projection to line AB parallel to line AC we get $\mathbf{u}=\mathbf{0}$.

14.33. Making use of the result of Problem 14.31 it is easy to verify that the answer is as follows: a) $(\sin 2\alpha:\sin 2\beta:\sin 2\gamma)$; b) $(a:b:c)$; c) $(\tan\alpha:\tan\beta:\tan\gamma)$.

14.34. Adding vector $(\beta+\gamma)\overrightarrow{XA}$ to both sides of the equality $\alpha\overrightarrow{XA}+\beta\overrightarrow{XB}+\gamma\overrightarrow{XC}=\overrightarrow{0}$ we get

$$\overrightarrow{XA}=(\beta+\gamma)\overrightarrow{XA}+\beta\overrightarrow{BX}+\gamma\overrightarrow{CX}=\beta\overrightarrow{BA}+\gamma\overrightarrow{CA}.$$

14.35. By Problem 14.1 b) we have $3\overrightarrow{XM}=\overrightarrow{XA}+\overrightarrow{XB}+\overrightarrow{XC}$. Moreover, $\overrightarrow{XA}=\beta\overrightarrow{BA}+\gamma\overrightarrow{CA}$, $\overrightarrow{XB}=\alpha\overrightarrow{AB}+\gamma\overrightarrow{CB}$ and $\overrightarrow{XC}=\alpha\overrightarrow{AC}+\beta\overrightarrow{BC}$ (see Problem 14.34).

14.36. Let the lines through point X parallel to AC and BC intersect the line AB at points K and L, respectively. If $(\alpha:\beta:\gamma)$ are the barycentric coordinates of X and $\alpha+\beta+\gamma=1$, then

$$2\overrightarrow{XC_1}=\overrightarrow{XK}+\overrightarrow{XL}=\gamma\overrightarrow{CA}+\gamma\overrightarrow{CB}$$

(see the solution of Problem 14.42). Therefore,

$$3\overrightarrow{XM_1}=\overrightarrow{XA_1}+\overrightarrow{XB_1}+\overrightarrow{XC_1}=$$
$$\tfrac{1}{2}(\alpha(\overrightarrow{AB}+\overrightarrow{AC})+\beta(\overrightarrow{BA}+\overrightarrow{BC})+\gamma\overrightarrow{CA}+\overrightarrow{CB})=\tfrac{3}{2}\overrightarrow{XM}$$

(see Problem 14.35).

14.37. Let X be an arbitrary point, O the center of the circumscribed circle of the given triangle, $\mathbf{e}_i = \overrightarrow{OA_i}$ and $\mathbf{a} = \overrightarrow{XO}$. If the barycentric coordinates of X are $(x_1 : x_2 : x_3)$, then $\sum x_i(\mathbf{a} + \mathbf{e}_i) = \sum x_i\overrightarrow{XA_i} = \mathbf{0}$ because X is the center of mass of points A_1, A_2, A_3 with masses x_1, x_2, x_3. Therefore, $(\sum x_i)\mathbf{a} = -\sum x_i\mathbf{e}_i$.

Point X belongs to the circumscribed circle of the triangle if and only if $|\mathbf{a}| = XO = R$, where R is the radius of this circle. Thus, the circumscribed circle of the triangle is given in the barycentric coordinates by the equation

$$R^2(\sum x_i)^2 = (\sum x_i\mathbf{e}_i)^2,$$

i.e.,

$$R\overset{2}{\sum} x_i^2 + 2R^2\sum_{i<j} x_ix_j = R^2\sum x_i^2 + 2\sum_{i<j} x_ix_j(\mathbf{e}_i, \mathbf{e}_j)$$

because $|\mathbf{e}_i| = R$. This equation can be rewritten in the form

$$\sum_{i<j} x_ix_j(R^2 - (\mathbf{e}_i, \mathbf{e}_j)) = 0.$$

Now notice that $2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)) = a_{ij}^2$, where a_{ij} is the length of side A_iA_j. Indeed,

$$a_{ij}^2 = |\mathbf{e}_i - \mathbf{e}_j|^2 = |\mathbf{e}_i|^2 + |\mathbf{e}_j|^2 - 2(\mathbf{e}_i, \mathbf{e}_j) = 2(R^2 - (\mathbf{e}_i, \mathbf{e}_j)).$$

As a result we see that the circumscribed circle of triangle $A_1A_2A_3$ is given in the barycentric coordinates by the equation $\sum_{i<j} x_ix_ja_{ij} = 0$, where a_{ij} is the length of side A_iA_j.

14.38. a) Let X and Y be the points with barycentric coordinates $(\alpha : \beta : \gamma)$ and $(\alpha^{-1} : \beta^{-1} : \gamma^{-1})$ and let lines CX and CY intersect line AB at points X_1 and Y_1, respectively. Then

$$\overline{AX_1} : \overline{BX_1} = \beta : \alpha = \alpha^{-1} : \beta^{-1} = \overline{BY_1} : \overline{AY_1}.$$

Similar arguments for lines AX and BX show that points X and Y are isotomically conjugate with respect to triangle ABC.

b) Let X be the point with barycentric coordinates $(\alpha : \beta : \gamma)$. We may assume that $\alpha + \beta + \gamma = 1$. Then by Problem 14.34 we have

$$\overrightarrow{AX} = \beta\overrightarrow{AB} + \gamma\overrightarrow{AC} = \beta c(\frac{\overrightarrow{AB}}{c}) + \gamma b(\frac{\overrightarrow{AC}}{b}).$$

Let Y be the point symmetric to X through the bisector of angle $\angle A$ and $(\alpha' : \beta' : \gamma')$ the barycentric coordinates of Y. It suffices to verify that $\beta' : \gamma' = \frac{b^2}{\beta} : \frac{c^2}{\gamma}$. The symmetry through the bisector of angle $\angle A$ interchanges unit vectors $\frac{\overrightarrow{AB}}{c}$ and $\frac{\overrightarrow{AC}}{b}$, consequently, $\overrightarrow{AY} = \beta c\frac{\overrightarrow{AC}}{b} + \gamma b\frac{\overrightarrow{AB}}{c}$. It follows that

$$\beta' : \gamma' = \frac{\gamma b}{c} : \beta cb = \frac{b^2}{\beta} : \frac{c^2}{\gamma}.$$

Chapter 15. PARALLEL TRANSLATIONS

Background

1. The *parallel translation* by vector $\overrightarrow{AB}$ is the transformation which sends point X into point X' such that $\overrightarrow{XX'} = \overrightarrow{AB}$.

2. The composition (i.e., the consecutive execution) of two parallel translations is, clearly, a parallel translation.

Introductory problems

1. Prove that every parallel translation turns any circle into a circle.

2. Two circles of radius R are tangent at point K. On one of them we take point A,n on the other one we take point B such that $\angle AKB = 90^\circ$. Prove that $AB = 2R$.

3. Two circles of radius R intersect at points M and N. Let A and B be the intersection points of these circles with the perpendicular erected at the midpoint of segment MN. It so happens that the circles lie on one side of line MN. Prove that $MN^2 + AB^2 = 4R^2$.

4. Inside rectangle $ABCD$, point M is taken. Prove that there exists a convex quadrilateral with perpendicular diagonals of the same length as AB and BC whose sides are equal to AM, BM, CM, DM.

§1. Solving problems with the aid of parallel translations

15.1. Where should we construct bridge MN through the river that separates villages A and B so that the path $AMNB$ from A to B was the shortest one? (The banks of the river are assumed to be parallel lines and the bridge perpendicular to the banks.)

15.2. Consider triangle ABC. Point M inside the triangle moves parallel to side BC to its intersection with side CA, then parallel to AB to its intersection with BC, then parallel to AC to its intersection with AB, and so on. Prove that after a number of steps the trajectory of the point becomes a closed one.

15.3. Let K, L, M and N be the midpoints of sides AB, BC, CD and DA, respectively, of convex quadrilateral $ABCD$.

a) Prove that $KM \leq \frac{1}{2}(BC + AD)$ and the equality is attained only if $BC \parallel AD$.

b) For given lengths of the sides of quadrilateral $ABCD$ find the maximal value of the lengths of segments KM and LN.

15.4. In trapezoid $ABCD$, sides BC and AD are parallel, M the intersection point of the bisectors of angles $\angle A$ and $\angle B$, and N the intersection point of the bisectors of angles $\angle C$ and $\angle D$. Prove that $2MN = |AB + CD - BC - AD|$.

15.5. From vertex B of parallelogram $ABCD$ heights BK and BH are drawn. It is known that $KH = a$ and $BD = b$. Find the distance from B to the intersection point of the heights of triangle BKH.

15.6. In the unit square a figure is placed such that the distance between any two of its points is not equal to 0.001. Prove that the area of this figure does not exceed a) 0.34; b) 0.287.

§2. Problems on construction and loci

15.7. Consider angle $\angle ABC$ and straight line l. Construct a line parallel to l on which the legs of angle $\angle ABC$ intercept a segment of given length a.

15.8. Consider two circles S_1, S_2 and line l. Draw line l_1 parallel to l so that:

a) the distance between the intersection points of l_1 with circles S_1 and S_2 is of a given value a;

b) S_1 and S_2 intercept on l_1 equal chords;

c) S_1 and S_2 intercept on l_1 chords the sum (or difference) of whose lengths is equal to a given value.

15.9. Consider nonintersecting chords AB and CD on a circle. Construct a point X on the circle so that chords AX and BX would intercept on chord CD a segment, EF, of a given length a.

15.10. Construct quadrilateral $ABCD$ given the quadrilateral's angles and the lengths of sides $AB = a$ and $CD = b$.

15.11. Given point A and circles S_1 and S_2. Through A draw line l so that S_1 and S_2 intercept on l equal chords.

15.12. a) Given circles S_1 and S_2 intersect at points A and B. Through point A draw line l so that the intercept of this line between circles S_1 and S_2 were of a given length.

b) Consider triangle ABC and triangle PQR. In triangle ABC inscribe a triangle equal to PQR.

15.13. Construct a quadrilateral given its angles and diagonals.

* * *

15.14. Find the loci of the points for which the following value is given: a) the sum, b) the difference of the distances from these points to the two given straight lines.

15.15. An angle made of a transparent material moves so that two nonintersecting circles are tangent to its legs from the inside. Prove that on the angle a point circumscribing an arc of a circle can be marked.

Problems for independent study

15.16. Consider two pairs of parallel lines and point P. Through P draw a line on which both pairs of parallel lines intercept equal segments.

15.17. Construct a parallelogram given its sides and an angle between the diagonals.

15.18. In convex quadrilateral $ABCD$, sides AB and CD are equal. Prove that

a) lines AB and CD form equal angles with the line that connects the midpoints of sides AC and BD;

b) lines AB and CD form equal angles with the line that connects the midpoints of diagonals BC and AD.

15.19. Among all the quadrilaterals with given lengths of the diagonals and an angle between them find the one of the least perimeter.

15.20. Given a circle and two neighbouring vertices of a parallelogram. Construct the parallelogram if it is known that its other two (not given) vertices belong to the given circle.

Solutions

15.1. Let A' be the image of point A under the parallel translation by $\overrightarrow{MN}$. Then $A'N = AM$ and, therefore, the length of path $AMNB$ is equal to $A'N + NB + MN$. Since the length of segment MN is a constant, we have to find point N for which the sum

$A'N + NB$ is the least one. It is clear that the sum is minimal if N belongs to segment $A'B$, i.e., N is the closest to B intersection point of the bank and segment $A'B$.

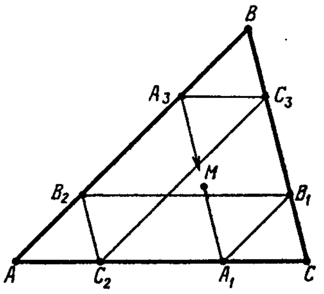


Figure 147 (Sol. 15.2)

15.2. Denote the consecutive points of the trajectory on the sides of the triangle as on Fig. 147:

$$A_1,\ B_1,\ B_2,\ C_2,\ C_3,\ A_3,\ A_4,\ B_4,\dots$$

Since $A_1B_1 \parallel AB_2$, $B_1B_2 \parallel CA_1$ and $B_1C \parallel B_2C_2$, it is clear that triangle AB_2C_2 is the image of triangle A_1B_1C under a parallel translation. Similarly, triangle A_3BC_3 is the image of triangle AB_2C_2 under a parallel translation and A_4B_4C is obtained in the same way from A_3BC_3. But triangle A_1B_1C is also the image of triangle A_3BC_3 under a parallel translation, hence, $A_1 = A_4$, i.e., after seven steps the trajectory becomes closed. (It is possible for the trajectory to become closed sooner. Under what conditions?)

15.3. a) Let us complement triangle CBD to parallelogram $CBDE$. Then $2KM = AE \le AD + DE = AD + BC$ and the equality is attained only if $AD \parallel BC$.

b) Let $a = AB$, $b = BC$, $c = CD$ and $d = DA$. If $|a - c| = |b - d| \neq 0$ then by heading a) the maximum is attained in the degenerate case when all points A, B, C and D belong to one line. Now suppose that, for example, $|a - c| < |b - d|$. Let us complement triangles ABL and LCD to parallelograms $ABLP$ and $LCDQ$, respectively; then $PQ \ge |b - d|$ and, therefore,

$$LN^2 = \frac{1}{4}(2LP^2 + 2LQ^2 - PQ^2) \le \frac{1}{4}(2(a^2 + c^2) - (b - d)^2).$$

Moreover, by heading a) $KM \le \frac{1}{2}(b + d)$. Both equalities are attained when $ABCD$ is a trapezoid with bases AD and BC.

15.4. Let us construct circle S tangent to side AB and rays BC and AD; translate triangle CND parallelly (in the direction of bases BC and AD) until N' coincides with point M, i.e., side $C'D'$ becomes tangent to circle S (Fig. 148).

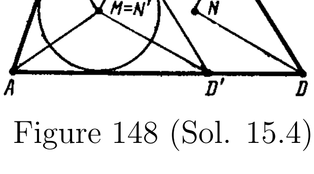


Figure 148 (Sol. 15.4)

For the circumscribed trapezoid $ABC'D'$ the equality $2MN' = |AB+C'D'-BC'-AD'|$ is obvious because $N' = M$. Under the passage from trapezoid $ABC'D'$ to trapezoid $ABCD$ the left-hand side of this equality accrues by $2N'N$ and the right-hand side accrues by $CC' + DD' = 2NN'$. Hence, the equality is preserved.

15.5. Denote the intersection point of heights of triangle BKH by H_1. Since $HH_1 \perp BK$ and $KH_1 \perp BH$, it follows that $HH_1 \parallel AD$ and $KH_1 \parallel DC$, i.e., H_1HDK is a parallelogram. Therefore, under the parallel translation by vector $\overrightarrow{H_1H}$ point K passes to point D and point B passes to point P (Fig. 149). Since $PD \parallel BK$, it follows that $BPDK$ is a rectangle and $PK = BD = b$. Since $BH_1 \perp KH$, it follows that $PH \perp KH$. It is also clear that $PH = BH_1$.

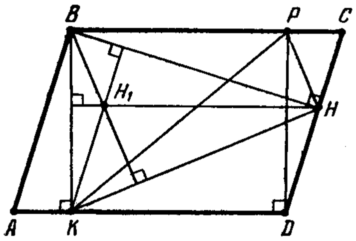


Figure 149 (Sol. 15.5)

In right triangle PKH, hypothenuse $KP = b$ and the leg $KH = a$ are known; therefore, $BH_1 = PH = \sqrt{b^2 - a^2}$.

15.6. a) Denote by F the figure that lies inside the unit square $ABCD$; let S be its area. Let us consider two vectors $\overrightarrow{AA_1}$ and $\overrightarrow{AA_2}$, where point A_1 belongs to side AD and $AA_1 = 0.001$ and where point A_2 belongs to the interior of angle $\angle BAD$, $\angle A_2AA_1 = 60°$ and $AA_2 = 0.001$ (Fig. 150).

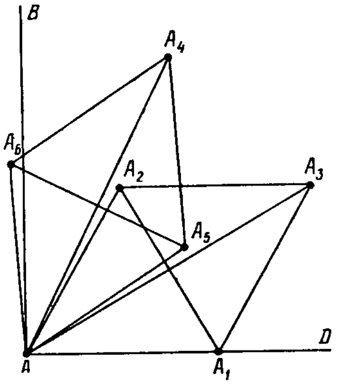


Figure 150 (Sol. 15.6 a))

Let F_1 and F_2 be the images of F under the parallel translations by vectors $\overrightarrow{AA_1}$ and $\overrightarrow{AA_2}$, respectively. The figures F, F_1 and F_2 have no common points and belong to the interior of the square with side 1.001. Therefore, $2S < 1.001^2$, i.e., $S < 0.335 < 0.34$.

b) Consider vector $\overrightarrow{AA_3} = \overrightarrow{AA_1} + \overrightarrow{AA_2}$. Let us rotate $\overrightarrow{AA_3}$ about point A through an acute angle counterclockwise so that point A_3 turns into point A_4 such that $A_3A_4 = 0.001$. Let us also consider vectors $\overrightarrow{AA_5}$ and $\overrightarrow{AA_6}$ of length 0.001 each constituting an angle of 30° with vector $\overrightarrow{AA_4}$ and situated on both sides of it (Fig. 151).

Denote by F_i the image of figure F under the parallel translation by the vector $\overrightarrow{AA_i}$. Denote the area of the union of figures A and B by $S(A \cup B)$ and by $S(A \cap B)$ the area of their intersection.

For definiteness, let us assume that $S(F_4 \cap F) \leq S(F_3 \cap F)$. Then $S(F_4 \cap F) \leq \frac{1}{2}S$ and, therefore, $S(F_4 \cup F) \geq \frac{3}{2}S$. The figures F_5 and F_6 do not intersect either each other or figures F or F_4 and, therefore, $S(F \cup F_4 \cup F_5 \cup F_6) \geq \frac{7}{2}S$. (If it would have been that

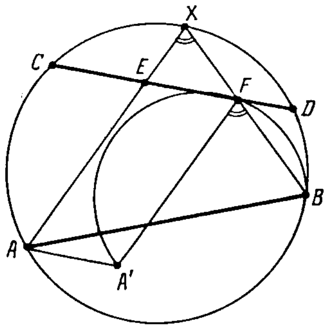


Figure 151 (Sol. 15.6 b))

$S(F_3 \cap F) \leq S(F_4 \cap F)$, then instead of figures F_5 and F_6 we should have taken F_1 and F_2.) Since the lengths of vectors $\overrightarrow{AA_i}$ do not exceed $0.001\sqrt{3}$, all the figures considered lie inside a square with side $1 + 0.002\sqrt{3}$. Therefore, $7S/2 \leq (1 + 0.002\sqrt{3})^2$ and $S < 0.287$.

15.7. Given two vectors $\pm\mathbf{a}$ parallel to l and of given length a. Consider the images of ray BC under the parallel translations by these vectors. Their intersection point with ray BA belongs to the line to be constructed (if they do not intersect, then the problem has no solutions).

15.8. a) Let S_1' be the image of circle S_1 under the parallel translation by a vector of length a parallel to l (there are two such vectors). The desired line passes through the intersection point of circles S_1' and S_2.

b) Let O_1 and O_2 be the projections of the centers of circles S_1 and S_2 to line l; let S_1' be the image of the circle S_1 under the parallel translation by vector $\overrightarrow{O_1O_2}$. The desired line passes through the intersection point of circles S_1' and S_2.

c) Let S_1' be the image of circle S_1 under the parallel translation by a vector parallel to l. Then the lengths of chords cut by the line l_1 on circles S_1 and S_1' are equal. If the distance between the projections of the centers of circles S_1' and S_2 to line l is equal to $\frac{1}{2}a$, then the sum of difference of the lengths of chords cut by the line parallel to l and passing through the intersection point of circles S_1' and S_2 is equal to a. Now it is easy to construct circle S_1'.

15.9. Suppose that point X is constructed. Let us translate point A by vector $\overrightarrow{EF}$, i.e., let us construct point A' such that $\overrightarrow{EF} = \overrightarrow{AA'}$. This construction can be performed since we know vector $\overrightarrow{EF}$: its length is equal to a and it is parallel to CD.

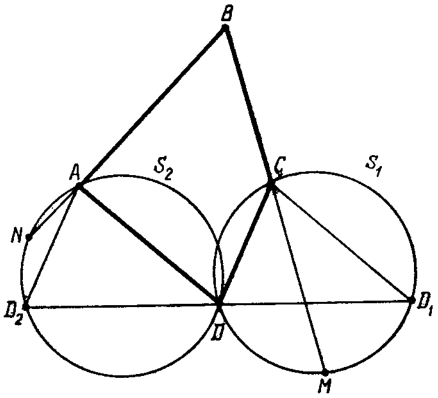


Figure 152 (Sol. 15.9)

Since $AX \parallel A'F$, it follows that $\angle A'FB = \angle AXB$ and, therefore, angle $\angle A'FB$ is known. Thus, point F belongs to the intersection of two figures: segment CD and an arc of the circle whose points are vertices of the angles equal to $\angle AXB$ that subtend segment $A'B$, see Fig. 152.

15.10. Suppose that quadrilateral $ABCD$ is constructed. Denote by D_1 the image of point D under the parallel translation by vector $\overrightarrow{CB}$. In triangle ABD_1, sides AB, BD_1 and angle $\angle ABD_1$ are known. Hence, the following construction.

Let us arbitrarily construct ray BC' and then draw rays BD_1' and BA' so that $\angle D_1'BC' = 180^\circ - \angle C$, $\angle A'BC' = \angle B$ and these rays lie in the half plane on one side of ray BC'.

On rays BA' and BD_1', draw segments $BA = a$ and $BD_1 = b$, respectively. Let us draw ray AD' so that $\angle BAD' = \angle A$ and rays BC', AD' lie on one side of line AB. Vertex D is the intersection point of ray AD' and the ray drawn from D_1 parallel to ray BC'. Vertex C is the intersection point of BC' and the ray drawn from D parallel to ray D_1B.

15.11. Suppose that points M and N at which line l intersects circle S_2 are constructed. Let O_1 and O_2 be the centers of circles S_1 and S_2; let O_1' be the image of point O_1 under the parallel translation along l such that $O_1'O_2 \perp MN$; let S_1' be the image of circle S_1 under the same translation.

Let us draw tangents AP and AQ to circles S_1' and S_2, respectively. Then $AQ^2 = AM \cdot AN = AP^2$ and, therefore, $O_1'A^2 = AP^2 + R^2$, where R is the radius of circle S_1'. Since segment AP can be constructed, we can also construct segment AO_1'. It remains to notice that point O_1' belongs to both the circle of radius AO_1' with the center at A and to the circle with diameter O_1O_2.

15.12. a) Let us draw through point A line PQ, where P belongs to circle S and Q belongs to circle S_2. From the centers O_1 and O_2 of circles S_1 and S_2, respectively, draw perpendiculars O_1M and O_2N to line PQ. Let us parallelly translate segment MN by a vector $\overrightarrow{MO_1}$. Let C be the image of point N under this translation.

Triangle O_1CO_2 is a right one and $O_1C = MN = \frac{1}{2}PQ$. It follows that in order to construct line PQ for which $PQ = a$ we have to construct triangle O_1CO_2 of given hypothenuse O_1O_2 and leg $O_1C = \frac{1}{2}a$ and then draw through A the line parallel to O_1C.

b) It suffices to solve the converse problem: around the given triangle PQR circumscribe a triangle equal (?) to the given triangle ABC. Suppose that we have constructed triangle ABC whose sides pass through given points P, Q and R. Let us construct the arcs of circles whose points serve as vertices for angles $\angle A$ and $\angle B$ that subtend segments RP and QP, respectively. Points A and B belong to these arcs and the length of segment AB is known.

By heading a) we can construct line AP through P whose intercept between circles S_1 and S_2 is of given length. Draw lines AR and BQ; we get triangle ABC equal to the given triangle since these triangles have by construction equal sides and the angles adjacent to it.

15.13. Suppose that the desired quadrilateral $ABCD$ is constructed. Let D_1 and D_2 be the images of point D under the translations by vectors $\overrightarrow{AC}$ and $\overrightarrow{CA}$, respectively. Let us circumscribe circles S_1 and S_2 around triangles DCD_1 and DAD_2, respectively. Denote the intersection points of lines BC and BA with circles S_1 and S_2 by M and N, respectively, see Fig. 153. It is clear that $\angle DCD_1 = \angle DAD_2 = \angle D$, $\angle DCM = 180^\circ - \angle C$ and $\angle DAN = 180^\circ - \angle A$.

This implies the following construction. On an arbitrary line l, take a point, D, and construct points D_1 and D_2 on l so that $DD_1 = DD_2 = AC$. Fix one of the half planes Π determined by line l and assume that point B belongs to this half plane. Let us construct a circle S_1 whose points belonging to Π serve as vertices of the angles equal to $\angle D$ that subtend segment DD_1.

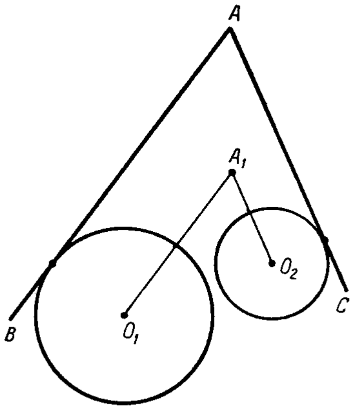


Figure 153 (Sol. 15.13)

We similarly construct circle S_2. Let us construct point M on S_1 so that all the points of the part of the circle that belongs to Π serve as vertices of the angles equal to $180^\circ - \angle C$ that subtend segment DM.

Point N is similarly constructed. Then segment MN subtends angle $\angle B$, i.e., B is the intersection point of the circle with center D of radius DB and the arc of the circle serve as vertices of the angles equal to $\angle B$ that subtend segment MN (it also belongs to the half plane Π). Points C and A are the intersection points of lines BM and BN with circles S_1 and S_2, respectively.

15.14. From a point X draw perpendiculars XA_1 and XA_2 to given lines l_1 and l_2, respectively. On ray A_1X, take point B so that $A_1B = a$. Then if $XA_1 \pm XA_2 = a$, we have $XB = XA_2$. Let l_1' be the image of line l_1 under the parallel translation by vector $\overrightarrow{A_1B}$ and M the intersection point of lines l_1' and l_2. Then in the indicated cases ray MX is the bisector of angle $\angle A_2MB$. As a result we get the following answer.

Let the intersection points of lines l_1 and l_2 with the lines parallel to lines l_1 and l_2 and distant from them by a form rectangle $M_1M_2M_3M_4$. The locus to be found is either a) the sides of this rectangle; or b) the extensions of these sides.

15.15. Let leg AB of angle $\angle BAC$ be tangent to the circle of radius r_1 with center O_1 and leg AC be tangent to the circle of radius r_2 with center O_2. Let us parallelly translate line AB inside angle $\angle BAC$ by distance r_1 and let us parallelly translate line AC inside angle $\angle BAC$ by distance r_2. Let A_1 be the intersection point of the translated lines (Fig. 154).

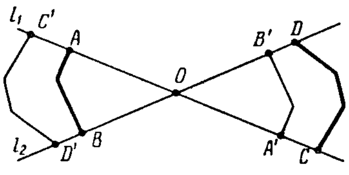


Figure 154 (Sol. 15.15)

Then $\angle O_1A_1O_2 = \angle BAC$. The constant(?) angle $O_1A_1O_2$ subtends fixed segment O_1O_2 and, therefore, point A_1 traverses an arc of a(?) circle.

Chapter 16. CENTRAL SYMMETRY

Background

1. The *symmetry through point A* is the transformation of the plane which sends point X into point X' such that A is the midpoint of segment XX'. The other names of such a transformation: the *central symmetry with center A* or just the *symmetry with center A*.

Notice that the symmetry with center A is a particular case of two other transformations: it is the rotation through an angle of 180° with center A and also the homothety with center A and coefficient -1.

2. If a figure turns into itself under the symmetry through point A, then A is called the *center of symmetry* of this figure.

3. The following notations for transformations are used in this chapter:

S_A — the symmetry with center A;

$T_{\mathbf{a}}$ — the translation by vector $\mathbf{a}$.

4. We will denote the composition of symmetries through points A and B by $S_B \circ S_A$; here we assume that we first perform symmetry S_A and then symmetry S_B. This notation might look unnatural at first glance, but it is, however, justified by the identity $(S_B \circ S_A)(X) = S_B(S_A(X))$.

The composition of maps is associative: $F \circ (G \circ H) = (F \circ G) \circ H$. Therefore, the order of the compositions is inessential and we may simply write $F \circ G \circ H$.

5. The compositions of two central symmetries or of a symmetry with a parallel translation are calculated according to the following formulas (Problem 16.9):

a) $S_B \circ S_A = T_{2\overrightarrow{AB}}$;

b) $T_{\mathbf{a}} \circ S_A = S_B$ and $S_B \circ T_{\mathbf{a}} = S_A$, where $\mathbf{a} = 2\overrightarrow{AB}$.

Introductory problems

1. Prove that under any central symmetry any circle turns into a circle.

2. Prove that a quadrilateral with a center of symmetry is a parallelogram.

3. The opposite sides of a convex hexagon are equal and parallel. Prove that the hexagon has a center of symmetry.

4. Consider parallelogram $ABCD$ and point M. The lines parallel to lines MC, MD, MA and MB are drawn through points A, B, C and D, respectively. Prove that the lines drawn intersect at one point.

5. Prove that the opposite sides of a hexagon formed by the sides of a triangle and the tangents to its circumscribed circle parallel to the sides of the triangle are equal.

§1. Solving problems with the help of a symmetry

16.1. Prove that if in a triangle a median and a bisector coincide, then the triangle is an isosceles one.

16.2. Two players lay out nickels on a rectangular table taking turns. It is only allowed to place a coin onto an unoccupied place. The *loser* is the one who can not make any move. Prove that the first player can always win in finitely many moves.

16.3. A circle intersects sides BC, CA, AB of triangle ABC at points A_1 and A_2, B_1 and B_2, C_1 and C_2, respectively. Prove that if the perpendiculars to the sides of the triangle drawn through points A_1, B_1 and C_1 intersect at one point, then the perpendiculars to the sides drawn through A_2, B_2 and C_2 also intersect at one point.

16.4. Prove that the lines drawn through the midpoints of the circumscribed quadrilateral perpendicularly to the opposite sides intersect at one point.

16.5. Let P be the midpoint of side AB of convex quadrilateral $ABCD$. Prove that if the area of triangle PCD is equal to a half area of quadrilateral $ABCD$, then $BC \parallel AD$.

16.6. Unit circles S_1 and S_2 are tangent at point A; the center O of circle S of radius 2 belongs to S_1. Circle S_1 is tangent to circle S at point B. Prove that line AB passes through the intersection point of circles S_2 and S.

16.7. In triangle ABC medians AF and CE are drawn. Prove that if $\angle BAF = \angle BCE = 30^\circ$, then triangle ABC is an equilateral one.

16.8. Consider a convex n-gon with pairwise nonparallel sides and point O inside it. Prove that it is impossible to draw more than n lines through O so that each line divides the area of the n-gon in halves.

§2. Properties of the symmetry

16.9. a) Prove that the composition of two central symmetries is a parallel translation.

b) Prove that the composition of a parallel translation with a central symmetry (in either order) is a central symmetry.

16.10. Prove that if a point is reflected symmetrically through points O_1, O_2 and O_3 and then reflected symmetrically once again through the same points, then it assumes the initial position.

16.11. a) Prove that a bounded figure cannot have more than one center of symmetry.

b) Prove that no figure can have precisely two centers of symmetry.

c) Let M be a finite set of points on a plane. Point O will be called an "almost center of symmetry" of the set M if we can delete a point from M so that O becomes the center of symmetry of the remaining set. How many "almost centers of symmetry" can a set have?

16.12. On segment AB, consider n pairs of points symmetric through the midpoint; n of these $2n$ points are painted blue and the remaining are painted red. Prove that the sum of distances from A to the blue points is equal to the sum of distances from B to the red points.

§3. Solving problems with the help of a symmetry. Constructions

16.13. Through a common point A of circles S_1 and S_2 draw a straight line so that these circles would intercept on it equal chords.

16.14. Given point A, a line and a circle. Through A draw a line so that A divides the segment between the intersection points of the line drawn with the given line and the given circle in halves.

16.15. Given angle ABC and point D inside it. Construct a segment with the endpoints on the legs of the given angle and with the midpoint at D.

16.16. Consider an angle and points A and B inside it. Construct a parallelogram for which points A and B are opposite vertices and the two other vertices belong to the legs of the angle.

16.17. Given four pairwise nonparallel straight lines and point O not belonging to these lines. Construct a parallelogram whose center is O and the vertices lie on the given lines, one on each.

16.18. Consider two concentric circles S_1 and S_2. Draw a line on which these circles intercept three equal segments.

16.19. Consider nonintersecting chords AB and CD of a circle and point J on chord CD. Construct point X on the circle so that chords AX and BX would intercept on chord CD segment EF which J divides in halves.

16.20. Through a common point A of circles S_1 and S_2 draw line l so that the difference of the lengths of the chords intercepted by circles S_1 and S_2 on l were of given value a.

16.21. Given $m = 2n+1$ points — the midpoints of the sides of an m-gon — construct the vertices of the m-gon.

Problems for independent study

16.22. Construct triangle ABC given medians m_a, m_b and angle $\angle C$.

16.23. a) Given a point inside a parallelogram; the point does not belong to the segments that connect the midpoints of the opposite sides. How many segments divided in halves by the given point are there such that their endpoints are on the sides of the parallelogram?

b) A point inside the triangle formed by the midlines of a given triangle is given. How many segments divided in halves by the given point and with the endpoints on the sides of the given triangle are there?

16.24. a) Find the locus of vertices of convex quadrilaterals the midpoints of whose sides are the vertices of a given square.

b) Three points are given on a plane. Find the locus of vertices of convex quadrilaterals the midpoints of three sides of each of which are the given points.

16.25. Points A, B, C, D lie in the indicated order on a line and $AB = CD$. Prove that for any point P on the plane we have $AP + DP \geq BP + CP$.

Solutions

16.1. Let median BD of triangle ABC be a bisector as well. Let us consider point B_1 symmetric to B through point D. Since D is the midpoint of segment AC, the quadrilateral $ABCB_1$ is a parallelogram. Since $\angle ABB_1 = \angle B_1BC = \angle AB_1B$, it follows that triangle B_1AB is an isosceles one and $AB = AB_1 = BC$.

16.2. The first player places a nickel in the center of the table and then places nickels symmetrically to the nickels of the second player with respect to the center of the table. Using this strategy the first player has always a possibility to make the next move. It is also clear that the play will be terminated in a finite number of moves.

16.3. Let the perpendiculars to the sides drawn through points A_1, B_1 and C_1 intersect at point M. Denote the center of the circle by O. The perpendicular to side BC drawn through point A_1 is symmetric through point O to the perpendicular to side BC drawn through A_2. It follows that the perpendiculars to the sides drawn through points A_2, B_2 and C_2 intersect at the point symmetric to M through point O.

16.4. Let P, Q, R and S be the midpoints of sides AB, BC, CD and DA, respectively, and M the intersection point of segments PR and QS (i.e., the midpoint of both of these segments, see Problem 14.5); O the center of the circumscribed circle and O' the point symmetric to O through M. Let us prove that the lines mentioned in the formulation of the problem pass through O'. Indeed, $O'POR$ is a parallelogram and, therefore, $O'P \parallel OR$. Since R is the midpoint of chord CD, it follows that $OR \perp CD$, i.e., $O'P \perp CD$.

For lines $O'Q, O'R$ and $O'S$ the proof is similar.

16.5. Let point D' be symmetric to D through P. If the area of triangle PCD is equal to a half area of quadrilateral $ABCD$, then it is equal to$S_{PBC} + S_{PAD}$, i.e., it is equal to $S_{PBC} + S_{PBD'}$. Since P is the midpoint of segment DD', it follows that $S_{PCD'} = S_{PCD} = S_{PBC} + S_{PBD'}$ and, therefore, point B belongs to segment $D'C$. It remains to notice that $D'B \parallel AD$.

16.6. Circles S_1 and S_2 are symmetric through point A. Since OB is the diameter of circle S_1, it follows that $\angle BAO = 90^\circ$ and, therefore, under the symmetry through A point B becomes on the circle S again. It follows that under the symmetry through A point B turns into the intersection point of circles S_2 and S.

16.7. Since $\angle EAF = \angle ECF = 30^\circ$, we see that points A, E, F and C belong to one circle S and if O is its center, then $\angle EOF = 60^\circ$. Point B is symmetric to A through E and, therefore, B belongs to circle S_1 symmetric to circle S through E. Similarly, point B belongs to circle S_2 symmetric to circle S through point F. Since triangle EOF is an equilateral one, the centers of circles S, S_1 and S_2 form an equilateral triangle with side $2R$, where R is the radius of these circles. Therefore, circles S_1 and S_2 have a unique common point — B — and triangle BEF is an equilateral one. Thus, triangle ABC is also an equilateral one.

16.8. Consider a polygon symmetric to the initial one through point O. Since the sides of the polygons are pairwise nonparallel, the contours of these polygons cannot have common segments but could only have common points. Since the polygons are convex ones, each side has not more than two intersection points; therefore, there are not more than $2n$ intersection points of the contours (more precisely, not more than n pairs of points symmetric through O).

Let l_1 and l_2 be the lines passing through O and dividing the area of the initial polygon in halves. Let us prove that inside each of the four parts into which these lines divide the plane there is an intersection point of the contours.

Suppose that one of the parts has no such points between lines l_1 and l_2. Denote the intersection points of lines l_1 and l_2 with the sides of the polygon as indicated on Fig. 12.

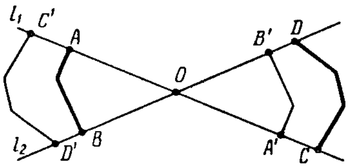


Figure 154 (Sol. 16.8)

Let points A', B', C' and D' be symmetric trough O to points A, B, C and D, respectively. For definiteness sake, assume that point A is closer to O than C'. Since segments AB and $C'D'$ do not intersect, point B is closer to O than D'. It follows that $S_{ABO} < S_{C'D'O} = S_{CDO}$, where ABO is a convex figure bounded by segments AO and BO and the part of the boundary of the n-gon between points A and B.

On the other hand, $S_{ABO} = S_{CDO}$ because lines l_1 and l_2 divide the area of the polygon in halves. Contradiction.

Therefore, between every pair of lines which divide the area of the polygon in halves there is a pair of symmetric intersection points of contours; in other words, there are not more than n such lines.

16.9. a) Let the central symmetry through O_1 send point A into A_1; let the central symmetry through O_2 send point A_1 into A_2. Then O_1O_2 is the midline of triangle AA_1A_2 and, therefore, $\overrightarrow{AA_2} = 2\overrightarrow{O_1O_2}$.

b) Let O_2 be the image of point O_1 under the translation by vector $\frac{1}{2}\mathbf{a}$. By heading a) we have $S_{O_1} \circ S_{O_2} = T_{\mathbf{a}}$. Multiplying this equality by S_{O_1} from the right or by S_{O_2} from the left and taking into account that $S_X \circ S_X$ is the identity transformation we get $S_{O_1} = S_{O_2} \circ T_{\mathbf{a}}$ and $S_{O_2} = T_{\mathbf{a}} \circ S_{O_1}$.

16.10. By the preceding problem $S_B \circ S_A = T_{2\overrightarrow{AB}}$; therefore,

$$S_{O_3} \circ S_{O_2} \circ S_{O_1} \circ S_{O_3} \circ S_{O_2} \circ S_{O_1} = T_{2(\overrightarrow{O_2O_3}+\overrightarrow{O_3O_1}+\overrightarrow{O_1O_2})}$$

is the identity transformation.

16.11. a) Suppose that a bounded figure has two centers of symmetry: O_1 and O_2. Let us introduce a coordinate system whose absciss axis is directed along ray O_1O_2. Since $S_{O_2} \circ S_{O_1} = T_{2\overrightarrow{O_1O_2}}$, the figure turns into itself under the translation by vector $2\overrightarrow{O_1O_2}$. A bounded figure cannot possess such a property since the image of the point with the largest absciss does not belong to the figure.

b) Let $O_3 = S_{O_2}(O_1)$. It is easy to verify that $S_{O_3} = S_{O_2} \circ S_{O_1} \circ S_{O_2}$ and, therefore, if O_1 and O_2 are the centers of symmetry of a figure, then O_3 is also a center of symmetry, moreover, $O_3 \neq O_1$ and $O_3 \neq O_2$.

c) Let us demonstrate that a finite set can only have 0, 1, 2 or 3 "almost centers of symmetry". The corresponding examples are given on Fig. 13. It only remains to prove that a finite set cannot have more than three "almost centers of symmetry".

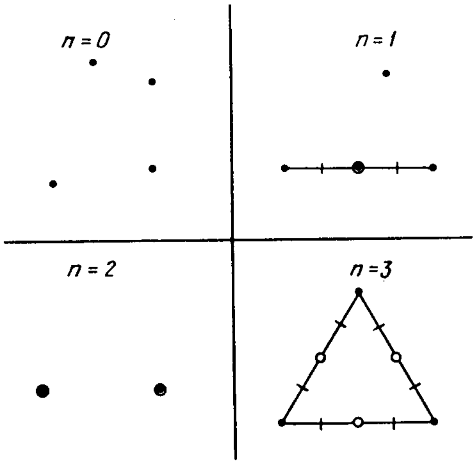


Figure 155 (Sol. 16.11)

There are finitely many "almost centers of symmetry" since they are the midpoints of the segments that connect the points of the set. Therefore, we can select a line such that the projections of "almost centers of symmetry" to the line are distinct. Therefore, it suffices to carry out the proof for the points which belong to one line.

Let n points on a line be given and $x_1 < x_2 < \dots < x_{n-1} < x_n$ be their coordinates. If we discard the point x_1, then only point $\frac{1}{2}(x_2 + x_n)$ can serve as the center of symmetry of the remaining set; if we discard x_n, then only point $\frac{1}{2}(x_1 + x_{n-1})$ can be the center of symmetry of the remaining set and if we discard any other point, then only point $\frac{1}{2}(x_1 + x_n)$ can be the center of symmetry of the remaining set. Therefore, there can not be more than 3 centers of symmetry.

16.12. A pair of symmetric points is painted different colours, therefore, it can be discarded from the consideration; let us discard all such pairs. In the remaining set of points the number of blue pairs is equal to the number of red pairs. Moreover, the sum of the distances from either of points A or B to any pair of symmetric points is equal to the length of segment AB.

16.13. Consider circle S_1' symmetric to circle S_1 through point A. The line to be found passes through the intersection points of S_1' and S_2.

16.14. Let l' be the image of line l under the symmetry through point A. The desired line passes through point A and an intersection point of line l' with the circle S.

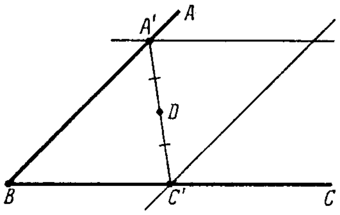


Figure 156 (Sol. 16.15)

16.15. Let us construct the intersection points A' and C' of the lines symmetric to the lines BC and AB through the point D with lines AB and BC, respectively, see Fig. 14. It is clear that point D is the midpoint of segment $A'C'$ because points A' and C' are symmetric through D.

16.16. Let O be the midpoint of segment AB. We have to construct points C and D that belong to the legs of the angle so that point O is the midpoint of segment CD. This construction is described in the solution of the preceding problem.

16.17. Let us first separate the lines into pairs. This can be done in three ways. Let the opposite vertices A and C of parallelogram $ABCD$ belong to one pair of lines, B and D to the other pair. Consider the angle formed by the first pair of lines and construct points A and C as described in the solution of Problem 16.15. Construct points B and D in a similar way.

16.18. On the smaller circle, S_1, take an arbitrary point, X. Let S_1' be the image of S_1 under the symmetry with respect to X, let Y be the intersection point of circles S_1' and S_2. Then XY is the line to be found.

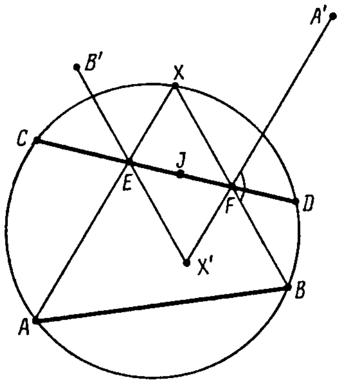


Figure 157 (Sol. 16.19)

16.19. Suppose X is constructed. Denote the images of points A, B and X under the symmetry through point J by A', B' and X', respectively, see Fig. 15. Angle $\angle A'FB = 180° - \angle AXB$ is known and, therefore, point F is the intersection point of segment CD with the arc of the circle whose points serve as vertices of angles of value $180° - \angle AXB$ that subtend segment BA'. Point X is the intersection point of line BF with the given circle.

16.20. Suppose that line l is constructed. Let us consider circle S_1' symmetric to circle S_1 through point A. Let O_1, O_1' and O_2 be the centers of circles S_1, S_1' and S_2, as shown on Fig. 16.

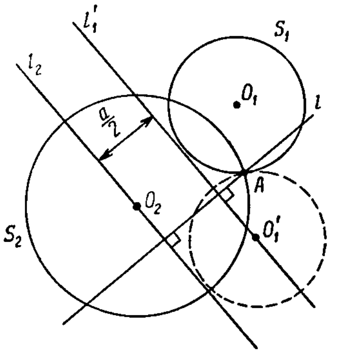


Figure 158 (Sol. 16.20)

Let us draw lines l_1' and l_2 through O_1' and O_2 perpendicularly to line l. The distance between lines l_1' and l_2 is equal to a half difference of the lengths of chords intercepted by l on circles S_1 and S_2. Therefore, in order to construct l, we have to construct the circle of radius $\frac{1}{2}a$ with center O_1'; line l_2 is tangent to this circle. Having constructed l_2, drop the perpendicular from point A to l_2; this perpendicular is line l.

16.21. Let B_1, B_2, $\dots$, B_m be the midpoints of sides A_1A_2, A_2A_3, $\dots$, A_mA_1 of polygon $A_1A_2\dots A_m$. Then $S_{B_1}(A_1) = A_2$, $S_{B_2}(A_2) = A_3, \dots, S_{B_m}(A_m) = A_1$. It follows that $S_{B_m}\circ\dots\circ S_{B_1}(A_1) = A_1$, i.e., A_1 is a fixed point of the composition of symmetries $S_{B_m}\circ S_{B_{m-1}}\circ\dots\circ S_{B_1}$. By Problem 16.9 the composition of an odd number of central symmetries is a central symmetry, i.e., has a unique fixed point. This point can be constructed as the midpoint of the segment that connects points X and $S_{B_m}\circ S_{B_{m-1}}\circ\dots\circ S_{B_1}(X)$, where X is an arbitrary point.

Chapter 17. THE SYMMETRY THROUGH A LINE

Background

1. The *symmetry through a line l* (notation: S_l) is a transformation of the plane which sends point X into point X' such that l is the midperpendicular to segment XX'. Such a transformation is also called *the axial symmetry* and l is called *the axis of the symmetry.*

2. If a figure turns into itself under the symmetry through line l, then l is called *the axis of symmetry* of this figure.

3. The composition of two symmetries through axes is a parallel translation, if the axes are parallel, and a rotation, if they are not parallel, cf. Problem 17.22.

Axial symmetries are a sort of "bricks" all the other motions of the plane are constructed from: any motion is a composition of not more than three axial symmetries (Problem 17.35). Therefore, the composition of axial symmetries give much more powerful method for solving problems than compositions of central symmetries. Moreover, it is often convenient to decompose a rotation into a composition of two symmetries with one of the axes of symmetry being a line passing through the center of the rotation.

Introductory problems

1. Prove that any axial symmetry sends any circle into a circle.

2. A quadrilateral has an axis of symmetry. Prove that this quadrilateral is either an equilateral trapezoid or is symmetric through a diagonal.

3. An axis of symmetry of a polygon intersects its sides at points A and B. Prove that either point A is a vertex of the polygon or the midpoint of a side perpendicular to the axis of symmetry.

4. Prove that if a figure has two perpendicular axes of symmetry, it has a center of symmetry.

§1. Solving problems with the help of a symmetry

17.1. Point M belongs to a diameter AB of a circle. Chord CD passes through M and intersects AB at an angle of 45°. Prove that the sum $CM^2 + DM^2$ does not depend on the choice of point M.

17.2. Equal circles S_1 and S_2 are tangent to circle S from the inside at points A_1 and A_2, respectively. An arbitrary point C of circle S is connected by segments with points A_1 and A_2. These segments intersect S_1 and S_2 at points B_1 and B_2, respectively. Prove that $A_1A_2 \parallel B_1B_2$.

17.3. Through point M on base AB of an isosceles triangle ABC a line is drawn. It intersects sides CA and CB (or their extensions) at points A_1 and B_1. Prove that $A_1A : A_1M = B_1B : B_1M$.

§2. Constructions

17.4. Construct quadrilateral $ABCD$ whose diagonal AC is the bisector of angle $\angle A$ knowing the lengths of its sides.

17.5. Construct quadrilateral $ABCD$ in which a circle can be inscribed knowing the lengths of two neighbouring sides AB and AD and the angles at vertices B and D.

17.6. Construct triangle ABC knowing a, b and the difference of angles $\angle A - \angle B$.

17.7. Construct triangle ABC given its side c, height h_c and the difference of angles $\angle A - \angle B$.

17.8. Construct triangle ABC given a) c, $a-b$ $(a>b)$ and angle $\angle C$; b) c, $a+b$ and angle $\angle C$.

17.9. Given line l and points A and B on one side of it. Construct point X on l such that $AX + XB = a$, where a is given.

17.10. Given acute angle $\angle MON$ and points A and B inside it. Find point X on leg OM such that triangle XYZ, where Y and Z are the intersection points of lines XA and XB with ON, were isosceles, i.e., $XY = XZ$.

17.11. Given line MN and two points A and B on one side of it. Construct point X on MN such that $\angle AXM = 2\angle BXN$.

* * *

17.12. Given three lines l_1, l_2 and l_3 intersecting at one point and point A_1 on l_1. Construct triangle ABC so that A_1 is the midpoint of its side BC and lines l_1, l_2 and l_3 are the midperpendiculars to the sides.

17.13. Construct triangle ABC given points A, B and the line on which the bisector of angle $\angle C$ lies.

17.14. Given three lines l_1, l_2 and l_3 intersecting at one point and point A on line l_1. Construct triangle ABC so that A is its vertex and the bisectors of the triangle lie on lines l_1, l_2 and l_3.

17.15. Construct a triangle given the midpoints of two of its sides and the line that contains the bisector drawn to one of these sides.

§3. Inequalities and extremals

17.16. On the bisector of the exterior angle $\angle C$ of triangle ABC point M distinct from C is taken. Prove that $MA + MB > CA + CB$.

17.17. In triangle ABC median AM is drawn. Prove that $2AM \geq (b+c)\cos(\frac{1}{2}\alpha)$.

17.18. The inscribed circle of triangle ABC is tangent to sides AC and BC at points B_1 and A_1. Prove that if $AC > BC$, then $AA_1 > BB_1$.

17.19. Prove that the area of any convex quadrilateral does not exceed a half-sum of the products of opposite sides.

17.20. Given line l and two points A and B on one side of it, find point X on line l such that the length of segment AXB of the broken line was minimal.

17.21. Inscribe a triangle of the least perimeter in a given acute triangle.

§4. Compositions of symmetries

17.22. a) Lines l_1 and l_2 are parallel. Prove that $S_{l_1} \circ S_{l_2} = T_{2\mathbf{a}}$, where $T_{\mathbf{a}}$ is the parallel translation that sends l_1 to l_2 and such that $\mathbf{a} \perp l_1$.

b) Lines l_1 and l_2 intersect at point O. Prove that $S_{l_2} \circ S_{l_1} = R_O^{2\alpha}$, where R_O^{α} is the rotation about O through the angle of α that sends l_1 to l_2.

17.23. On the plane, there are given three lines a, b, c. Let $T = S_a \circ S_b \circ S_c$. Prove that $T \circ T$ is a parallel translation (or the identity map).

17.24. Let $l_3 = S_{l_1}(l_2)$. Prove that $S_{l_3} = S_{l_1} \circ S_{l_2} \circ S_{l_1}$.

17.25. The inscribed circle is tangent to the sides of triangle ABC at points A_1, B_1 and C_1. Points A_2, B_2 and C_2 are symmetric to these points through the bisectors of the corresponding angles of the triangle. Prove that $A_2B_2 \parallel AB$ and lines AA_2, BB_2 and CC_2 intersect at one point.

17.26. Two lines intersect at an angle of γ. A grasshopper hops from one line to another one; the length of each jump is equal to 1 m and the grasshopper does not jump backwards whenever possible. Prove that the sequence of jumps is periodic if and only if γ/π is a rational number.

17.27. a) Given a circle and n lines. Inscribe into the circle an n-gon whose sides are parallel to given lines.

b) n lines go through the center O of a circle. Construct an n-gon circumscribed about this circle such that the vertices of the n-gon belong to these lines.

17.28. Given n lines, construct an n-gon for which these lines are a) the midperpendiculars to the sides; b) the bisectors of the inner or outer angles at the vertices.

17.29. Given a circle, a point and n lines. Into the circle inscribe an n-gon one of whose sides passes through the given point and the other sides are parallel to the given lines.

§5. Properties of symmetries and axes of symmetries

17.30. Point A lies at the distance of 50 cm from the center of the disk of radius 1 cm. It is allowed to reflect point A symmetrically through any line intersecting the disk. Prove that a) after 25 reflexions point A can be driven inside the given circle; b) it is impossible to perform this in 24 reflexions.

17.31. On a circle with center O points $A_1, \dots, A_n$ which divide the circle into equal archs and a point X are given. Prove that the points symmetric to X through lines $OA_1, \dots, OA_n$ constitute a regular polygon.

17.32. Prove that if a planar figure has exactly two axes of symmetry, then these axes are perpendicular to each other.

17.33. Prove that if a polygon has several (more than 2) axes of symmetry, then all of them intersect at one point.

17.34. Prove that if a polygon has an even number of axes of symmetry, then it has a center of symmetry.

§6. Chasles's theorem

A transformation which preserves distances between points (i.e., such that if A' and B' are the images of points A and B, respectively, then $A'B' = AB$) is called a *movement.* A movement of the plane that preserves 3 points which do not belong to one line preserves all the other points.

17.35. Prove that any movement of the plane is a composition of not more than three symmetries through lines.

A movement which is the composition of an even number of symmetries through lines is called a *first type movement* or a *movement that preserves the orientation of the plane.*

A movement which is the composition of an odd number of symmetries through lines is called a *second type movement* or a *movement inversing the orientation of the plane.*

We will not prove that *the composition of an odd number of symmetries through lines is impossible to represent in the form of the composition of an odd number of symmetries through lines and the other way round* because this fact, though true, is beyond the scope of our book.

17.36. Prove that any first type movement is either a rotation or a parallel translation.

The composition of a symmetry through line l and the translation by a vector parallel to l (this vector might be the zero one) is called a *transvection.*

17.37. Prove that any second type movement is a transvection.

Problems for independent study

17.38. Given a nonconvex quadrilateral of perimeter P. Prove that there exists a convex quadrilateral of the same perimeter but of greater area.

17.39. Can a bounded figure have a center of symmetry and exactly one axis of symmetry?

17.40. Point M belongs to the circumscribed circle of triangle ABC. Prove that the lines symmetric to the lines AM, BM and CM through the bisectors of angles $\angle A$, $\angle B$ and $\angle C$ are parallel to each other.

17.41. The vertices of a convex quadrilateral belong to different sides of a square. Prove that the perimeter of this quadrilateral is not shorter than $2\sqrt{2}a$, where a is the length of the square's side.

17.42. A ball lies on a rectangular billiard table. Construct a trajectory traversing along which the ball would return to the initial position after one reflexion from each side of the table.

Solutions

17.1. Denote the points symmetric to points C and D through line AB by C' and D', respectively. Since $\angle C'MD = 90°$, it follows that $CM^2 + MD^2 = C'M^2 + MD^2 = C'D^2$. Since $\angle C'CD = 45°$, chord $C'D$ is of constant length.

17.2. In circle S, draw the diameter which is at the same time the axis of symmetry of circles S_1 and S_2. Let points C' and B_2' be symmetric to points C and B_2 through this diameter: see Fig. 17.

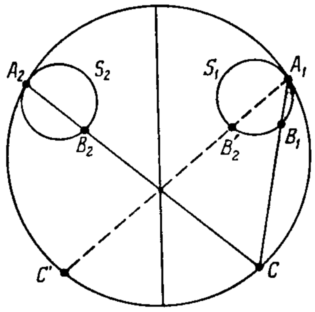


Figure 159 (Sol. 17.2)

Circles S_1 and S are homothetic with the center of homothety at point A_1; let this homothety send line B_1B_2' into line CC'. Therefore, these lines are parallel to each other. It is also clear that $B_2B_2' \parallel CC'$. Therefore, points B_1, B_2' and B_2 belong to one line and this line is parallel to line CC'.

17.3. Let the line symmetric to line A_1B_1 through line AB intersect sides CA and CB (or their extensions) at points A_2 and B_2, respectively. Since $\angle A_1AM = \angle B_2BM$ and $\angle A_1MA = \angle B_2MB$, it follows that $A_1AM \sim B_2BM$, i.e., $A_1A : A_1M = B_2B : B_2M$. Moreover, since MB is a bisector in triangle B_1MB_2, it follows that $B_2B : B_2M = B_1B : B_1M$.

17.4. Suppose that quadrilateral $ABCD$ is constructed. Let, for definiteness sake, $AD > AB$. Denote by B' the point symmetric to B through diagonal AC. Point B' belongs to side AD and $B'D = AD - AB$. In triangle $B'CD$, the lengths of all the sides are known: $B'D = AD - AB$ and $B'C = BC$. Constructing triangle $B'CD$ on the extension of side $B'D$ beyond B' let us construct point A.

Further construction is obvious.

17.5. Suppose that quadrilateral $ABCD$ is constructed. For definiteness sake, assume that $AD > AB$. Let O be the center of the circumscribed circle; let point D' be symmetric to D through line AO; let A' be the intersection point of lines AO and DC; let C' be the intersection point of lines BC and $A'D'$ (Fig. 18).

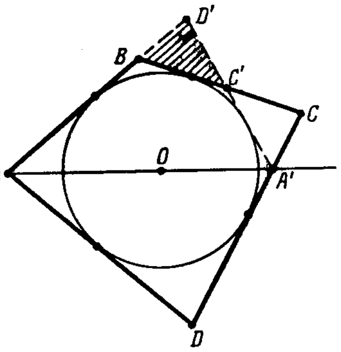


Figure 160 (Sol. 17.5)

In triangle $BC'D'$, side BD' and adjacent angles are known: $\angle D'BC' = 180^\circ - \angle B$ and $\angle BD'C' = \angle D$. Let us construct triangle $BC'D'$ given these elements. Since $AD' = AD$, we can construct point A. Further, let us construct O — the intersection point of bisectors of angles ABC' and $BD'C'$. Knowing the position of O we can construct point D and the inscribed circle. Point C is the intersection point of line BC' and the tangent to the circle drawn from D.

17.6. Suppose that triangle ABC is constructed. Let C' be the point symmetric to C through the midperpendicular to segment AB. In triangle ACC' there are known $AC = b$, $AC' = a$ and $\angle CAC' = \angle A - \angle B$. Therefore, the triangle can be constructed. Point B is symmetric to A through the midperpendicular to segment CC'.

17.7. Suppose that triangle ABC is constructed. Denote by C' the point symmetric to C through the midperpendicular to side AB and by B' the point symmetric to B through line CC'. For definiteness, let us assume that $AC < BC$. Then

$$\angle ACB' = \angle ACC' + \angle C'CB = 180^\circ - \angle A + \angle C'CB = 180^\circ - (\angle A - \angle B)$$

i.e., angle $\angle ACB'$ is known.

Triangle ABB' can be constructed because $AB = c$, $BB' = 2h_c$ and $\angle ABB' = 90^\circ$. Point C is the intersection point of the midperpendicular to segment BB' and the arc of the circle whose points serve as vertices of angles of value $180^\circ - (\angle A - \angle B)$ that subtend segment AB'.

17.8. a) Suppose triangle ABC is constructed. Let C' be the point symmetric to A through the bisector of angle $\angle C$. Then

$$\angle BC'A = 180^\circ - \angle AC'C = 180^\circ - \frac{1}{2}(180^\circ - \angle C) = 90^\circ + \frac{1}{2}\angle C$$

and $BC' = a - b$.

In triangle ABC', there are known $AB = c$, $BC' = a - b$ and $\angle C' = 90^\circ + \frac{1}{2}\angle C$. Since $\angle C' > 90^\circ$, triangle ABC' is uniquely constructed from these elements. Point C is the intersection point of the midperpendicular to segment AC' with line BC'.

b) The solution is similar to that of heading a). For C' we should take the point symmetric to A through the bisector of the outer angle $\angle C$ in triangle ABC.

Since $\angle AC'B = \frac{1}{2}\angle C < 90^\circ$, the problem can have two solutions.

17.9. Let S be the circle of radius a centered at B, let S' be the circle of radius AX with center X and A' the point symmetric to A through line l. Then circle S' is tangent to circle S and point A' belongs to circle S'. It remains to draw circle S' through the given points A and A' tangent to the given circle S and find its center X, cf. Problem 8.56 b).

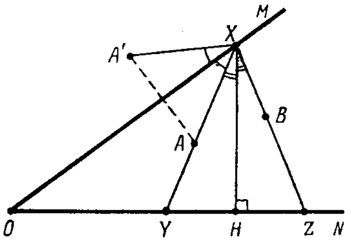


Figure 161 (Sol. 17.10)

17.10. Let the projection of point A to line ON be closer to point O than the projection of point B. Suppose that the isosceles triangle XYZ is constructed. Let us consider point A' symmetric to point A through line OM. Let us drop perpendicular XH from point X to line ON (Fig. 19). Since

$$\angle A'XB = \angle A'XO + \angle OXA + \angle YXH + \angle HXZ = \\ 2\angle OXY + 2\angle YXH = 2\angle OXH = 180^\circ - 2\angle MON,$$

angle $\angle A'XB$ is known. Point X is the intersection point of line OM and the arc whose points serve as vertices of angles of $180^\circ - 2\angle MON$ that subtend $A'B$. In addition, the projection of X onto ON must lie between the projections of A and B.

Conversely, if $\angle A'XB = 180^\circ - \angle MON$ and the projection of X to line ON lies between the projections of A and B, then triangle XYZ is an isosceles one.

17.11. Suppose that point X is constructed. Let B' be the point symmetric to point B through line MN; the circle of radius AB' with center B' intersects line MN at point A'. Then ray $B'X$ is the bisector of angle $\angle AB'A'$. It follows that X is the intersection point of lines $B'O$ and MN, where O is the midpoint of segment AA'.

17.12. Through point A_1 draw line BC perpendicular to line l_1. Vertex A of triangle ABC to be found is the intersection point of lines symmetric to line BC through lines l_2 and l_3.

17.13. Let point A' be symmetric to A through the bisector of angle $\angle C$. Then C is the intersection point of line $A'B$ and the line on which the bisector of angle $\angle C$ lies.

17.14. Let A_2 and A_3 be points symmetric to A through lines l_2 and l_3, respectively. Then points A_2 and A_3 belong to line BC. Therefore, points B and C are the intersection points of line A_2A_3 with lines l_2 and l_3, respectively.

17.15. Suppose that triangle ABC is constructed and N is the midpoint of AC, M the midpoint of BC and the bisector of angle $\angle A$ lies on the given line, l. Let us construct point N' symmetric to N through line l. Line BA passes through point N' and is parallel to MN. In this way we find vertex A and line BA. Having drawn line AN, we get line AC. It remains to construct a segment whose endpoints belong to the legs of angle $\angle BAC$ and whose midpoint is M, cf. the solution of Problem 16.15.

17.16. Let points A' and B' be symmetric to A and B, respectively, through line CM. Then $AM + MB = A'M + MB > A'B = A'C + CB = AC + CB$.

17.17. Let points B', C' and M' be symmetric to points B, C and M through the bisector of the outer angle at vertex A. Then

$$AM + AM' = MM' = \frac{1}{2}(BB' + CC') = (b + c)\sin(90^\circ - \frac{1}{2}\alpha) = (b + c)\cos(\frac{1}{2}\alpha).$$

17.18. Let point B' be symmetric to B through the bisector of angle $\angle ACB$. Then $B'A_1 = BB_1$, i.e., it remains to verify that $B'A_1 < AA_1$. To this end it suffices to notice that $\angle AB'A_1 > \angle AB'B > 90^\circ$.

17.19. Let D' be the point symmetric to D through the midperpendicular to segment AC. Then

$$S_{ABCD} = S_{ABCD'} = S_{BAD'} + S_{BCD'} \leq \frac{1}{2}AB \cdot AD' + \frac{1}{2}BC \cdot CD' = \frac{1}{2}(AB \cdot CD + BC \cdot AD).$$

17.20. Let point A' be symmetric to A through line l. Let X be a point on line l. Then $AX + XB = A'X + XB \geq A'B$ and the equality is attained only if X belongs to segment $A'B$. Therefore, the point to be found is the intersection point of line l with segment $A'B$.

17.21. Let PQR be the triangle determined by the bases of the heights of triangle ABC and let $P'Q'R'$ be any other triangle inscribed in triangle ABC. Further, let points P_1 and P_2 (respectively P_1' and P_2') be symmetric to point P (resp. P') through lines AB and AC, respectively (Fig. 20).

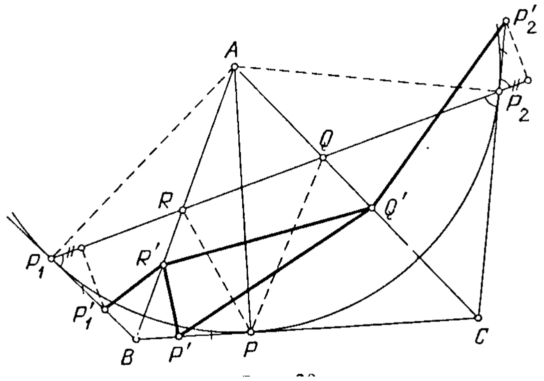


Figure 162 (Sol. 17.21)

Points Q and R belong to segment P_1P_2 (see Problem 1.57) and, therefore, the perimeter of triangle PQR is equal to the length of segment P_1P_2. The perimeter of triangle $P'Q'R'$ is, however, equal to the length of the broken segment $P_1'R'Q'P_2'$, i.e., it is not shorter than the length of segment $P_1'P_2'$. It remains to notice that $(P_1'P_2')^2 = P_1P_2^2 + 4d^2$, where d is the distance from point P_1' to line P_1P_2.

17.22. Let X be an arbitrary point, $X_1 = S_{l_1}(X)$ and $X_2 = S_{l_2}(X_1)$.

a) On line l_1, select an arbitrary point O and consider a coordinate system with O as the origin and the absciss axis directed along line l_1. Line l_2 is given in this coordinate system by the equation $y = a$. Let y, y_1 and y_2 be ordinates of points X, X_1 and X_2, respectively. It is clear that $y_1 = -y$ and $y_2 = (a - y_1) + a = y + 2a$. Since points X, X_1 and X_2 have identical abscisses, it follows that $X_2 = T_{2\mathbf{a}}(X)$, where $T_{\mathbf{a}}$ is the translation that sends l_1 to l_2, and $\mathbf{a} \perp l_1$.

b) Consider a coordinate system with O as the origin and the absciss axis directed along line l_1. Let the angle of rotation from line l_1 to l_2 in this coordinate system be equal to α and the angles of rotation from the absciss axes to rays OX, OX_1 and OX_2 be equal to φ, φ_1 and φ_2, respectively. Clearly, $\varphi_1 = -\varphi$ and $\varphi_2 = (\alpha - \varphi_1) + \alpha = \varphi + 2\alpha$. Since $OX = OX_1 = OX_2$, it follows that $X_2 = R_O^{2\alpha}(X)$, where R_O^{α} is the translation that sends l_1 to l_2.

17.23. Let us represent $T \circ T$ as the composition of three transformations:

$$T \circ T = (S_a \circ S_b \circ S_c) \circ (S_a \circ S_b \circ S_c) = (S_a \circ S_b) \circ (S_c \circ S_a) \circ (S_b \circ S_c).$$

Here $S_a \circ S_b$, $S_c \circ S_a$ and $S_b \circ S_c$ are rotations through the angles of $2\angle(b, a)$, $2\angle(a, c)$ and $2\angle(c, b)$, respectively. The sum of the angles of the rotations is equal to

$$2(\angle(b, a) + \angle(a, c) + \angle(c, b)) = 2\angle(b, b) = 0^\circ$$

and this value is determined up to $2 \cdot 180^\circ = 360^\circ$. It follows that this composition of rotations is a parallel translation, cf. Problem 18.33.

17.24. If points X and Y are symmetric through line l_3, then points $S_{l_1}(X)$ and $S_{l_1}(Y)$ are symmetric through line l_2, i.e., $S_{l_1}(X) = S_{l_2} \circ S_{l_1}(Y)$. It follows that $S_{l_1} \circ S_{l_3} = S_{l_2} \circ S_{l_1}$ and $S_{l_3} = S_{l_1} \circ S_{l_2} \circ S_{l_1}$.

17.25. Let O be the center of the inscribed circle; let a and b be lines OA and OB. Then $S_a \circ S_b(C_1) = S_a(A_1) = A_2$ and $S_b \circ S_a(C_1) = S_b(B_1) = B_2$. Points A_2 and B_2 are obtained from point C_1 by rotations with center O through opposite angles and, therefore, $A_2B_2 \parallel AB$.

Similar arguments show that the sides of triangles ABC and $A_2B_2C_2$ are parallel and, therefore, these triangles are homothetic. Lines AA_2, BB_2 and CC_2 pass through the center of homothety which sends triangle ABC to $A_2B_2C_2$. Notice that this homothety sends the circumscribed circle of triangle ABC into the inscribed circle, i.e., the center of homothety belongs to the line that connects the centers of these circles.

17.26. For every jump vector there are precisely two positions of a grasshopper for which the jump is given by this vector. Therefore, a sequence of jumps is periodic if and only if there exists but a finite number of distinct jump vectors.

Let $\mathbf{a}_1$ be the jump vector of the grasshopper from line l_2 to line l_1; let $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$, … be vectors of the successive jumps. Then $\mathbf{a}_2 = S_{l_2}(\mathbf{a}_1)$, $\mathbf{a}_3 = S_{l_1}(\mathbf{a}_2)$, $\mathbf{a}_4 = S_{l_2}(\mathbf{a}_3)$, … Since the composition $S_{l_1} \circ S_{l_2}$ is a rotation through an angle of 2γ (or $2\pi - 2\gamma$), it follows that vectors $\mathbf{a}_3$, $\mathbf{a}_5$, $\mathbf{a}_7$, … are obtained from $\mathbf{a}_1$ by rotations through angles of 2γ, 4γ, 6γ, … (or through angles of $2(\pi - \gamma)$, $4(\pi - \gamma)$, $6(\pi - \gamma)$, …). Therefore, the set $\mathbf{a}_1$, $\mathbf{a}_3$, $\mathbf{a}_5$, … contains a finite number of distinct vectors if and only if γ/π is a rational number. The set $\mathbf{a}_2$, $\mathbf{a}_4$, $\mathbf{a}_6$, … is similarly considered.

17.27. a) Suppose polygon $A_1A_2 \dots A_n$ is constructed. Let us draw through the center O of the circle the midperpendiculars l_1, l_2, … , l_n to chords A_1A_2, A_2A_3, … , A_nA_1, respectively. Lines l_1, … , l_n are known since they pass through O and are perpendicular to the given lines. Moreover, $A_2 = S_{l_1}(A_1)$, $A_3 = S_{l_2}(A_2)$, … , $A_1 = S_{l_n}(A_n)$, i.e., point A_1 is a fixed point of the composition of symmetries $S_{l_n} \circ \dots \circ S_{l_1}$. For n odd there are precisely two fixed points on the circle; for n even there are either no fixed points or all the points are fixed.

b) Suppose the desired polygon $A_1 \dots A_n$ is constructed. Consider polygon $B_1 \dots B_n$ formed by the tangent points of the circumscribed polygon with the circle. The sides of polygon $B_1 \dots B_n$ are perpendicular to the given lines, i.e., they have prescribed directions and, therefore, the polygon can be constructed (see heading a)); it remains to draw the tangents to the circle at points $B_1, \dots, B_n$.

17.28. Consider the composition of consecutive symmetries through given lines $l_1, \dots, l_n$. In heading a) for vertex A_1 of the desired n-gon we have to take a fixed point of this composition, and in heading b) for line A_1A_n we have to take the(a) fixed line.

17.29. The consecutive symmetries through lines $l_1, \dots, l_{n-1}$ perpendicular to given lines and passing through the center of the circle send vertex A_1 of the desired polygon to vertex A_n.

If n is odd, then the composition of these symmetries is a rotation through a known angle and, therefore, we have to draw through point M chord A_1A_n of known length.

If n is even, then the considered composition is a symmetry through a line and, therefore, from M we have to drop perpendicular to this line.

17.30. Let O be the center of the given disk, D_R the disk of radius R with center O. Let us prove that the symmetries through the lines passing through D_1 send the set of images of points of D_R into disk D_{R+2}. Indeed, the images of point O under the indicated symmetries fill in disk D_2 and the disks of radius R with centers in D_2 fill in disk D_{R+2}.

It follows that after n reflexions we can obtain from points of D_1 any point of D_{2n+1} and only them. It remains to notice that point A can be “herded” inside D_R after n reflexions if and only if we can transform any point of D_R into A after n reflexions.

17.31. Denote symmetries through lines $OA_1, \dots, OA_n$ by $S_1, \dots, S_n$, respectively. Let $X_k = S_k(X)$ for $k = 1, \dots, n$. We have to prove that under a rotation through point O the system of points $X_1, \dots, X_n$ turns into itself. Clearly,

$$S_{k+1} \circ S_k(X_k) = S_{k+1} \circ S_k \circ S_k(X) = X_{k+1}.$$

Transformations $S_{k+1} \circ S_k$ are rotations about O through an angle of $\frac{4\pi}{n}$, see Problem 17.22 b).

Remark. For n even we get an $\frac{n}{2}$-gon.

17.32. Let lines l_1 and l_2 be axes of symmetry of a plane figure. This means that if point X belongs to the figure, then points $S_{l_1}(X)$ and $S_{l_2}(X)$ also belong to the figure. Consider line $l_3 = S_{l_1}(l_2)$. Thanks to Problem 17.24 $S_{l_3}(X) = S_{l_1} \circ S_{l_2} \circ S_{l_1}(X)$ and, therefore, l_3 is also an axis of symmetry.

If the figure has precisely two axes of symmetry, then either $l_3 = l_1$ or $l_3 = l_2$. Clearly, $l_3 \neq l_1$ and, therefore, $l_3 = l_2$ i.e., line l_2 is perpendicular to line l_1.

17.33. Suppose that the polygon has three axes of symmetry which do not intersect at one point, i.e., they form a triangle. Let X be the point of the polygon most distant from an inner point M of this triangle. Points X and M lie on one side of one of the considered axes of symmetry, l. If X' is the point symmetric to X through l, then $M'X > MX$ and point X' is distant from M further than X. The obtained contradiction implies that all the axes of symmetry of a polygon intersect at one point.

17.34. All the axes of symmetry pass through one point O (Problem 17.33). If l_1 and l_2 are axes of symmetry, then $l_3 = S_{l_1}(l_2)$ is also an axis of symmetry, see Problem 17.24.

Select one of the axes of symmetry l of our polygon. The odd axes of symmetry are divided into pairs of lines symmetric through l. If line l_1 perpendicular to l and passing through O is not an axis of symmetry, then there is an odd number of axes of symmetry. Therefore, l_1 is an axis of symmetry. Clearly, $S_{l_1} \circ S_l = R_O^{180^\circ}$ is a central symmetry i.e., O is the center of symmetry.

17.35. Let F be a movement sending point A into A' and such that A and A' are distinct; S the symmetry through the midperpendicular l to segment AA'. Then $S \circ F(A) = A$, i.e., A is a fixed point of $S \circ F$. Moreover, if X is a fixed point of transformation F, then $AX = A'X$, i.e., point X belongs to line l; hence, X is a fixed point of $S \circ F$. Thus, point A and all the fixed points of F are fixed points of the transformation $S \circ F$.

Take points A, B and C not on one line and consider their images under the given movement G. We can construct transformations S_1, S_2 and S_3 which are either symmetries through lines or identity transformations such that $S_3 \circ S_2 \circ S_1 \circ G$ preserves points A, B and C, i.e., is the identity transformation E. Multiplying the equality $S_3 \circ S_2 \circ S_1 \circ G = E$ from the left consecutively by S_3, S_2 and S_1 and taking into account that $S_i \circ S_i = E$ we get $G = S_1 \circ S_2 \circ S_3$.

17.36. Thanks to Problem 17.35 any first type movement is a composition of two symmetries through lines. It remains to make use of the result of Problem 17.22.

17.37. By Problem 17.35 any second type movement can be represented in the form $S_3 \circ S_2 \circ S_1$, where S_1, S_2 and S_3 are symmetries through lines l_1, l_2 and l_3, respectively. First, suppose that the lines l_2 and l_3 are not parallel. Then under the rotation of the lines l_2 and l_3 about their intersection point through any angle the composition $S_3 \circ S_2$ does not change (see Problem 17.22 b)), consequently, we can assume that $l_2 \perp l_1$. It remains to rotate lines l_1 and l_2 about their intersection point so that line l_2 became parallel to line l_3.

Now, suppose that $l_2 \parallel l_3$. If line l_1 is not parallel to these lines, then it is possible to rotate l_1 and l_2 about their intersection point so that lines l_2 and l_3 become nonparallel. If $l_1 \parallel l_2$, then it is possible to perform a parallel transport of l_1 and l_2 so that lines l_2 and l_3 coincide.

Chapter 18. ROTATIONS

Background

1. We will not give a rigorous definition of a rotation. To solve the problems it suffices to have the following idea on the notion of the rotation: a *rotation with center* O (or *about the point* O) through an angle of φ is the transformation of the plane which sends point X into point X' such that:

a) $OX' = OX$;

b) the angle from vector $\overrightarrow{OX}$ to vector $\overrightarrow{OX'}$ is equal to φ.

2. In this chapter we make use of the following notations for the transformations and their compositions:

$T_{\mathbf{a}}$ is a translation by vector $\mathbf{a}$;

S_O is the symmetry through point O;

S_l is the symmetry through line l;

R_O^{φ} is the rotation with center O through an angle of φ;

$F \circ G$ is the composition of transformations F and G defined as $(F \circ G)(X) = F(G(X))$.

3. The problems solvable with the help of rotations can be divided into two big classes: problems which do not use the properties of compositions of rotations and properties which make use of these properties. To solve the problems which make use of the properties of the compositions of rotations the following result of Problem 18.33 is handy: $R_B^{\beta} \circ R_A^{\alpha} = R_C^{\gamma}$, where $\gamma = \alpha + \beta$ and $\angle BAC = \frac{1}{2}\alpha$, $\angle ABC = \frac{1}{2}\beta$.

Introductory problems

1. Prove that any rotation sends any circle into a circle.

2. Prove that a convex n-gon is a regular one if and only if it turns into itself under the rotation through an angle of $\frac{360^\circ}{n}$ about a point.

3. Prove that triangle ABC is an equilateral one if and only if under the rotation through 60° (either clockwise or counterclockwise) about point A vertex B turns into vertex C.

4. Prove that the midpoints of the sides of a regular polygon determine a regular polygon.

5. Through the center of a square two perpendicular lines are drawn. Prove that their intersection points with the sides of the square determine a square.

§1. Rotation by 90°

18.1. On sides BC and CD of square $ABCD$ points M and K, respectively, are taken so that $\angle BAM = \angle MAK$. Prove that $BM + KD = AK$.

18.2. In triangle ABC median CM and height CH are drawn. Through an arbitrary point P of the plane in which ABC lies the lines are drawn perpendicularly to CA, CM and CB. They intersect CH at points A_1, M_1 and B_1, respectively. Prove that $A_1M_1 = B_1M_1$.

18.3. Two squares $BCDA$ and $BKMN$ have a common vertex B. Prove that median BE of triangle ABK and height BF of triangle CBH belong to one line.

The vertices of each square are counted clockwise.

18.4. Inside square $A_1A_2A_3A_4$ point P is taken. From vertex A_1 we drop the perpendicular on A_2P; from A_2 on A_3P; from A_3 on A_4P and from A_4 on A_1P. Prove that all four perpendiculars (or their extensions) intersect at one point.

18.5. On sides CB and CD of square $ABCD$ points M and K are taken so that the perimeter of triangle CMK is equal to the doubled length of the square's side. Find the value of angle $\angle MAK$.

18.6. On the plane three squares (with same orientation) are given: $ABCD$, $AB_1C_1D_1$ and $A_2B_2CD_2$; the first square has common vertices A and C with the two other squares. Prove that median BM of triangle BB_1B_2 is perpendicular to segment D_1D_2.

18.7. Triangle ABC is given. On its sides AB and BC squares $ABMN$ and $BCPQ$ are constructed outwards. Prove that the centers of these squares and the midpoints of segments MQ and AC form a square.

18.8. A parallelogram is circumscribed about a square. Prove that the perpendiculars dropped from the vertices of the parallelograms to the sides of the square form a square.

§2. Rotation by 60°

18.9. On segment AE, on one side of it, equilateral triangles ABC and CDE are constructed; M and P are the midpoints of segments AD and BE. Prove that triangle CPM is an equilateral one.

18.10. Given three parallel lines. Construct an equilateral triangle so that its vertices belong to the given lines.

18.11. Geven a square, consider all possible equilateral triangles PKM with fixed vertex P and vertex K belonging to the square. Find the locus of vertices M.

18.12. On sides BC and CD of parallelogram $ABCD$, equilateral triangles BCP and CDQ are constructed outwards. Prove that triangle APQ is an equilateral one.

18.13. Point M belongs to arc $\smile AB$ of the circle circumscribed about an equilateral triangle ABC. Prove that $MC = MA + MB$.

18.14. Find the locus of points M that lie inside equilateral triangle ABC and such that $MA^2 = MB^2 + MC^2$.

18.15. Hexagon $ABCDEF$ is a regular one, K and M are the midpoints of segments BD and EF, respectively. Prove that triangle AMK is an equilateral one.

18.16. Let M and N be the midpoints of sides CD and DE, respectively, of regular hexagon $ABCDEF$, let P be the intersection point of segments AM and BN.

a) Find the value of the angle between lines AM and BN.

b) Prove that $S_{ABP} = S_{MDNP}$.

18.17. On sides AB and BC of an equilateral triangle ABC points M and N are taken so that $MN \parallel AC$; let E be the midpoint of segment AN and D the center of mass of triangle BMN. Find the values of the angles of triangle CDE.

18.18. On the sides of triangle ABC equilateral triangles ABC_1, AB_1C and A_1BC are constructed outwards. Let P and Q be the midpoints of segments A_1B_1 and A_1C_1. Prove that triangle APQ is an equilateral one.

18.19. On sides AB and AC of triangle ABC equilateral triangles ABC' and $AB'C$ are constructed outwards. Point M divides side BC in the ratio of $BM : MC = 3 : 1$; points K and L are the midpoints of sides AC' and $B'C$, respectively. Prove that the angles of triangle KLM are equal to 30°, 60° and 90°.

18.20. Equilateral triangles ABC, CDE, EHK (vertices are circumvent counterclockwise) are placed on the plane so that $\overrightarrow{AD} = \overrightarrow{DK}$. Prove that triangle BHD is also an equilateral one.

18.21. a) Inside an acute triangle find a point the sum of distances from which to the vertices is the least one.

b) Inside triangle ABC all the angles of which are smaller than 120° a point O is taken; it serves as vertex of the angles of 120° that subtend the sides. Prove that the sum of distances from O to the vertices is equal to $\frac{1}{2}(a^2+b^2+c^2)+2\sqrt{3}S$.

18.22. Hexagon $ABCDEF$ is inscribed in a circle of radius R and $AB=CD=EF=R$. Prove that the midpoints of sides BC, DE and FA determine an equilateral triangle.

18.23. On sides of a convex centrally symmetric hexagon $ABCDEF$ equilateral triangles are constructed outwards. Prove that the midpoints of the segments connecting the vertices of neighbouring triangles determine a regular hexagon.

§3. Rotations through arbitrary angles

18.24. Given points A and B and circle S construct points C and D on S so that $AC \parallel BD$ and the value of arc $\smile CD$ is a given quantity α.

18.25. A rotation with center O transforms line l_1 into line l_2 and point A_1 on l_1 into point A_2. Prove that the intersection point of lines l_1 and l_2 belongs to the circle circumscribed about triangle A_1OA_2.

18.26. Two equal letters Γ lie on the plane. Denote by A and A' the endpoints of the shorter segments of these letters. Points $A_1, \dots, A_{n-1}$ and $A'_1, \dots, A'_{n-1}$ divide the longer segments into n equal parts (the division points are numbered starting from the outer endpoints of longer segments). Lines AA_i and $A'A'_i$ intersect at point X_i. Prove that points $X_1, \dots, X_{n-1}$ determine a convex polygon.

18.27. Along two lines that intersect at point P two points are moving with the same speed: point A along one line and point B along the other one. They pass P not simultaneously. Prove that at all times the circle circumscribed about triangle ABP passes through a fixed point distinct from P.

18.28. Triangle $A_1B_1C_1$ is obtained from triangle ABC by a rotation through an angle of α ($\alpha < 180^\circ$) about the center of its circumscribed circle. Prove that the intersection points of sides AB and A_1B_1, BC and B_1C_1, CA and C_1A_1 (or their extensions) are the vertices of a triangle similar to triangle ABC.

18.29. Given triangle ABC construct a line which divides the area and perimeter of triangle ABC in halves.

18.30. On vectors $\overrightarrow{A_iB_i}$, where $i=1,\dots,k$ similarly oriented regular n-gons $A_iB_iC_iD_i\dots$ ($n \geq 4$) are constructed (a given vector serving as a side). Prove that k-gons $C_1\dots C_k$ and $D_1\dots D_k$ are regular and similarly oriented ones if and only if the k-gons $A_1\dots A_k$ and $B_1\dots B_k$ are regular and similarly oriented ones.

18.31. Consider a triangle. Consider three lines symmetric through the triangles sides to an arbitrary line passing through the intersection point of the triangle's heights. Prove that the three lines intersect at one point.

18.32. A lion runs over the arena of a circus which is a disk of radius 10 m. Moving along a broken line the lion covered 30 km. Prove that the sum of all the angles of his turns is not less than 2998 radian.

§4. Compositions of rotations

18.33. Prove that the composition of two rotations through angles whose sum is not proportional to 360° is a rotation. In which point is its center and what is the angle of the rotation equal to? Investigate also the case when the sum of the angles of rotations is a multiple of 360°.

* * *

18.34. On the sides of an arbitrary convex quadrilateral squares are constructed outwards. Prove that the segments that connect the centers of opposite squares have equal lengths and are perpendicular to each other.

18.35. On the sides of a parallelogram squares are constructed outwards. Prove that their centers form a square.

18.36. On sides of triangle ABC squares with centers P, Q and R are constructed outwards. On the sides of triangle PQR squares are constructed inwards. Prove that their centers are the midpoints of the sides of triangle ABC.

18.37. Inside a convex quadrilateral $ABCD$ isosceles right triangles ABO_1, BCO_2, CDO_3 and DAO_4 are constructed. Prove that if $O_1 = O_3$, then $O_2 = O_4$.

* * *

18.38. a) On the sides of an arbitrary triangle equilateral triangles are constructed outwards. Prove that their centers form an equilateral triangle.

b) Prove a similar statement for triangles constructed inwards.

c) Prove that the difference of the areas of equilateral triangles obtained in headings a) and b) is equal to the area of the initial triangle.

18.39. On sides of triangle ABC equilateral triangles $A'BC$ and $B'AC$ are constructed outwards and $C'AB$ inwards; M is the center of mass of triangle $C'AB$. Prove that $A'B'M$ is an isosceles triangle such that $\angle A'MB' = 120^\circ$.

18.40. Let angles α, β, γ be such that $0 < \alpha, \beta, \gamma < \pi$ and $\alpha + \beta + \gamma = \pi$. Prove that if the composition of rotations $R_C^{2\gamma} \circ R_B^{2\beta} \circ R_A^{2\alpha}$ is the identity transformation, then the angles of triangle ABC are equal to α, β, γ.

18.41. Construct an n-gon given n points which are the vertices of isosceles triangles constructed on the sides of this n-gon and such that the angles of these triangles at the vertices are equal to $\alpha_1, \dots, \alpha_n$.

18.42. On the sides of an arbitrary triangle ABC isosceles triangles $A'BC$, $AB'C$ and ABC' are constructed outwards with angles α, β and γ at vertices A', B' and C', respectively, such that $\alpha + \beta + \gamma = 2\pi$. Prove that the angles of triangle $A'B'C'$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$ and $\frac{1}{2}\gamma$.

18.43. Let AKL and AMN be similar isosceles triangles with vertex A and angle α at the vertex; GNK and $G'LM$ similar isosceles triangles with angle $\pi - \alpha$ at the vertex. Prove that $G = G'$. (All the triangles are oriented ones.)

18.44. On sides AB, BC and CA of triangle ABC points P, Q and R, respectively, are taken. Prove that the centers of the circles circumscribed about triangles APR, BPQ and CQR constitute a triangle similar to triangle ABC.

Problems for independent study

18.45. On the plane, the unit circle with center at O is drawn. Two neighbouring vertices of a square belong to this circle. What is the maximal distance from point O that the two other of the square's vertices can have?

18.46. On the sides of convex quadrilateral $ABCD$, equilateral triangles ABM, CDP are constructed outwards and BCN, ADK inwards. Prove that $MN = AC$.

18.47. On the sides of a convex quadrilateral $ABCD$, squares with centers M, N, P, Q are constructed outwards. Prove that the midpoints of the diagonals of quadrilaterals $ABCD$ and $MNPQ$ form a square.

18.48. Inside an equilateral triangle ABC lies point O. It is known that $\angle AOB = 113^\circ$, $\angle BOC = 123^\circ$. Find the angles of the triangle whose sides are equal to segments OA, OB, OC.

18.49. On the plane, there are drawn n lines ($n > 2$) so that no two of them are parallel and no three intersect at one point. It is known that it is possible to rotate the plane about a point O through an angle of α ($\alpha < 180^\circ$) so that each of the drawn lines coincides with some other of the drawn lines. Indicate all n for which this is possible.

18.50. Ten gears of distinct shapes are placed so that the first gear is meshed with the second one, the second one with the third one, etc., the tenth is meshed with the first one. Is it possible for such a system to rotate? Can a similar system of 11 gears rotate?

18.51. Given a circle and a point. a) Construct an equilateral triangle whose heights intersect at the given point and two vertices belong to the given circle.

b) Construct a square two vertices of which belong to the given circle and the diagonals intersect at the given point.

Solutions

18.1. Let us rotate square $ABCD$ about point A through 90° so that B turns into D. This rotation sends point M into point M' and point K into point K'. It is clear that $\angle BMA = \angle DM'A$. Since $\angle MAK = \angle MAB = \angle M'AD$, it follows that $\angle MAD = \angle M'AK$. Therefore,

$$\angle MA'K = \angle MAD = \angle BMA = \angle DM'A.$$

Hence, $AK = KM' = KD + DM' = KD + BM$.

18.2. Under the rotation through 90° about point P lines PA_1, PB_1, PM_1 and CH turn into lines parallel to CA, CB, CM and AB, respectively. It follows that under such a rotation of triangle PA_1B_1 segment PM_1 turns into a median of the (rotated) triangle.

18.3. Consider a rotation through 90° about point B which sends vertex K into vertex N and vertex C into A. This rotation sends point A into point A' and point E into point E'. Since E' and B are the midpoints of sides $A'N$ and $A'C$ of triangle $A'NC$, it follows that $BE' \parallel NC$. But $\angle EBE' = 90^\circ$ and, therefore, $BE \perp NC$.

18.4. A rotation through an angle of 90° about the center of the square sends point A_1 to point A_2. This rotation sends the perpendiculars dropped from points A_1, A_2, A_3 and A_4 into lines A_2P, A_3P, A_4P and A_1P, respectively. Therefore, the intersection point is the image of point P under the inverse rotation.

18.5. Let us turn the given square through an angle of 90° about point A so that vertex B would coincide with D. Let M' be the image of M under this rotation. Since by the hypothesis

$$MK + MC + CK = (BM + MC) + (KD + CK),$$

it follows that $MK = BM + KD = DM' + KD = KM'$. Moreover, $AM = AM'$; hence, $\triangle AMK = \triangle AM'K$, consequently, $\angle MAK = \angle M'AK = \frac{1}{2}\angle MAM' = 45^\circ$.

18.6. Let R be the rotation through an angle of 90° that sends $\overrightarrow{BC}$ to $\overrightarrow{BA}$. Further, let $\overrightarrow{BC} = \mathbf{a}$, $\overrightarrow{CB_2} = \mathbf{b}$ and $\overrightarrow{AB_1} = \mathbf{c}$. Then $\overrightarrow{BA} = R\mathbf{a}$, $\overrightarrow{D_2C} = R\mathbf{b}$ and $\overrightarrow{AD_1} = R\mathbf{c}$. Hence, $\overrightarrow{D_2D_1} = R\mathbf{b} - \mathbf{a} + R\mathbf{a} + R\mathbf{c}$ and $2\overrightarrow{BM} = \mathbf{a} + \mathbf{b} + R\mathbf{a} + \mathbf{c}$. Therefore, $R(2\overrightarrow{BM}) = \overrightarrow{D_2D_1}$ because $R(R\mathbf{a}) = -\mathbf{a}$.

18.7. Let us introduce the following notations: $\mathbf{a} = \overrightarrow{BM}$, $\mathbf{b} = \overrightarrow{BC}$; let $R\mathbf{a}$ and $R\mathbf{b}$ be the vectors obtained from vectors $\mathbf{a}$ and $\mathbf{b}$ under a rotation through an angle of 90°, i.e., $R\mathbf{a} = \overrightarrow{BA}$, $R\mathbf{b} = \overrightarrow{BQ}$. Let O_1, O_2, O_3 and O_4 be the midpoints of segments AM, MQ, QC

and CA, respectively. Then

$$\overrightarrow{BO_1}=\frac{(\mathbf{a}+R\mathbf{a})}{2},\ \overrightarrow{BO_2}=\frac{(\mathbf{a}+R\mathbf{b})}{2},$$

$$\overrightarrow{BO_3}=\frac{(\mathbf{b}-R\mathbf{b})}{2},\ \overrightarrow{BO_4}=\frac{(\mathbf{b}+R\mathbf{a})}{2}.$$

Therefore, $\overrightarrow{O_1O_2}=\frac{1}{2}(R\mathbf{b}-R\mathbf{a})=-\overrightarrow{O_3O_4}$ and $\overrightarrow{O_2O_3}=\frac{1}{2}(\mathbf{b}-\mathbf{a})=-O_4O_1$. Moreover, $\overrightarrow{O_1O_2}=R(\overrightarrow{O_2O_3})$.

18.8. Parallelogram $A_1B_1C_1D_1$ is circumscribed around square $ABCD$ so that point A belongs to side A_1B_1, B to side B_1C_1, etc. Let us drop perpendiculars l_1, l_2, l_3 and l_4 from vertices A_1, B_1, C_1 and D_1, respectively to the sides of the square. To prove that these perpendiculars form a square, it suffices to verify that under a rotation through an angle of 90° about the center O of square $ABCD$ lines l_1, l_2, l_3 and l_4 turn into each other. Under the rotation about O through an angle of 90° points A_1, B_1, C_1 and D_1 turn into points A_2, B_2, C_2 and D_2 (Fig. 21).

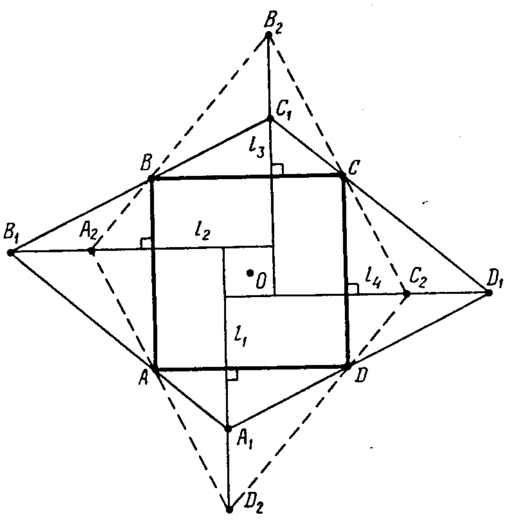


Figure 163 (Sol. 18.8)

Since $AA_2 \perp B_1B$ and $BA_2 \perp B_1A$, it follows that $B_1A_2 \perp AB$. This means that line l_1 turns under the rotation through an angle of 90° about O into l_2. For the other lines the proof is similar.

18.9. Let us consider a rotation through an angle of 60° about point C that turns E into D. Under this rotation B turns into A, i.e., segment BE turns into AD. Therefore, the midpoint P of segment BE turns into the midpoint M of segment AD, i.e., triangle CPM is an equilateral one.

18.10. Suppose that we have constructed triangle ABC so that its vertices A, B and C lie on lines l_1, l_2 and l_3, respectively. Under the rotation through an angle of 60° with center A point B turns into point C and, therefore, C is the intersection point of l_3 and the image of l_2 under the rotation through an angle of 60° about A.

18.11. The locus to be found consists of two squares obtained from the given one by rotations through angles of ±60° about P.

18.12. Under the rotation through an angle of 60° vectors $\overrightarrow{QC}$ and $\overrightarrow{CP}$ turn into $\overrightarrow{QD}$ and $\overrightarrow{CB} = \overrightarrow{DA}$, respectively. Therefore, under this rotation vector $\overrightarrow{QP} = \overrightarrow{QC} + \overrightarrow{CP}$ turns into vector $\overrightarrow{QD} + \overrightarrow{DA} = \overrightarrow{QA}$.

18.13. Let M' be the image of M under the rotation through an angle of 60° about B that turns A into C. Then $\angle CM'B = \angle AMB = 120^\circ$. Triangle $MM'B$ is an equilateral one and, therefore, $\angle BM'M = 60^\circ$. Since $\angle CM'B + \angle BM'M = 180^\circ$, point M' belongs to segment MC. Therefore, $MC = MM' + M'C = MB + MA$.

18.14. Under the rotation through an angle of 60° about A sending B to C point M turns into point M' and point C into point D. The equality $MA^2 = MB^2 + MC^2$ is equivalent to the equality $M'M^2 = M'C^2 + MC^2$, i.e., $\angle MCM' = 90^\circ$ and, therefore,

$$\angle MCB + \angle MBC = \angle MCB + \angle M'CD = 120^\circ - 90^\circ = 30^\circ$$

that is $\angle BMC = 150^\circ$. The locus to be found is the arc of the circle situated inside the triangle and such that the pionts of the arc serve as vertices of angles of 150° subtending segment BC.

18.15. Let O be the center of a hexagon. Consider a rotation about A through an angle of 60° sending B to O. This rotation sends segment OC into segment FE. Point K is the midpoint of diagonal BD of parallelogram $BCDO$ because it is the midpoint of diagonal CO. Therefore, point K turns into M under our rotation; in other words, triangle AMK is an equilateral one.

18.16. There is a rotation through an angle of 60° about the center of the given hexagon that sends A into B. It sends segment CD into DE and, therefore, sends M into N. Therefore, this rotation sends AM into BN, that is to say, the angle between these segments is equal to 60°. Moreover, this rotation turns pentagon $AMDEF$ into $BNEFA$; hence, the areas of the pentagons are equal. Cutting from these congruent pentagons the common part, pentagon $APNEF$, we get two figures of the same area: triangle ABP and quadrilateral $MDNP$.

18.17. Consider the rotation through an angle of 60° about C sending B to A. It sends points M, N and D into M', N' and D', respectively. Since $AMNN'$ is a parallelogram, the midpoint E of diagonal AN is its center of symmetry. Therefore, under the symmetry through point E triangle BMN turns into $M'AN'$ and, therefore, D turns into D'. Hence, E is the midpoint of segment DD'. Since triangle CDD' is an equilateral one, the angles of triangle CDE are equal to 30°, 60° and 90°.

18.18. Consider a rotation about A sending point C_1 into B. Under this rotation equilateral triangle A_1BC turns into triangle A_2FB_1 and segment A_1C_1 into segment A_2B. It remains to notice that $BA_1A_2B_1$ is a parallelogram, i.e., the midpoint of segment A_2B coincides with the midpoint of segment A_1B_1.

18.19. Let $\overrightarrow{AB} = 4\mathbf{a}$, $\overrightarrow{CA} = 4\mathbf{b}$. Further, let R be the rotation sending vector $\overrightarrow{AB}$ into $\overrightarrow{AC'}$ (and, therefore, sending $\overrightarrow{CA}$ into $\overrightarrow{CB'}$). Then $\overrightarrow{LM} = (\mathbf{a} + \mathbf{b}) - 2R\mathbf{b}$ and $\overrightarrow{LK} = -2R\mathbf{b} + 4\mathbf{b} + 2R\mathbf{a}$. It is easy to verify that $\mathbf{b} + R^2\mathbf{b} = R\mathbf{b}$. Hence, $2R(\overrightarrow{LM}) = \overrightarrow{LK}$ which implies the required statement.

18.20. Under the rotation about point C through an angle of 60° counterclockwise point A turns into B and D into E and, therefore, vector $\overrightarrow{DK} = \overrightarrow{AD}$ turns into $\overrightarrow{BE}$. Since the rotation about point H through an angle of 60° counterclockwise sends K into E and $\overrightarrow{DK}$ into $\overrightarrow{BE}$, it sends D into B which means that triangle BHD is an equilateral one.

18.21. a) Let O be a point inside triangle ABC. The rotation through an angle of 60° about A sends B, C and O into some points B', C' and O', respectively, see Fig. 22. Since

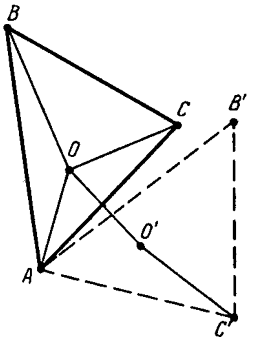


Figure 164 (Sol. 18.21)

$AO = OO'$ and $OC = O'C'$, we have:

$$BO + AO + CO = BO + OO' + O'C'.$$

The length of the broken line $BOO'C'$ is minimal if and only if this broken line is a segment, i.e., if $\angle AOB = \angle AO'C' = \angle AOC = 120°$. To construct the desired point, we can make use of the result of Problem 2.8.

b) The sum of distances from O to the vertices is equal to the length of segment BC' obtained in heading a). It is also clear that

$$(BC')^2 = b^2 + c^2 - 2bc\cos(\alpha + 60°) = \\ b^2 + c^2 - bc\cos\alpha + bc\sqrt{3}\sin\alpha = \\ \tfrac{1}{2}(a^2 + b^2 + c^2) + 2\sqrt{3}S.$$

18.22. Let P, Q and R be the midpoints of sides BC, DE and FA; let O be the center of the circumscribed circle. Suppose that triangle PQR is an equilateral one. Let us prove then that the midpoints of sides BC, DE' and $F'A$ of hexagon $ABCDE'F'$ in which vertices E' and F' are obtained from vertices E and F after a rotation through an angle about point O also form an equilateral triangle.

This will complete the proof since for a regular hexagon the midpoints of sides BC, DE and FA constitute an equilateral triangle and any of the considered hexagons can be obtained from a regular one with the help of rotations of triangles OCD and OEF.

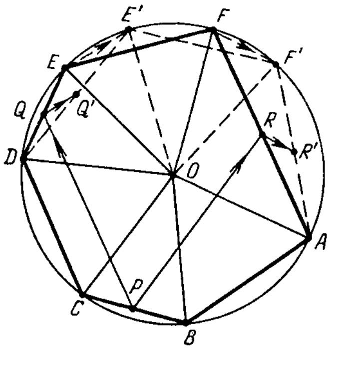


Figure 165 (Sol. 18.22)

Let Q' and R' be the midpoints of sides DE' and AF', see Fig. 23. Under the rotation through an angle of 60° vector $\overrightarrow{EE'}$ turns into $\overrightarrow{FF'}$. Since $\overrightarrow{QQ'} = \frac{1}{2}\overrightarrow{EE'}$ and $\overrightarrow{RR'} = \frac{1}{2}\overrightarrow{FF'}$, this rotation sends $\overrightarrow{QQ'}$ into $\overrightarrow{RR'}$. By hypothesis, triangle PQR is an equilateral one, i.e., under the rotation through an angle of 60° vector $\overrightarrow{PQ}$ turns into $\overrightarrow{PR}$. Therefore, vector

$\overrightarrow{PQ'} = \overrightarrow{PQ} + \overrightarrow{QQ'}$ turns into vector $\overrightarrow{PR'} = \overrightarrow{PR} + \overrightarrow{RR'}$ under a rotation through an angle of 60°. This means that triangle $PQ'R'$ is an equilateral one.

18.23. Let K, L, M and N be vertices of equilateral triangles constructed (wherewards?) on sides BC, AB, AF and FE, respectively; let also B_1, A_1 and F_1 be the midpoints of segments KL, LM and MN (see Fig. 24).

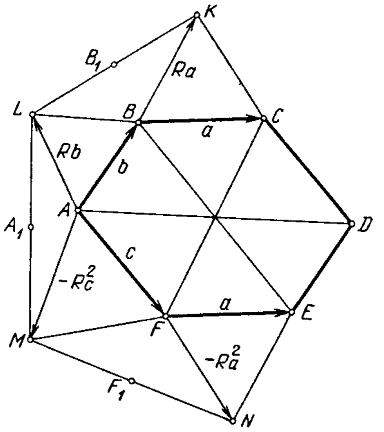


Figure 166 (Sol. 18.23)

Further, let $\mathbf{a} = \overrightarrow{BC} = \overrightarrow{FE}$, $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AF}$; let R be the rotation through an angle of 60° that sends $\overrightarrow{BC}$ into $\overrightarrow{BK}$. Then $\overrightarrow{AM} = -R^2\mathbf{c}$ and $\overrightarrow{FN} = -R^2\mathbf{a}$. Therefore, $2\overrightarrow{A_1B_1} = R^2\mathbf{c} + R\mathbf{a} + \mathbf{b}$ and $2\overrightarrow{F_1A_1} = R^2\mathbf{a} - \mathbf{c} + R\mathbf{b}$, i.e., $\overrightarrow{F_1A_1} = R(\overrightarrow{A_1B_1})$.

18.24. Suppose a rotation through an angle of α about the center of circle S sends C into D. This rotation sends point A into point A'. Then $\angle(BD, DA') = \alpha$, i.e., point D belongs to the arc of the circle whose points serve as vertices of the angles of α that subtend segment $A'B$.

18.25. Let P be the intersection point of lines l_1 and l_2. Then

$$\angle(OA_1, A_1P) = \angle(OA_1, l_1) = \angle(OA_2, l_2) = \angle(OA_2, A_2P).$$

Therefore, points O, A_1, A_2 and P belong to one circle.

18.26. It is possible to identify similar letters Γ after a rotation about O (unless they can be identified by a parallel translation in which case $AA_i \parallel A'A_i'$). Thanks to Problem 18.25 point X_i belongs to the circle circumscribed about triangle $A'OA$. It is clear that the points that belong to one circle constitute a convex polygon.

18.27. Let O be the center of rotation R that sends segment $A(t_1)A(t_2)$ into segment $B(t_1)B(t_2)$, where t_1 and t_2 are certain time moments. Then this rotation sends $A(t)$ into $B(t)$ at any moment t. Therefore, by Problem 18.25 point O belongs to the circle circumscribed about triangle APB.

18.28. Let A and B be points on the circle with center O; let A_1 and B_1 be the images of these points under the rotation through an angle of α about O. Let P and P_1 be the midpoints of segments AB and A_1B_1; let M be the intersection point of lines AB and A_1B_1. The right triangles POM and P_1OM have a common hypothenuse and equal legs $PO = P_1O$, therefore, these triangles are equal and $\angle MOP = \angle MOP_1 = \frac{1}{2}\alpha$. Point M is obtained from point P under a rotation through an angle of $\frac{1}{2}\alpha$ and a subsequent homothety with coefficient $\frac{1}{\cos(\frac{1}{2}\alpha)}$ and center O.

The intersection points of lines AB and A_1B_1, AC and A_1C_1, BC and B_1C_1 are the vertices of a triangle which is homothetic with coefficient $\frac{1}{\cos(\frac{1}{2}\alpha)}$ to the triangle determined by the midpoints of the sides of triangle ABC. It is clear that the triangle determined by the midpoints of the sides of triangle ABC is similar to triangle ABC.

18.29. By Problem 5.50 the line which divides in halves both the area and the perimeter of a triangle passes through the center of its inscribed circle. It is also clear that if the line passes through the center of the inscribed circle of a triangle and divides its perimeter in halves, then it divides in halves its area as well. Therefore, we have to draw a line passing through the center of the inscribed circle of the triangle and dividing its perimeter in halves.

Suppose we have constructed points M and N on sides AB and AC of triangle ABC so that line MN passes through the center O of the inscribed circle and divides the perimeter of the triangle in halves. On ray AC construct point D so that $AD = p$, where p is a semiperimeter of triangle ABC. Then $AM = ND$. Let Q be the center of rotation R that sends segment AM into segment DN (so that A goes to D and M to N). Since the angle between lines AM and CN is known, it is possible to construct Q: it is the vertex of isosceles triangle AQD, where $\angle AQD = 180^\circ - \angle A$ and points B and Q lie on one side of line AD. The rotation R sends segment OM into segment $O'N$. We can now construct point O'. Clearly, $\angle ONO' = \angle A$ because the angle between lines OM and $O'N$ is equal to $\angle A$. Therefore, point N is the intersection point of line AC and the arc of the circle whose points serve as vertices for the angles equal to $\angle A$ that subtend segment OO'. Constructing point N, draw line ON and find point M.

It is easy to verify that if the constructed points M and N belong to sides AB and AC, then MN is the desired line. The main point of the proof is the proof of the fact that the rotation about Q through an angle of $180^\circ - \angle A$ sends M into N. To prove this fact, one has to make use of the fact that $\angle ONO' = \angle A$, i.e., this rotation sends line OM into line $O'N$.

18.30. Suppose that the k-gons $C_1 \dots C_k$ and $D_1 \dots D_k$ are regular and similarly oriented. Let C and D be the centers of these k-gons; let $\mathbf{c}_i = \overrightarrow{CC_i}$ and $\mathbf{d}_i = \overrightarrow{DD_i}$. Then

$$\overrightarrow{C_iD_i} = \overrightarrow{C_iC} + \overrightarrow{CD} + \overrightarrow{DD_i} = -\mathbf{c}_i + \overrightarrow{CD} + \mathbf{d}_i.$$

The rotation R^φ, where φ is the angle at a vertex of a regular n-gon, sends $\overrightarrow{C_iD_i}$ into $\overrightarrow{C_iB_i}$. Therefore,

$$\overrightarrow{XB_i} = \overrightarrow{XC} + \mathbf{c}_i + \overrightarrow{C_iB_i} = \overrightarrow{XC} + \mathbf{c}_i + R^\varphi(-\mathbf{c}_i + \overrightarrow{CD} + \mathbf{d}_i).$$

Let us select point X so that $\overrightarrow{XC} + R^\varphi(\overrightarrow{CD}) = \overrightarrow{0}$. Then $\overrightarrow{XB_i} = \mathbf{c}_i + R^\varphi(\mathbf{d}_i - \mathbf{c}_i) = R^{i\psi}\mathbf{u}$, where $\mathbf{u} = \mathbf{c}_k + R^\varphi(\mathbf{d}_k - \mathbf{c}_k)$ and R^ψ is the rotation sending $\mathbf{c}_k$ to $\mathbf{c}_1$. Hence, $B_1 \dots B_k$ is a regular k-gon with center X.

We similarly prove that $A_1 \dots A_k$ is a regular k-gon.

The converse statement is similarly proved.

18.31. Let H be the intersection point of heights of triangle ABC; let H_1, H_2 and H_3 be points symmetric to H through sides BC, CA and AB, respectively. Points H_1, H_2 and H_3 belong to the circle circumscribed about triangle ABC (Problem 5.9). Let l be a line passing through H. The line symmetric to l through BC (resp. through CA and AB) intersects the circumscribed circle at point H_1 (resp. H_2 and H_3) and at a point P_1 (resp. P_2 and P_3).

Consider another line l' passing through H. Let φ be the angle between l and l'. Let us construct points P_1', P_2' and P_3' for line l' in the same way as points P_1, P_2 and P_3 were constructed for line l. Then $\angle P_iH_iP_i' = \varphi$, i.e., the value of arc $\smile P_iP_i'$ is equal to 2φ (the direction of the rotation from P_i to P_i' is opposite to that of the rotation from l to l'). Therefore, points P_1', P_2' and P_3' are the images of points P_1, P_2 and P_3 under a certain rotation. It is clear that if for l' we take the height of the triangle dropped from vertex A, then $P_1' = P_2' = P_3' = A$, and, therefore, $P_1 = P_2 = P_3$.

18.32. Suppose that the lion ran along the broken line $A_1A_2\dots A_n$. Let us rectify the lion's trajectory as follows. Let us rotate the arena of the circus and all(?) the further trajectory about point A_2 so that point A_3 would lie on ray A_1A_2. Then let us rotate the arena and the further trajectory about point A_3 so that point A_4 were on ray A_1A_2, and so on. The center O of the arena turns consecutively into points $O_1 = O$, O_2, ... , O_{n-1}; and points A_1, ... , A_n into points A'_1, ... , A'_n all on one line (Fig. 25).

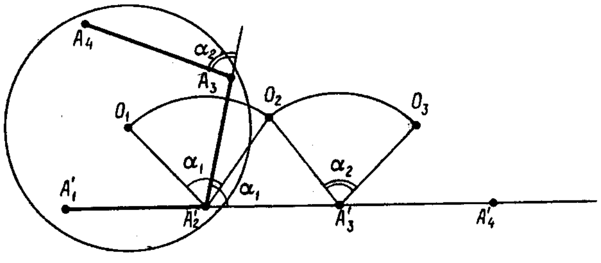


Figure 167 (Sol. 18.32)

Let α_{i-1} be the angle of through which the lion turned at point A'_i. Then $\angle O_{i-1}A'_iO_i = \alpha_{i-1}$ and $A'_iO_{i-1} = A'_iO_i \leq 10$; hence, $O_iO_{i-1} \leq 10\alpha_{i-1}$. Hence,

$$30000 = A'_1A'_n \leq A'_1O_1 + O_1O_2 + \dots + O_{n-2}O_{n-1} + O_{n-1}A'_n \leq 10 + 10(\alpha_1 + \dots + \alpha_{n-2}) + 10$$

i.e., $\alpha_1 + \dots + \alpha_{n-2} \geq 2998$.

18.33. Consider the composition of the rotations $R_B^\beta \circ R_A^\alpha$. If $A = B$, then the statement of the problem is obvious and, therefore, let us assume that $A \neq B$. Let $l = AB$; let lines a and b pass through points A and B, respectively, so that $\angle(a, l) = \frac{1}{2}\alpha$ and $\angle(l, b) = \frac{1}{2}\beta$. Then

$$R_B^\beta \circ R_A^\alpha = S_b \circ S_l \circ S_l \circ S_a = S_b \circ S_a.$$

If $a \parallel b$, then $S_a \circ S_b = T_{2\mathbf{u}}$, where $T_\mathbf{u}$ is a parallel translation sending a into b and such that $\mathbf{u} \perp a$. If lines a and b are not parallel and O is their intersection point, then $S_a \circ S_b$ is the rotation through an angle of $\alpha + \beta$ with center O. It is also clear that $a \parallel b$ if and only if $\frac{1}{2}\alpha + \frac{1}{2}\beta = k\pi$, i.e., $\alpha + \beta = 2k\pi$.

18.34. Let P, Q, R and S be the centers of squares constructed outwards on sides AB, BC, CD and DA, respectively. On segments QR and SP, construct inwards isosceles right triangles with vertices O_1 and O_2. Then $D = R_R^{90^\circ} \circ R_Q^{90^\circ}(B) = R_{O_1}^{180^\circ}(B)$ and $B = R_P^{90^\circ} \circ R_S^{90^\circ}(D) = R_{O_2}^{180^\circ}(D)$, i.e., $O_1 = O_2$ is the midpoint of segment BD.

The rotation through an angle of 90° about point $O = O_1 = O_2$ that sends Q into R sends point S into P, i.e., it sends segment QS into RP and, therefore, these segments are equal and perpendicular to each other.

18.35. Let P, Q, R and S be the centers of squares constructed outwards on the sides AB, BC, CD and DA of parallelogram $ABCD$. By the previous problem $PR = QS$ and $PR \perp QS$. Moreover, the center of symmetry of parallelogram $ABCD$ is the center of symmetry of quadrilateral $PQRS$. This means that $PQRS$ is a parallelogram with equal and perpendicular diagonals, hence, a square.

18.36. Let P, Q and R be the centers of squares constructed outwards on sides AB, BC and CA. Let us consider a rotation through an angle of 90° with center R that sends C to A. Under the rotation about P through an angle of 90° in the same direction point A turns into B. The composition of these two rotations is a rotation through an angle of 180° and, therefore, the center of this rotation is the midpoint of segment BC. On the other hand, the center of this rotation is a vertex of an isoscceles right triangle with base PR, i.e., it is the center of a square constructed on PR. This square is constructed inwards on a side of triangle PQR.

18.37. If $O_1 = O_3$, then $R_D^{90^\circ} \circ R_C^{90^\circ} \circ R_B^{90^\circ} \circ R_A^{90^\circ} = R_{O_3}^{180^\circ} \circ R_{O_1}^{180^\circ} = E$. Therefore,

$$E = R_A^{90^\circ} \circ E \circ R_A^{-90^\circ} = R_A^{90^\circ} \circ R_D^{90^\circ} R_C^{90^\circ} \circ R_B^{90^\circ} = R_{O_4}^{180^\circ} \circ R_{O_2}^{180^\circ},$$

where E is the identity transformation, i.e., $O_4 = O_2$.

18.38. a) See solution of a more general Problem 18.42 (it suffices to set $\alpha = \beta = \gamma = 120^\circ$). In case b) proof is analogous.

b) Let Q and R (resp. Q_1 and R_1) be the centers of equilateral triangles constructed outwards (resp. inwards) on sides AC and AB. Since $AQ = \frac{1}{\sqrt{3}}b$, $AR = \frac{1}{\sqrt{3}}c$ and $\angle QAR = 60^\circ + \alpha$, it follows that $3QR^2 = b^2 + c^2 - 2bc\cos(\alpha + 60^\circ)$. Similarly, $3Q_1R_1^2 = b^2 + c^2 - 2bc\cos(\alpha - 60^\circ)$. Therefore, the difference of areas of the obtained equilateral triangles is equal to

$$\frac{(QR^2 - Q_1R_1^2)\sqrt{3}}{4} = \frac{bc\sin\alpha\sin 60^\circ}{\sqrt{3}} = S_{ABC}.$$

18.39. The combination of a rotation through an angle of 60° about A' that sends B to C, a rotation through an angle of 60° about B' that sends C to A and a rotation through an angle of 120° about M that sends A to B has B as a fixed point. Since the first two rotations are performed in the direction opposite to the direction of the last rotation, it follows that the composition of these rotations is a parallel translation with a fixed point, i.e., the identity transformation:

$$R_M^{-120^\circ} \circ R_{B'}^{60^\circ} \circ R_{A'}^{60^\circ} = E.$$

Therefore, $R_{B'}^{60^\circ} \circ R_{A'}^{60^\circ} = R_M^{120^\circ}$, i.e., M is the center of the rotation $R_{B'}^{60^\circ} \circ R_{A'}^{60^\circ}$. It follows that $\angle MA'B' = \angle MB'A' = 30^\circ$, i.e., $A'B'M$ is an isosceles triangle and $\angle A'MB' = 120^\circ$.

18.40. The conditions of the problem imply that $R_C^{-2\gamma} = R_B^{2\beta} \circ R_A^{2\alpha}$, i.e., point C is the center of the composition of rotations $R_B^{2\beta} \circ R_A^{2\alpha}$. This means that $\angle BAC = \alpha$ and $\angle ABC = \beta$ (see Problem 18.33). Therefore, $\angle ACB = \pi - \alpha - \beta = \gamma$.

18.41. Denote the given points by $M_1, \dots, M_n$. Suppose that we have constructed polygon $A_1A_2\dots A_n$ so that triangles $A_1M_1A_2$, $A_2M_2A_3, \dots, A_nM_nA_1$ are isosceles, where $\angle A_iM_iA_{i+1} = \alpha_i$ and the sides of the polygon are bases of these isosceles triangles. Clearly, $R_{M_n}^{\alpha_n} \circ \dots \circ R_{M_1}^{\alpha_1}(A_1) = A_1$. If $\alpha_1 + \dots + \alpha_n \neq k \cdot 360^\circ$, then point A_1 is the center of the rotation $R_{M_n}^{\alpha_n} \circ \dots \circ R_{M_1}^{\alpha_1}$.

We can construct the center of the composition of rotations. The construction of the other vertices of the polygon is done in an obvious way. If $\alpha_1 + \dots + \alpha_n = k \cdot 360^\circ$, then the problem is ill-posed: either an arbitrary point A_1 determines a polygon with the required property or there are no solutions.

18.42. Since $R_{C'}^{\gamma} \circ R_{B'}^{\beta} \circ R_{A'}^{\alpha}(B) = R_{C'}^{\gamma} \circ R_{B'}^{\beta}(C) = R_{C'}^{\gamma}(A) = B$, it follows that B is a fixed point of the composition $R_{C'}^{\gamma} \circ R_{B'}^{\beta} \circ R_{A'}^{\alpha}$. Since $\alpha + \beta + \gamma = 2\pi$, it follows that this composition is a parallel translation with a fixed point, i.e., the identity transformation. It remains to make use of the result of Problem 18.40.

18.43. Since $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha}(N) = L$ and $R_G^{\pi-\alpha} \circ R_A^{\alpha}(L) = N$, it follows that the transformations $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha}$ and $R_G^{\pi-\alpha} \circ R_A^{\alpha}$ are central symmetries with respect to the midpoint of segment LN, i.e., $R_{G'}^{\pi-\alpha} \circ R_A^{\alpha} = R_G^{\pi-\alpha} \circ R_A^{\alpha}$. Therefore, $R_{G'}^{\pi-\alpha} = R_G^{\pi-\alpha}$ and $G' = G$.

18.44. Let A_1, B_1 and C_1 be the centers of the circumscribed circles of triangles APR, BPQ and CQR. Under the successive rotations with centers A_1, B_1 and C_1 through angles

2α, 2β and 2γ point R turns first into P, then into Q, and then returns home. Since $2\alpha+2\beta+2\gamma = 360^\circ$, the composition of the indicated rotations is the identity transformation. It follows that the angles of triangle $A_1B_1C_1$ are equal to α, β and γ (see Problem 18.40).

Chapter 19. HOMOTHETY AND ROTATIONAL HOMOTHETY

Background

1. A *homothety* is a transformation of the plane sending point X into point X' such that $\overrightarrow{OX'} = k\overrightarrow{OX}$, where point O and the number k are fixed. Point O is called *the center of homothety* and the number k *the coefficient of homothety.*

We will denote the homothety with center O and coefficient k by H_O^k.

2. Two figures are called *homothetic* if one of them turns into the other one under a homothety.

3. A *rotational homothety* is the composition of a homothety and a rotation with a common center. The order of the composition is inessential since $R_O^{\varphi} \circ H_O^k = H_O^k \circ R_O^{\varphi}$.

We may assume that the coefficient of a rotational homothety is positive since $R_O^{180^\circ} \circ H_O^k = H_O^{-k}$.

4. The composition of two homotheties with coefficients k_1 and k_2, where $k_1k_2 \neq 1$, is a homothety with coefficient k_1k_2 and its center belongs to the line that connects the centers of these homotheties (see Problem 19.23).

5. The center of a rotational homothety that sends segment AB into segment CD is the intersection point of the circles circumscribed about triangles ACP and BDP, where P is the intersection point of lines AB and CD (see Problem 19.41).

Introductory problems

1. Prove that a homothety sends a circle into a circle.

2. Two circles are tangent at point K. A line passing through K intersects these circles at points A and B. Prove that the tangents to the circles through A and B are parallel to each other.

3. Two circles are tangent at point K. Through K two lines are drawn that intersect the first circle at points A and B and the second one at points C and D. Prove that $AB \parallel CD$.

4. Prove that points symmetric to an arbitrary point with respect to the midpoints of a square's sides are vertices of a square.

5. Two points A and B and a line l on the plane are given. What is the trajectory of movement of the intersection point of medians of triangle ABC when C moves along l?

§1. Homothetic polygons

19.1. A quadrilateral is cut by diagonals into four triangles. Prove that the intersection points of their medians form a parallelogram.

19.2. The extensions of the lateral sides AB and CD of trapezoid $ABCD$ intersect at point K and its diagonals intersect at point L. Prove that points K, L, M and N, where M and N are the midpoints of bases BC and AD, respectively, belong to one line.

19.3. The intersection point of diagonals of a trapezoid is equidistant from the lines to which the sides of the trapezoid belong. Prove that the trapezoid is an isosceles one.

19.4. Medians AA_1, BB_1 and CC_1 of triangle ABC meet at point M; let P be an arbitrary point. Line l_a passes through point A parallel to line PA_1; lines l_b and l_c are similarly defined. Prove that:

a) lines l_a, l_b and l_c meet at one point, Q;

b) point M belongs to segment PQ and $PM : MQ = 1 : 2$.

19.5. Circle S is tangent to equal sides AB and BC of an isosceles triangle ABC at points P and K, respectively, and is also tangent from the inside to the circle circumscribed about triangle ABC. Prove that the midpoint of segment PK is the center of the circle inscribed into triangle ABC.

19.6. A convex polygon possesses the following property: if all its sides are pushed by distance 1 outwards and extended, then the obtained lines form a polygon similar to the initial one. Prove that this polygon is a circumscribed one.

19.7. Let R and r be the radii of the circumscribed and inscribed circles of a triangle. Prove that $R \geq 2r$ and the equality is only attained for an equilateral triangle.

19.8. Let M be the center of mass of an n-gon $A_1 \dots A_n$; let $M_1, \dots, M_n$ be the centers of mass of the $(n-1)$-gons obtained from the given n-gon by discarding vertices $A_1, \dots,$ A_n, respectively. Prove that polygons $A_1 \dots A_n$ and $M_1 \dots M_n$ are homothetic to each other.

19.9. Prove that any convex polygon Φ contains two nonintersecting polygons Φ_1 and Φ_2 similar to Φ with coefficient $\frac{1}{2}$.

See also Problem 5.87.

§2. Homothetic circles

19.10. On a circle, points A and B are fixed and point C moves along this circle. Find the locus of the intersection points of the medians of triangles ABC.

19.11. a) A circle inscribed into triangle ABC is tangent to side AC at point D, and DM is its diameter. Line BM intersects side AC at point K. Prove that $AK = DC$.

b) In the circle, perpendicular diameters AB and CD are drawn. From point M outside the circle there are drawn tangents to the circle that intersect AB at points E and H and also lines MC and MD that intersect AB at points F and K, respectively. Prove that $EF = KH$.

19.12. Let O be the center of the circle inscribed into triangle ABC, let D be the point where the circle is tangent to side AC and B_1 the midpoint of AC. Prove that line B_1O divides segment BD in halves.

19.13. The circles α, β and γ are of the same radius and are tangent to the sides of angles A, B and C of triangle ABC, respectively. Circle δ is tangent from the outside to all the three circles α, β and γ. Prove that the center of δ belongs to the line passing through the centers of the circles inscribed into and circumscribed about triangle ABC.

19.14. Consider triangle ABC. Four circles of the same radius ρ are constructed so that one of them is tangent to the three other ones and each of those three is tangent to two sides of the triangle. Find ρ given the radii r and R of the circles inscribed into and circumscribed about the triangle.

§3. Costructions and loci

19.15. Consider angle $\angle ABC$ and point M inside it. Construct a circle tangent to the legs of the angle and passing through M.

19.16. Inscribe two equal circles in a triangle so that each of the circles were tangent to two sides of the triangle and the other circle.

19.17. Consider acute triangle ABC. Construct points X and Y on sides AB and BC, respectively, so that a) $AX = XY = YC$; b) $BX = XY = YC$.

19.18. Construct triangle ABC given sides AB and AC and bisector AD.

19.19. Solve Problem 16.18 with the help of homothety.

19.20. On side BC of given triangle ABC, construct a point such that the line that connects the bases of perpendiculars dropped from this point to sides AB and AC is parallel to BC.

* * *

19.21. Right triangle ABC is modified so that vertex A of the right angle is fixed whereas vertices B and C slide along fixed circles S_1 and S_2 tangent to each other at A from the outside. Find the locus of bases D of heights AD of triangles ABC.

See also problems 7.26–7.29, 8.15, 8.16, 8.70.

§4. Composition of homotheties

19.22. A transformation f has the following property: if A' and B' are the images of points A and B, then $\overrightarrow{A'B'} = k\overrightarrow{AB}$, where k is a constant. Prove that:

a) if $k = 1$, then f is a parallel translation;

b) if $k \neq 1$, then f is a homothety.

19.23. Prove that the composition of two homotheties with coefficients k_1 and k_2, where $k_1k_2 \neq 1$, is a homothety with coefficient k_1k_2 and its center belongs to the line that connects the centers of these homotheties. Investigate the case $k_1k_2 = 1$.

19.24. Common outer tangents to the pairs of circles S_1 and S_2, S_2 and S_3, S_3 and S_1 intersect at points A, B and C, respectively. Prove that points A, B and C belong to one line.

19.25. Trapezoids $ABCD$ and $APQD$ have a common base AD and the length of all their bases are distinct. Prove that the intersections points of the following pairs of lines belong to one line:

a) AB and CD, AP and DQ, BP and CQ;

b) AB and CD, AQ and DP, BQ and CP.

§5. Rotational homothety

19.26. Circles S_1 and S_2 intersect at points A and B. Lines p and q passing through point A intersect circle S_1 at points P_1 and Q_1 and circle S_2 at points P_2 and Q_2. Prove that the angle between lines P_1Q_1 and P_2Q_2 is equal to the angle between circles S_1 and S_2.

19.27. Circles S_1 and S_2 intersect at points A and B. Under the rotational homothety P with center A that sends S_1 into S_2 point M_1 from circle S_1 turns into M_2. Prove that line M_1M_2 passes through B.

19.28. Circles $S_1, \dots, S_n$ pass through point O. A grasshopper hops from point X_i on circle S_i to point X_{i+1} on circle S_{i+1} so that line X_iX_{i+1} passes through the intersection point of circles S_i and S_{i+1} distinct from O. Prove that after n hops (from S_1 to S_2 from S_2 to $S_3, \dots$, from S_n to S_1) the grasshopper returns to the initial position.

19.29. Two circles intersect at points A and B and chords AM and AN are tangent to these circles. Let us complete triangle MAN to parallelogram $MANC$ and divide segments BN and MC by points P and Q in equal proportions. Prove then that $\angle APQ = \angle ANC$.

19.30. Consider two nonconcentric circles S_1 and S_2. Prove that there exist precisely two rotational homotheties with the angle of rotation of 90° that send S_1 into S_2.

* * *

19.31. Consider square $ABCD$ and points P and Q on sides AB and BC, respectively, so that $BP = BQ$. Let H be the base of the perpendicular dropped from B on PC. Prove that $\angle DHQ = 90^\circ$.

19.32. On the sides of triangle ABC similar triangles are constructed outwards: $\triangle A_1BC \sim \triangle B_1CA \sim \triangle C_1AB$. Prove that the intersection points of medians of triangles ABC and $A_1B_1C_1$ coincide.

19.33. The midpoints of sides BC and B_1C_1 of equilateral triangles ABC and $A_1B_1C_1$ coincide (the vertices of both triangles are listed clockwise). Find the value of the angle between lines AA_1 and BB_1 and also the ratio of the lengths of segments AA_1 and BB_1.

19.34. Triangle ABC turns under a rotational homothety into triangle $A_1B_1C_1$; let O be an arbitrary point. Let A_2 be the vertex of parallelogram OAA_1A_2; let points B_2 and C_2 be similarly defined. Prove that $\triangle A_2B_2C_2 \sim \triangle ABC$.

19.35. On top of a rectangular map lies a map of the same locality but of lesser scale. Prove that it is possible to pierce by a needle both maps so that the points where both maps are pierced depict the same point of the locality.

19.36. Rotational homotheties P_1 and P_2 with centers A_1 and A_2 have the same angle of rotation and the product of their coefficients is equal to 1. Prove that the composition $P_2 \circ P_1$ is a rotation and its center coincides with the center of another rotation that sends A_1 into A_2 and whose angle of rotation is equal to $2\angle(\overrightarrow{MA_1}, \overrightarrow{MN})$, where M is an arbitrary point and $N = P_1(M)$.

19.37. Triangles MAB and MCD are similar but have opposite orientations. Let O_1 be the center of rotation through an angle of $2\angle(\overrightarrow{AB}, \overrightarrow{BM})$ that sends A to C and O_2 the center of rotation through an angle of $2\angle(\overrightarrow{AB}, \overrightarrow{AM})$ that sends B to D. Prove that $O_1 = O_2$.

* * *

19.38. Consider a half circle with diameter AB. For every point X on this half circle a point Y is placed on ray XA so that $XY = kXB$. Find the locus of points Y.

19.39. Consider point P on side AB of (unknown?) triangle ABC and triangle LMN. Inscribe triangle PXY similar to LMN into triangle ABC.

19.40. Construct quadrilateral $ABCD$ given $\angle B + \angle D$ and the lengths $a = AB$, $b = BC$, $c = CD$ and $d = DA$.

See also Problem 5.122.

§6. The center of a rotational homothety

19.41. a) Let P be the intersection point of lines AB and A_1B_1. Prove that if no points among A, B, A_1, B_1 and P coincide, then the common point of circles circumscribed about triangles PAA_1 and PBB_1 is the center of a rotational homothety that sends A to A_1 and B to B_1 and that such a rotational homothety is unique.

b) Prove that the center of a rotational homothety that sends segment AB to segment BC is the intersection point of circles passing through point A and tangent to line BC at point B and the circle passing through C and tangent to line AB at point B.

19.42. Points A and B move along two intersecting lines with constant but distinct speeds. Prove that there exists a point, P, such that at any moment $AP : BP = k$, where k is the ratio of the speeds.

19.43. Construct the center O of a rotational homothety with a given coefficient $k \neq 1$ that sends line l_1 into line l_2 and point A_1 that belongs to l_1 into point A_2. (?)

19.44. Prove that the center of a rotational homothety that sends segment AB into segment A_1B_1 coincides with the center of a rotational homothety that sends segment AA_1 into segment BB_1.

19.45. Four intersecting lines form four triangles. Prove that the four circles circumscribed about these triangles have one common point.

19.46. Parallelogram $ABCD$ is not a rhombus. Lines symmetric to lines AB and CD through diagonals AC and DB, respectively, intersect at point Q. Prove that Q is the center of a rotational homothety that sends segment AO into segment OD, where O is the center of the parallelogram.

19.47. Consider wo regular pentagons with a common vertex. The vertices of each pentagon are numbered 1 to 5 clockwise so that the common vertex has number 1. Vertices with equal numbers are connected by straight lines. Prove that the four lines thus obtained intersect at one point.

19.48. On sides BC, CA and AB of triangle ABC points A_1, B_1 and C_1 are taken so that $\triangle ABC \sim \triangle A_1B_1C_1$. Pairs of segments BB_1 and CC_1, CC_1 and AA_1, AA_1 and BB_1 intersect at points A_2, B_2 and C_2, respectively. Prove that the circles circumscribed about triangles ABC_2, BCA_2, CAB_2, $A_1B_1C_2$, $B_1C_1A_2$ and $C_1A_1B_2$ intersect at one point.

§7. The similarity circle of three figures

Let F_1, F_2 and F_3 be three similar figures, O_1 the center of a rotational homothety that sends F_2 to F_3. Let points O_2 and O_3 be similarly defined. If O_1, O_2 and O_3 do not belong to one line, then triangle $O_1O_2O_3$ is called *the similarity triangle* of figures F_1, F_2 and F_3 and its circumscribed circle is called *the similarity circle* of these figures. In case points O_1, O_2 and O_3 coincide the similarity circle degenerates into *the center of similarity* and in case when not all these points coincide but belong to one line the similarity circle degenerates into *the axis of similarity.*

In the problems of this section we assume that the similarity circle of the figures considered is not degenerate.

19.49. Lines A_2B_2 and A_3B_3, A_3B_3 and A_1B_1, A_1B_1 and A_2B_2 intersect at points P_1, P_2, P_3, respectively.

a) Prove that the circumscribed circles of triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$ intersect at one point that belongs to the similarity circle of segments A_1B_1, A_2B_2 and A_3B_3.

b) Let O_1 be the center of rotational homothety that sends segment A_2B_2 into segment A_3B_3; points O_2 and O_3 be similarly defined. Prove that lines P_1O_1, P_2O_2 and P_3O_3 intersect at one point that belongs to the similarity circle of segments A_1B_1, A_2B_2 and A_3B_3.

Points A_1 and A_2 are called *correspondent* points of similar figures F_1 and F_2 if the rotational symmetry that sends F_1 to F_2 transforms A_1 into A_2. *Correspondent lines* and *correspondent segments* are analogously defined.

19.50. Let A_1B_1, A_2B_2 and A_3B_3 and also A_1C_1, A_2C_2 and A_3C_3 be correspondent segments of similar figures F_1, F_2 and F_3. Prove that the triangle formed by lines A_1B_1, A_2B_2 and A_3B_3 is similar to the triangle formed by lines A_1C_1, A_2C_2 and A_3C_3 and the center of the rotational homothety that sends one of these triangles into another one belongs to the similarity circle of figures F_1, F_2 and F_3.

19.51. Let l_1, l_2 and l_3 be the correspondent lines of similar figures F_1, F_2 and F_3 and let the lines intersect at point W.

a) Prove that W belongs to the similarity circle of F_1, F_2 and F_3.

b) Let J_1, J_2 and J_3 be distinct from W intersection points of lines l_1, l_2 and l_3 with the similarity circle. Prove that these points only depend on figures F_1, F_2 and F_3 and do not depend on the choice of lines l_1, l_2 and l_3.

Points J_1, J_2 and J_3 are called *constant points of similar figures* F_1, F_2 and F_3 and triangle $J_1J_2J_3$ is called *the constant triangle of similar figures.*

19.52. Prove that the constant triangle of three similar figures is similar to the triangle formed by their correspondent lines and these triangles have opposite orientations.

19.53. Prove that constant points of three similar figures are their correspondent points.

The *similarity circle of triangle ABC* is the similarity circle of segments AB, BC and CA (or of any three similar triangles constructed from these segments). *Constant points of a triangle* are the constant points of the three figures considered.

19.54. Prove that the similarity circle of triangle ABC is the circle with diameter KO, where K is Lemoin's point and O is the center of the circumscribed circle.

19.55. Let O be the center of the circumscribed circle of triangle ABC, K Lemoin's point, P and Q Brokar's points, φ Brokar's angle (see Problems 5.115 and 5.117). Prove that points P and Q belong to the circle of diameter KO and $OP = OQ$ and $\angle POQ = 2\varphi$.

Problems for independent study

19.56. Given triangles ABC and KLM. Inscribe triangle $A_1B_1C_1$ into triangle ABC so that the sides of $A_1B_1C_1$ wre parallel to the respective sides of triangle KLM.

19.57. On the plane, there are given points A and E. Construct a rhombus $ABCD$ with a given height for which E is the midpoint of BC.

19.58. Consider a quadrilateral. Inscribe a rombus in it so that the sides of the rombus are parallel to the diagonals of the quandrangle.

19.59. Consider acute angle $\angle AOB$ and point C inside it. Find point M on leg OB equidistant from leg OA and from point C.

19.60. Consider acute triangle ABC. Let O be the intersection point of its heights; ω the circle with center O situated inside the triangle. Construct triangle $A_1B_1C_1$ circumscribed about ω and inscribed in triangle ABC.

19.61. Consider three lines a, b, c and three points A, B, C each on the respective line. Construct points X, Y, Z on lines a, b, c, respectively, so that $BY : AX = 2$, $CZ : AX = 3$ and so that X, Y, Z are all on one line.

Solutions

19.1. A homothety with the center at the intersection point of the diagonals of the quadrilateral and with coefficient 3/2 sends the intersection points of the medians of the triangles in question into the midpoints of the sides of the quadrilateral. It remains to make use of the result of Problem 1.2.

19.2. The homothety with center K that sends $\triangle KBC$ into $\triangle KAD$ sends point M into N and, therefore, K belongs to line MN. The homothety with center L that sends $\triangle LBC$ into $\triangle LDA$ sends M into N. Therefore, L belongs to line MN.

19.3. Suppose the continuations of the lateral sides AB and CD intersect at point K and the diagonals of the trapezoid intersect at point L. By the preceding problem line KL passes through the midpoint of segment AD and by the hypothesis this line divides angle $\angle AKD$ in halves. Therefore, triangle AKD is an isosceles one (see Problem 16.1); hence, so is trapezoid $ABCD$.

19.4. The homothety with center M and coefficient -2 sends lines PA_1, PB_1 and PC_1 into lines l_a, l_b and l_c, respectively, and, therefore, the point Q to be found is the image of P under this homothety.

19.5. Consider homothety H_B^k with center B that sends segment AC into segment $A'C'$ tangent to the circumscribed circle of triangle ABC. Denote the midpoints of segments PK and $A'C'$ by O_1 and D, respectively, and the center of S by O.

Circle S is the inscribed circle of triangle $A'BC'$ and, therefore, it suffices to show that homothety H_B^k sends O_1 to O. To this end it suffices to verify that $BO_1 : BO = BA : BA'$. This equality follows from the fact that PO_1 and DA are heights of similar right triangles BPO and BDA'.

19.6. Let k be the similarity coefficient of polygons and $k < 1$. Shifting the sides of the initial polygon inside consecutively by k, k^2, k^3, $\dots$ units of length we get a contracting system of embedded convex polygons similar to the initial one with coefficients k, k^2, k^3, $\dots$. The only common point of these polygons is the center of the inscribed circle of the initial polygon.

19.7. Let A_1, B_1 and C_1 be the midpoints of sides BC, AC and AB, respectively. The homothety with center at the intersection point of the medians of triangle ABC and with coefficient $-\frac{1}{2}$ sends the circumscribed circle S of triangle ABC into the circumscribed circle S_1 of triangle $A_1B_1C_1$. Since S_1 passes through all the vertices of triangle ABC, we can construct triangle $A'B'C'$ whose sides are parallel to the respective sides of triangle ABC and for which S_1 is the inscribed circle, see Fig. 26.

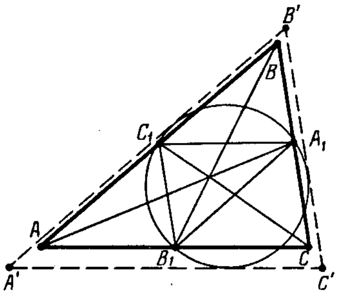


Figure 68 (Sol. 19.7)

Let r and r' be the radii of the inscribed circles of triangles ABC and $A'B'C'$; let R and R_1 be the radii of S and S_1, respectively. Clearly, $r \leq r' = R_1 = R/2$. The equality is attained if triangles $A'B'C'$ and ABC coincide, i.e., if S_1 is the inscribed circle of triangle ABC. In this case $AB_1 = AC_1$ and, therefore, $AB = AC$. Similarly, $AB = BC$.

19.8. Since

$$\overrightarrow{MM_i} = \frac{\overrightarrow{MA_1} + \dots + \overrightarrow{MA_n} - \overrightarrow{MA_i}}{n-1} = -\frac{\overrightarrow{MA_i}}{n-1},$$

it follows that the homothety with center M and coefficient $-\frac{1}{n-1}$ sends A_i into M_i.

19.9. Let A and B be a pair of most distant from each other points of polygon Φ. Then $\Phi_1 = H_A^{1/2}(\Phi)$ and $\Phi_2 = H_B^{1/2}(\Phi)$ are the required figures.

Indeed, Φ_1 and Φ_2 do not intersect because they lie on different sides of the midperpendicular to segment AB. Moreover, Φ_1 and Φ_2 are contained in Φ because Φ is a convex polygon.

19.10. Let M be the intersection point of the medians of triangle ABC, O the midpoints of segment AB. Clearly, $3\overrightarrow{OM} = \overrightarrow{OC}$ and, therefore, points M fill in the circle obtained from the initial circle under the homothety with coefficient $\frac{1}{3}$ and center O.

19.11. a) The homothety with center B that sends the inscribed circle into the escribed circle tangent to side AC sends point M into point M'. Point M' is the endpoint of the diameter perpendicular to AC and, therefore, M' is the tangent point of the inscribed circle with AC, hence, it is the intersection point of BM with AC. Therefore, $K = M'$ and K is the tangent point of the escribed circle with side AC. Now it is easy to compute that $AK = \frac{1}{2}(a + b - c) = CD$, where a, b and c are the lengths of the sides of triangle ABC.

b) Consider a homothety with center M that sends line EH into a line tangent to the given circle. This homothety sends points E, F, K and H into points E', F', K' and H', respectively. By heading a) $E'F' = K'H'$; hence, $EF = KH$.

19.12. Let us make use of the solution and notations of Problem 19.11 a). Since $AK = DC$, then $B_1K = B_1D$ and, therefore, B_1O is the midline of triangle MKD.

19.13. Let O_α, O_β, O_γ and O_δ be the centers of circles α, β, γ and δ, respectively, O_1 and O_2 the centers of the inscribed and circumscribed circles, respectively, of triangle ABC. A homothety with center O_1 sends triangle $O_\alpha O_\beta O_\gamma$ into triangle ABC. This homothety sends point O_2 into the center of the circumscribed circle of triangle $O_\alpha O_\beta O_\gamma$; this latter center coincides with O_δ. Therefore, points O_1, O_2 and O_δ belong to one line.

19.14. Let A_1, B_1 and C_1 be the centers of the given circles tangent to the sides of the triangle, O the center of the circle tangent to these circles, O_1 and O_2 the centers of the inscribed and circumscribed circles of triangle ABC. Lines AA_1, BB_1 and CC_1 are the bisectors of triangle ABC and, therefore, they intersect at point O_1. It follows that triangle $A_1B_1C_1$ turns into triangle ABC under a homothety with center O_1 and the coefficient of the homothety is equal to the ratio of distances from O_1 to the sides of triangles ABC and $A_1B_1C_1$, i.e., is equal to $\frac{r-\rho}{r}$.

Under this homothety the circumscribed circle of triangle ABC turns into the circumscribed circle of triangle $A_1B_1C_1$. Since $OA_1 = OB_1 = OC_1 = 2\rho$, the radius of the circumscribed circle of triangle $A_1B_1C_1$ is equal to 2ρ. Hence, $R\frac{r-\rho}{r} = 2\rho$, i.e., $\rho = \frac{rR}{2r+R}$.

19.15. On the bisector of angle $\angle ABC$ take an arbitrary point O and construct a circle S with center O tangent to the legs of the angle. Line BM intersects circle S at points M_1 and M_2. The problem has two solutions: circle S turns into the circles passing through M and tangent to the legs of the angle under the homothety with center B that sends M_1 into M and under the homothety with center B that sends M_2 into M.

19.16. Clearly, both circles are tangent to one of the triangle's sides. Let us show how to construct circles tangent to side AB. Let us take line c' parallel to line AB. Let us construct circles S_1' and S_2' of the same radius tangent to each other and to line c'. Let us construct tangents a' and b' to these circles parallel to lines BC and AC, respectively. The sides of triangle $A'B'C'$ formed by lines a', b' and c' are parallel to respective sides of triangle ABC. Therefore, there exists a homothety sending triangle $A'B'C'$ into triangle ABC. The desired circles are the images of circles S_1' and S_2' with respect to this homothety.

19.17. a) On sides AB and BC of triangle ABC fix segments AX_1 and CY_1 of equal length a.Through point Y_1 draw a line l parallel to side AC. Let Y_2 be the intersection point of l and the circle of radius a with center X_1 situated(who?) inside the triangle. Then point Y to be found is the intersection point of line AY_2 with side BC and X is a point on ray AB such that $AX = CY$.

b) On side AB, take an arbitrary point X_1 distinct from B. The circle of radius BX_1 with center X_1 intersects ray BC at points B and Y_1. Construct point C_1 on line BC such that $Y_1C_1 = BX_1$ and such that Y_1 lies between B and C_1. The homothety with center B that sends point C_1 into C sends X_1 and Y_1 into points X and Y to be found.

19.18. Take segment AD and draw circles S_1 and S_2 with center A and radii AB and AC, respectively. Vertex B is the intersection point of S_1 with the image of S_2 under the homothety with center D and coefficient $-\frac{DB}{DC} = -\frac{AB}{AC}$.

19.19. On the great circle S_2 take an arbitrary point X. Let S_2' be the image of S_2 under the homothety with center X and coefficient $\frac{1}{3}$, let Y be the intersection point of S_2' and S_1. Then XY is the line to be found.

19.20. From points B and C draw perpendiculars to lines AB and AC and let P be their intersection point. Then the intersection point of lines AP and BC is the desired one.

19.21. Let us draw common exterior tangents l_1 and l_2 to circles S_1 and S_2, respectively. Lines l_1 and l_2 intersect at a point K which is the center of a homothety H that sends S_1 to S_2. Let $A_1 = H(A)$. Points A and K lie on a line that connects the centers of the circles and, therefore, AA_1 is a diameter of S_2, i.e., $\angle ACA_1 = 90^\circ$ and $A_1C \parallel AB$. It follows that segment AB goes into A_1C under H. Therefore, line BC passes through K and $\angle ADK = 90^\circ$. Point D belongs to circle S with diameter AK. It is also clear that point D lies inside the angle formed by lines l_1 and l_2. Therefore, the locus of points D is the arc of S cut off by l_1 and l_2.

19.22. The hypothesis of the problem implies that the map f is one-to-one.

a) Suppose f sends point A to point A' and B to B'. Then

$$\overrightarrow{BB'} = \overrightarrow{BA} + \overrightarrow{AA'} + \overrightarrow{A'B'} = -\overrightarrow{AB} + \overrightarrow{AA'} + \overrightarrow{AB} = \overrightarrow{AA'},$$

i.e., f is a parallel translation.

b) Consider three points A, B and C not on one line. Let A', B' and C' be their images under f. Lines AB, BC and CA cannot coincide with lines $A'B'$, $B'C'$ and $C'A'$, respectively, since in this case $A = A'$, $B = B'$ and $C = C'$. Let $AB \neq A'B'$. Lines AA' and BB' are not parallel because otherwise quandrilateral $ABB'A'$ would have been a parallelogram and $\overrightarrow{AB} = \overrightarrow{A'B'}$. Let O be the intersection point of AA' and BB'. Triangles AOB and $A'OB'$ are similar with similarity coefficient k and, therefore, $\overrightarrow{OA'} = k\overrightarrow{OA}$, i.e., O is a fixed point of the transformation f. Therefore,

$$\overrightarrow{Of(X)} = \overrightarrow{f(O)f(X)} = k\overrightarrow{OX}$$

for any X which means that f is a homothety with coefficient k and center O.

19.23. Let $H = H_2 \circ H_1$, where H_1 and H_2 are homotheties with centers O_1 and O_2 and coefficients k_1 and k_2, respectively. Denote:

$$A' = H_1(A),\ B' = H_1(B),\ A'' = H_2(A'),\ B'' = H_2(B').$$

Then $\overrightarrow{A'B'} = k_1\overrightarrow{AB}$ and $\overrightarrow{A''B''} = k_2\overrightarrow{A'B'}$, i.e., $\overrightarrow{A''B''} = k_1k_2\overrightarrow{AB}$. With the help of the preceding problem this implies that for $k_1k_2 \neq 1$ the transformation H is a homothety with coefficient k_1k_2 and if $k_1k_2 = 1$, then H is a parallel translation.

It remains to verify that the fixed point of H belongs to the line that connects the centers of homotheties H_1 and H_2. Since $\overrightarrow{O_1A'} = k_1\overrightarrow{O_1A}$ and $\overrightarrow{O_2A''} = k_2\overrightarrow{O_2A'}$, it follows that

$$\overrightarrow{O_2A''} = k_2(\overrightarrow{O_2O_1} + \overrightarrow{O_1A'}) = k_2(\overrightarrow{O_2O_1} + k_1\overrightarrow{O_1A}) = \\ k_2\overrightarrow{O_2O_1} + k_1k_2\overrightarrow{O_1O_2} + k_1k_2\overrightarrow{O_2A}.$$

For a fixed point X we get the equation

$$\overrightarrow{O_2X} = (k_1k_2 - k_2)\overrightarrow{O_1O_2} + k_1k_2\overrightarrow{O_2X}$$

and, therefore, $\overrightarrow{O_2X} = \lambda\overrightarrow{O_1O_2}$, where $\lambda = \frac{k_1k_2 - k_2}{1 - k_1k_2}$.

19.24. Point A is the center of homothety that sends S_1 to S_2 and B is the center of homothety that sends S_2 to S_3. The composition of these homotheties sends S_1 to S_3 and its center belongs to line AB. On the other hand, the center of homothety that sends S_1 to S_3 is point C. Indeed, to the intersection point of the outer tangents there corresponds a homothety with any positive coefficient and a composition of homotheties with positive coefficients is a homothety with a positive coefficient.

19.25. a) Let K, L, M be the intersection points of lines AB and CD, AP and DQ, BP and CQ, respectively. These points are the centers of homotheties H_K, H_L and H_M with positive coefficients that consequtively send segments BC to AD, AD to PQ and BC to PQ. Clearly, $H_L \circ H_K = H_M$. Therefore, points K, L and M belong to one line.

b) Let K, L, M be the intersection points of lines AB and CD, AQ and DP, BQ and CP, respectively. These points are the centers of homotheties, H_K, H_L and H_M that consequtively send segments BC to AD, AD to QP, BC to QP; the coefficient of the first homothety is a positive one those of two other homotheties are negative ones. Clearly, $H_L \circ H_K = H_M$. Therefore, points K, L and M belong to one line.

19.26. Since $\angle(P_1A, AB) = \angle(P_2A, AB)$, the oriented angle values of arcs $\smile BP_1$ and $\smile BP_2$ are equal. Therefore, the rotational homothety with center B that sends S_1 to S_2 sends point P_1 to P_2 and line P_1Q_1 into line P_2Q_2.

19.27. Oriented angle values of arcs $\smile AM_1$ and $\smile AM_2$ are equal, consequently, $\angle(M_1B, BA) = \angle(M_2B, BA)$ and, therefore, points M_1, M_2 and B belong to one line.

19.28. Let P_i be a rotational homothety with center O that sends circle S_i to S_{i+1}. Then $X_{i+1} = P_i(X_i)$ (see Problem 19.27). It remains to observe that the composition $P_n \circ \cdots \circ P_2 \circ P_1$ is a rotational homothety with center O that sends S_1 to S_1, i.e., is an identity transformation.

19.29. Since $\angle AMB = \angle NAB$ and $\angle BAM = \angle BNA$, we have $\triangle AMB \sim \triangle NAB$ and, therefore, $AN : AB = MA : MB = CN : MB$. Moreover, $\angle ABM = 180^\circ - \angle MAN = \angle ANC$. It follows that $\triangle AMB \sim \triangle ACN$, i.e., the rotational homothety with center A sending M to B sends C to N and, therefore, it maps Q to P.

19.30. Let O_1 and O_2 be the centers of given circles, r_1 and r_2 be their radii. The coefficient k of the rotational homothety which maps S_1 to S_2 is equal to r_1/r_2 and its center O belongs to the circle with diameter O_1O_2. Moreover, $OO_1 : OO_2 = k = r_1/r_2$. It remains to verify that the circle with diameter O_1O_2 and the locus of points O such that $OO_1 : OO_2 = k$ have precisely two common points. For $k = 1$ it is obvious and for $k \neq 1$ the locus in question is described in the solution of Problem 7.14: it is the(A?) circle and one of its intersection points with line O_1O_2 is an inner point of segment O_1O_2 whereas the other intersection point lies outside the segment.

19.31. Consider a transformation which sends triangle BHC to triangle PHB, i.e., the composition of the rotation through an angle of 90° about point H and the homothety with coefficient $BP : CB$ and center H. Since this transformation maps the vertices of any square into vertices of a square, it maps points C and B to points B and P, respectively. Then it maps point D to Q, i.e., $\angle DHQ = 90^\circ$.

19.32. Let P be a rotational homothety that sends $\overrightarrow{CB}$ to $\overrightarrow{CA_1}$. Then

$$\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} =$$

$$\overrightarrow{AC} + P(\overrightarrow{CB}) + \overrightarrow{CB} + P(\overrightarrow{BA}) + \overrightarrow{BA} + P(\overrightarrow{AC}) = \overrightarrow{0}.$$

Hence, if M is the center of mass of triangle ABC, then

$$\overrightarrow{MA_1}+\overrightarrow{MB_1}+\overrightarrow{MC_1}=(\overrightarrow{MA}+\overrightarrow{MB}+\overrightarrow{MC})+(\overrightarrow{AA_1}+\overrightarrow{BB_1}+\overrightarrow{CC_1})=\overrightarrow{0}.$$

19.33. Let M be the common midpoint of sides BC and $B_1C_1, \mathbf{x}=\overrightarrow{MB}$ and $\mathbf{y}=\overrightarrow{MB_1}$. Further, let P be the rotational homothety with center M, the angle of rotation 90° and coefficient $\sqrt{3}$ that sends B to A and B_1 to A_1. Then $\overrightarrow{BB_1}=\mathbf{y}-\mathbf{x}$ and $\overrightarrow{AA_1}=P(\mathbf{y})-P(\mathbf{x})=P(\overrightarrow{BB_1})$. Therefore, the angle between vectors $\overrightarrow{AA_1}$ and $\overrightarrow{BB_1}$ is equal to 90° and $AA_1 : BB_1=\sqrt{3}$.

19.34. Let P be the rotational homothety that sends triangle ABC to triangle $A_1B_1C_1$. Then

$$\overrightarrow{A_2B_2}=\overrightarrow{A_2O}+\overrightarrow{OB_2}=\overrightarrow{A_1A}+\overrightarrow{BB_1}=\overrightarrow{BA}+\overrightarrow{A_1B_1}=-\overrightarrow{AB}+P(\overrightarrow{AB}).$$

Similarly, the transformation $f(\mathbf{a})=-\mathbf{a}+P(\mathbf{a})$ sends the other vectors of the sides of triangle ABC to the vectors of the sides of triangle $A_2B_2C_2$.

19.35. Let the initial map be rectangle K_0 on the plane, the smaller map rectangle K_1 contained in K_0. Let us consider a rotational homothety f that maps K_0 to K_1. Let $K_{i+1}=f(K_i)$ for $i>1$. Since the sequence K_i for $i=1, 2, \dots$ is a contracting sequence of embedded polygons, there exists (by Helly's theorem) a unique fixed point X that belongs to all the rectangles K_i.

Let us prove that X is the required point, i.e., $f(X)=X$. Indeed, since X belongs to K_i, point $f(X)$ belongs to K_{i+1}, i.e., point $f(X)$ belongs also to all rectangles K_i. Since there is just one point that belongs to all rectangles, we deduce that $f(X)=X$.

19.36. Since the product of coefficients of rotational homotheties P_1 and P_2 is equal to 1, their composition is a rotation (cf. Problem 17.36). Let O be the center of rotation $P_2\circ P_1$ and $R=P_1(O)$. Since $P_2\circ P_1(O)=O$, it follows that $P_2(R)=O$. Therefore, by hypothesis $A_1O : A_1R=A_2O : A_2R$ and $\angle OA_1R=\angle OA_2R$, i.e., $\triangle OA_1R\sim\triangle OA_2R$. Moreover, OR is a common side of these similar triangles; hence, $\triangle OA_1R=\triangle OA_2R$. Therefore, $OA_1=OA_2$ and

$$\angle(\overrightarrow{OA_1},\ \overrightarrow{OA_2})=2\angle(\overrightarrow{OA_1},\ \overrightarrow{OR})=2\angle(\overrightarrow{MA_1},\overrightarrow{MN}),$$

i.e., O is the center of rotation through an angle of $2\angle(\overrightarrow{MA_1},\overrightarrow{MN})$ that maps A_1 to A_2.

19.37. Let P_1 be the rotational homothety with center B sending A to M and P_2 be rotational homothety with center D sending M to C. Since the product of coefficients of these rotational homotheties is equal to $(BM : BA)\cdot(DC : DM)=1$, their composition $P_2\circ P_1$ is a rotation (sending A to C) through an angle of

$$\angle(\overrightarrow{AB},\overrightarrow{BM})+\angle(\overrightarrow{DM},\overrightarrow{DC})=2\angle(\overrightarrow{AB},\overrightarrow{BM}).$$

On the other hand, the center of the rotation $P_2\circ P_1$ coincides with the center of the rotation through an angle of $2\angle(\overrightarrow{AB},\overrightarrow{AM})$ that sends B to D (cf. Problem 19.36).

19.38. It is easy to verify that $\tan\angle XBY=k$ and $BY : BX=\sqrt{k^2+1}$, i.e., Y is obtained from X under the rotational homothety with center B and coefficient $\sqrt{k^2+1}$, the angle of rotation being of value $\arctan k$. The locus to be found is the image of the given half circle under this rotational homothety.

19.39. Suppose that triangle PXY is constructed and points X and Y belong to sides AC and CB, respectively. We know a transformation that maps X to Y, namely, the rotational homothety with center P, the angle of rotation $\varphi = \angle XPY = \angle MLN$ and the homothety coefficient $k = PY : PX = LN \cdot LM$. Point Y to be found is the intersection point of segment BC and the image of segment AC under this transformation.

19.40. Suppose that rectangle $ABCD$ is constructed. Consider the rotational homothety with center A that sends B to D. Let C' be the image of point C under this homothety. Then $\angle CDC' = \angle B + \angle D$ and $DC' = \frac{BC \cdot AD}{AB} = \frac{bd}{a}$.

We can recover triangle CDC' from CD, DC' and $\angle CDC'$. Point A is the intersection point of the circle of radius d with center D and the locus of points X such that $C'X : CX = d : a$ (this locus is a circle, see Problem 7.14). The further construction is obvious.

19.41. a) If O is the center of a rotational homothety that sends segment AB to segment A_1B_1, then

$$\angle(PA, AO) = \angle(PA_1, A_1O) \quad \text{and} \quad \angle(PB, BO) = \angle(PB_1, B_1O) \tag{1}$$

and, therefore, point O is the intersection point of the inscribed circles of triangles PAA_1 and PBB_1.

The case when these circles have only one common point P is clear: this is when segment AB turns into segment A_1B_1 under a homothety with center P.

If P and O are two intersection points of the circles considered, then equalities (1) imply that $\triangle OAB \sim \triangle OA_1B_1$ and, therefore, O is the center of a rotational homothety that maps segment AB into segment A_1B_1.

b) It suffices to notice that point O is the center of a rotational homothety that maps segment AB to segment BC if and only if $\angle(BA, AO) = \angle(CB, BO)$ and $\angle(AB, BO) = \angle(BC, CO)$.

19.42. Let A_1 and B_1 be the positions of the points at one moment, A_2 and B_2 the position of the points at another moment. Then for point P we can take the center of a rotational homothety that maps segment A_1A_2 to segment B_1B_2.

19.43. Let P be the intersection point of lines l_1 and l_2. By Problem 19.41 point O belongs to the circumscribed circle S_1 of triangle A_1A_2P. On the other hand, $OA_2 : OA_1 = k$. The locus of points X such that $XA_2 : XA_1 = k$ is circle S_2 (by Problem 7.14). Point O is the intersection point of circles S_1 and S_2 (there are two such points).

19.44. Let O be the center of a rotational homothety that maps segment AB to segment A_1B_1. Then $\triangle ABO \sim \triangle A_1B_1O$, i.e., $\angle AOB = \angle A_1OB_1$ and $AO : BO = A_1O : B_1O$. Therefore, $\angle AOA_1 = \angle BOB_1$ and $AO : A_1O = BO : B_1O$, i.e., $\triangle AA_1O \sim \triangle BB_1O$. Hence, point O is the center of the rotational homothety that maps segment AA_1 to segment BB_1.

19.45. Let lines AB and DE intersect at point C and lines BD and AE intersect at point F. The center of rotational homothety that maps segment AB to segment ED is the distinct from C intersection point of the circumscribed circles of triangles AEC and BDC (see Problem 19.41) and the center of rotational homothety sending AE to BD is the intersection point of circles circumscribed about triangles ABF and EDF. By Problem 19.44 the centers of these rotational homotheties coincide, i.e., all the four circumscribed circles have a common point.

19.46. The center O of parallelogram $ABCD$ is equidistant from the following pairs of lines: AQ and AB, AB and CD, CD and DQ and, therefore, QO is the bisector of angle $\angle AQD$. Let $\alpha = \angle BAO$, $\beta = \angle CDO$ and $\varphi = \angle AQO = \angle DQO$. Then $\alpha + \beta = \angle AOD = 360^\circ - \alpha - \beta - 2\varphi$, i.e., $\alpha + \beta + \varphi = 180^\circ$ and, therefore, $\triangle QAO \sim \triangle QOD$.

19.47. Let us solve a slightly more general problem. Suppose point O is taken on circle S and H is a rotational homothety with center O. Let us prove that then all lines XX', where X is a point from S and $X' = H(X)$, intersect at one point.

Let P be the intersection point of lines X_1X_1' and X_2X_2'. By Problem 19.41 points O, P, X_1 and X_2 lie on one circle and points O, P, X_1' and X_2' also belong to one circle. Therefore, P is an intersection point of circles S and $H(S)$, i.e., all lines XX' pass through the distinct from O intersection point of circles S and $H(S)$.

19.48. Let O be the center of a rotational homothety sending triangle $A_1B_1C_1$ to triangle ABC. Let us prove that, for instance, the circumscribed circles of triangles ABC_2 and $A_1B_1C_2$ pass through point O. Under the considered homothety segment AB goes into segment A_1B_1; therefore, point O coincides with the center of the rotational homothety that maps segment AA_1 to segment BB_1 (see Problem 19.44). By problem 19.41 the center of the latter homothety is the second intersection point of the circles circumscribed about triangles ABC_2 and $A_1B_1C_2$ (or is their tangent point).

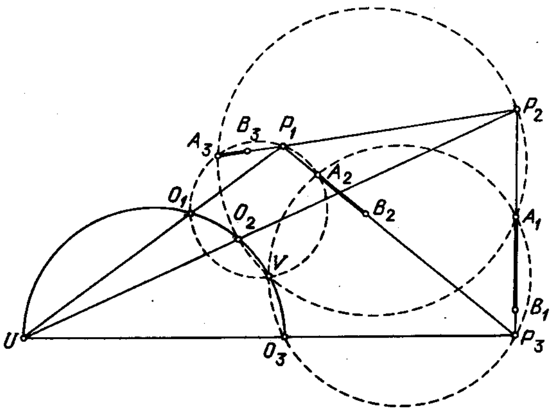


Figure 169 (Sol. 19.48)

19.49. Points A_1, A_2 and A_3 belong to lines P_2P_3, P_3P_1 and P_1P_2 (Fig. 27). Therefore, the circles circumscribed about triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$ have a common point V (see Problem 2.80 a)), and points O_3, O_2 and O_1 lie on these circles (see Problem 19.41). Similarly, the circles circumscribed about triangles $B_1B_2P_3$, $B_1B_3P_2$ and $B_2B_3P_1$ have a common point V'. Let U be the intersection point of lines P_2O_2 and P_3O_3. Let us prove that point V belongs to the circle circumscribed about triangle O_2O_3U. Indeed,

$$\angle(O_2V, VO_3) = \angle(VO_2, O_2P_2) + \angle(O_2P_2, P_3O_3) + \angle(P_3O_3, O_3V) =$$
$$\angle(VA_1, A_1P_2) + \angle(O_2U, UO_3) + \angle(P_3A_1, A_1V) = \angle(O_2U, UO_3).$$

Analogous arguments show that point V' belongs to the circle circumscribed about triangle O_2O_3U. In particular, points O_2, O_3, V and V' belong to one circle. Similarly, points O_1, O_2, V and V' belong to one circle and, therefore, points V and V' belong to the circle circumscribed about triangle $O_1O_2O_3$; point U also belongs to this circle.

We can similarly prove that lines P_1O_1 and P_2O_2 intersect at one point that belongs to the similarity circle. Line P_2O_2 intersects the similarity circle at points U and O_2 and, therefore, line P_1O_1 passes through point U.

19.50. Let P_1 be the intersection point of lines A_2B_2 and A_3B_3, let P_1' be the intersection point of lines A_2C_2 and A_3C_3; let points P_2, P_3, P_2' and P_3' be similarly defined. The rotational homothety that sends F_1 to F_2 sends lines A_1B_1 and A_1C_1 to lines A_2B_2 and A_2C_2, respectively, and, therefore, $\angle(A_1B_1, A_2B_2) = \angle(A_1C_1, A_2C_2)$. Similar arguments show that $\triangle P_1P_2P_3 \sim \triangle P_1'P_2'P_3'$.

The center of the rotational homothety that maps segment P_2P_3 to $P_2'P_3'$ belongs to the circle circumscribed about triangle $A_1P_3P_3'$ (see Problem 19.41). Since

$$\angle(P_3A_1, A_1P_3') = \angle(A_1B_1, A_1C_1) = \angle(A_2B_2, A_2C_2) = \angle(P_3A_2, A_2P_3'),$$

the circle circumscribed about triangle $A_1P_3P_3'$ coincides with the circle circumscribed about triangle $A_1A_2P_3$. Similar arguments show that the center of the considered rotational homothety is the intersection point of the circles circumscribed about triangles $A_1A_2P_3$, $A_1A_3P_2$ and $A_2A_3P_1$; this point belongs to the similarity circle of figures F_1, F_2 and F_3 (see Problem 19.49 a)).

19.51. a) Let l_1', l_2' and l_3' be the corresponding lines of figures F_1, F_2 and F_3 such that $l_i' \parallel l_i$. These lines form triangle $P_1P_2P_3$. The rotational homothety with center O_3 that maps F_1 to F_2 sends lines l_1 and l_1' to lines l_2 and l_2', respectively, and, therefore, the homothety with center O_3 that maps l_1 to l_1' sends line l_2 to l_2'. Therefore, line P_3O_3 passes through point W.

Similarly, lines P_1O_1 and P_2O_2 pass through point W; hence, W belongs to the similarity circle of figures F_1, F_2 and F_3 (see Problem 19.49 b)).

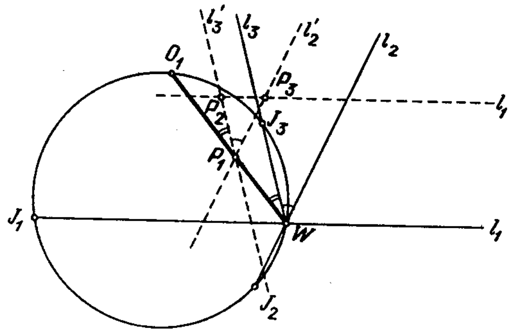


Figure 170 (Sol. 19.51 a))

b) The ratio of the distances from point O_1 to lines l_2' and l_3' is equal to the coefficient of the rotational homothety that maps F_2 to F_3 and the angle $\angle P_1$ of triangle $P_1P_2P_3$ is equal to the angle of the rotation. Therefore, $\angle(O_1P_1, P_1P_2)$ only depends on figures F_2 and F_3. Since $\angle(O_1W, WJ_3) = \angle(O_1P_1, P_1P_2)$, arc $\smile O_1J_3$ is fixed (see Fig. 28) and, therefore, point J_3 is fixed. We similarly prove that points J_1 and J_2 are fixed.

19.52. Let us make use of notations from Problem 19.51. Clearly,

$$\angle(J_1J_2, J_2J_3) = \angle(J_1W, WJ_3) = \angle(P_3P_2, P_2P_1).$$

For the other angles of the triangle the proof is similar.

19.53. Let us prove, for instance, that under the rotational homothety with center O_1 that maps F_2 to F_3 point J_2 goes to J_3. Indeed, $\angle(J_2O_1, O_1J_3) = \angle(J_2W, WJ_3)$. Moreover, lines J_2W and J_3W are the corresponding lines of figures F_2 and F_3 and, therefore, the distance from lines J_2W and J_3W to point O_1 is equal to the similarity coefficient k_1; hence, $\frac{O_1J_2}{O_1J_3} = k_1$.

19.54. Let O_a be the intersection point of the circle passing through point B and tangent to line AC at point A and the circle passing through point C and tangent to line AB at point A.

By Problem 19.41 b) point O_a is the center of rotational homothety that sends segment BA to segment AC. Having similarly defined points O_b and O_c and making use of the result of Problem 19.49 b) we see that lines AO_a, BO_b and CO_c intersect at a point that belongs to the similarity circle S. On the other hand, these lines intersect at Lemoin's point K (see Problem 5.128).

The midperpendiculars to the sides of the triangle are the corresponding lines of the considered similar figures. The midperpendiculars intersect at point O; hence, O belongs to the similarity circle S (see Problem 19.51 a)). Moreover, the midperpendiculars intersect S at fixed points A_1, B_1 and C_1 of triangle ABC (see Problem 19.51 b)). On the other hand, the lines passing through point K parallel to BC, CA and AB are also corresponding lines of the considered figures (see solution to Problem 5.132), therefore, they also intersect circle S at points A_1, B_1 and C_1. Hence, $OA_1 \perp A_1K$, i.e., OK is a diameter of S.

19.55. If P is the first of Brokar's points of triangle ABC, then CP, AP and BP are the corresponding lines for similar figures constructed on segments BC, CA and AB. Therefore, point P belongs to the similarity circle S (see Problem 19.51 a)). Similarly, point Q belongs to S. Moreover, lines CP, AP and BP intersect S at fixed points A_1, B_1 and C_1 of triangle ABC (cf. Problem 19.51 b)). Since $KA_1 \parallel BC$ (see the solution of Problem 19.54), it follows that $\angle(PA_1, A_1K) = \angle(PC, CB) = \varphi$, i.e., $\smile PK = 2\varphi$. Similarly, $\smile KQ = 2\varphi$. Therefore, $PQ \perp KO$; hence, $OP = OQ$ and $\angle POQ = \frac{1}{2} \smile PQ = 2\varphi$.

Chapter 20. THE PRINCIPLE OF AN EXTREMAL ELEMENT

Background

1. Solving various problems it is often convenient to consider a certain extremal or "boundary" element, i.e., an element at which a certain function takes its maximal or minimal value. For instance, the longest or the shortest side a triangle, the greatest or the smallest angle, etc. This method for solving problems is sometimes called *the principle* (or the *rule*) *of an extremal element*; this term, however, is not conventional.

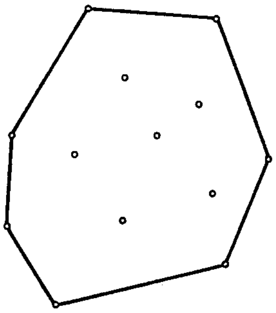


Figure 171

2. Let O be the intersection point of the diagonals of a convex quadrilateral. Its vertices can be denoted so that $CO \leq AO$ and $BO \leq DO$ (see Fig. *). Then under symmetries with respect to point O triangle BOC is mapped inside triangle AOD, i.e., in a certain sense triangle BOC is the smallest and triangle AOD is the greatest (see §4).

3. The vertices of the convex hull and the basic lines are also extremal elements; to an extent these notions are used in §5 where they are defined and where their main properties are listed.

§1. The least and the greatest angles

20.1. Prove that if the lengths of all the sides of a triangle are smaller than 1, then its area is smaller than $\frac{1}{4}\sqrt{3}$.

20.2. Prove that the disks constructed on the sides of a convex quadrilateral as on diameters completely cover this quadrilateral.

20.3. In a country, there are 100 airports such that all the pairwise distances between them are distinct. From each airport a plane lifts up and flies to the nearest airport. Prove that there is no airport to which more than five planes can arrive.

20.4. Inside a disk of radius 1, eight points are placed. Prove that the distance between some two of them is smaller than 1.

20.5. Six disks are placed on the plane so that point O is inside each of them. Prove that one of these disks contains the center of some other disk.

20.6. Inside an acute triangle point P is taken. Prove that the greatest distance from P to the vertices of this triangle is smaller than twice the shortest of the distances from P to the sides of the triangle.

20.7. The lengths of a triangle's bisectors do not exceed 1. Prove that the area of the triangle does not exceed $\frac{1}{\sqrt{3}}$.

§2. The least and the greatest distances

20.8. Given $n \geq 3$ points on the plane not all of them on one line. Prove that there is a circle passing through three of the given points such that none of the remaining points lies inside the circle.

20.9. Several points are placed on the plane so that all the pairwise distances between them are distinct. Each of these points is connected with the nearest one by a line segment. Do some of these segments constitute a closed broken line?

20.10. Prove that at least one of the bases of perpendiculars dropped from an interior point of a convex polygon to its sides is on the side itself and not on its extension.

20.11. Prove that in any convex pentagon there are three diagonals from which one can construct a triangle.

20.12. Prove that it is impossible to cover a polygon with two polygons which are homothetic to the given one with coefficient k for $0 < k < 1$.

20.13. Given finitely many points on the plane such that any line passing through two of the given points contains one more of the given points. Prove that all the given points belong to one line.

20.14. In plane, there are given finitely many pairwise non-parallel lines such that through the intersection point of any two of them one more of the given lines passes. Prove that all these lines pass through one point.

20.15. In plane, there are given n points. The midpoints of all the segments with both endpoints in these points are marked, the given points are also marked. Prove that there are not less than $2n - 3$ marked points.

See also Problems 9.17, 9.19.

§3. The least and the greatest areas

20.16. In plane, there are n points. The area of any triangle with vertices in these points does not exceed 1. Prove that all these points can be placed in a triangle whose area is equal to 4.

20.17. Polygon M' is homothetic to a polygon M with homothety coefficient equal to $-\frac{1}{2}$. Prove that there exists a parallel translation that sends M' inside M.

§4. The greatest triangle

20.18. Let O be the intersection point of diagonals of convex quadrilateral $ABCD$. Prove that if the perimeters of triangles ABO, BCO, CDO and DAO are equal, then $ABCD$ is a rhombus.

20.19. Prove that if the center of the inscribed circle of a quadrilateral coincides with the intersection point of the diagonals, then this quadrilateral is a rhombus.

20.20. Let O be the intersection point of the diagonals of convex quadrilateral $ABCD$. Prove that if the radii of inscribed circles of triangles ABO, BCO, CDO and DAO are equal, then $ABCD$ is a rhombus.

§5. The convex hull and the base lines

While solving problems of this section we will consider convex hulls of systems of points and base lines of convex polygons.

The *convex hull* of a finite set of points is the least convex polygon which contains all these points. The word "least" means that the polygon is not contained in any other such polygon. Any finite system of points possesses a unique convex hull (Fig. 29).

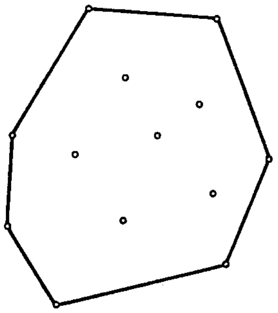


Figure 171

A *base line* of a convex polygon is a line passing through its vertex and with the property that the polygon is situated on one side of it. It is easy to verify that for any convex polygon there exist precisely two base lines parallel to a given line (Fig. 30).

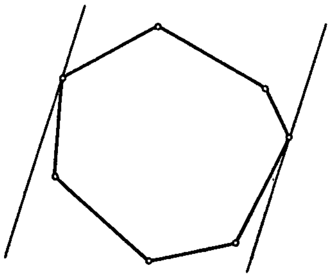


Figure 172

20.21. Solve Problem 20.8 making use of the notion of the convex hull.

20.22. Given $2n+3$ points on a plane no three of which belong to one line and no four of which belong to one circle. Prove that one can select three points among these so that n of the remaining points lie inside the circle drawn through the selected points and n of the points lie outside the circle.

20.23. Prove that any convex polygon of area 1 can be placed inside a rectangle of area 2.

20.24. Given a finite set of points in plane prove that there always exists a point among them for which not more than three of the given points are the nearest to it.

20.25. On the table lie n cardboard and n plastic squares so that no two cardboard and no two plastic squares have common points, the boundary points included. It turned out that the set of vertices of the cardboard squares coincides with that of the plastic squares. Is it necessarily true that every cardboard square coincides with a plastic one?

20.26. Given $n \geq 4$ points in plane so that no three of them belong to one line. Prove that if for any 3 of them there exists a fourth (among the given ones) together with which they form vertices of a parallelogram, then $n = 4$.

§6. Miscellaneous problems

20.27. In plane, there are given a finite set of (not necessarily convex) polygons each two of which have a common point. Prove that there exists a line having a common point with all these polygons.

20.28. Is it possible to place 1000 segments on the plane so that the endpoints of every segment are interior points of certain other of these segments?

20.29. Given four points in plane not on one line. Prove that at least one of the triangles with vertices in these points is not an acute one.

20.30. Given an infinite set of rectangles in plane. The vertices of each of the rectangles lie in points with coordinates $(0,0)$, $(0,m)$, $(n,0)$, (n,m), where n and m are positive integers (each rectangle has its own numbers). Prove that among these rectangles one can select such a pair that one is contained inside the other one.

20.31. Given a convex polygon $A_1\dots A_n$, prove that the circumscribed circle of triangle $A_iA_{i+1}A_{i+2}$ contains the whole polygon.

Solutions

20.1. Let α be the least angle of the triangle. Then $\alpha\leq 60^\circ$. Therefore, $S=\frac{bc\sin\alpha}{2}\leq\frac{\sin 60^\circ}{2}=\frac{\sqrt{3}}{4}$.

20.2. Let X be an arbitrary point inside a convex quadrilateral. Since

$$\angle AXB+\angle BXC+\angle CXD+\angle AXD=360^\circ,$$

the maximal of these angles is not less than 90°. Let, for definiteness sake, $\angle AXB\geq 90^\circ$. Then point X is inside the circle with diameter AB.

20.3. If airplanes from points A and B arrived to point O, then AB is the longest side of triangle AOB, i.e., $\angle AOB>60^\circ$. Suppose that airplanes from points $A_1, \dots, A_n$ arrived to point O. Then one of the angles $\angle A_iOA_j$ does not exceed $\frac{360^\circ}{n}$. Therefore, $\frac{360^\circ}{n}>60^\circ$, i.e., $n<6$.

20.4. At least seven points are distinct from the center O of the circle. Therefore, the least of the angles $\angle A_iOA_j$, where A_i and A_j are given points, does not exceed $\frac{360^\circ}{7}<60^\circ$. If A and B are points corresponding to the least angle, then $AB<1$ because $AO\leq 1$, $BO\leq 1$ and angle $\angle AOB$ cannot be the largest angle of triangle AOB.

20.5. One of the angles between the six segments that connect point O with the centers of the disks does not exceed $\frac{360^\circ}{6}=60^\circ$. Let $\angle O_1OO_2\leq 60^\circ$, where O_1 and O_2 are the centers of the disks of radius r_1 and r_2, respectively. Since $\angle O_1OO_2\leq 60^\circ$, this angle is not the largest angle in triangle O_1OO_2 and, therefore, either $O_1O_2\leq O_1O$ or $O_1O_2\leq O_2O$. Let, for definiteness, $O_1O_2\leq O_1O$. Since point O is inside the circles, $O_1O<r_1$. Therefore, $O_1O_2\leq O_1O<r_1$, i.e., point O_2 is inside the disk of radius r_1 with center O_1.

20.6. Let us drop perpendiculars PA_1, PB_1 and PC_1 from point P to sides BC, CA and AB, respectively, and select the greatest of the angles formed by these perpendiculars and rays PA, PB and PC. Let, for definiteness sake, this be angle $\angle APC_1$. Then $\angle APC_1\geq 60^\circ$; hence, $PC_1:AP=\cos APC_1\leq\cos 60^\circ=\frac{1}{2}$, i.e., $AP\geq 2PC_1$. Clearly, the inequality still holds if AP is replaced with the greatest of the numbers AP, BP and CP and PC_1 is replaced with the smallest of the numbers PA_1, PB_1 and PC_1.

20.7. Let, for definiteness, α be the smallest angle of triangle ABC; let AD be the bisector. One of sides AB and AC does not exceed $AD/\cos(\alpha/2)$ since otherwise segment BC does not pass through point D. Let, for definiteness,

$$AB\leq\frac{AD}{\cos(\alpha/2)}\leq\frac{AD}{\cos 30^\circ}\leq\frac{2}{\sqrt{3}}.$$

Then $S_{ABC}=\frac{1}{2}h_cAB\leq\frac{1}{2}l_cAB\leq\frac{1}{\sqrt{3}}$.

20.8. Let A and B be those of the given points for which the distance between them is minimal. Then inside the circle with diameter AB there are no given points. Let C be the remaining point — the vertex of the greatest angle that subtends segment AB. Then inside the circle passing through points A, B and C there are no given points.

20.9. Suppose that we have obtained a closed broken line. Then AB is the longest link of this broken line and AC and BD are the links neighbouring to AB. Then $AC < AB$, i.e., B is not the point closest to A and $BD < AB$, i.e., A is not the point closest to B. Therefore, points A and B cannot be connected. Contradiction.

20.10. Let O be the given point. Let us draw lines containing the sides of the polygon and select among them the one which is the least distant from point O. Let this line contain side AB. Let us prove that the base of the perpendicular dropped from O to AB belongs to side AB itself. Suppose that the base of the perpendicular dropped from O to line AB is point P lying outside segment AB. Since O belongs to the interior of the convex polygon, segment OP intersects side CD at point Q. Clearly, $OQ < OP$ and the distance from O to line CD is smaller than OQ. Therefore, line CD is less distant from point O than line AB. This contradicts the choice of line AB.

20.11. Let BE be the longest diagonal of pentagon $ABCDE$. Let us prove then that from segments BE, EC and BD one can construct a triangle. To this end, it suffices to verify that $BE < EC + BD$. Let O be the intersection point of diagonals BD and EC. Then

$$BE < BO + OE < BD + EC.$$

20.12. Let O_1 and O_2 be the centers of homotheties, each with coefficient k, sending polygon M to polygons M_1 and M_2, respectively. Then a point from M the most distant from line O_1O_2 is not covered by polygons M_1 and M_2.

20.13. Suppose that not all of the given points lie on one line. Through every pair of given points draw a line (there are finitely many of such lines) and select the least nonzero distance from the given points to these lines. Let the least distance be the one from point A to line BC, where points B and C are among given ones.

On line BC, there lies one more of the given points, D. Drop perpendicular AQ from point A to line BC. Two of the points B, C and D lie to one side of point Q, let these be C and D. Let, for definiteness, $CQ < DQ$ (Fig. 31).

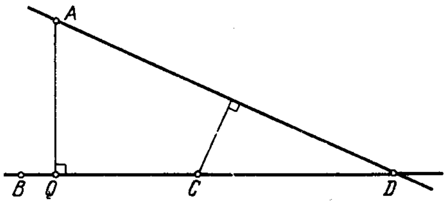


Figure 173 (Sol. 20.13)

Then the distance from point C to line AD is smaller than that from A to line BC which contradicts to the choice of point A and line BC.

20.14. Suppose that not all lines pass through one point. Consider the intersection points of lines and select the least nonzero distance from these points to the given lines. Let the least distance be the one from point A to line l. Through point A at least three of given lines pass. Let them intersect line l at points B, C and D. From point A drop perpendicular AQ to line l.

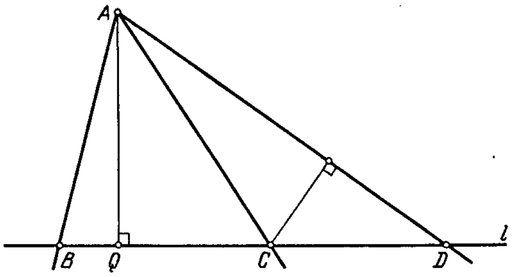


Figure 174 (Sol. 20.14)

Two of the points B, C and D lie on one side of point Q, let them be C and D. Let, for definiteness, $CQ < DQ$ (Fig. 32). Then the distance from point C to line AD is smaller than the distance from point A to line l which contradicts the choice of A and l.

20.15. Let A and B be the most distant from each other given points. The midpoints of the segments that connect point A (resp. B) with the other points are all distinct and lie inside the circle of radius $\frac{1}{2}AB$ with center A (resp. B). The two disks obtained have only one common point and, therefore, there are no less than $2(n-1)-1 = 2n-3$ distinct fixed points.

20.16. Among all the triangles with vertices in the given points select a triangle of the greatest area. Let this be triangle ABC. Let us draw through vertex C line l_c so that $l_c \parallel AB$. If points X and A lie on different sides of line l_c, then $S_{ABX} > S_{ABC}$. Therefore, all the given points lie on one side of l_c.

Similarly, drawing lines l_b and l_a through points B and A so that $l_b \parallel AC$ and $l_a \parallel BC$ we see that all given points lie inside (or on the boundary of) the triangle formed by lines l_a, l_b and l_c. The area of this triangle is exactly four times that of triangle ABC and, therefore, it does not exceed 4.

20.17. Let ABC be the triangle of the greatest area among these with vertices in the vertices of polygon M. Then M is contained inside triangle $A_1B_1C_1$ the midpoints of whose sides are points A, B and C. The homothety with center in the center of mass of triangle ABC and with coefficient $-\frac{1}{2}$ sends triangle $A_1B_1C_1$ to triangle ABC and, therefore, sends polygon M inside triangle ABC.

20.18. For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points B_1 and C_1 be symmetric to points B and C through point O (Fig. 33).

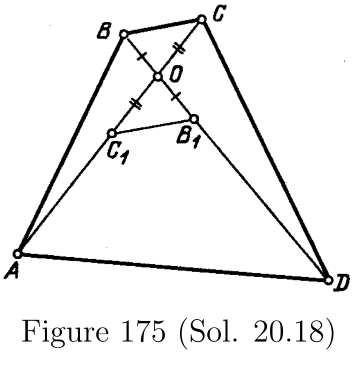


Figure 175 (Sol. 20.18)

Since triangle B_1OC_1 lies inside triangle AOD, it follows that $P_{AOD} \geq P_{B_1OC_1} = P_{BOC}$ and the equality is attained only if $B_1 = D$ and $C_1 = A$ (see Problem 9.27 b)). Therefore, $ABCD$ is a parallelogram. Therefore, $AB - BC = P_{ABO} - P_{BCO} = 0$, i.e., $ABCD$ is a rhombus.

20.19. Let O be the intersection point of the diagonals of quadrilateral $ABCD$. For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points B_1 and C_1 be symmetric to points B and C, respectively, through point O. Since O is the center of the circle inscribed into the quadrilateral, we see that segment B_1C_1 is tangent to this circle. Therefore, segment AD can be tangent to this circle only if $B_1 = D$ and $C_1 = A$, i.e., if $ABCD$ is a parallelogram. One can inscribe a circle into this parallelogram since this parallelogram is a rhombus.

20.20. For definiteness, we may assume that $AO \geq CO$ and $DO \geq BO$. Let points B_1 and C_1 be symmetric to points B and C through point O. Then triangle C_1OB_1 is contained inside triangle AOD and, therefore, the inscribed circle S of triangle C_1OB_1 is contained inside triangle AOD. Suppose that segment AD does not coincide with segment C_1B_1. Then circle S turns into the inscribed circle of triangle AOD under the homothety with center O and coefficient greater than 1, i.e., $r_{AOD} > r_{C_1OB_1} = r_{COB}$. We have got a contradiction; hence, $A = C_1$ and $D = B_1$, i.e., $ABCD$ is a parallelogram.

In parallelogram $ABCD$, the areas of triangles AOB and BOC are equal and, therefore, if the inscribed circles have equal radii, then they have equal perimeters since $S = pr$. It follows that $AB = BC$, i.e., $ABCD$ is a rhombus.

20.21. Let AB be the side of the convex hull of the given points, B_1 be the nearest to A of all the given points that lie on AB. Select the one of the remaining points that is the vertex of the greatest angle that subtends segment AB_1. Let this be point C. Then the circumscribed circle of triangle AB_1C is the one to be found.

20.22. Let AB be one of the sides of the convex hull of the set of given points. Let us enumerate the remaining points in the order of increase of the angles with vertex in these points that subtend segment AB, i.e., denote them by $C_1, C_2, \dots, C_{2n+1}$ so that

$$\angle AC_1B < \angle AC_2B \leq \dots < \angle AC_{2n+1}B.$$

Then points $C_1, \dots, C_n$ lie outside the circle circumscribed about triangle ABC_{n+1} and points $C_{n+2}, \dots, C_{2n+1}$ lie inside it, i.e., this is the circle to be constructed.

20.23. Let AB be the greatest diagonal (or side) of the polygon. Through points A and B draw lines a and b perpendicular to line AB. If X is a vertex of the polygon, then $AX \leq AB$ and $XB \leq AB$, therefore, the polygon lies inside the band formed by lines a and b.

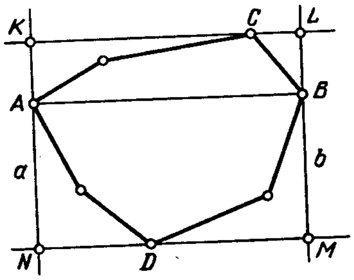


Figure 176 (Sol. 20.23)

Draw the base lines of the polygon parallel to AB. Let these lines pass through vertices C and D and together with a and b form rectangle $KLMN$ (see Fig. 34). Then

$$S_{KLMN} = 2S_{ABC} + 2S_{ABD} = 2S_{ACBD}.$$

Since quadrilateral $ACBD$ is contained in the initial polygon whose area is equal to 1, $S_{KLMN} \leq 2$.

20.24. Select the least of all the distances between the given points and consider points which have neighbours at this distance. Clearly, it suffices to prove the required statement for these points. Let P be the vertex of the convex hull of these points. If A_i and A_j are the points nearest to P, then $A_iA_j \geq A_iP$ and $A_iA_j \geq A_jP$ and, therefore, $\angle A_iPA_j \geq 60^\circ$. It follows that P cannot have four nearest neighbours since otherwise one of the angles $\angle A_iPA_j$ would have been smaller than $\frac{180^\circ}{3} = 60^\circ$. Therefore, P is the point to be found.

20.25. Suppose that there are cardboard squares that do not coincide with the plastic ones. Let us discard all the coinciding squares and consider the convex hull of the vertices of the remaining squares. Let A be a vertex of this convex hull. Then A is a vertex of two distinct squares, a cardboard one and a plastic one. It is easy to verify that one of the vertices of the smaller of these squares lies inside the larger one (Fig. 35).

Let, for definiteness, vertex B of the cardboard square lie inside the plastic one. Then point B lies inside a plastic square and is a vertex of another plastic square, which is impossible. This is a contradiction, hence, every cardboard square coincides with a plastic one.

20.26. Let us consider the convex hull of the given points. The two cases are possible:

1) The convex hull is a parallelogram, $ABCD$. If point M lies inside parallelogram $ABCD$, then the vertices of all three parallelograms with vertices at A, B, and M lie outside $ABCD$ (Fig. 36). Hence, in this case there can be no other points except A, B, C and D.

2) The convex hull is not a parallelogram. Let AB and BC be edges of the convex hull. Let us draw base lines parallel to AB and BC. Let these base lines pass through vertices P and Q. Then the vertices of all the three parallelograms with vertices at B, P and Q lie outside the convex hull (Fig. 37).

They even lie outside the parallelogram formed by the base lines except for the case when P and Q are vertices of this parallelogram. In this last case the fourth vertex of the parallelogram does not belong to the convex hull since the convex hull is not a parallelogram.

20.27. In plane, take an arbitrary straight line l and project all the polygons to it. We will get several segments any two of which have a common point. Let us order line l; consider left endpoints of the segments-projections and select the right-most left endpoint. The point belongs to all the segments and, therefore, the perpendicular drawn through it to l intersects all the given polygons.

20.28. Let 1000 segments lie in plane. Take an arbitrary line l not perpendicular to any of them and consider the projections of the endpoints of all these segments on l. It is clear that the endpoint of the segment whose projection is the left-most of the obtained points cannot belong to the interior of another segment.

20.29. Two variants of disposition of these four points are possible:

(1) The points are vertices of a convex quadrilateral, $ABCD$. Take the largest of the angles of its vertices. Let this be angle $\angle ABC$. Then $\angle ABC \geq 90^\circ$, i.e., triangle ABC is not an acute one.

(2) Point D lies inside triangle ABC. Select the greatest of the angles $\angle ADB$, $\angle BDC$ and $\angle ADC$. Let this be angle $\angle ADB$. Then $\angle ADB \geq 120^\circ$, i.e., triangle ADB is an obtuse one.

We can prove in the following way that there are no other positions of the four points. The lines that pass through three of given points divide the plane into seven parts (Fig. 38). If the fourth given point belongs to the 2nd, 4th or 6th part, then we are in situation (1); if it belongs to the 1st, 3rd, 5th or 7th part, then we are in situation (2).

20.30. The rectangle with vertices at points $(0, 0)$, $(0, m)$, $(n, 0)$ and (n, m) the horizontal side is equal to n and vertical side is equal to m. From the given set select a rectangle with

Figure 177 (Sol. 20.25)

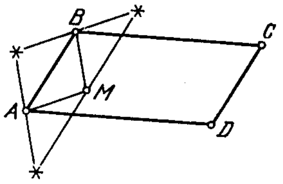


Figure 178 (Sol. 20.26)

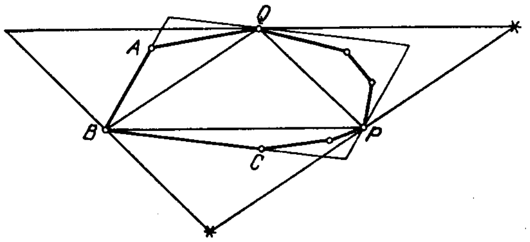


Figure 179 (Sol. 20.26)

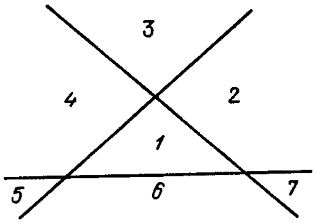


Figure 180 (Sol. 20.29)

the least horizontal side. Let the length of its vertical side be equal to m_1. Consider any side m_1 of the remaining rectangles. The two cases are possible:

1) The vertical sides of two of these m_1-rectangles are equal. Then one of them is contained in another one.

2) The vertical sides of all these rectangles are distinct. Then the vertical side of one of them is greater than m_1 and, therefore, it contains the rectangle with the least horizontal side.

20.31. Consider all the circles passing through two neighbouring vertices A_i and A_{i+1} and a vertex A_j such that $\angle A_iA_jA_{i+1} < 90^\circ$. At least one such circle exists. Indeed, one of the angles $\angle A_iA_{i+2}A_{i+1}$ and $\angle A_{i+1}A_iA_{i+2}$ is smaller than 90°; in the first case set $A_j = A_{i+2}$ and in the second case set $A_j = A_i$. Among all such circles (for all i and j) select a circle S of the largest radius; let, for definiteness, it pass through points A_1, A_2 and A_k.

Suppose that vertex A_p lies outside S. Then points A_p and A_k lie on one side of line A_1A_2 and $\angle A_1A_pA_2 < \angle A_1A_kA_2 \leq 90^\circ$. The law of sines implies that the radius of the circumscribed circle of triangle $A_1A_pA_2$ is greater than that of $A_1A_kA_2$. This is a contradiction and, therefore, S contains the whole polygon $A_1 \dots A_n$.

Let, for definiteness sake, $\angle A_2A_1A_k \leq \angle A_1A_2A_k$. Let us prove then that A_2 and A_k are neighbouring vertices. If $A_k \neq A_3$, then

$$180^\circ - \angle A_2A_3A_k \leq \angle A_2A_1A_k \leq 90^\circ$$

and, therefore, the radius of the circumscribed circle of triangle $A_2A_3A_k$ is greater than the radius of the circumscribed circle of triangle $A_1A_2A_k$. Contradiction implies that S passes through neighbouring vertices A_1, A_2 and A_3.

Chapter 21. DIRICHLET'S PRINCIPLE

Background

1. The most popular (Russian) formulation of *Dirichlet's* or *pigeonhole principle* is the following one: "If m rabbits sit in n hatches and $m > n$, then at least one hatch contains at least two rabbits."

It is even unclear at first glance why this absolutely transparent remark is a quite effective method for solving problems. The point is that in every concrete problem it is sometimes difficult to see what should we designate as the rabbits and the hatches and why there are more rabbits than the hatches. The choice of rabbits and hatches is often obscured; and from the formulation of the problem it is not often clear how to immediately deduce that one should apply Dirichlet's principle. What is very important is that this method gives a *nonconstructive* proof (naturally, we cannot say which precisely hatch contains two rabbits and only know that such a hatch exists) and an attempt to give a constructive proof, i.e., the proof by explicitly constructing or indicating the desired object can lead to far greater difficulties (and more profound results).

2. Certain problems are also solved by methods in a way similar to Dirichlet's principle. Let us formulate the corresponding statements (all of them are easily proved by the rule of contraries).

a) If several segments the sum of whose lengths is greater than 1 lie on a segment of length 1, then at least two of them have a common point.

b) If several arcs the sum of whose lengths is greater than 2π lie on the circle of radius 1, then at least two of them have a common point.

c) If several figures the sum of whose areas is greater than 1 are inside a figure of area 1, then at least two of them have a common point.

§1. The case when there are finitely many points, lines, etc.

21.1. The nodes of an infinite graph paper are painted two colours. Prove that there exist two horizontal and two vertical lines on whose intersection lie points of the same colour.

21.2. Inside an equilateral triangle with side 1 five points are placed. Prove that the distance between certain two of them is shorter than 0.5.

21.3. In a 3×4 rectangle there are placed 6 points. Prove that among them there are two points the distance between which does not exceed $\sqrt{5}$.

21.4. On an 8×8 checkboard the centers of all the cells are marked. Is it possible to divide the board by 13 straight lines so that in each part there are not more than 1 of marked points?

21.5. Given 25 points in plane so that among any three of them there are two the distance between which is smaller than 1, prove that there exists a circle of radius 1 that contains not less than 13 of the given points.

21.6. In a unit square, there are 51 points. Prove that certain three of them can be covered by a disk of radius $\frac{1}{7}$.

21.7. Each of two equal disks is divided into 1985 equal sectors and on each of the disks some 200 sectors are painted (one colour). One of the disks was placed upon the other one and they began rotating one of the disks through multiples of $\frac{360^\circ}{1985}$. Prove that there exists at least 80 positions for which not more than 20 of the painted sectors of the disks coincide.

21.8. Each of 9 straight lines divides a square into two quadrilaterals the ratio of whose areas is 2 : 3. Prove that at least three of those nine straight lines pass through one point.

21.9. In a park, there grow 10, 000 trees planted by a so-called square-cluster method (100 rows of 100 trees each). What is the largest number of trees one has to cut down in order to satisfy the following condition: if one stands on any stump, then no other stump is seen (one may assume the trees to be sufficiently thin).

21.10. What is the least number of points one has to mark inside a convex n-gon in order for the interior of any triangle with the vertices at vertices of the n-gon to contain at least one of the marked points?

21.11. Point P is taken inside a convex $2n$-gon. Through every vertex of the polygon and P a line is drawn. Prove that there exists a side of the polygon which has no common interior points with neither of the drawn straight lines.

21.12. Prove that any convex $2n$-gon has a diagonal non-parallel to either of its sides.

21.13. The nodes of an infinite graph paper are painted three colours. Prove that there exists an isosceles right triangle with vertices of one colour.

§2. Angles and lengths

21.14. Given n pairwise nonparallel lines in plane. Prove that the angle between certain two of them does not exceed $\frac{180^\circ}{n}$.

21.15. In a circle of radius 1 several chords are drawn. Prove that if every diameter intersects not more than k chords, then the sum of the length of the chords is shorter than $k\pi$.

21.16. In plane, point O is marked. Is it possible to place in plane a) five disks; b) four disks that do not cover O and so that any ray with the beginning in O would intersect not less than two disks? ("Intersect" means has a common point.)

21.17. Given a line l and a circle of radius n. Inside the circle lie $4n$ segments of length 1. Prove that it is possible to draw a line which is either parallel or perpendicular to the given line and intersects at least two of the given segments.

21.18. Inside a unit square there lie several circles the sum of their lengths being equal to 10. Prove that there exists a straight line intersecting at least four of these circles.

21.19. On a segment of length 1 several segments are marked so that the distance between any two marked points is not equal to 0.1. Prove that the sum of the lengths of the marked segments does not exceed 0.5.

21.20. Given two circles the length of each of which is equal to 100 cm. On one of them 100 points are marked, on the other one there are marked several arcs with the sum of their lengths less than 1 cm. Prove that these circles can be identified so that no one of the marked points would be on a marked arc.

21.21. Given are two identical circles; on each of them k arcs are marked, the angle value of each of the arcs is $> \frac{1}{k^2-k+1} \cdot 180^\circ$. The circles can be identified(?) so that the marked arcs of one circle would coincide with the marked arcs of the other one. Prove that these circles can be identified so that all the marked arcs would lie on unmarked arcs.

§3. Area

21.22. In square of side 15 there lie 20 pairwise nonintersecting unit squares. Prove that it is possible to place in the large square a unit disk so that it would not intersect any of the small squares.

21.23. Given an infinite graph paper and a figure whose area is smaller than the area of a small cell prove that it is possible to place this figure on the paper without covering any of the nodes of the mesh.

21.24. Let us call the figure formed by the diagonals of a unit square (Fig. 39) a *cross*. Prove that it is possible to place only a finite number of nonintersecting crosses in a disk of radius 100.

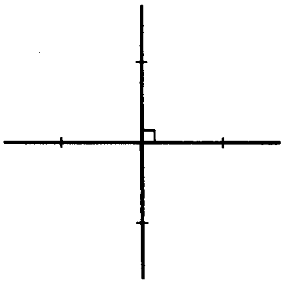


Figure 181 (21.24)

21.25. Pairwise distances between points $A_1, \dots, A_n$ is greater than 2. Prove that any figure whose area is smaller than π can be shifted by a vector not longer than 1 so that it would not contain points $A_1, \dots, A_n$.

21.26. In a circle of radius 16 there are placed 650 points. Prove that there exists a ring (annulus) of inner radius 2 and outer radius 3 which contains not less than 10 of the given points.

21.27. There are given n figures in plane. Let $S_{i_1 \dots i_k}$ be the area of the intersection of figures indexed by $i_1, \dots, i_k$ and S be the area of the part of the plane covered by the given figures; M_k the sum of all the $S_{i_1 \dots i_k}$. Prove that:

a) $S = M_1 - M_2 + M_3 - \dots + (-1)^{n+1}M_n$;

b) $S \geq M_1 - M_2 + M_3 - \dots + (-1)^{m+1}M_m$ for m even and $S \leq M_1 - M_2 + M_3 - \dots + (-1)^{m+1}M_m$ for m odd.

21.28. a) In a square of area 6 there are three polygons of total area 3. Prove that among them there are two polygons such that the area of their intersection is not less than 1.

b) In a square of area 5 there are nine polygons of total area 1. Prove that among them there are two polygons the area of whose intersection is not less than $\frac{1}{9}$.

21.29. On a rug of area 1 there are 5 patches the area of each of them being not less than 0.5. Prove that there are two patches such that the area of their intersection is not less than 0.2.

Solutions

21.1. Let us take three vertical lines and nine horizontal lines. Let us consider only intersection points of these lines. Since there are only $2^3 = 8$ variants to paint three points two colours, there are two horizontal lines on which lie similarly coloured triples of points. Among three points painted two colours there are, by Dirichlet's principle, two similarly coloured points. The vertical lines passing through these points together with the two horizontal lines selected earlier are the ones to be found.

21.2. The midlines of an equilateral triangle with side 1 separate it into four equilateral triangles with side 0.5. Therefore, one of the triangles contains at least two of the given points and these points cannot be vertices of the triangle. The distance between these points is less than 0.5.

21.3. Let us cut the rectangle into five figures as indicated on Fig. 40. One of the figures contains at least two points and the distance between any two points of each of the figures does not exceed $\sqrt{5}$.

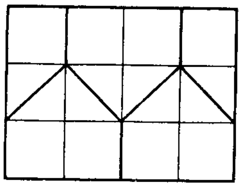


Figure 182 (Sol. 21.3)

21.4. 28 fields are adjacent to a side of an 8×8 chessboard. Let us draw 28 segments that connect the centers of neighbouring end(?) fields. Every line can intersect not more than 2 such segments and, therefore, 13 lines can intersect not more than 26 segments, i.e., there are at least 2 segments that do not intersect any of 13 drawn lines. Therefore, it is impossible to split the chessboard by 13 lines so that in each part there would be not more than 1 marked point since both endpoints of the segment that does not intersect with the lines belongs to one of the parts.

21.5. Let A be one of the given points. If all the remaining points lie in disk S_1 of radius 1 with center A, then we have nothing more to prove.

Now, let B be a given point that lies outside S_1, i.e., $AB > 1$. Consider disk S_2 of radius 1 with center B. Among points A, B and C, where C is any of the given points, there are two at a distance less than 1 and these cannot be points A and B. Therefore, disks S_1 and S_2 contain all the given points, i.e., one of them contains not less than 13 points.

21.6. Let us divide a given square into 25 similar small squares with side 0.2. By Dirichlet's principle one of them contains no less than 3 points. The radius of the circumscribed circle of the square with side 0.2 is equal to $\frac{1}{5}\sqrt{2} < \frac{1}{7}$ and, therefore, it can be covered by a disk of radius $\frac{1}{7}$.

21.7. Let us take 1985 disks painted as the second of our disks and place them upon the first disk so that they would take all possible positions. Then over every painted sector of the first disk there lie 200 painted sectors, i.e., there are altogether200^2 pairs of coinciding painted sectors. Let there be n positions of the second disk when not less 21 pairs of painted sectors coincide. Then the number of coincidences of painted sectors is not less than $21n$. Therefore, $21n \leq 200^2$, i.e., $n \leq 1904.8$. Since n is an integer, $n \leq 1904$. Therefore, at least for $1985 - 1904 = 81$ positions not more than 20 pairs of painted sectors coincide.

21.8. The given lines cannot intersect neighbouring sides of square $ABCD$ since otherwise we would have not two quadrilaterals but a triangle and a pentagon. Let a line intersect sides BC and AD at points M and N, respectively. Trapezoids $ABMN$ and $CDNM$ have equal heights, and, therefore, the ratio of their areas is equal to that of their midlines, i.e., MN divides the segment that connects the midpoints of sides AB and CD in the ratio of $2 : 3$. There are precisely 4 points that divide the midlines of the square in the ratio of $2 : 3$. Since the given nine lines pass through these four points, then through one of the points at least three lines pass.

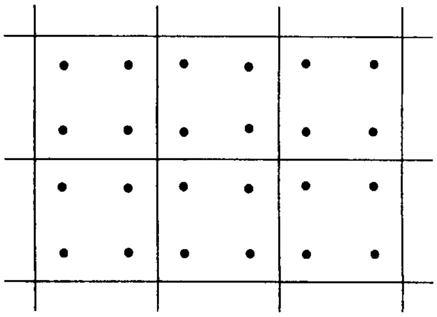


Figure 183 (Sol. 21.8)

21.9. Let us divide the trees into 2500 quadruples as shown in Fig. 41. In each such quadruple it is impossible to chop off more than 1 tree. On the other hand, one can chop off all the trees that grow in the left upper corners of the squares formed by our quadruples. Therefore, the largest number of trees that can be chopped off is equal to 2500.

21.10. Since any diagonal that goes out of one vertex divides an n-gon into $n-2$ triangles, then $n-2$ points are necessary.

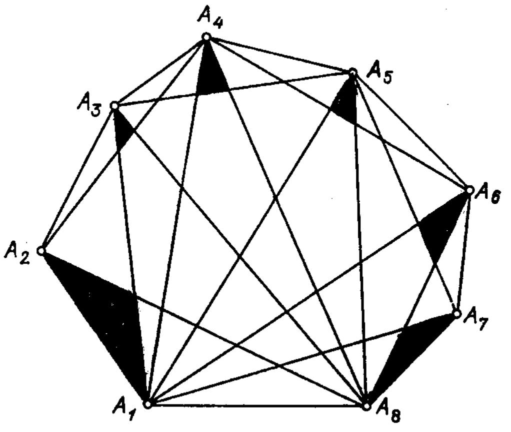


Figure 184 (Sol. 21.10)

From Fig. 42 one can deduce that $n-2$ points are sufficient: it suffices to mark one points in each shaded triangle. Indeed, inside triangle $A_pA_qA_r$, where $p<q<r$, there is always contained a shaded triangle adjacent to vertex A_q.

21.11. The two cases are possible.

(1) Point P lies on diagonal AB. Then lines PA and PB coincide and do not intersect the sides. There remain $2n-2$ lines; they intersect not more than $2n-2$ sides.

(2) Point P does not belong to a diagonal of polygon $A_1A_2\dots A_{2n}$. Let us draw diagonal A_1A_{n+1}. On both sides of it there lie n sides. Let, for definiteness, point P be inside polygon $A_1\dots A_{n+1}$ (Fig. 43).

Then lines PA_{n+1}, PA_{n+2}, … , PA_{2n}, PA_1 (there are $n+1$ such lines) cannot intersect sides $A_{n+1}A_{n+2}$, $A_{n+2}A_{n+3}$, … , $A_{2n}A_1$, respectively. Therefore, the remaining straight lines can intersect not more than $n-1$ of these n sides.

21.12. The number of diagonals of a $2n$-gon is equal to $\frac{2n(2n-3)}{2}=n(2n-3)$. It is easy to verify that there are not more than $n-2$ diagonals parallel to the given one. Therefore, there are not more than $2n(n-2)$ diagonals parallel to the sides. Since $2n(n-2)<n(2n-3)$, there exists a diagonal which is not parallel to any side.

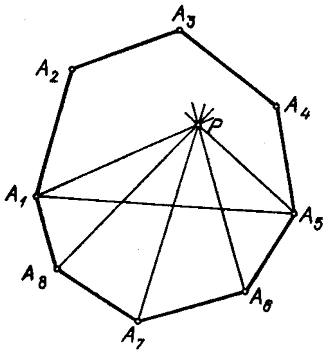


Figure 185 (Sol. 21.11)

21.13. Suppose that there does not exist an equilateral right triangle whose legs are parallel to the sides of the cells and with vertices of the same colour. For convenience we may assume that it is the cells which are painted, not the nodes.

Let us divide the paper into squares of side 4; then on the diagonal of each such square there are two cells of the same colour. Let n be greater than the number of distinct colorings of the square of side 4. Consider a square consisting of n^2 squares of side 4. On its diagonal we can find two similarly painted squares of side 4. Finally, take square K on whose diagonal we can find two similarly painted squares of side $4n$.

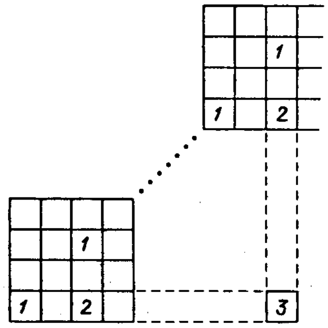


Figure 186 (Sol. 21.13)

Considering the square with side $4n$ and in it two similarly painted squares with side 4 we get four cells of the first colour, two cells of the second colour and one cell of the third colour, see Fig. 44. Similarly, considering square K we get a cell which cannot be of the first, or second, or third colour.

21.14. In plane, take an arbitrary point and draw through it lines parallel to the given ones. They divide the plane into $2n$ angles whose sum is equal to 360°. Therefore, one of these angles does not exceed $\frac{180^\circ}{n}$.

21.15. Suppose the sum of the length of the chords is not shorter than πk. Let us prove that then there exists a diameter which intersects with at least $k+1$ chords. Since the length of the arc corresponding to the chord is greater than the length of this chord, the sum of the lengths of the archs corresponding to given chords is longer than πk. If we add to these arcs the arcs symmetric to them through the center of the circle, then the sum of the lengths of all these arcs becomes longer than $2\pi k$. Therefore, there exists a point covered by at least $k+1$ of these arcs. The diameter drawn through this point intersects with at least $k+1$ chord.

21.16. a) It is possible. Let O be the center of regular pentagon $ABCDE$. Then the disks inscribed in angles $\angle AOC$, $\angle BOD$, $\angle COD$, $\angle DOA$ and $\angle EOB$ possess the required property.

b) It is impossible. For each of the four disks consider the angle formed by the tangents to the disk drawn through point O. Since each of these four angles is smaller than 180°, their sum is less than $2 \cdot 360^\circ$. Therefore, there exists a point on the plane covered by not more than 1 of these angles. The ray drawn through this point intersects with not more than one disk.

21.17. Let l_1 be an arbitrary line perpendicular to l. Denote the lengths of the projections of the i-th segment to l and l_1 by a_i and b_i, respectively. Since the length of each segment is equal to 1, we have $a_i + b_i \geq 1$. Therefore,

$$(a_1 + \cdots + a_{4n}) + (b_1 + \cdots + b_{4n}) \geq 4n.$$

Let, for definiteness,

$$a_1 + \cdots + a_{4n} \geq b_1 + \cdots + b_{4n}.$$

Then $a_1 + \cdots + a_{4n} \geq 2n$. The projection of any of the given segment is of length $2n$ because all of them lie inside the circle of radius n. If the projections of the given segments to l would have had no common points, then we would had $a_1 + \cdots + a_{4n} < 2n$. Therefore, on l there exists a point which is the image under the projection of at least two of the given segments. The perpendicular to l drawn through this point intersects with at least two of given segments.

21.18. Let us project all the given circles on side AB of square $ABCD$. The projection of the circle of length l is a segment of length $\frac{l}{\pi}$. Therefore, the sum of the lengths of the projections of all the given circles is equal to $\frac{10}{\pi}$. Since $\frac{10}{\pi} > 3 = 3AB$, on segment AB there is a point which belongs to projections of at least four circles. The perpendicular to AB drawn through this point intersects at least four circles.

21.19. Let us cut the segment into ten segments of length 0.1, stack them in a pile and consider their projection to a similar segment as shown on Fig. 45.

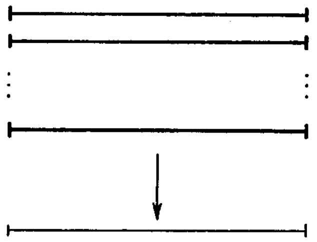


Figure 187 (Sol. 21.19)

Since the distance between any two painted points is not equal to 0.1, the painted points of neighbouring segments cannot be projected into one point. Therefore, neither of the points can be the image under the projection of painted points of more than 5 segments. It follows that the sum of the lengths of the projections of the painted segments (equal to the sum of their lengths) does not exceed $5 \cdot 0.1 = 0.5$.

21.20. Let us identify the given circles and let us place a painter in a fixed point of one of them. Let us rotate this circle and let the painter paint a point of the other circle each time when it is a marked point that belongs to a marked arc. We have to prove that after a complete revolution a part of the circle would remain unpainted.

The final result of the painter's job will be the same as if he were rotated 100 times and (s)he was asked to paint the other circle on the i-th revolution so that (s)he would have to paint the i-th marked point that belongs to one of the marked arcs. Since in this case at each revolution less than 1 cm is being painted, it follows that after 100 revolutions there will be painted less than 100 cm. Therefore, a part of the circle will be unpainted.

21.21. Let us identify(?) our circles and place a painter into a fixed point of one of them. Let us rotate this circle and let the painter paint the point of the other circle against which he moves each time when some of the marked arcs intersect. We have to prove that after a full revolution a part of the circle will be unpainted.

The final result of the painter's job would be the same as if (s)he were rotated k times and was asked to paint the circle on the i-th revolution when the i-th marked arc on which the painter resides would intersect with a marked arc of the other circle.

Let $\varphi_1, \dots, \varphi_n$ be the angle parameters of the marked arcs. By the hypothesis $\varphi_1 < \alpha$, $\dots, \varphi_n < \alpha$, where $\alpha = \frac{180^\circ}{k^2-k+1}$. During the time when the marked arcs with counters i and j intersect the painter paints an arc of length $\varphi_i + \varphi_j$.

Therefore, the sum of the angle values of the arcs painted during the i-th revolution does not exceed $k\varphi_i(\varphi_1 + \dots + \varphi_k)$ and the sum of the angle values of the arcs painted during all k revolutions does not exceed $2k(\varphi_1 + \dots + \varphi_k)$. Observe that during all this we have actually counted the intersection of arcs with similar(?) counters k times.

In particular, point A across which the painter moves at the moment when the marked arcs coincide has, definitely, k coats of paint. Therefore, it is desirable to disregard the arcs that the painter paints at the moment when some of the marked arcs with similar counters intersect. Since all these arcs contain point A, we actually disregard only one arc and the angle value of this arc does not exceed 2α.

The sum of the angle values of the remaining part of the arcs painted during the i-th revolution does not exceed $(k-1)\varphi_1 + (\varphi_1 + \dots + \varphi_k - \varphi_i)$ and the sum of the angle values of the remaining part of the arcs painted through all k revolutions does not exceed

$$(2k-2)\cdot(\varphi_1 + \dots + \varphi_k) < (2k^2 - 2k)\alpha.$$

A part of the circle will be unpainted if $(2k^2-2k)\alpha \le 360^\circ - 2\alpha$, i.e., $\alpha \le \frac{180^\circ}{k^2-k+1}$.

21.22. Let us consider a figure consisting of all the points whose distance from the small unit square is not greater than 1 (Fig. 46).

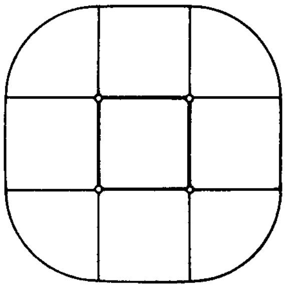


Figure 188 (Sol. 21.22)

It is clear that no unit disk whose center is outside this figure intersects the small square. The area of such a figure is equal to $\pi + 5$. The center of the needed disk should also lie at a distance greater than 1 from the sides of the large square, i.e., inside the square of side 13. Obviously, 20 figures of total area $\pi + 5$ cannot cover a square of side 13 because $20(\pi + 5) < 13^2$. The disk with the center in an uncovered point possesses the desired property.

21.23. Let us paint the figure to(?) the graph paper arbitrarily, cut the paper along the cells of the mesh and stack them in a pile moving them parallelly with themselves and without turning. Let us consider the projection of this stack on a cell. The projections of parts of the figure cannot cover the whole cell since their area is smaller. Now, let us recall how the figure was placed on the graph paper and move the graph paper parallelly with itself so that its vertices would be in the points whose projection is an uncovered point. As a result we get the desired position of the figure.

21.24. For every cross consider a disk of radius $\frac{1}{2}\sqrt{2}$ with center in the center of the cross. Let us prove that if two such disks intersect, then the crosses themselves also intersect. The distance between the centers of equal intersecting disks does not exceed the doubled radius of any of them and, therefore, the distance between the centers of the corresponding crosses does not exceed $\frac{1}{\sqrt{2}}$. Let us consider a rectangle given by bars of the first cross and the center of the second one (Fig. 47).

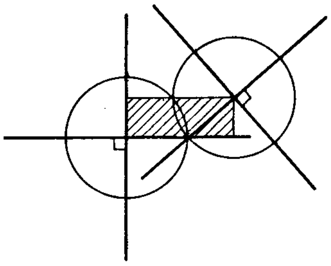


Figure 189 (Sol. 21.24)

One of the bars of the second cross passes through this rectangle and, therefore, it intersects the first cross since the length of the bar is equal to $\frac{1}{\sqrt{2}}$ and the length of the diagonal of the rectangle does not exceed $\frac{1}{\sqrt{2}}$. In the disk of a finite radius one can only place finitely many non-intersecting disks of radius $\frac{1}{2}\sqrt{2}$.

21.25. Let Φ be a given figure, $S_1, \dots , S_n$ unit disks with centers at points $A_1, \dots , A_n$. Since disks $S_1, \dots , S_n$ do not intersect, then neither do figures $V_i = \Phi \cap S_i$, consequently, the sum of their areas does not exceed the area of figure Φ, i.e., it is smaller than π. Let O be an arbitrary point and W_i the image of V_i under the translation by vector $\overrightarrow{A_iO}$. The figures W_i lie inside the unit disk S centered at O and the sum of their areas is smaller than the area of this disk. Therefore, point B of disk S does not belong to any of the figures W_i. It is clear that the translation by vector $\overrightarrow{BO}$ is the desired one.

21.26. First, notice that point X belongs to the ring with center O if and only if point O belongs to a similar ring centered at X. Therefore, it suffices to show that if we construct rings with centers at given points, then not less than 10 rings will cover one of the points of the considered disk. The considered rings lie inside a disk of radius $16 + 3 = 19$ whose area is equal to 361π. It remains to notice that $9 \cdot 361\pi = 3249\pi$ and the total area of the rings is equal to $650 \cdot 5\pi = 3250\pi$.

21.27. a) Let $\binom{n}{k}$ be the number of ways to choose k elements from n indistinguishable ones. One can verify the following Newton binomial formula

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Denote by W_m the area of the part of the plane covered by exactly m figures. This part consists of pieces each of which is covered by certain m figures. The area of each such piece has been counted $\binom{n}{k}$ times in calculation of M_k because from m figures we can form $\binom{n}{k}$

intersections of k figures. Therefore,

$$M_k = \binom{n}{k} W_k + \binom{k+1}{k} W_{k+1} + \dots + \binom{n}{k} W_n.$$

It follows that

$$\begin{aligned} &M_1 - M_2 + M_3 - \dots = \\ &\binom{1}{1} W_1 + (\binom{2}{1} - \binom{2}{2}) W_2 + \dots + (\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \dots) W_n = \\ &W_1 + \dots + W_n \end{aligned}$$

since

$$\begin{aligned} &\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \dots - (-1)^m \binom{m}{m} = \\ &(-1 + \binom{m}{1} - \binom{m}{2} + \dots) + 1) = -(1-1)^m + 1 = 1. \end{aligned}$$

It remains to observe that $S = W_1 + \dots + W_n$.

b) According to heading a)

$$\begin{aligned} &S - (M_1 - M_2 + \dots + (-1)^{m+1} M_m) = \\ &(-1)^{m+2} M_{m+1} + (-1)^{m+3} M_{m+2} + \dots \dots + (-1)^{n+1} M_n = \\ &\sum_{i=1}^{n} ((-1)^{m+2} \binom{i}{m+1} + \dots + (-1)^{n+1} \binom{i}{n}) W_i \end{aligned}$$

(it is convenient to assume that $\binom{n}{k}$ is defined for $k > n$ so that $\binom{n}{k} = 0$). Therefore, it suffices to verify that

$$\binom{i}{m+1} - \binom{i}{m+2} + \binom{i}{m+3} - \dots + (-1)^{m+n+1} \binom{i}{n} \geq 0 \text{ for } i \leq n.$$

The identity

$$(x+y)^i = (x+y)^{i-1}(x+y)$$

implies that $\binom{i}{j} = \binom{i-1}{j-1} + \binom{i-1}{j}$. Hence,

$$\binom{i}{m+1} - \binom{i}{m+2} + \dots + (-1)^{m+n+1} \binom{i}{n} = \binom{i-1}{m} \pm \binom{i-1}{n}.$$

It remains to notice that $\binom{i-1}{n} = 0$ for $i \leq n$.

21.28. a) By Problem 21.27 a) we have

$$6 = 9 - (S_{12} + S_{23} + S_{13}) + S_{123},$$

i.e.,

$$S_{12} + S_{23} + S_{13} = 3 + S_{123} \geq 3.$$

Hence, one of the numbers S_{12}, S_{23}, S_{13} is not less than 1.

b) By Problem 21.27 b) $5 \geq 9 - M_2$, i.e., $M_2 \geq 4$. Since from 9 polygons one can form $9 \cdot \frac{8}{2} = 36$ pairs, the area of the common part of one of such pairs is not less than $\frac{M_2}{36} \geq \frac{1}{9}$.

21.29. Let the area of the rug be equal to M, the area of the intersection of the patches indexed by $i_1, \dots, i_k$ is equal to $S_{i_1\dots i_k}$ and $M_k = \sum S_{i_1\dots i_k}$. By Problem 21.27 a)

$$M - M_1 + M_2 - M_3 + M_4 - M_5 \geq 0$$

since $M \geq S$. One can write similar inequalities not only for the whole rug but also for every patch: if we consider the patch S_1 as the rug with patches S_{12}, S_{13}, S_{14}, S_{15} we get

$$S_1 - \sum_i S_{1i} + \sum_{i<j} S_{1ij} - \sum_{i<j<k} S_{1ijk} + S_{12345} \geq 0.$$

Adding such inequalities for all five patches we get

$$M_1 - 2M_2 + 3M_3 - 4M_4 + 5M_5 \geq 0$$

(the summand $S_{i_1\dots i_k}$ enters the inequality for patches $i_1, \dots, i_k$ and, therefore, it enters the sum of all inequalities with coefficient k). Adding the inequalities

$$3(M - M_1 + M_2 - M_3 + M_4 - M_5) \geq 0 \text{ and } M_1 - 2M_2 + 3M_3 - 4M_4 + 5M_5 \geq 0$$

we get

$$3M - 2M_1 + M_2 - M_4 + 2M_5 \geq 0.$$

Adding to this the inequality $M_4 - 2M_5 \geq 0$ (which follows from the fact that S_{12345} enters every $S_{i_1i_2i_3i_4}$, i.e., $M_4 \geq 5M_5 \geq 2M_5$) we get $3M - 2M_1 + M_2 \geq 0$, i.e., $M_2 \geq 2M_1 - 3M \geq 5 - 3 = 2$.

Since from five patches we can form ten pairs, the area of the intersection of patches from one of these pairs is not less than $\frac{1}{10}M_2 \geq 0.2$.

Chapter 22. CONVEX AND NONCONVEX POLYGONS

Background

1. There are several different (nonequivalent) definitions of a convex polygon. Let us give the most known and most often encountered definitions. A polygon is called *convex* if one of the following conditions is satisfied:

a) the polygon lies on one side of any of its sides (i.e., the intersections of the sides of the polygon do not intersect its other sides);

b) the polygon is the intersection (i.e., the common part) of several half planes;

c) any segment whose endpoints belong to the polygon wholly belongs to the polygon.

2. A figure is called a *convex* one if any segment with the endpoints in the points of a figure belongs to the figure.

3. In solutions of several problems of this chapter we make use of the notion of the *convex hull* and the *basic line*.

§1. Convex polygons

22.1. Given n points in plane such that any four of them are the vertices of a convex quadrilateral, prove that these points are the vertices of a convex n-gon.

22.2. Given five points in plane no three of which belong to one line, prove that four of these points are placed in the vertices of a convex quadrilateral.

22.3. Given several regular n-gons in plane prove that the convex hull of their vertices has not less than n angles.

22.4. Among all numbers n such that any convex 100-gon can be represented as an intersection (i.e., the common part) of n triangles find the least number.

22.5. A convex heptagon will be called *singular* if three of its diagonals intersect at one point. Prove that by a slight movement of one of the vertices of a singular heptagon one can obtain a nonsingular heptagon.

22.6. In plane lie two convex polygons, F and G. Denote by H the set of midpoints of the segments one endpoint of each of which belongs to F and the other one to G. Prove that H is a convex polygon.

a) How many sides can H have if F and G have n_1 and n_2 sides, respectively?

b) What value can the perimeter of H have if the perimeters of F and G are equal to P_1 and P_2, respectively?

22.7. Prove that there exists a number N such that among any N points no three of which lie on one line one can select 100 points which are vertices of a convex polygon.

* * *

22.8. Prove that in any convex polygon except parallelogram one can select three sides whose extensions form a triangle which is ambient(?) with respect to the given polygon.

22.9. Given a convex n-gon no two sides of which are parallel, prove that there are not less than $n - 2$ distinct triangles such as discussed in Problem 22.8.

22.10. A point O is inside a convex n-gon, $A_1 \dots A_n$. Prove that among the angles $\angle A_iOA_j$ there are not fewer than $n-1$ acute ones.

22.11. Convex n-gon $A_1 \dots A_n$ is inscribed in a circle and among the vertices of the polygon there are no diametrically opposite points. Prove that among the triangles $A_pA_qA_r$ there is at least one acute triangle, then there are not fewer than $n-2$ such acute triangles.

§2. Helly's theorem

22.12. a) Given four convex figures in plane such that any three of them have a common point, prove that all of them have a common point.

b) (*Helly's theorem.*) Given n convex figures in plane such that any three of them have a common point, prove that all n figures have a common point.

22.13. Given n points in plane such that any three of them can be covered by a unit disk, prove that all n points can be covered by a unit disk.

22.14. Prove that inside any convex heptagon there is a point that does not belong to any of quadrilaterals formed by quadruples of its neighbouring vertices.

22.15. Given several parallel segments such that for any three of them there is a line that intersects them, prove that there exists a line that intersects all the points.

§3. Non-convex polygons

In this section all polygons considered are non-convex unless otherwise mentioned.

22.16. Is it true that any pentagon lies on one side of not fewer than two of its sides?

22.17. a) Draw a polygon and point O inside it so that the polygon's angle with vertex in O would not subtend any side without intersecting some of the other of the polygon's sides.

b) Draw a polygon and point O outside it so that the polygon's angle with vertex in O would not subtend any side without intersecting some of the other of the polygon's sides.

22.18. Prove that if a polygon is such that point O is the vertex of an angle that subtends its entire contour, then any point of the plane is the vertex of an angle that entirely subtends at least one of its sides.

22.19. Prove that for any polygon the sum of the outer angles adjacent to the inner ones that are smaller than 180° is $\geq 360^\circ$.

22.20. a) Prove that any n-gon ($n \geq 4$) has at least one diagonal that completely lies inside it.

b) Find out what is the least number of such diagonals for an n-gon.

22.21. What is the maximal number of vertices of an n-gon from which one cannot draw a diagonal?

22.22. Prove that any n-gon can be cut into triangles by nonintersecting diagonals.

22.23. Prove that the sum of the inner angles of any n-gon is equal to $(n-2)180^\circ$.

22.24. Prove that the number of triangles into which an n-gon is cut by nonintersecting diagonals is equal to $n-2$.

22.25. A polygon is cut by nonintersecting diagonals into triangles. Prove that at least two of these diagonals cut triangles off it.

22.26. Prove that for any 13-gon there exists a line containing exactly one of its sides; however, for any $n > 13$ there exists an n-gon for which the similar statement is false.

22.27. What is the largest number of acute angles in a nonconvex n-gon?

22.28. The following operations are done over a nonconvex non-selfintersecting polygon. If it lies on one side of line AB, where A and B are non-neighbouring vertices, then we reflect one of the parts into which points A and B divide the contour of the polygon through the midpoint of segment AB. Prove that after several such operations the polygon becomes a convex one.

22.29. The numbers $\alpha_1, \dots , \alpha_n$ whose sum is equal to $(n-2)\pi$ satisfy inequalities $0 < \alpha_i \leq 2\pi$. Prove that there exists an n-gon $A_1 \dots A_n$ with anagles $\alpha_1, \dots , \alpha_n$ at vertices $A_1, \dots , A_n$, respectively.

Solutions

22.1. Consider the convex hull of given points. It is a convex polygon. We have to prove that all the given points are its vertices. Suppose one of the given points (point A) is not a vertex, i.e., it lies inside or on the side of the polygon. The diagonals that go out of this vertex cut the convex hull into triangles; point A belongs to one of the triangles. The vertices of this triangle and point A cannot be vertices of a convex quadrilateral. Contradiction.

22.2. Consider the convex hull of given points. If it is a quadrilateral or a pentagon, then all is clear. Now, suppose that the convex hull is triangle ABC and points D and E lie inside it. Point E lies inside one of the triangles ABD, BCD, CAD; let for definiteness sake it belong to the interior of triangle ABC. Let H be the intersection point of lines CD and AB. Point E lies inside one of the triangles ADH and BDH. If, for example, E lies inside triangle ADH, then $AEDC$ is a convex quadrilateral (Fig. 48).

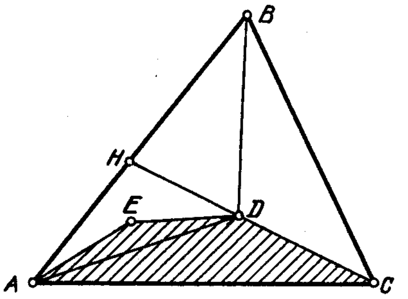


Figure 190 (Sol. 22.2)

22.3. Let the convex hull of the vertices of the given n-gons be an m-gon and $\varphi_1, \dots ,$ φ_m its angles. Since to every angle of the convex hull an angle of a regular n-gon is adjacent, $\varphi_i \geq (1-(\frac{2}{n}))\pi$ (in the right-hand side there stands the value of an angle of a regular n-gon). Therefore,

$$\varphi_1 + \dots + \varphi_m \geq m(1-(\frac{2}{n}))\pi = (m-(\frac{2m}{n}))\pi.$$

On the other hand, $\varphi_1 + \dots + \varphi_m = (m-2)\pi$; hence, $(m-2)\pi \geq (m-(\frac{2m}{n}))\pi$, i.e., $m \geq n$.

22.4. First, notice that it suffices to take 50 triangles. Indeed, let Δ_k be the triangle whose sides lie on rays A_kA_{k-1} and A_kA_{k+1} and which contains convex polygon $A_1 \dots A_{100}$. Then this polygon is the intersection of the triangles $\Delta_2, \Delta_4, \dots , \Delta_{100}$.

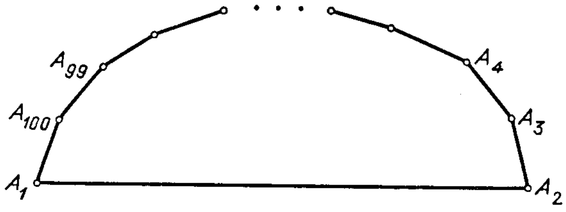


Figure 191 (Sol. 22.4)

On the other hand, the 100-gon depicted on Fig. 49 cannot be represented as the intersection of less than 50 triangles. Indeed, if three of its sides lie on the sides of one triangle, then one of these sides is side A_1A_2. All the sides of this polygon lie on the sides of n triangles and, therefore, $2n + 1 \geq 100$, i.e., $n \geq 50$.

22.5. Let P be the intersection point of diagonals A_1A_4 and A_2A_5 of convex heptagon $A_1 \dots A_7$. One of the diagonals A_3A_7 or A_3A_6, let, for definiteness, this be A_3A_6, does not pass through point P. There are finitely many intersection points of the diagonals of hexagon $A_1 \dots A_6$ and, therefore, in a vicinity of point A_7 one can select a point A_7' such that lines A_1A_7', … , A_6A_7' do not pass through these points, i.e., heptagon $A_1 \dots A_7'$ is a nonsingular one.

22.6. First, let us prove that H is a convex figure. Let points A and B belong to H, i.e., A and B be the midpoints of segments C_1D_1 and C_2D_2, where C_1 and C_2 belong to F and D_1, respectively, and D_2 belong to G. We have to prove that the whole segment AB belongs to H. It is clear that segments C_1C_2 and D_1D_2 belong to F and G, respectively. The locus of the midpoints of segments with the endpoints on segments C_1C_2 and D_1D_2 is the parallelogram with diagonal AB (Fig. 50); this follows from the fact that the locus of the midpoints of segments CD, where C is fixed and D moves along segment D_1D_2, is the midline of triangle CD_1D_2.

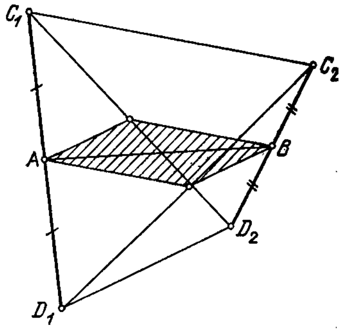


Figure 192 (Sol. 22.6)

In plane, take an arbitrary coordinate axis Ox. The set of all the points of the polygon whose projections to the axis have the largest value (Fig. 51) will be called the *basic set* of the polygon with respect to axis Ox.

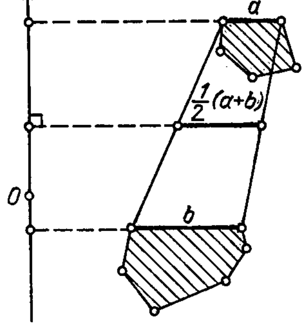


Figure 193 (Sol. 22.6)

The convex polygon is given by its basic sets for all possible axes Ox. If basic sets F and G with respect to an axis are segments of length a and b, then the basic set of H with respect to the same axis is a segment of length $\frac{a+b}{2}$ (here we assume that a point is *segment of zero length*). Therefore, the perimeter of H is equal to $\frac{P_1+P_2}{2}$ and the number of H's sides can take any value from the largest — n_1 or n_2 — to n_1+n_2 depending on for how many axes both basic sets of F and G are sides and not vertices simultaneously.

22.7. We will prove a more general statement. Recall that *cardinality* of a set is (for a finite set) the number of its element.

(*Ramsey's theorem.*) *Let p, q and r be positive integers such that $p,\ q \geq r$. Then there exists a number $N = N(p,q,r)$ with the following property: if r-tuples from a set S of cardinality N are divided at random into two nonintersecting families α and β, then either there exists a p-tuple of elements from S all subsets of cardinality r of which are contained in α or there exists a q-tuple all subsets of cardinality r of which are contained in β.*

The desired statement follows easily from Ramsey's theorem. Indeed, let $N = N(p,5,4)$ and family α consist of quadruples of elements of an N-element set of points whose convex hulls are quadrilaterals. Then there exists a subset of n elements of the given set of points the convex hulls of any its four-elements subset being quadrilaterals because there is no five-element subset such that the convex hulls of any four-element subsets of which are triangles (see Problem 22.2). It remains to make use of the result of Problem 22.1.

Now, let us prove Ramsey's theorem. It is easy to verify that for $N(p,q,1)$, $N(r,q,r)$ and $N(p,r,r)$ one can take numbers $p+q-1$, q and p, respectively.

Now, let us prove that if $p > r$ and $q > r$, then for $N(p,q,r)$ one can take numbers $N(p_1,q_1,r-1)+1$, where $p_1 = N(p-1,q,r)$ and $q_1 = N(p,q-1,r)$. Indeed, let us delete from the $N(p,q,r)$-element set S one element and divide the $(r-1)$-element subsets of the obtained set S' into two families: family α' (resp. β') consists of subsets whose union with the deleted element enters α (resp. β). Then either (1) there exists a p_1-element subset of S' all $(r-1)$-element subsets of which are contained in α' or (2) there exists a q_1-element subset all whose $(r-1)$ element subsets are contained in family β'.

Consider case (1). Since $p_1 = N(p-1,q,r)$, it follows that either there exists a q-element subset of S' all r-element subsets of which belong to β (then these q elements are the desired one) or there exists a $(p-1)$-element subset of S' all the r-element subsets of which are contained in α (then these $p-1$ elements together with the deleted element are the desired ones).

Case (2) is treated similarly.

Thus, the proof of Ramsey's theorem can be carried out by induction on r, where in the proof of the inductive step we make use of induction on $p+q$.

22.8. If the polygon is not a triangle or parallelogram, then it has two nonparallel non-neighbouring sides. Extending them until they intersect, we get a new polygon which contains the initial one and has fewer number of sides. After several such operations we get a triangle or a parallelogram.

If we have got a triangle, then everything is proved; therefore, let us assume that we have got a parallelogram, $ABCD$. On each of its sides there lies a side of the initial polygon and one of its vertices, say A, does not belong to the initial polygon (Fig. 52). Let K be a vertex of the polygon nearest to A and lying on AD; let KL be the side that does not lie on AD. Then the polygon is confined inside the triangle formed by lines KL, BC and CD.

22.9. The proof will be carried out by induction on n. For $n = 3$ the statement is obvious. Let $n \geq 4$. By Problem 22.8 there exist lines a, b and c which are extensions of the sides of the given n-gon that constitute triangle T which contains the given n-gon. Let line l be the extension of some other side of the given n-gon. The extensions of all the sides

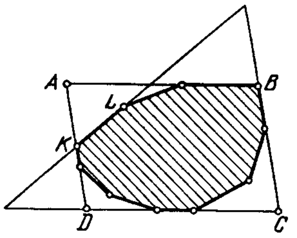


Figure 194 (Sol. 22.8)

of the n-gon except the side which lies on line l form a convex $(n-1)$-gon that lies inside triangle T.

By the inductive hypothesis for this $(n-1)$-gon there exist $n-3$ required triangles. Moreover, line l and two of the lines a, b and c also form a required triangle.

REMARK. If points $A_2, \dots, A_n$ belong to a circle with center at A_1, where $\angle A_2A_1A_n < 90^\circ$ and the n-gon $A_1 \dots A_n$ is a convex one, then for this n-gon there exist precisely $n-2$ triangles required.

22.10. Proof will be carried out by induction on n. For $n = 3$ the proof is obvious. Now, let us consider n-gons $A_1 \dots A_n$, where $n \geq 4$. Point O lies inside triangle $A_pA_qA_r$. Let A_k be a vertex of the given n-gon distinct from points A_p, A_q and A_r. Selecting vertex A_k in n-gon $A_1 \dots A_n$ we get a $(n-1)$-gon to which the inductive hypothesis is applicable. Moreover, the angles $\angle A_kOA_p$, $\angle A_kOA_q$ and $\angle A_kOA_r$ cannot all be acute ones because the sum of certain two of them is greater than 180°.

22.11. Proof will be carried out by induction on n. For $n = 3$ the statement is obvious. Let $n \geq 4$. Fix one acute triangle $A_pA_qA_r$ and let us discard vertex A_k distinct from the vertices of this triangle. The inductive hypothesis is applicable to the obtained $(n-1)$-gon. Moreover, if, for instance, point A_k lies on arc A_pA_q and $\angle A_kA_pA_r \leq \angle A_kA_qA_r$, then triangle $A_kA_pA_r$ is an acute one.

Indeed, $\angle A_pA_kA_r = \angle A_pA_qA_r$, $\angle A_pA_rA_k < \angle A_pA_rA_q$ and $\angle A_kA_pA_r \leq 90^\circ$; hence, $\angle A_kA_pA_r < 90^\circ$.

22.12. a) Denote the given figures by M_1, M_2, M_3 and M_4. Let A_i be the intersection point of all the figures except M_i. Two variants of arrangements of points A_i are possible.

1) One of the points, for example, A_4 lies inside the triangle formed by the remaining points. Since points A_1, A_2, A_3 belong to the convex figure M_4, all points of $A_1A_2A_3$ also belong to M_4. Therefore, point A_4 belongs to M_4 and it belongs to the other figures by its definition.

2) $A_1A_2A_3A_4$ is a convex quadrilateral. Let C be the intersection point of diagonals A_1A_3 and A_2A_4. Let us prove that C belongs to all the given figures. Both points A_1 and A_3 belong to figures M_2 and M_4, therefore, segment A_1A_3 belongs to these figures. Similarly, segment A_2A_4 belongs to figures M_1 and M_3. It follows that the intersection point of segments A_1A_3 and A_2A_4 belongs to all the given figures.

b) Proof will be carried out by induction on the number of figures. For $n = 4$ the statement is proved in the preceding problem. Let us prove that if the statement holds for $n \geq 4$ figures, then it holds also for $n+1$ figures. Given convex figures $\Phi_1, \dots, \Phi_n, \Phi_{n+1}$ every three of which have a common point, consider instead of them figures $\Phi_1, \dots, \Phi_{n-1}$, Φ'_n, where Φ'_n is the intersection of Φ_n and Φ_{n+1}. It is clear that Φ'_n is also a convex figure.

Let us prove that any three of the new figures have a common point. One can only doubt this for the triple of figures that contain Φ'_n but the preceding problem implies that figures

Φ_i, Φ_j, Φ_n and Φ_{n+1} always have a common point. Therefore, by the inductive hypothesis $\Phi_1, \dots, \Phi_{n-1}, \Phi'_n$ have a common point; hence, $\Phi_1, \dots, \Phi_n, \Phi_{n+1}$ have a common point.

22.13. A unit disk centered at O covers certain points if and only if unit disks centered at these points contain point O. Therefore, our problem admits the following reformulation:

Given n points in plane such that any three unit disks centered at these points have a common point, prove that all these disks have a common point.

This statement clearly follows from Helley's theorem.

22.14. Consider pentagons that remain after deleting pairs of neighbouring vertices of a heptagon. It suffices to verify that any three of the pentagons have a common point. For three pentagons we delete not more than 6 distinct vertices, i.e., one vertex remains. If vertex A is not deleted, then the triangle shaded in Fig. 53 belongs to all three pentagons.

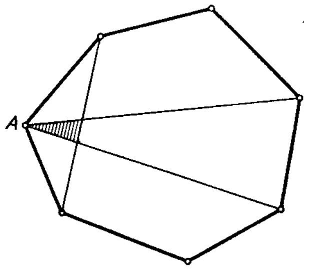


Figure 195 (Sol. 22.14)

22.15. Let us introduce the coordinate system with Oy-axis parallel to the given segments. For every segment consider the set of all points (a, b) such that the line $y = ax + b$ intersects it. It suffices to verify that these sets are convex ones and apply to them Helley's theorem. For the segment with endpoints (x_0, y_1) and (x_0, y_2) the considered set is a band between parallel lines $ax_0 + b = y_1$ and $ax_0 + b = y_2$.

22.16. Wrong. A counterexample is given on Fig. 54.

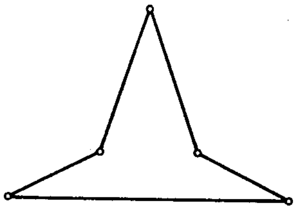


Figure 196 (Sol. 22.16)

22.17. The required polygons and points are drawn on Fig. 55.

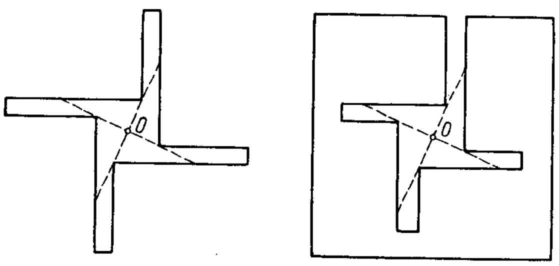


Figure 197 (Sol. 22.17)

22.18. Let the whole contour of polygon $A_1 \dots A_n$ subtend an angle with vertex O. Then no other side of the polygon except A_iA_{i+1} lies inside angle $\angle A_iOA_{i+1}$; hence, point O lies inside the polygon (Fig. 56). Any point X in plane belongs to one of the angles $\angle A_iOA_{i+1}$ and, therefore, side A_iA_{i+1} subtends an angle with vertex in X.

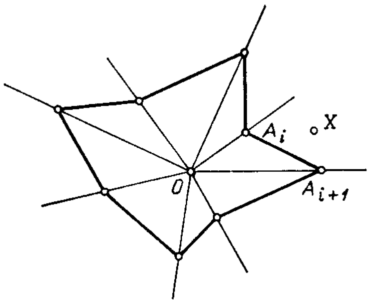


Figure 198 (Sol. 22.18)

22.19. Since all the inner angles of a convex n-gon are smaller than 180° and their sum is equal to $(n-2) \cdot 180°$, the sum of the exterior angles is equal to 360°, i.e., for a convex polygon we attain the equality.

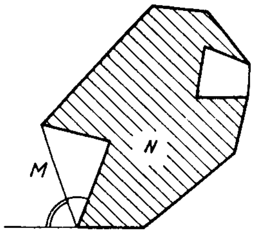


Figure 199 (Sol. 22.19)

Now, let M be the convex hull of polygon N. Each angle of M contains an angle of N smaller than 180° and the angle of M can be only greater than the angle of N, i.e., the exterior angle of N is not less than the exterior angle of M (Fig. 57). Therefore, even restricting to the angles of N adjacent to the angles of M we will get not less than 360°.

22.20. a) If the polygon is a convex one, then the statement is proved. Now, suppose that the exterior angle of the polygon at vertex A is greater than 180°. The visible part of the side subtends an angle smaller than 180° with vertex at point A, therefore, parts of at least two sides subtend an angle with vertex at A. Therefore, there exist rays exiting point A and such that on these rays the change of (parts of) sides visible from A occurs (on Fig. 58 all such rays are depicted). Each of such rays determines a diagonal that lies entirely inside the polygon.

b) On Fig. 59 it is plotted how to construct an n-gon with exactly $n-3$ diagonals inside it. It remains to demonstrate that any n-gon has at least $n-3$ diagonals. For $n=3$ this statement is obvious.

Suppose the statement holds for all k-gons, where $k < n$ and let us prove it for an n-gon. By heading a) it is possible to divide an n-gon by its diagonal into two polygons: a $(k+1)$-gon and an $(n-k+1)$-gon, where $k+1 < n$ and $n-k+1 < n$. These parts have at least $(k+1)-3$ and $(n-k+1)-3$ diagonals, respectively, that lie inside these parts. Therefore, the n-gon has at least $1+(k-2)+(n-k-2) = n-3$ diagonals that lie inside it.

22.21. First, let us prove that if A and B are neighbouring vertices of the n-gon, then either from A or from B it is possible to draw a diagonal. The case when the inner angle

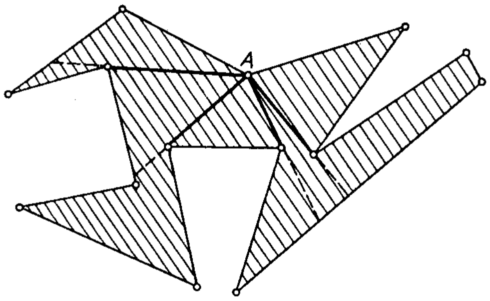


Figure 200 (Sol. 22.20 a))

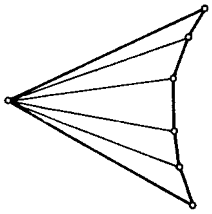


Figure 201 (Sol. 22.20 b))

of the polygon at A is greater than 180° is considered in the solution of Problem 22.20 a). Now, suppose that the angle at vertex A is smaller than 180°. Let B and C be vertices neighbouring A.

If inside triangle ABC there are no other vertices of the polygon, then BC is the diagonal and if P is the nearest to A vertex of the polygon lying inside triangle ABC, then AP is the diagonal. Hence, the number of vertices from which it is impossible to draw the diagonal does not exceed $[\frac{n}{2}]$ (the integer part of $\frac{n}{2}$). On the other hand, there exist n-gons for which this estimate is attained, see Fig. 60.

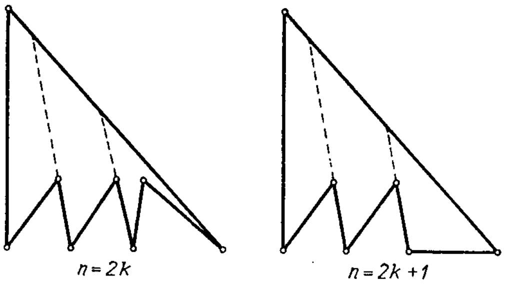


Figure 202 (Sol. 22.21)

22.22. Let us prove the statement by induction on n. For $n = 3$ it is obvious. Let $n \geq 4$. Suppose the statement is proved for all k-gons, where $k < n$; let us prove it for an n-gon. Any n-gon can be divided by a diagonal into two polygons (see Problem 22.20 a)) and the number of vertices of every of the smaller polygons is strictly less than n, i.e., they can be divided into triangles by the inductive hypothesis.

22.23. Let us prove the statement by induction. For $n = 3$ it is obvious. Let $n \geq 4$. Suppose it is proved for all k-gons, where $k < n$, and let us prove it for an n-gon. Any n-gon can be divided by a diagonal into two polygons (see Problem 22.20 a)). If the number of sides of one of the smaller polygons is equal to $k + 1$, then the number of sides of the other one is equal to $n - k + 1$ and both numbers are smaller than n. Therefore, the sum of the angles of these polygons are equal to $(k-1)\cdot 180^\circ$ and $(n-k-1)\cdot 180^\circ$, respectively. It is also clear that the sum of the angles of a n-gon is equal to the sum of the angles of these polygons, i.e., it is equal to

$$(k-1+n-k-1)\cdot 180^\circ = (n-2)\cdot 180^\circ.$$

22.24. The sum of all the angles of the obtained triangles is equal to the sum of the angles of the polygon, i.e., it is equal to $(n-2)\cdot 180^\circ$, see Problem 22.23. Therefore, the number of triangles is equal to $n-2$.

22.25. Let k_i be the number of triangles in the given partition for which precisely i sides are the sides of the polygon. We have to prove that $k_2 \geq 2$. The number of sides of the n-gon is equal to n and the number of the triangles of the partition is equal to $n-2$, see Problem 22.24. Therefore, $2k_2 + k_1 = n$ and $k_2 + k_1 + k_0 = n-2$. Subtracting the second equality from the first one we get $k_2 = k_0 + 2 \geq 2$.

22.26. Suppose that there exists a 13-gon for which on any line that contains its side there lies at least one side. Let us draw lines through all the sides of this 13-gon. Since the number of sides is equal to 13, it is clear that one of the lines contains an odd number of sides, i.e., one of the lines has at least 3 sides. On these sides lie 6 vertices and through each vertex a line passes on which there lie at least 2 sides. Therefore, this 13-gon has not less than $3 + 2\cdot 6 = 15$ sides but this is impossible.

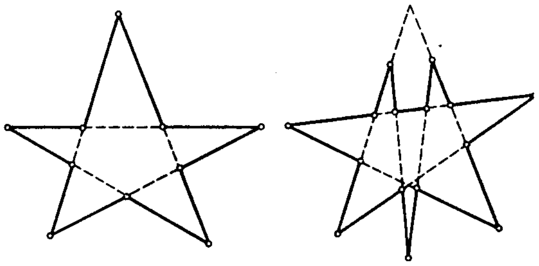
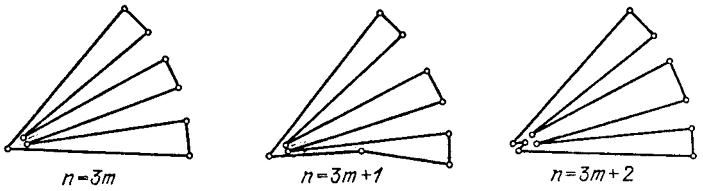


Figure 203 (Sol. 22.26)

For n even, $n \geq 10$, the required example is the contour of a "star" (Fig. 61 a)) and an idea of how to construct an example for n odd is illustrated on Fig. 61 b).

22.27. Let k be the number of acute angles of the n-gon. Then the number of its angles is smaller than $k\cdot 90^\circ + (n-k)\cdot 360^\circ$. On the other hand, the sum of the angles of an n-gon is equal to $(n-2)\cdot 180^\circ$ (see Problem 22.23) and, therefore, $k\cdot 90^\circ + (n-k)\cdot 360^\circ > (n-2)\cdot 180^\circ$, i.e., $3k < 2n+4$. It follows that $k \leq [\frac{2n}{3}] + 1$, where $[x]$ denotes the largest integer not exceeding x.



$n=3m$ $n=3m+1$ $n=3m+2$

Figure 204 (Sol. 22.27)

Examples of n-gons with $[\frac{2n}{3}] + 1$ acute angles are given on Fig. 62.

22.28. Under these operations the vectors of the sides of a polygon remain the same only their order changes (Fig. 63). Therefore, there exists only a finite number of polygons that

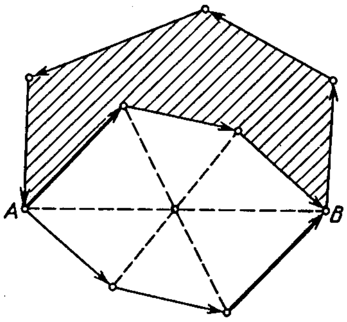


Figure 205 (Sol. 22.28)

may be obtained. Moreover, after each operation the area of the polygon strictly increases. Hence, the process terminates.

22.29. Let us carry out the proof by induction on n. For $n = 3$ the statement is obvious. Let $n \geq 4$. If one of the numbers α_i is equal to π, then the inductive step is obvious and, therefore, we may assume that all the numbers α_i are distinct from π. If $n \geq 4$, then

$$\frac{1}{n}\sum_{i=1}^{n}(\alpha_i + \alpha_{i+1}) = 2(n-2)\frac{\pi}{n} \geq \pi,$$

where the equality is only attained for a quadrilateral. Hence, in any case except for a parallelogram ($\alpha_1 = \pi - \alpha_2 = \alpha_3 = \pi - \alpha_4$), and (?) there exist two neighbouring numbers whose sum is greater than π. Moreover, there exist numbers α_i and α_{i+1} such that $\pi < \alpha_i + \alpha_{i+1} < 3\pi$. Indeed, if all the given numbers are smaller than π, then we can take the above-mentioned pair of numbers; if $\alpha_j > \pi$, then we can take numbers α_i and α_{i+1} such that $\alpha_i < \pi$ and $\alpha_{i+1} > \pi$. Let $\alpha_i^* = \alpha_i + \alpha_{i+1} - 1$. Then $0 < \alpha_i^* < 2\pi$ and, therefore, by the inductive hypothesis there exists an $(n-1)$-gon M with angles $\alpha_1, \dots, \alpha_{i-1}, \alpha_i^*, \alpha_{i+2}, \dots, \alpha_n$.

Three cases might occur: 1) $\alpha_i^* < \pi$, 2) $\alpha_i^* = \pi$, 3) $\pi < \alpha_i^* < 2\pi$.

In the first case $\alpha_i + \alpha_{i+1} < 2\pi$ and, therefore, one of these numbers, say α_i, is smaller than π. If $\alpha_{i+1} < \pi$, then let us cut from M a triangle with angles $\pi - \alpha_i$, $\pi - \alpha_{i+1}$, α_i^* (Fig. 64 a)). If $\alpha_{i+1} > \pi$, then let us juxtapose to M a triangle with angles α_i, $\alpha_{i+1} - \pi$, $\pi - \alpha_i^*$ (Fig. 64 b)).

In the second case let us cut from M a trapezoid with the base that belongs to side $A_{i-1}A_i^*A_{i+2}$ (Fig. 64 c)).

In the third case $\alpha_i + \alpha_{i+1} > \pi$ and, therefore, one of these numbers, say α_i, is greater than π. If $\alpha_{i+1} > \pi$, then let us juxtapose to M a triangle with angles $\alpha_i - \pi$, $\alpha_{i+1} - \pi$, $2\pi - \alpha_i^*$ (Fig. 64 d)), and if $\alpha_{i+1} < \pi$ let us cut off M a triangle with angles $2\pi - \alpha_i$, $\pi - \alpha_{i+1}$ and $\alpha_i^* - \pi$ (Fig.

64 e)).

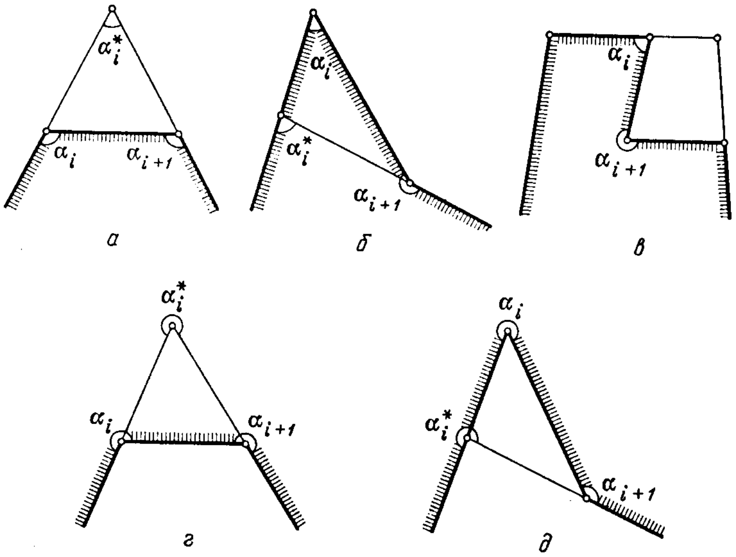


Figure 206 (Sol. 22.29)

Chapter 23. DIVISIBILITY, INVARIANTS, COLORINGS

Background

1. In a number of problems we encounter the following situation. A certain system consecutively changes its state and we have to find out something at its final state. It might be difficult or impossible to trace the whole intermediate processes but sometimes it is possible to answer the question with the help of a quantity that characterizes the state of the system and is preserved during all the transitions (such a quantity is sometimes called an *invariant* of the system considered). Clearly, in the final state the value of the invariant is the same as in the initial one, i.e., the system cannot occur in any state with another value of the invariant.

2. In practice this method reduces to the following. A quantity is calculated in two ways: first, it is simply calculated in the initial and final states and then its variation is studied under consecutive elementary transitions.

3. The simplest and most often encountered invariant is the parity of a number; the residue after a division not only by 2 but some other number can also be an invariant.

In the construction of invariants certain auxiliary colorings are sometimes convenient, i.e., partitions of considered objects into several groups, where each group consists of the objects of the same colour.

§1. Even and odd

23.1. Can a line intersect (in inner points) all the sides of a nonconvex a) $(2n+1)$-gon; b) $2n$-gon?

23.2. Given a closed broken plane line with a finite number of links and a line l that intersects it at 1985 points, prove that there exists a line that intersects this broken line in more than 1985 points.

23.3. In plane, there lie three pucks A, B and C. A hockey player hits one of the pucks so that it passes (along the straight line) between the other two and stands at some point. Is it possible that after 25 hits all the pucks return to the original places?

23.4. Is it possible to paint 25 small cells of the graph paper so that each of them has an odd number of painted neighbours? (Riddled cells are called *neighbouring* if they have a common side).

23.5. A circle is divided by points into $3k$ arcs so that there are k arcs of length 1, 2, and 3. Prove that there are 2 diametrically opposite division points.

23.6. In plane, there is given a non-selfintersecting closed broken line no three vertices of which lie on one line. A pair of non-neighbouring links of the broken will be called a singular one if the extension of one of them intersects the other one. Prove that the number of singular pairs is always even.

23.7. (*Sperner's lemma.*) The vertices of a triangle are labeled by figures 0, 1 and 2. This triangle is divided into several triangles so that no vertex of one triangle lies on a side of the other one. The vertices of the initial triangle retain their old labels and the additional vertices get labels 0, 1, 2 so that any vertex on a side of the initial triangle should be labelled by one of the vertices of this side, see Fig. 65. Prove that there exists a triangle in the partition labelled by 0, 1, 2.

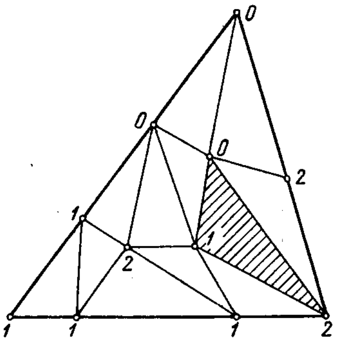


Figure 207 (23.6)

23.7. The vertices of a regular $2n$-gon $A_1 \dots A_{2n}$ are divided into n pairs. Prove that if $n = 4m+2$ or $n = 4m+3$, then the two pairs of vertices are the endpoints of equal segments.

§2. Divisibility

23.9. On Fig. 66 there is depicted a hexagon divided into black and white triangles so that any two triangles have either a common side (and then they are painted different colours) or a common vertex, or they have no common points and every side of the hexagon is a side of one of the black triangles. Prove that it is impossible to find a similar partition for a 10-gon.

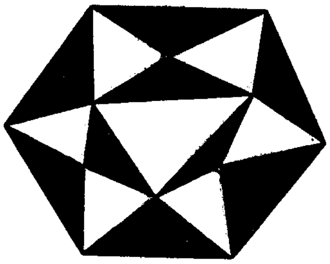


Figure 208 (23.9)

23.10. A square sheet of graph paper is divided into smaller squares by segments that follow the sides of the small cells. Prove that the sum of the lengths of these segments is divisible by 4. (The length of a side of a small cell is equal to 1).

§3. Invariants

23.11. Given a chess board, it is allowed to simultaneously repaint into the opposite colour either all the cells of one row or those of a column. Can we obtain in this way a board with precisely one black small cell?

23.12. Given a chess board, it is allowed to simultaneously repaint into the opposite colour all the small cells situated inside a 2×2 square. Is it possible that after such repaintings there will be exactly one small black cell left?

23.13. Given a convex $2m$-gon $A_1 \dots A_{2m}$ and point P inside it not belonging to any of the diagonals, prove that P belongs to an even number of triangles with vertices at points $A_1, \dots, A_{2m}$.

23.14. In the center of every cell of a chess board stands a chip. Chips were interchanged so that the pairwise distances between them did not diminish. Prove that the pairwise distances did not actually alter at all.

23.15. A polygon is cut into several polygons so that the vertices of the obtained polygons do not belong to the sides of the initial polygon nor to the sides of the obtained polygons. Let p be the number of the obtained smaller polygons, q the number of segments which serve as the sides of the smaller polygons, r the number of points which are their vertices. Prove that

$$p - q + r = 1. \qquad (\textit{Euler's formula})$$

23.16. A square field is divided into 100 equal square patches 9 of which are overgrown with weeds. It is known that during a year the weeds spread to those patches that have not less than two neighbouring (i.e., having a common side) patches that are already overgrown with weeds and only to them. Prove that the field will never overgrow completely with weeds.

23.17. Prove that there exist polygons of equal size and impossible to divide into polygons (perhaps, nonconvex ones) which can be translated into each other by a parallel translation.

23.18. Prove that it is impossible to cut a convex polygon into finitely many nonconvex quadrilaterals.

23.18. Given points $A_1, \dots, A_n$. We considered a circle of radius R encircling some of them. Next, we constructed a circle of radius R with center in the center of mass of points that lie inside the first circle, etc. Prove that this process eventually terminates, i.e., the circles will start to coincide.

§4. Auxiliary colorings

23.20. In every small cell of a 5×5 chess board sits a bug. At certain moment all the bugs crawl to neighbouring (via a horizontal or a vertical) cells. Is it necessary that some cell to become empty at the next moment?

23.21. Is it possible to tile by 1×2 domino chips a 8×8 chess board from which two opposite corner cells are cut out?

23.22. Prove that it is impossible to cut a 10×10 chess board into T-shaped figures consisting of four cells.

23.23. The parts of a toy railroad's line are of the form of a quarter of a circle of radius R. Prove that joining them consecutively so that they would smoothly turn into each other it is impossible to construct a closed path whose first and last links form the dead end depicted on Fig. 67.

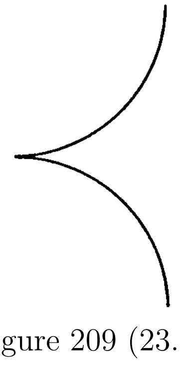


Figure 209 (23.23)

23.24. At three vertices of a square sit three grasshoppers playing the leap frog as follows. If a grasshopper A jumps over a grasshopper B, then after the jump it lands at the same distance from B but, naturally, on the other side and on the same line. Is it possible that after several jumps one of the grasshoppers gets to the fourth vertex of the square?

23.25. Given a square sheet of graph paper of size 100×100 cells. Several nonselfintersecting broken lines passing along the sides of the small cells and without common points are drawn. These broken lines are all strictly inside the square but their endpoints are invariably on the boundary. Prove that apart from the vertices of the square there will be one more node (of the graph paper inside the square or on the boundary) that does not belong to any of the broken lines.

§5. More auxiliary colorings

23.26. An equilateral triangle is divided into n^2 equal equilateral triangles (Fig. 68). Some of them are numbered by numbers 1, 2, ... , m and consecutively numbered triangles have adjacent sides. Prove that $m \leq n^2 - n + 1$.

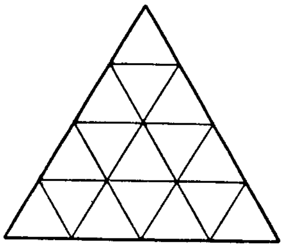


Figure 210 (23.26)

23.27. The bottom of a parallelepipedal box is tiled with tiles of size 2×2 and 1×4. The tiles had been removed from the box and in the process one tile of size 2×2 was lost. We replaced it with a tile of size 1×4. Prove that it will be impossible to tile now the bottom of the box.

23.28. Of a piece of graph paper of size 29×29 (of unit cells) 99 squares of size 2×2 were cut. Prove that it is still possible to cut off one more such square.

23.29. Nonintersecting diagonals divide a convex n-gon into triangles and at each of the n-gon's vertex an odd number of triangles meet. Prove that n is divisible by 3.

* * *

23.30. Is it possible to tile a 10×10 graph board by tiles of size 2×4?

23.31. On a graph paper some arbitrary n cells are fixed. Prove that from them it is possible to select not less than $\frac{n}{4}$ cells without common points.

23.32. Prove that if the vertices of a convex n-gon lie in the nodes of graph paper and there are no other nodes inside or on the sides of the n-gon, then $n \leq 4$.

23.33. From 16 tiles of size 1×3 and one tile of size 1×1 one constructed a 7×7 square. Prove that the 1×1 tile either sits in the center of the square or is adjacent to its boundary.

23.34. A picture gallery is of the form of a nonconvex n-gon. Prove that in order to overview the whole gallery $[\frac{n}{3}]$ guards suffices.

§6. Problems on colorings

23.35. A plane is painted two colours. Prove that there exist two points of the same colour the distance between which is equal to 1.

23.36. A plane is painted three colours. Prove that there are two points of the same colour the distance between which is equal to 1.

23.37. The plane is painted seven colours. Are there necessarily two points of the same colour the distance between which is equal to 1?

(?)**23.38.** The points on sides of an equilateral triangle are painted two colours. Prove that there exists a right triangle with vertices of the same colour.

* * *

A *triangulation* of a polygon is its partition into triangles with the property that these triangles have either a common side or a common vertex or have no common points (i.e., the vertex of one triangle cannot belong to a side of the other one).

23.39. Prove that it is possible to paint the triangles of a triangulation three colours so that the triangles with a common side would be of different colours.

23.40. A polygon is cut by nonintersecting diagonals into triangles. Prove that the vertices of the polygon can be painted three colours so that all the vertices of each of the obtained triangles would be of different colours.

23.41. Several disks of the same radius were put on the table so that no two of them overlap. Prove that it is possible to paint disks four colours so that any two tangent disks would be of different colours.

Solutions

23.1. a) Let a line intersect all the sides of the polygon. Consider all the vertices on one side of the line. To each of these vertices we can assign a pair of sides that intersect at it. Thus we get a partition of all the sides of the polygon into pairs. Therefore, if a line intersects all the sides of an m-gon, then m is even.

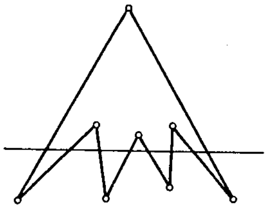


Figure 211 (Sol. 23.1)

b) It is clear from Fig. 69 how to construct $2n$-gon and a line that intersects all its sides for any n.

23.2. A line l determines two half planes; one of them will be called *upper* the other one *lower*. Let n_1 (resp. n_2) be the number of the vertices of the broken line that lie on l for which both links that intersect at this point belong to the upper (resp. lower) half plane and m the number of all the remaining intersection points of l and the broken line. Let us circumvent the broken line starting from a point that does not lie on l (and returning to the same point). In the process we pass from one half plane to the other one only passing through any of m intersection points. Since we will have returned to the same point from which we have started, m is even.

By the hypothesis $n_1 + n_2 + m = 1985$ and, therefore, $n_1 + n_2$ is odd, i.e., $n_1 \neq n_2$.

Let for definiteness $n_1 > n_2$. Then let us draw in the upper halfplane a line l_1 parallel to l and distant from it by a distance smaller than any nonzero distance from l to any of the vertices of the broken line (Fig. 70). The number of intersection points of the broken line with l_1 is equal to $2n_1 + m > n_1 + n_2 + m = 1985$, i.e., l_1 is the desired line.

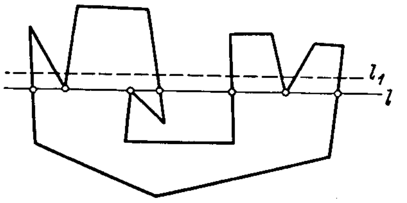


Figure 212 (Sol. 23.2)

23.3. No, they cannot. After each hit the orientation (i.e., the direction of the circumventing pass) of triangle ABC changes.

23.4. Let on a graph paper several cells be painted and n_k be the number of painted cells with exactly k painted neighbours. Let N be the number of common sides of painted cells. Since each of them belongs to exactly two painted cells,

$$N = \frac{n_1 + 2n_2 + 3n_3 + 4n_4}{2} = \frac{n_1 + n_3}{2} + n_2 + n_3 + 2n_4.$$

Since N is an integer, $n_1 + n_3$ is even.

(?) We have proved that the number of painted cells with an odd number of painted cells is always even. Therefore, it is impossible to paint 25 cells so that each of them would have had an odd number of painted neighbours.

23.5. Suppose that the circle is divided into arcs as indicated and there are no diametrically opposite division points. Then against the endpoints of any arc of length 1 there are no division points and, therefore, against it there lies an arc of length 3. Let us delete one of the arcs of length 1 and the opposite arc of length 3. Then the circle is divided into two arcs.

If on one of them there lie m arcs of length 1 and n arcs of length 3, then on the other one there lie m arcs of length 3 and n arcs of length 1. The total number of arcs of length 1 and 3 lying on these two "great" arcs is equal to $2(k-1)$ and, therefore, $n + m = k - 1$.

Since beside arcs of length 1 and 3 there are only arcs of even length, the parity of the length of each of the considered arcs coincides with the parity of $k-1$. On the other hand, the length of each of them is equal to $\frac{6k-1-3}{2} = 3k - 2$. We have obtained a contradiction since numbers $k-1$ and $3k-2$ are of opposite parities.

23.6. Take neighbouring links AB and BC and call the angle symmetric to angle $\angle ABC$ through point B a *little angle* (on Fig. 71 the little angle is shaded).

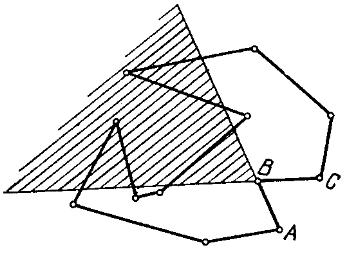


Figure 213 (Sol. 23.6)

We can consider similar little angles for all vertices of the broken line. It is clear that the number of singular pairs is equal to the number of intersection points of links with little angles. It remains to notice that the number of links of the broken line which intersect one angle is even because during the passage from A to C the broken line goes into the little angle as many times as it goes out of it.

23.7. Let us consider segments into which side 01 is divided. Let a be the number of segments of the form 00 and b the number of segments of the form 01. For every segment consider the number of zeros at its ends and add all these numbers. We get $2a+b$. On the other hand, all the "inner" zeros enter this sum twice and there is one more zero at a vertex of the initial triangle. Consequently, the number $2a+b$ is odd, i.e., b is odd.

Let us now divide the triangle. Let a_1 be the total number of triangles of the form 001 and 011 and b_1 the total number of triangles of the form 012. For every triangle consider the number of its sides of the form 01 and add all these numbers. We get $2a_1+b_1$. On the other hand all "inner" sides enter twice the sum and all the "boundary" sides lie on the side 01 of the initial triangle and their number is odd by above arguments. Therefore, the number $2a_1+b_1$ is odd in particular $b_1 \neq 0$.

23.8. Suppose that all the pairs of vertices determine segments of distinct lengths. Let us assign to segment A_pA_q the least of the numbers $|p-q|$ and $2n-|p-q|$. As a result, for the given n pairs of vertices we get numbers 1, 2, … , n; let among these numbers there be k even and $n-k$ odd ones. To odd numbers segments A_pA_q, where numbers p and q are of opposite parity, correspond. Therefore, among vertices of the other segments there are k vertices with even numbers and k vertices with odd numbers and the segments connect vertices with numbers of the same parity. Therefore, k is even. For numbers n of the form $4m$, $4m+1$, $4m+2$ and $4m+3$ the number k of even numbers is equal to $2m$, $2m$, $2m+1$ and $2m+1$, respectively, and therefore, either $n=4m$ or $n=4m+1$.

23.9. Suppose we have succeded to cut the decagon as required. Let n be the number of sides of black triangles, m the number of sides of white triangles. Since every side of an odd triangle (except the sides of a polygon) is also a side of a white triangle, then $n-m=10$. On the other hand, both n and m are divisible by 3. Contradiction.

23.10. Let Q be a square sheet of paper, $L(Q)$ the sum of lengths of the sides of the small cells that lie inside it. Then $L(Q)$ is divisible by 4 since all the considered sides split into quadruples of sides obtained from each other by rotations through angles of $\pm 90^\circ$ and 180° about the center of the square.

If Q is divided into squares $Q_1, \dots, Q_n$, then the sum of the lengths of the segments of the partition is equal to $L(Q)-L(Q_1)-\dots-L(Q_n)$. Clearly, this number is divisible by 4 since the numbers $L(Q)$, $L(Q_1)$, … , $L(Q_n)$ are divisible by 4.

23.11. Repainting the horizontal or vertical containing k black and $8-k$ white cells we get $8-k$ black and k white cells. Therefore, the number of black cells changes by $(8-k)-k=8-2k$, i.e., by an even number. Since the parity of the number of black cells is preserved, we cannot get one black cell from the initial 32 black cells.

23.12. After repainting the 2×2 square containing k black and $4-k$ white cells we get $4-k$ black and k white cells. Therefore, the number of black cells changes by $(4-k)-k=4-2k$, i.e., by an even number. Since the parity of the number of black cells is preserved, we cannot get one black cell from the initial 32 black cells.

23.13. The diagonals divide a polygon into several parts. Parts that have a common side are called *neighbouring.* Clearly, from any inner point of the polygon we can get into any other point passing each time only from a neighbouring part to a neighbouring part. A part of the plane that lies outside the polygon can also be considered as one of these parts. The number of the considered triangles for the points of this part is equal to zero and, therefore, it suffices to prove that under the passage from a neighbouring part to a neighbouring one the parity of the number of triangles is preserved.

Let the common side of two neighbouring parts lie on diagonal (or side) PQ. Then for all the triangles considered, except the triangles with PQ as a side, both these parts either simultaneously belong to or do not belong to. Therefore, under the passage from one part to the other one the number of triangles changes by $k_1 - k_2$, where k_1 is the number of vertices of the polygon situated on one side of PQ and k_2 is the number of vertices situated on the other side of PQ. Since $k_1 + k_2 = 2m - 2$, it follows that $k_1 - k_2$ is even.

23.14. If at least one of the distances between chips would increase, then the sum of the pairwise distances between chips would have also increased but the sum of all pairwise distances between chips does not vary under any permutation.

23.15. Let n be the number of vertices of the initial polygon, $n_1, \dots, n_p$ the number of vertices of the obtained polygons. On the one hand, the sum of angles of all the obtained polygons is equal to

$$\sum_{i=1}^{p}(n_i - 2)\pi = \sum_{i=1}^{p} n_i\pi - 2p\pi.$$

On the other hand, it is equal to

$$2(r - n)\pi + (n - 2)\pi.$$

It remains to observe that

$$\sum_{i=1}^{p} n_i = 2(q - n) + n.$$

23.16. It is easy to verify that the length of the boundary of the whole patch (of several patches) overgrown with weeds does not increase. Since in the initial moment it did not surpass $9 \cdot 4 = 36$, then at the final moment it cannot be equal to 40.

23.17. In plane, fix ray AB. To any polygon M assign a number $F(M)$ (depending on AB) as follows. Consider all the sides of M perpendicular to AB and to each of them assign the number $\pm l$, where l is the length of this side and the sine "plus" is taken if following this side in the direction of ray AB we get inside M and "minus" if we get outside M, see Fig. 72.

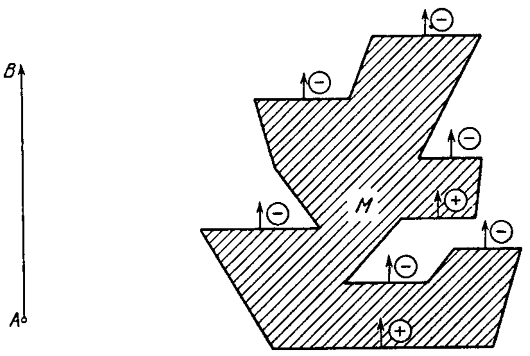


Figure 214 (Sol. 23.17)

Let us denote the sum of all the obtained numbers by $F(M)$; if M has no sides perpendicular to AB, then $F(M) = 0$.

It is easy to see that if polygon M is divided into the union of polygons M_1 and M_2, then $F(M) = F(M_1) + F(M_2)$ and if M' is obtained from M by a parallel translation, then $F(M') = F(M)$. Therefore, if M_1 and M_2 can be cut into parts that can be transformed into each other by a parallel translation, then $F(M_1) = F(M_2)$.

On Fig. 73 there are depicted congruent equilateral triangles PQR and PQS and ray AB perpendicular to side PQ. It is easy to see that $F(PQR) = a$ and $F(PQS) = -a$, where a is the length of the side of these equilateral triangles. Therefore, it is impossible to divide

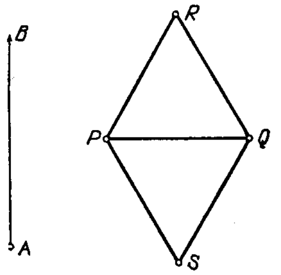


Figure 215 (Sol. 23.17)

congruent triangles PQR and PQS into parts that can be translated into each other by a parallel translation.

23.18. Suppose that a convex polygon M is divided into nonconvex quadrilaterals M_1, … , M_n. To every polygon N assign the number $f(N)$ equal to the difference between the sum of its inner angles smaller than 180° and the sum of the angles that complements its angles greater than 180° to 360°. Let us compare the numbers $A = f(M)$ and $B = f(M_1) + \dots + f(M_n)$. To this end consider all the points that are vertices of triangles M_1, … , M_n. These points can be divided into four types:

1) The (inner?) points of M. These points contribute equally to A and to B.

2) The points on sides of M or M_i. The contribution of each such point to B exceeds the contribution to A by 180°.

(?)3) The inner points of the polygon in which the angles of the quadrilateral smaller than 180° in it. The contribution of every such point to B is smaller than that to A by 360°.

4) The inner points of polygon M in which the angles of the quadrilaterals meet and one of the angles is greater than 180°. Such points give zero contribution to both A and B.

As a result we see that $A \leq B$. On the other hand, $A > 0$ and $B = 0$. The inequality $A > 0$ is obvious and to prove that $B = 0$ it suffices to verify that if N is a nonconvex quadrilateral, then $f(N) = 0$. Let the angles of N be equal to α, β, γ and δ, where $\alpha \geq \beta \geq \gamma \geq \delta$. Any nonconvex quadrilateral has exactly one angle greater than 180° and, therefore,

$$f(N) = \beta + \gamma + \delta - (360^\circ - \alpha) = \alpha + \beta + \gamma + \delta - 360^\circ = 0^\circ.$$

We have obtained a contradiction and, therefore, it is impossible to cut a convex polygon into a finite number of nonconvex quadrilaterals.

23.19. Let S_n be the circle constructed at the n-th step; O_n its center. Consider the quantity $F_n = \sum(R^2 - O_nA_i^2)$, where the sum runs over points that are inside S_n only. Let us denote the points lying inside circles S_n and S_{n+1} by letters B with an index; the points that lie inside S_n but outside S_{n+1} by letters C with an index and points lying inside S_{n+1} but outside S_n by letters D with an index. Then

$$F_n = \sum(R^2 - O_nB_i^2) + \sum(R^2 - O_nC_i^2)$$

and

$$F_{n+1} = \sum(R^2 - O_{n+1}B_i^2) + \sum(R^2 - O_{n+1}D_i^2).$$

Since O_{n+1} is the center of mass of the system of points B and C, it follows that

$$\sum O_nB_i^2 + \sum O_nC_i^2 = qO_nO_{n+1}^2 + \sum O_{n+1}B_i^2 + \sum O_{n+1}C_i^2,$$

where q is the total number of points of type B and C. It follows that

$$F_{n+1} - F_n = qO_nO_{n+1}^2 + \sum(R^2 - O_{n+1}D_i^2) - \sum(R^2 - O_{n+1}C_i^2).$$

All the three summands are nonnegative and, therefore, $F_{n+1} \geq F_n$. In particular, $F_n \geq F_1 > 0$, i.e., $q > 0$.

There is a finite number of centers of mass of distinct subsets of given points and, therefore, there is also only finitely many distinct positions of circles S_i. Hence, $F_{n+1} = F_n$ for some n and, therefore, $qO_nO_{n+1}^2 = 0$, i.e., $O_n = O_{n+1}$.

23.20. Since the total number of cells of a 5×5 chessboard is odd, the number of black fields cannot be equal to the number of white fields. Let, for definiteness, there be more black fields than white fields. Then there are less bugs that sit on white fields than there are black fields. Therefore, at least one of black fields will be empty since only bugs that sit on white fields crawl to black fields.

23.21. Since the fields are cut of one colour only, say, of black colour, there remain 32 white and 30 black fields. Since a domino piece always covers one white and one black field, it is impossible to tile with domino chips a 8×8 chessboard without two opposite corner fields.

23.22. Suppose that a 10×10 chessboard is divided into such tiles. Every tile contains either 1 or 3 black fields, i.e., always an odd number of them. The total number of figures themselves should be equal to $\frac{100}{4} = 25$. Therefore, they contain an odd number of black fields and the total of black fields is $\frac{100}{2} = 50$ copies. Contradiction.

(?)**23.23.** Let us divide the plane into equal squares with side $2R$ and paint them in a staggered order. Let us inscribe a circle into each of them. Then the details of the railway can be considered placed on these circles and the movement of the train that follows from the beginning to the end is performed clockwise on white fields and counterclockwise on black fields (or the other way round, see Fig. 74).

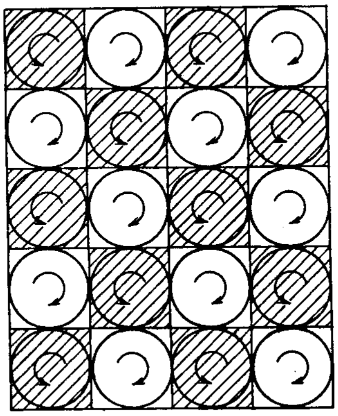


Figure 216 (Sol. 23.23)

Therefore, a deadend cannot arise since along both links of the deadend the movement is performed in the same fashion (clockwise or counterclockwise).

23.24. Let us consider the lattice depicted on Fig. 75 and paint it two colours as indicated in Fig. (white nodes are not painted on this Fig. and the initial square is shaded so that the grasshoppers sit in its white vertices). Let us prove that the grasshoppers can only reach white nodes, i.e., under the symmetry through a white node any white node turns into a white one. To prove this, it suffices to prove that under a symmetry through a white node a black node turns into a black one.

Let A be a black node, B a white one and A_1 the image of A under the symmetry through B. Point A_1 is a black node if and only if $\overrightarrow{AA_1} = 2m\mathbf{e}_1 + 2n\mathbf{e}_2$, where m and n are integers.

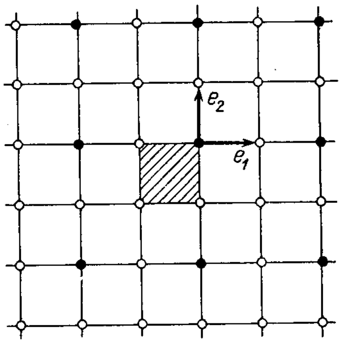


Figure 217 (Sol. 23.24)

It is clear that

$$\overrightarrow{AA_1} = 2\overrightarrow{AB} = 2(m\mathbf{e}_1 + n\mathbf{e}_2)$$

and, therefore, A_1 is a black node. Therefore, a grasshopper cannot reach the fourth vertex of the square.

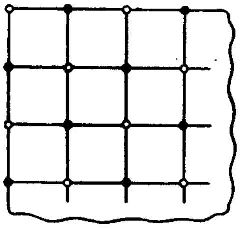


Figure 218 (Sol. 23.25)

23.25. Let us paint the nodes of the graph paper in a (?)chess order (Fig. 76). Since the endpoints of any unit segment are of different colours, the broken line with the endpoints of the same colour contains an odd number of nodes and an even number of nodes if its endpoints are of the same colour. Suppose that broken lines go out of all the nodes of the boundary (except for the vertices of the square). Let us prove then that all the broken lines together contain an even number of nodes. To this end it suffices to show that the number of broken lines with the endpoints of the same colour is even.

Let $4m$ white and $4n$ black nodes (the vertices of the square are not counted) are placed on the boundary of the square. Let k be the number of broken lines with both endpoints white. Then there are $4m - 2k$ broken lines with endpoints of different colours and $\frac{4n-(4m-2k)}{2} = 2(n-m)+k$ broken lines with black endpoints. It follows that there are $k + 2(n-m) + k = 2(n-m+k)$ — an even number — of broken lines with the endpoints of the same colour. It remains to notice that a 100 × 100 piece of paper contains an odd number of nodes. Therefore, the broken lines with an even number of nodes cannot pass through all the nodes.

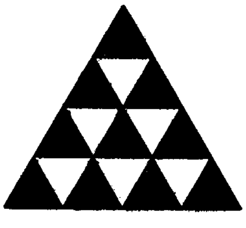


Figure 219 (Sol. 23.25)

23.26. Let us paint the triangles as shown on Fig. 77. Then there are $1+2+\cdots+n=\frac{1}{2}n(n+1)$ black triangles and $1+2+\cdots+(n-1)=\frac{1}{2}n(n-1)$ white triangles. It is clear that two triangles with consecutive indices are of distinct colours. Hence, among the numbered triangles the number of black triangles is only by 1 greater than that of white ones.

Therefore, the total number of numbered triangles does not exceed $n(n-1)+1$.

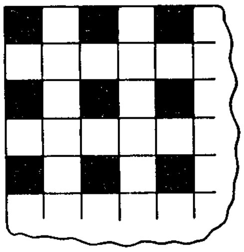


Figure 220 (Sol. 23.27)

23.27. Let us paint the bottom of the box two colours as shown on Fig. 78. Then every 2×2 tile covers exactly one black cell and a 1×4 tile covers 2 or 0 of them. Hence, the parity of the number of odd cells on the bottom of the box coincides with the parity of the number of 2×2 tiles. Since under the change of a 2×2 tile by a 1×4 tile the parity of the number of 2×2 tiles changes, we will not be able to tile the bottom of the box.

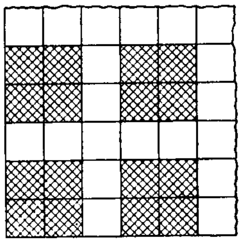


Figure 221 (Sol. 23.28)

23.28. In the given square piece of graph paper, let us shade 2×2 squares as shown on Fig. 79. We thus get 100 shaded squares. Every cut off square touches precisely one shaded square and therefore, at least one shaded square remains intact and can be cut off(?).

23.29. If a polygon is divided into parts by several diagonals, then these parts can be painted two colours so that parts with a common side were of distinct colours. This can be done as follows.

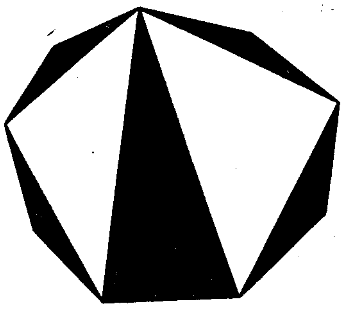


Figure 222 (Sol. 23.29)

Let us consecutively draw diagonals. Every diagonal splits the polygon into two parts. In one of them retain its painting and repaint the other one changing everywhere the white colour to black and black to white. Performing this operation under all the needed diagonals, we get the desired coloring.

Since in the other case at every vertex an odd number of triangles meet, then under such a coloring all the sides of the polygon would belong to triangles of the same colour, for example, black, Fig. 80.

Denote the number of sides of white triangles by m. It is clear that m is divisible by 3. Since every side of a white triangle is also a side of a black triangle and all the sides of the polygon are sides of the black triangles, it follows that the number of sides of black triangles is equal to $n+m$. Hence, $n+m$ is divisible by 3 and since m is divisible by 3, then n is divisible by 3.

23.30. Let us paint the chessboard four colours as shown on Fig. 81. It is easy to count the number of cells of the second colour: it is 26; that of the fourth is 24.

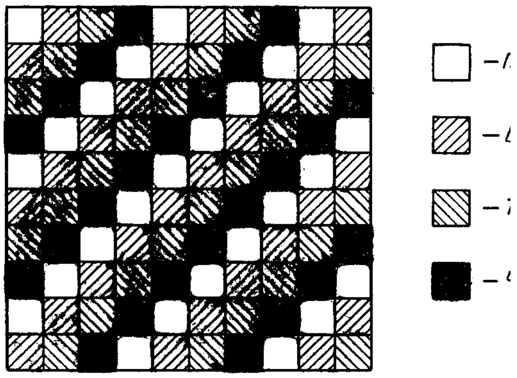


Figure 223 (Sol. 23.30)

Every 1×4 tile covers one cell of each colour. Therefore, it is impossible to tile a 10×10 chessboard with tiles of size 1×4 since otherwise there would have been an equal number of cells of every colour.

23.31. Let us paint the graph paper four colours as shown on Fig. 82. Among the given n cells there are not less than $\frac{n}{4}$ cells of the same colour and such cells do not have common points.

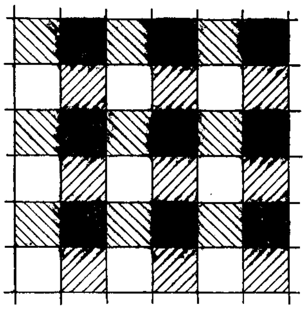


Figure 224 (Sol. 23.32)

23.32. Let us paint the nodes of graph paper four colours in the same order as the cells on Fig. 82 are painted. If $n \geq 5$, then there exist two vertices of an n-gon of the same colour. The midpoint of the segment with the endpoints in the nodes of the same colour is a node itself. Since the n-gon is a convex one, then the midpoint of the segment with the endpoints at its nodes lies either inside it or on its side.

23.33. Let us divide the obtained square into cells of size 1×1 and paint them three colours as shown on Fig. 83. It is easy to verify that it is possible to divide tiles of size 1×3 into two types: a tile of the first type covers one cell of the first colour and two cells of the second colour and a tile of the second type covers one cell of the second colour and two cells of the third colour.

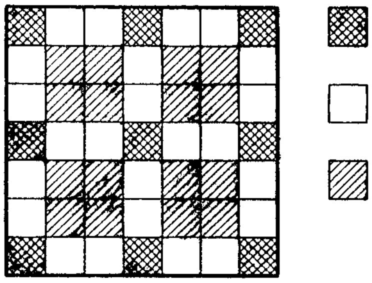


Figure 225 (Sol. 23.33)

Suppose that all the cells of the first colour are covered by tiles 1×3. Then there are 9 tiles of the first type and 7 tiles of the second type. Hence, they cover $9 \cdot 2 + 7 = 25$ cells of the second colour and $7 \cdot 2 = 14$ cells of the third colour. We have reached a contradiction and, therefore, one of the cells of the first colour is covered by the tile of size 1×1.

23.34. Let us cut the given n-gon by nonintersecting diagonals into triangles (cf. Problem 22.22). The vertices of the n-gon can be painted 3 colours so that all the vertices of each of the obtained triangles are of distinct colours (see Problem 23.40). There are not more than $[\frac{n}{3}]$ vertices of any colour; and it suffices to place guards at these points.

23.35. Let us consider an equilateral triangle with side 1. All of its three vertices cannot be of distinct colours and, therefore, two of the vertices are of the same colour; the distance between them is equal to 1.

23.36. Suppose that any two points situated at distance 1 are painted distinct colours. Consider an equilateral triangle ABC with side 1; all its vertices are of distinct colours. Let point A_1 be symmetric to A through line BC. Since $A_1B = A_1C = 1$, the colour of A_1 is distinct from that of B and C and A_1 is painted the same colour as A.

These arguments show that if $AA_1 = \sqrt{3}$, then points A and A_1 are of the same colour. Therefore, all the points on the circle of radius $\sqrt{3}$ with center A are of the same colour. It is clear that on this circle there are two points the distance between which is equal to 1. Contradiction.

23.37. Let us give an example of a seven-colour coloring of the plane for which the distance between any two points of the same colour is not equal to 1. Let us divide the plane into equal hexagons with side a and paint them as shown on Fig. 84 (the points belonging to two or three hexagons can be painted any of the colours of these hexagons).

The greatest distance between points of the same colour that belong to one hexagon does not exceed $2a$ and the least distance between points of the same colour lying in distinct hexagons is not less than the length of segment AB (see Fig. 84). It is clear that

$$AB^2 = AC^2 + BC^2 = 4a^2 + 3a^2 = 7a^2 > (2a)^2.$$

Therefore, if $2a < 1 < \sqrt{7}a$, i.e., $\frac{1}{\sqrt{7}} < a < \frac{1}{2}$, then the distance between points of the same colour cannot be equal to 1.

23.38. Suppose there does not exist a right triangle with vertices of the same colour. Let us divide every side of an equilateral triangle into three parts by two points. These points form a right hexagon. If two of its opposite vertices are of the same colour, then all the other

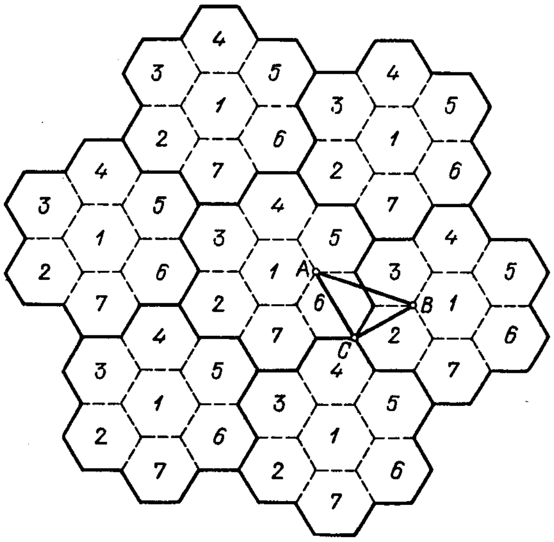


Figure 226 (Sol. 23.37)

vertices are of the second colour and therefore, there exists a right triangle with vertices of the second colour. Hence, the opposite vertices of the hexagon must be of distinct colours.

Therefore, there exist two neighbouring vertices of distinct colours; the vertices opposite to them are also of distinct colours. One of these pairs of vertices of distinct colours lies on a side of the triangle. The points of this side distinct from the vertices of the hexagon cannot be of either first or second colour. Contradiction.

23.39. Let us prove this statement by induction on the number of triangles of the triangulation. For one triangle the needed coloring exists. Now, let us suppose that it is possible to paint in the required way any triangulation consisting of less than n triangles; let us prove that then we can paint any triangulation consisting of n triangles.

Let us delete a triangle one of the sides of which lies on a side of the triangulated figure. The remaining part can be painted by the inductive hypothesis. (It is clear that this part can consist of several disjoint pieces but this does not matter.) Only two sides of the deleted triangle can be neighbouring with the other triangles. Therefore, it can be coloured the colour distinct from the colours of its two neighbouring triangles.

23.40. Proof is similar to that of Problem 23.39. The main difference is in that one must delete a triangle with two sides of the boundary of the polygon (cf. Problem 22.25).

23.41. Proof will be carried out by induction on the number of disks n. For $n = 1$ the statement is obvious. Let M be any point, O the most distant from M center of a(?) given disk. Then the disk centered at O is tangent to not more than 3 other given disks. Let us delete it and paint the other disks; this is possible thanks to the inductive hypothesis. Now, let us paint the deleted disk the colour distinct from the colours of the disks tangent to it.

Chapter 24. INTEGER LATTICES

In plane, consider a system of lines given by equations $x = m$ and $y = n$, where m and n are integers. These lines form a lattice of squares or an *integer lattice*. The vertices of these squares, i.e., the points with integer coordinates, are called *the nodes* of the integer lattice.

§1. Polygons with vertices in the nodes of a lattice

24.1. Is there an equilateral triangle with vertices in the nodes of an integer lattice?

24.2. Prove that for $n \neq 4$ a regular n-gon is impossible to place so that its vertices would lie in the nodes of an integer lattice.

24.3. Is it possible to place a right triangle with integer sides (i.e., with sides of integer length) so that its vertices would be in nodes of an integer lattice but none of its sides would pass along the lines of the lattice?

24.4. Is there a closed broken line with an odd number of links of equal length all vertices of which lie in the nodes of an integer lattice?

24.5. The vertices of a polygon (not necessarily convex one) are in nodes of an integer lattice. Inside the polygon lie n nodes of the lattice and m nodes lie on the polygon's boundary. Prove that the polygon's area is equal to $n + \frac{m}{2} - 1$. (*Pick's formula.*)

24.6. The vertices of triangle ABC lie in nodes of an integer lattice and there are no other nodes on its sides whereas inside it there is precisely one node, O. Prove that O is the intersection point of the medians of triangle ABC.

See also Problem 23.32.

§2. Miscellaneous problems

24.7. On an infinite sheet of graph paper N, cells are painted black. Prove that it is possible to cut off a finite number of squares from this sheet so that the following two conditions are satisfied:

1) all black cells belong to the cut-off squares;

2) in any cut-off square K, the area of black cells constitutes not less than 0.2 and not more than 0.8 of the area of K.

24.8. The origin is the center of symmetry of a convex figure whose area is greater than 4. Prove that this figure contains at least one distinct from the origin point with integer coordinates. (*Minkowski's theorem.*)

24.9. In all the nodes of an integer lattice except one, in which a hunter stands, trees are growing and the trunks of these trees are of radius r each. Prove that the hunter will not be able to see a hare that sits further than $\frac{1}{r}$ of the unit length from it.

24.10. Inside a convex figure of area S and semiperimeter p there are n nodes of a lattice. Prove that $n > S - p$.

24.11. Prove that for any n there exists a circle inside which there are exactly (not more nor less) n integer points.

24.12. Prove that for any n there exists a circle on which lies exactly (not more nor less) n integer points.

Solutions

24.1. Suppose that the vertices of an equilateral triangle ABC are in nodes of an integer lattice. Then the tangents of all the angles formed by sides AB and AC with the lines of the lattice are rational. For any position of triangle ABC either the sum or the difference of certain two of such angles α and β is equal to 60°. Hence,

$$\sqrt{3} = \tan 60^\circ = \tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

is a rational number. Contradiction.

24.2. For $n = 3$ and $n = 6$ the statement follows from the preceding problem and, therefore, in what follows we will assume that $n \neq 3$, 4, 6. Suppose that there exist regular n-gons with vertices in nodes of an integer lattice ($n \neq 3$, 4, 6). Among all such n-gons we can select one with the shortest side. (To prove that we can do it, it suffices to observe that if a is the length of a segment with the endpoints in nodes of the lattice, then $a = \sqrt{n^2 + m^2}$, where n and m are integers, i.e., there is only a finite number of distinct length of segments with the endpoints in nodes of the lattice shorter than the given length.) Let $\overrightarrow{A_iB_i} = \overrightarrow{A_{i+1}A_{i+2}}$. Then $B_1 \dots B_n$ is a regular n-gon whose vertices lie in nodes of the integer lattice and its side is shorter than any side of the n-gon $A_1 \dots A_n$. For $n = 5$ this is clear from Fig. 85 and for $n \geq 7$ look at Fig. 86. We have arrived to a contradiction with the choice of the n-gon $A_1 \dots A_n$.

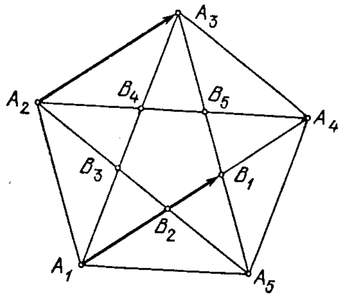


Figure 227 (Sol. 24.2)

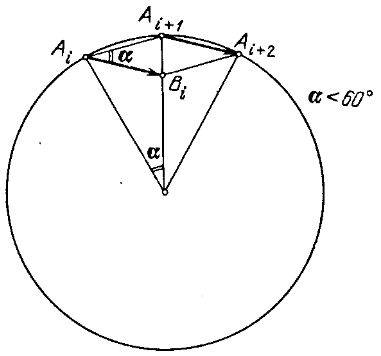


Figure 228 (Sol. 24.2)

24.3. It is easy to verify that the triangle with the vertices at points with coordinates $(0, 0)$, $(12, 16)$ and $(-12, 9)$ possesses the required properties.

24.4. Suppose that there exists a closed broken line $A_1 \dots A_n$ with an odd number of links of equal length all the vertices of which lie in nodes of an integer lattice. Let a_i and

b_i be coordinates of the projections of vector $\overrightarrow{A_iA_{i+1}}$ to the horizontal and vertical axes, respectively. Let c be the length of the link of the broken line. Then $c^2 = a_i^2 + b_i^2$.

Hence, the residue after the division of c^2 by 4 is equal to 0, 1 or 2. If c^2 is divisible by 4, then a_i and b_i are divisible by 4 (this is proved by a simple case-by-case checking of all possible residues after the division of a_i and b_i by 4). Therefore, the homothety centered at A_1 with coefficient 0.5 sends our broken line into a broken line with a shorter links but whose vertices are also in the nodes of the lattice. After several such operations we get a broken line for which c^2 is not divisible by 4, i.e., the corresponding residue is equal to either 1 or 2.

Let us consider these variants, but first observe that

$$a_1 + \cdots + a_m = b_1 + \cdots + b_m = 0.$$

1) The residue after division of c^2 by 4 is equal to 1. Then one of the numbers a_i and b_i is odd and the other one is even; hence, the number $a_1 + \cdots + a_m$ is odd and cannot equal to zero.

2) The residue after division of c^2 by 4 is equal to 2. Then the numbers a_i and b_i are both odd; hence, $a_1 + \cdots + a_m + b_1 + \cdots + b_m$ is odd and cannot equal to zero.

24.5. To every polygon N with vertices in nodes of an integer lattice assign the number $f(N) = n + \frac{m}{2} - 1$. Let polygon M be cut into polygons M_1 and M_2 with vertices in nodes of the lattice. Let us prove that if Pick's formula holds for two of the polygons M, M_1 and M_2, then it is true for the third one as well.

To this end it suffices to prove that $f(M) = f(M_1) + f(M_2)$. The nodes which lie outside the line of cut contribute equally to $f(M)$ and $f(M_1) + f(M_2)$. "Nonterminal" nodes of the cut contribute 1 to $f(M)$ and 0.5 to $f(M_1)$ and $f(M_2)$. Each of the two terminal nodes of the cut contributes 0.5 to each of $f(M), f(M_1)$ and $f(M_2)$ and, therefore, the contribution of the terminal nodes to $f(M)$ is by 1 less than to $f(M_1) + f(M_2)$. Since we have to deduct 1 from each contribution to $f(M)$ and two from each contribution to $f(M_1) + f(M_2)$, it follows that $f(M) = f(M_1) + f(M_2)$.

Now, let us prove the validity of Pick's formula for an arbitrary triangle. If M is a rectangle with sides of length p and q directed along the lines of the lattice, then

$$f(M) = (p-1)(q-1) + \frac{2(p+q)}{2} - 1 = pq,$$

i.e., Pick's formula holds for M. Cutting triangle M into triangles M_1 and M_2 by a diagonal and making use of the fact that $f(M) = f(M_1) + f(M_2)$ and $f(M_1) = f(M_2)$ it is easy to prove the validity of Pick's formula for any right triangle with legs directed along the lines of the lattice. Cutting several such triangles from the rectangle we can get any triangle (Fig. 87).

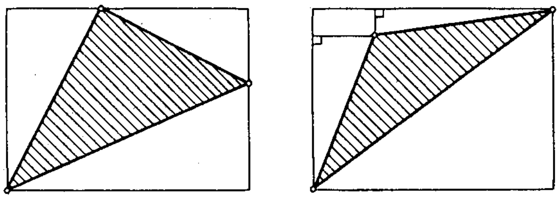


Figure 229 (Sol. 24.5)

To complete the proof of Pick's formula, it remains to notice that any polygon can be cut by diagonals into triangles.

24.6. Thanks to Pick's formula $S_{AOB} = S_{BOC} = S_{COA} = \frac{1}{2}$; hence, O is the intersection point of medians of triangle ABC (cf. Problem 4.2).

24.7. Take a sufficiently large square with side 2^n so that all the black cells are inside it and constitute less than 0.2 of its area. Let us divide this square into four identical squares. The painted area of each of them is less than 0.8 of the total. Let us leave those of them whose painted part constitutes more than 0.2 of the total and cut the remaining ones in the same way.

The painted area of the obtained 2×2 squares will be $\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$ of the total or they will not be painted at all. Now, we have to cut off those of the obtained squares which contain painted cells.

24.8. Consider all the convex figures obtained from the given one by translations by vectors with both coordinates even. Let us prove that at least two of these figures intersect. The initial figure can be squeezed in the disk of radius R centered in the origin, where for R we can take an integer. Take those of the considered figures the coordinates of whose centers are nonnegative integers not greater than $2n$.

There are precisely $(n+1)^2$ of such figures and all of them lie inside a square with side $2(n+R)$. If they do not intersect, then for any n we would have had $(n+1)^2S < 4(n+R)^2$, where S is the area of the given figure. Since $S > 4$, we can select n so that the inequality $\frac{n+R}{n+1} < \sqrt{\frac{S}{4}}$ holds.

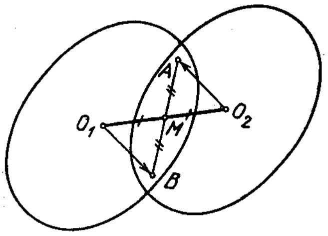


Figure 230 (Sol. 24.8)

Let now figures with centers O_1 and O_2 have a common point A (Fig. 88). Let us prove that then the midpoint M of segment O_1O_2 belongs to both figures (it is clear that the coordinates of M are integers). Let $\overrightarrow{O_1B} = -\overrightarrow{O_2A}$.

Since the given figure is centrally symmetric, point B belongs to the figure with center O_1. This figure is convex and points A and B belong to it and, therefore, the midpoint of segment AB also belongs to it. Clearly, the midpoint of segment AB coincides with the midpoint of segment O_1O_2.

24.9. Let the hunter sit at point O and the hare at point A; let A_1 be the point symmetric to A with respect to O. Consider figure Φ that contains all the points the distance from which to segment AA_1 does not exceed r (Fig. 89).

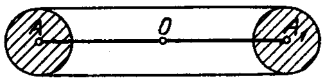


Figure 231 (Sol. 24.9)

It suffices to prove that Φ contains at least one node of the lattice (if the node gets into the shaded part, then point A belongs to the trunk).

The area of Φ is equal to $4rh + \pi r^2$, where h is the distance from the hunter to the hare. If $h > \frac{1}{r}$, then $4rh + \pi r^2 > 4$. By Minkowski's theorem Φ contains an integer point.

24.10. Consider the integer lattice given by equations $x = k + \frac{1}{2}$ and $y = l + \frac{1}{2}$, where k and l are integers. Let us prove that each small square of this lattice gives a nonnegative contribution to $n - S + p$. Consider two cases:

1) The figure contains the center of the square. Then $n' = 1$ and $S' \leq 1$; hence, $n' - S' + p' \geq 0$.

2) The figure intersects the square but does not contain its center. Let us prove that in this case $S' \leq p'$ and we can confine ourselves with the study of the cases depicted on Fig. 90 (i.e., we may assume that the center O of the square lies on the boundary of the figure). Since the distances from the center of the square to its sides are equal to $\frac{1}{2}$, it follows that $p' \geq \frac{1}{2}$. Draw the base line through O to this figure; we get $S' \leq \frac{1}{2}$.

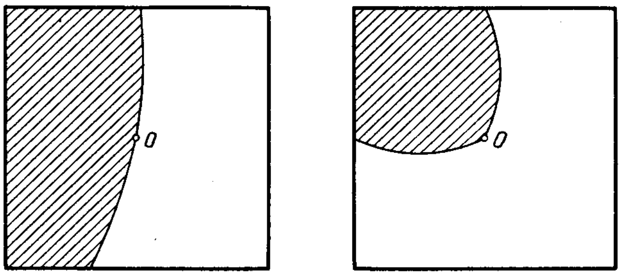


Figure 232 (Sol. 24.10)

It is also clear that all the contributions of the squares cannot be zero simultaneously.

24.11. First, let us prove that on the circle with center $A = (\sqrt{2}, \frac{1}{3})$ there cannot lie more than one integer point. If m and n are integers, then

$$(m - \sqrt{2})^2 + (n - \frac{1}{3})^2 = q - 2m\sqrt{2},$$

where q is a rational number. Therefore, the equality

$$(m_1 - \sqrt{2})^2 + (n_1 - \frac{1}{3})^2 = (m_2 - \sqrt{2})^2 + (n_2 - \frac{1}{3})^2$$

implies that $m_1 = m_2$. By Vièta's theorem the sum of roots of equation $(n - \frac{1}{3})^2 = d$ is equal to $\frac{2}{3}$; hence, at least one root can be integer.

Now, let us arrange the radii of the circles with center A passing through integer points in the increasing order: $R_1 < R_2 < R_3 < \dots$. If $R_n < R < R_{n+1}$, then inside the circle of radius R with center A there are n integer points.

24.12. First. let us prove that the equation $x^2 + y^2 = 5^k$ has $4(k+1)$ integer solutions. For $k = 0$ and $k = 1$ this statement is obvious. Let us prove that the equation $x^2 + y^2 = 5^k$ has exactly 8 solutions (x, y) such that x and y are not divisible by 5. Together with $4(k-1)$ solutions of the form $(5a, 5b)$, where (a, b) is a solution of the equation $a^2 + b^2 = 5^{k-2}$, they give the needed number of solutions.

These solutions are obtained from each other by permutations of x and y and changes of signs; we will call them *nontrivial solutions.*

Let $x^2 + y^2$ be divisible by 5. Then $(x + 2y)(x - 2y) = x^2 + y^2 - 5y^2$ is also divisible by 5. Hence, one of the numbers $x + 2y$ and $x - 2y$ is divisible by 5. It is also easy to verify that if $x + 2y$ and $x - 2y$ are divisible by 5, then both x and y are divisible by 5.

If (x, y) is a nontrivial solution of equation $x^2 + y^2 = 5^k$, then $(x + 2y, 2x - y)$ and $(x - 2y, 2x + y)$ are solutions of equation $\xi^2 + \eta^2 = 5^{k+1}$ and precisely one of them is nontrivial. It remains to prove that a nontrivial solution is unique up to permutations of x and y and changes of signs.

Let (x, y) be a nontrivial solution of the equation $x^2 + y^2 = 5^k$. Then the pairs

$$\left(\pm\frac{2x - y}{5}, \pm\frac{x + 2y}{5}\right) \quad \text{and} \quad \left(\pm\frac{x + 2y}{5}, \pm\frac{2x - y}{5}\right) \tag{1}$$

together with the pairs

$$\left(\pm\frac{2x + y}{5}, \pm\frac{x - 2y}{5}\right) \quad \text{and} \quad \left(\pm\frac{x - 2y}{5}, \pm\frac{2x + y}{5}\right) \tag{2}$$

are solutions of the equation $\xi^2 + \eta^2 = 5^{k-1}$ but the pairs of exactly one of these types will be integer since exactly one of the numbers $x + 2y$ and $x - 2y$ is divisible by 5. Thus, we will get a nontrivial solution because

$$(x + 2y)(x - 2y) = (x^2 + y^2) - 5y^2$$

for $k \geq 2$ is divisible by 5 but is not divisible by 25.

Therefore, each of the 8 nontrivial solutions of the equation $x^2+y^2 = 5^k$ yields 8 nontrivial solutions of the equation $\xi^2 + \eta^2 = 5^{k-1}$ where for one half of the solutions we have to make use of formulas (1) and for the other half of the formulas (2).

Now, let us pass directly to the solution of the problem. Let $n = 2k + 1$. Let us prove that on the circle of radius $\frac{5^k}{3}$ with center $(\frac{1}{3}, 0)$ there lie exactly (not more nor less) n integer points. The equation $x^2 + y^2 = 5^{2k}$ has $4(2k + 1)$ integer solutions. Moreover, after the division of 5^{2k} by 3 we have residue 1; hence, one of the numbers x and y is divisible by 3 and the residue after the division of the other one by 3 is equal to ± 1. Therefore, in precisely one of the pairs (x, y), $(x, -y)$, (y, x) and $(-y, x)$ the residues after the divisionof the first and the second number by 3 are equal to -1 and 0, respectively. Hence, the equation $(3z - 1)^2 + (3t)^2 = 5^{2k}$ has precisely $2k + 1$ integer solutions.

Let $n = 2k$. Let us prove that on the circle of radius $\frac{5^{(k-1)/2}}{2}$ with center $(\frac{1}{2}, 0)$ there lie n integer points. The equation $x^2 + y^2 = 5^{k-1}$ has $4k$ integer solutions; for them one of the numbers x and y is even and the other one is odd. Hence, the equation $(2z-1)^2+(2t)^2 = 5^{k-1}$ has $2k$ integer solutions.

Chapter 25. CUTTINGS

§1. Cuttings into parallelograms

25.1. Prove that the following properties of convex polygon F are equivalent:

1) F has a center of symmetry;

2) F can be cut into parallelograms.

25.2. Prove that if a convex polygon can be cut into centrally symmetric polygons, then it has a center of symmetry.

25.3. Prove that any regular $2n$-gon can be cut into rhombuss.

25.4. A regular octagon with side 1 is cut into parallelograms. Prove that among the parallelograms there is at least two rectangles and the sum of areas of all the rectangles is equal to 2.

§2. How lines cut the plane

In plane, let there be drawn n pairwise nonparallel lines no three of which intersect at one point. In Problems 25.5–25.9 we will consider properties of figures into which these lines cut the plane. A figure is called an *n-linked* one if it is bounded by n links (i.e., a link is a line segment or a ray).

25.5. Prove that for $n = 4$ among the obtained parts of the plane there is a quadrilateral.

25.6. a) Find the total number of all the obtained figures.

b) Find the total number of *bounded* figures, i.e., of polygons.

25.7. a) Prove that for $n = 2k$ there are not more than $2k-1$ angles among the obtained figures.

b) Is it possible that for $n = 100$ there are only three angles among the obtained figures?

25.8. Prove that if among the obtained figures there is a p-linked and a q-linked ones, then $p + q \leq n + 4$.

25.9. Prove that for $n \geq 3$ there are not less than $\frac{2n-2}{3}$ triangles among the obtained parts.

Now, let us abandon the assumption that no three of the considered lines intersect at one point. If P is the intersection point of two or several lines, then the number of lines of the given system passing through point P will be denoted by $\lambda(P)$.

25.10. Prove that the number of segments into which the given lines are divided by their intersection points is equal to $n + \sum \lambda(P)$.

25.11. Prove that the number of parts into which given lines divide the plane is equal to $1 + n + \sum(\lambda(P) - 1)$ and among these parts there are $2n$ unbounded ones.

25.12. The parts into which the plane is cut by lines are painted red and blue so that the neighbouring parts are of distinct colours (cf. Problem 27.1). Let r be the number of red parts, b the number of blue parts. Prove that

$$r \leq 2b - 2 - \sum(\lambda(P) - 2)$$

where the equality is attained if and only if the red parts are triangles and angles.

Solutions

25.1. Consider a convex polygon $A_1 \dots A_n$. Prove that each of the properties 1) and 2) is equivalent to the following property:

3) For any vector $\overrightarrow{A_iA_{i+1}}$ there exists a vector $\overrightarrow{A_jA_{j+1}} = -\overrightarrow{A_iA_{i+1}}$.

Property 1) clearly implies property 3). Let us prove that property 3) implies property 1). If a convex polygon $A_1 \dots A_n$ possesses property 3), then $n = 2m$ and $\overrightarrow{A_iA_{i+1}} = -\overrightarrow{A_{m+i}A_{m+i+1}}$. Let O_i be the midpoint of segment A_iA_{m+i}. Since $A_iA_{i+1}A_{m+i}A_{m+i+1}$ is a parallelogram, we have $O_i = O_{i+1}$. Hence, all the points O_i coincide and this point is the center of symmetry of the polygon.

Let us prove that property 2) implies property 3). Let a convex polygon F be divided into parallelograms. We have to prove that for any side of F there exists another side parallel and equal to it. From every side of F a chain of parallelograms departs, i.e., this side sort of moves along them parallelly so that it can be split into several parts (Fig. 91).

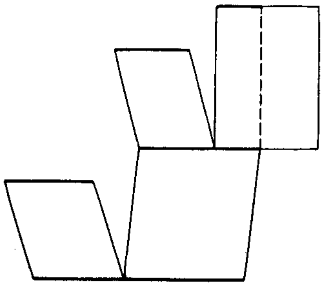


Figure 233 (Sol. 25.1)

Since a convex polygon can have only one more side parallel to the given one, all the bifurcations of the chain terminate in the same side which is not shorter than the side from which the chain starts. We can equally well begin the chain of parallelograms from the first side to the second one or from the second one to the first one; hence, the lengths of these sides are equal.

It remains to prove that property 3) implies property 2). A way of cutting a polygon with equal and parallel opposite sides is indicated on Fig. 92.

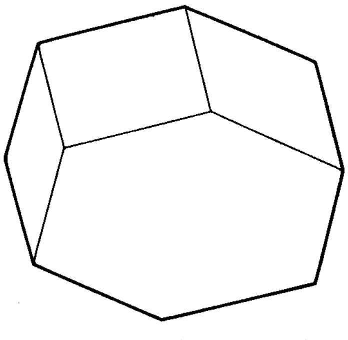


Figure 234 (Sol. 25.1)

After each such operation we get a polygon with a lesser number of sides which still possesses property 3) and by applying the same process to this polygon we eventually get a parallelogram.

25.2. Let us make use of the result of the preceding problem. If a convex polygon M is cut into convex centrally symmetric polygons, then they can be cut into parallelograms. Therefore, M can be cut into parallelograms, i.e., M has a center of symmetry.

25.3. Let us prove by induction on n that any $2n$-gon whose sides have the same length and opposite sides are parallel can be cut into rhombs. For $n = 2$ this is obvious and from Fig. 92 it is clear how to perform the inductive step.

25.4. Let us single out two perpendicular pairs of opposite sides in a regular octagon and, as in Problem 25.1, consider chains of parallelograms that connect the opposite sides. On the intersection of these chains rectangles stand. By considering two other pairs of opposite sides we will get at least one more rectangle.

It is possible to additionally cut parallelograms from every chain so that the chain would split into several "passes" and in each pass the neighbouring parallelograms are neighboring to each other along the whole sides, not a part of a side. The union of rectangles of a new partition coincides with the union of rectangles of the initial partition and, therefore, it suffices to carry out the proof for the new partition.

Every pass has a constant width; hence, the length of one side of each rectangle that enters a path is equal to the width of the path, and the sum of length of all the other sides is equal to the sum of the widths of the passes corresponding to the other pair of sides.

Therefore, the area of all the rectangles that constitute one path is equal to the product of the width of the path by the length of the side of the polygon, i.e., its value is equal to the width of the path. Hence, the area of all the rectangles corresponding to two perpendicular pairs of opposite sides is equal to 1 and the area of the union of the rectangles is equal to 2.

25.5. Denote the intersections points of one of the given lines with the other ones by A, B and C. For definiteness, let us assume that point B lies between A and C. Let D be the intersection point of lines through A and C. Then any line passing through point B and not passing through D cuts triangle ACD into a triangle and a quadrilateral.

25.6. a) Let n lines divide the plane into a_n parts. Let us draw one more line. This will increase the number of parts by $n + 1$ since the new line has n intersection points with the already drawn lines. Therefore, $a_{n+1} = a_n + n + 1$. Since $a_1 = 2$, it follows that $a_n = 2 + 2 + 3 + \dots + n = \frac{n^2+n+2}{2}$.

b) Encircle all the intersection points of the given lines. It is easy to verify that the number of unbounded figures is equal to $2n$. Therefore, the number of bounded figures is equal to

$$\frac{n^2 + n + 2}{2} - 2n = \frac{n^2 - 3n + 2}{2}.$$

25.7. a) All intersection points of given lines can be encircled in a circle. Lines divide this circle into $4k$ arcs. Clearly, two neighbouring arcs cannot simultaneously belong to angles; hence, the number of angles does not exceed $2k$, where the equality can only be attained if the arcs belonging to the angles alternate. It remains to prove that the equality cannot be attained. Suppose that the arcs belonging to angles alternate. Since on both sides from any of the given lines lie $2k$ arcs, the opposite arcs (i.e., the arcs determined by two lines) must belong to angles (Fig. 93) which is impossible.

b) For any n there can be three angles among the obtained figures. On Fig. 94 it is shown how to construct the corresponding division of the plane.

25.8. Let us call a line which is the continuation of a segment or a ray that bounds a figure a (border?)*bounding* line of the figure. It suffices to show that two considered figures cannot have more than 4 common bounding lines. If two figures have 4 common bounding lines, then one of the figures lies in domain 1 and the other one lies in domain 2 (Fig. 95).

The fifth bounding line of the figure that lies in domain 1 must intersect two neighbouring sides of the quadrilateral 1; but then it cannot be bounding line for the figure that belongs to domain 2.

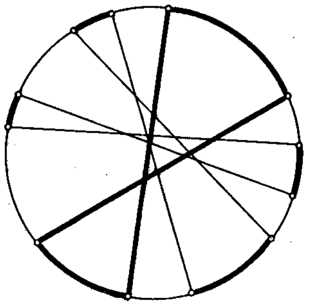


Figure 235 (Sol. 25.7 a))

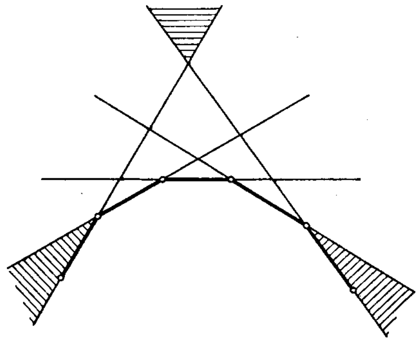


Figure 236 (Sol. 25.7 b))

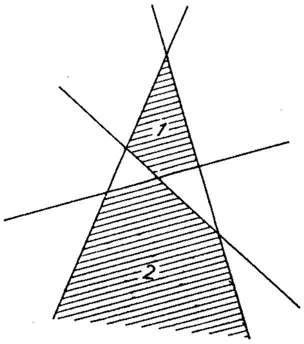


Figure 237 (Sol. 25.8)

25.9. Consider all the intersection points of the given lines. Let us prove that these points may lie on one side of not more than two given lines. Suppose that all the intersection points lie on one side of three given lines. These lines constitute triangle ABC. The fourth line cannot intersect the sides of this triangle only, i.e., it intersects at least one extension of a side. Let, for definiteness, it intersect the continuation of side AB beyond point B; let the intersection point be M. Then points A and M lie on distinct sides of line BC. Contradiction. Hence, there exist at least $n-2$ lines on both sides of which there are intersection points.

If in the half plane given by line l we select the nearest l intersection point, then this point is a vertex of a triangle adjacent to l. Thus, there exists not less than $n-2$ lines to each of which at least two triangles are adjacent and there are two lines to each of which at least one triangle is adjacent. Since every triangle is adjacent to exactly 3 lines, there are not less than $\frac{2(n-2)+2}{3}$ triangles.

25.10. If P is the intersection point of given lines, then $2\lambda(P)$ segments or rays go out of P. Moreover, each of x segments have two boundary points and each of $2n$ rays has one boundary point. Hence, $2x+2n=2\sum\lambda(P)$, i.e., $x=-n+\sum\lambda(P)$.

25.11. Let us carry out the proof by induction on n. For two lines the statement is obvious. Suppose that the statement holds for $n-1$ line and consider a system consisting of n lines. Let f be the number of parts into which the given n lines divide the plane; $g=1+n+\sum(\lambda(P)-1)$. Let us delete one line from the given system and define similarly numbers f' and g' for the system obtained. If on the deleted line there lie k intersection points of lines, then $f'=f-k-1$ and $g'=1+(n-1)+\sum(\lambda'(P)-1)$. It is easy to verify that $\sum(\lambda(P)-1)=-k+\sum(\lambda'(P)-1)$. By inductive hypothesis $f'=g'$.

Therefore, $f=f'+k+1=g'+k+1=g$. It is also clear that the number of unbounded parts is equal to $2n$.

25.12. Let r'_k be the number of red k-gons, r' the number of bounded red domains and the number of segments into which the given lines are divided by their intersection points be equal to $\sum\lambda(P)-n$, cf. Problem 25.10. Each segment is a side of not more than 1 red polygon, hence, $3r'\leq\sum_{k\geq3}kr'_k\leq\sum\lambda(P)-n$, where the left inequality is only attained if and only if there are no red k-gons for $k>3$, and the right inequality is only attained if and only if any segment is a side of a red k-gon, i.e., any unbounded red domain is an angle.

The number of bounded domains is equal to $1-n+\sum(\lambda(P)-1)=c$ (see Problem 25.11), hence, the number b' of bounded blue domaions is equal to

$$c-r'\geq 1-n+\sum(\lambda(P)-1)-\frac{\sum\lambda(P)-n}{3}=1-\frac{2n}{3}+\sum(\frac{2\lambda(P)}{3}-1).$$

The colours of $2n$ unbounded domains alternate; hence,

$$b=b'+n\geq 1+\frac{n}{3}+\sum(\frac{2\lambda(P)}{3}-1)$$

and

$$r=r'+n\leq\frac{2n+\sum\lambda(P)}{3}$$

and, therefore, $2b-r\geq 2+\sum(\lambda(P)-2)$.

Chapter 26. SYSTEMS OF POINTS AND SEGMENTS. EXAMPLES AND COUNTEREXAMPLES

§1. Systems of points

26.1. a) An architect wants to place four sky-scrapers so that any sightseer can see their spires in an arbitrary order. In other words, if the sky-scrapers are numbered, then for any ordered set (i, j, k, l) of sky-scrapers one can stand at an arbitrary place in the town and by turning either clockwise or counterclockwise see first the spire of the sky-scraper i, next, that of j, k and, lastly, l. Is it possible for the architect to perform this?

b) The same question for five sky-scrapers.

26.2. In plane, there are given n points so that from any foresome of these points one can delete one point so that the remaining three points lie on one line. Prove that it is possible to delete one of the given points so that all the remaining points lie on one line.

26.3. Given 400 points in plane, prove that there are not fewer than 15 distinct distances between them.

26.4. In plane, there are given $n \geq 3$ points. Let d be the greatest distance between any two of these points. Prove that there are not more than n pairs of points with the distance between the points of any pair equal to d.

26.5. In plane, there are given 4000 points no three of which lie on one line. Prove that there are 1000 nonintersecting quadrilaterals (perhaps, nonconvex ones) with vertices at these points.

26.6. In plane, there are given 22 points no three of which lie on one line. Prove that it is possible to divide them into pairs so that the segments determined by pairs intersect at least at 5 points.

26.7. Prove that for any positive integer N there exist N points no three of which lie on one line and all the pairwise distances between them are integers.

See also Problems 20.13-20.15, 22.7.

§2. Systems of segments, lines and circles

26.8. Construct a closed broken line of six links that intersects each of its links precisely once.

26.9. Is it possible to draw six points in plane and to connect them with nonintersecting segments so that each point is connected with precisely four other ones?

26.10. Point O inside convex polygon $A_1 \dots A_n$ possesses a property that any line OA_i contains one more vertex A_j. Prove that no point except O possesses such a property.

26.11. On a circle, $4n$ points are marked and painted alternatedly red and blue. Points of the same colour are divided into pairs and points from each pair are connected by segments of the same colour. Prove that if no three segments intersect at one point, then there exist at least n intersection points of red segments with blue segments.

26.12. In plane, $n \geq 5$ circles are placed so that any three of them have a common point. Prove that then all the circles have a common point.

§3. Examples and counterexamples

There are many wrong statements that at first glance seem to be true. To refute such statements we have to construct the corresponding example; such examples are called *counterexamples.*

26.13. Is there a triangle all the hights of which are shorter than 1 cm and the area is greater than 1 m ^{2}?

26.14. In a convex quandrilateral $ABCD$ sides AB and CD are equal and angles A and C are equal. Must this quandrilateral be a parallelogram?

26.15. The list of sides and diagonals of a convex quandrilateral ordered with respect to length coincides with a similar list for another quandrilateral. Must these quandrilaterals be equal?

26.16. Let $n \geq 3$. Do there exist n points that do not belong to one line and such that pairwise distances between which are irrational while the areas of all the triangles with vertices in these points are rational?

26.17. Do there exist three points A, B and C in plane such that for any point X the length of at least one of the segments XA, XB and XC is irrational?

26.18. In an acute triangle ABC, median AM, bisector BK and hight CH are drawn. Can the area of the triangle formed by the intersection points of these segments be greater than $0.499 \cdot S_{ABC}$?

26.19. On an infinite list of graph paper (with small cells of size 1×1) the domino chips of size 1×2 are placed so that they cover all the cells. Is it possible to make it so that any line that follows the lines of the mash cuts only a finite number of chips?

26.20. Is it possible for a finite set of points to contain for every of its points precisely 100 points whose distance from the point is equal to 1?

26.21. In plane, there are several nonintersecting segments. Is it always possible to connect the endpoints of some of them by segments so that we get a closed nonselfintersecting broken line?

26.22. Consider a triangle. Must the triangle be an isosceles one if the center of its inscribed circle is equidistant from the midpoints of two of its sides?

26.23. The arena of a circus is illuminated by n distinct spotlights. Each spotlight illuminates a convex figure. It is known that if any of the spotlights is turned off, then the arena is still completely illuminated, but if any two spotlights are turned off, then the arena is not completely illuminated. For which n this is possible?

See also problems 22.16–22.18, 22.26, 22.27, 22.29, 23.37, 24.11, 24.12.

Solutions

26.1. a) It is easy to verify that constructing the fourth building inside the triangle formed by the three other buildings we get the desired position of the buildings.

b) It is impossible to place in the desired way five buildings. Indeed, if we consecutively see buildings A_1, A_2, … , A_n, then $A_1A_2 \dots A_n$ is a nonselfintersecting broken line. Therefore, if $ABCD$ is a convex quandrilateral, then it is impossible to see its vertices in the following order: A, C, D, B. It remains to notice that of five points no three of which lie on one line it is always possible to select four points which are vertices of a convex quandrilateral (Problem 22.2).

26.2. It is possible to assume that $n \geq 4$ and not all the points lie on one line. Then we can select four points A, B, C and D not on one line. By the hypothesis, three of them lie on one line. Let, for definiteness, points A, B and C lie on line l and D does not lie on l.

We have to prove that all the points except for D lie on l. Suppose that a point E does not belong to l. Let us consider points A, B, D, E. Both triples A, B, D and A, B, E do not lie on one line. Therefore, on one line there lies either triple (A, D, E) or triple (B, D, E). Let, for definiteness, points A, D and E lie on one line. Then no three of the points B, C, D, E lie on one line. Contradiction.

26.3. Let the number of distinct distances between points be equal to k. Fix two points. Then all the other points are intersection points of two families of concentric circles containing k circles each. Hence, the total number of points does not exceed $2k^2 + 2$. It remains to notice that $2 \cdot 14^2 + 2 = 394 < 400$.

26.4. A segment of length d connecting a pair of given points will be called a *diameter*. The endpoints of all the diameters that begin at point A lie on the circle centered in A and of radius d. Since the distance between any two points does not exceed d, the endpoints of all the diameters beginning in A belong to an arc whose angle value does not exceed 60°. Therefore, if three diameters AB, AC and AD begin in point A, then one of the endpoints of these diameters lies inside the angle formed by the other two endpoints.

Let, for definiteness, point C lie inside angle $\angle BAD$. Let us prove that then not more than one diameter begins in point C. Suppose that there is another diameter, CP, and points B and P lie on different sides of line AC (Fig. 96). Then $ABCP$ is a convex quadrilateral; hence, $AB+CP < AC+BP$ (see Problem 9.14), i.e., $d+d < d+BP$ and, therefore, $BP > d$ which is impossible.

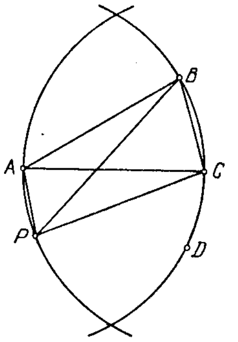


Figure 238 (Sol. 26.4)

As a result we see that either from each point there goes not more than two diameters or there exists a point from which there goes not more than one diameter. Now, the required statement can be proved by induction on the number of points. For $n = 3$ it is obvious.

Suppose the statement is proved for any system of n points; let us prove it for a system of $n + 1$ points. In this system either there is a point from which there goes not more than one diameter or from each point there goes not more than two diameters. In the first case we delete this point and, making use of the fact that in the remaining system there are not more than n diameters, get the desired.

The second case is obvious.

26.5. Let us draw all the lines that connect pairs of given points and select a line, l, not parallel to either of them. It is possible to divide the given points into quadruples with the help of lines parallel to l. The quadrilaterals with vertices in these quadruples of points are the desired ones (Fig. 97).

26.6. Let us divide the given points in an arbitrary way into six groups: four groups of four points each, a group of five points and a group of one point. Let us consider the group of five points. From these points we can select four points which are vertices of a convex

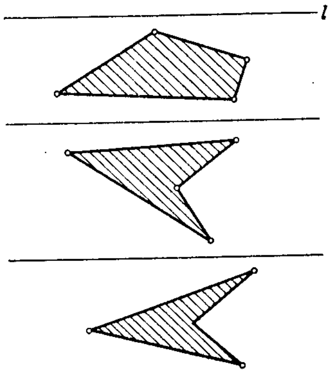


Figure 239 (Sol. 26.5)

quadrilateral $ABCD$ (see Problem 22.2). Let us unite points A, C and B, D into pairs. Then segments AC and BD given by pairs intersect. One of the five points is free. Let us adjoin it to the foursome of points and perform the same with the obtained 5-tuple of points, etc. After five of such operations there remain two points and we can unite them in a pair.

26.7. Since $\left(\frac{2n}{n^2+1}\right)^2 + \left(\frac{n^2-1}{n^2+1}\right)^2 = 1$, there exists an angle φ with the property that $\sin\varphi = \frac{2n}{n^2+1}$ and $\cos\varphi = \frac{n^2-1}{n^2+1}$, where $0 < 2N\varphi < \frac{\pi}{2}$ for a sufficiently large n. Let us consider the circle of radius R centered at O and points A_0, A_1, ... , A_{N-1} on it such that $\angle A_0OA_k = 2k\varphi$. Then $A_iA_j = 2R\sin(|i-j|\varphi)$. Making use of the formulas

$$\sin(m+1)\varphi = \sin m\varphi\cos\varphi + \sin\varphi\cos m\varphi,$$

$$\cos(m+1)\varphi = \cos m\varphi\cos\varphi - \sin m\varphi\sin\varphi$$

it is easy to prove that the numbers $\sin m\varphi$ and $\cos m\varphi$ are rational for all positive integers m. Let us take for R the greatest common divisor of all the denominators of the rational numbers $\sin\varphi$, ... , $\sin(N-1)\varphi$. Then A_0, ... , A_{N-1} is the required system of points.

26.8. An example is depicted on Fig. 98.

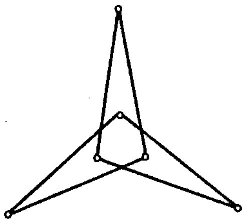


Figure 240 (Sol. 26.8)

26.9. It is possible. An example is plotted on Fig. 99.

26.10. The hypothesis implies that all the vertices of the polygon are divided into pairs that determine diagonals A_iA_j which pass through point O. Therefore, the number of vertices is even and on both parts of each of such diagonals A_iA_j there are an equal number of vertices. Hence, $j = i + m$, where m is a half of the total number of vertices. Therefore, point O is the intersection point of diagonals that connect opposite vertices. It is clear that the intersection point of these diagonals is unique.

26.11. If AC and BD are intersecting red segments, then the number of intersection points of any line with segments AB and CD does not exceed the number of intersection

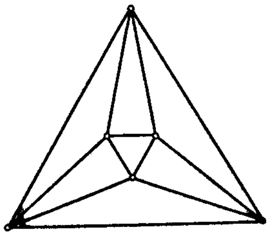


Figure 241 (Sol. 26.9)

points of this line with segments AC and BD. Therefore, by replacing red segments AC and BD with segments AB and CD we do not increase the number of intersection points of red segments with blue ones and diminish the number of intersection points of red segments with red ones because the intersection point of AC and BD vanishes. After several such operations all red segments become nonintersecting ones and it remains to prove that in this case the number of intersection points of red segments with blue ones is not smaller than n.

Let us consider an arbitrary red segment. Since the other red segments do not intersect it, we deduce that on both sides of it there lies an even number of red points or, equivalently, an odd number of blue points. Therefore, there exists a blue segment that intersects the given red segment. Therefore, the number of intersection points of red segments with blue ones is not fewer than the number of red segments i.e., is not less than n.

26.12. Let A be a common point of the first three circles S_1, S_2 and S_3. Denote the intersection points of S_1 and S_2, S_2 and S_3, S_3 and S_1 by B, C, D, respectively. Suppose there exists a circle S not passing through point A. Then S passes through points B, C and D. Let S' be the fifth circle. Each pair of points from the collection A, B, C, D is a pair of intersection points of two of the circles S_1, S_2, S_3, S. Therefore, S' passes through one point from each pair of points A, B, C, D. On the other hand, S' cannot pass through three points from the set A, B, C, D because each triple of these points determines one of the circles S_1, S_2, S_3, S. Hence, S' does not pass through certain two of these points. Contradiction.

26.13. Let us consider rectangle $ABCD$ with sides $AB = 1$ cm and $BC = 500$ m. Let O be the intersection point of the rectangle's diagonals. It is easy to verify that the area of AOD is greater than 1 m^2 and all its hights are shorter than 1 cm.

26.14. No, not necessarily. On Fig. 100 it is shown how to get the required quadrilateral $ABCD$.

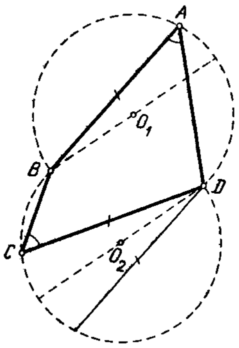


Figure 242 (Sol. 26.14)

26.15. Not necessarily. It is easy to verify that the list of the lengths of sides and diagonals for an isosceles trapezoid with height 1 and bases 2 and 4 coincides with the similar list for the quadrilateral with perpendicular diagonals of length 2 and 4 that are divided by their intersection point into segments of length 1 and 1 and 1 and 3, respectively (Fig. 101).

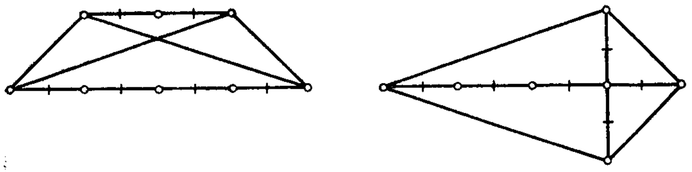


Figure 243 (Sol. 26.15)

26.16. Yes, there exist. Let us consider points $P_i = (i, i^2)$, where $i = 1, \dots, n$. The areas of all the triangles with vertices in the nodes of an integer lattice are rational (see Problem 24.5) and the numbers $P_iP_j = |i-j|\sqrt{1+(i+j)^2}$ are irrational.

26.17. Yes, there exist. Let C be the midpoint of segment AB. Then

$$XC^2 = \frac{2XA^2 + 2XB^2 - AB^2}{2}.$$

If the number AB^2 is irrational, then the numbers XA, XB and XC cannot simultaneously be rational.

26.18. It can. Consider right triangle ABC_1 with legs $AB = 1$ and $BC_1 = 2n$. In this triangle draw median AM_1, bisector BK_1 and hight C_1H_1. The area of the triangle formed by these segments is greater than $S_{ABM_1} - S_{ABK_1}$. Clearly, $S_{ABK_1} < \frac{1}{2}$ and $S_{ABM_1} = \frac{n}{2}$, i.e., $S_{ABM_1} - S_{ABK_1} > (\frac{S}{2}) - (\frac{S}{2n})$, where $S = S_{ABC_1}$. Hence, for a sufficiently large n the area of the triangle formed by segments AM_1, BK_1 and C_1H_1 will be greater than $0.499 \cdot S$.

Slightly moving point C_1 we turn triangle ABC_1 into an acute triangle ABC and the area of the triangle formed by the intersection points of segments remains greater than $0.499 \cdot S_{ABC}$.

26.19. It is possible. Let us pave, for instance, infinite angles illustrated on Fig. 102.

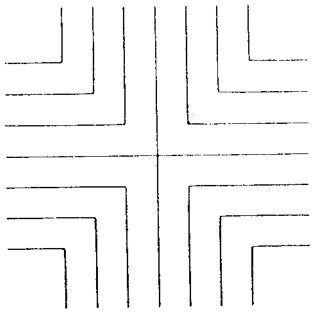


Figure 244 (Sol. 26.19)

26.20. Yes, it can. Let us prove the statement by induction replacing 100 with n.

For $n = 1$ we can take the endpoints of a segment of length 1. Suppose that the statement is proved for n and $A_1, \dots, A_k$ is the required set of points. Let $A'_1, \dots, A'_k$ be the images of points $A_1, \dots, A_k$ under the parallel transport by unit vector $\mathbf{a}$. To prove the inductive step it suffices to select the unit vector $\mathbf{a}$ so that $\mathbf{a} \neq \overrightarrow{A_iA_j}$ and $A_jA'_i \neq 1$ for $i \neq j$, i.e., $|\overrightarrow{A_jA_i} + \mathbf{a}| \neq 1$ for $i \neq j$. Each of these restrictions excludes from the unit circle not more than 1 point.

26.21. Not always. Consider the segments plotted on Fig. 103. The endpoints of each short segment can be connected with the endpoints of the nearest to it long segment only. It is clear that in this way we cannot get a closed nonselfintersecting broken line.

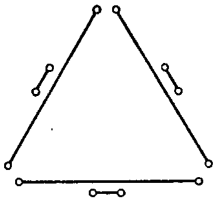


Figure 245 (Sol. 26.21)

26.22. Not necessarily. Let us prove that the center O of the circle inscribed in triangle ABC with sides $AB = 6$, $BC = 4$ and $CA = 8$ is equidistant from the midpoints of sides AC and BC. Denote the midpoints of sides AC and BC by B_1 and A_1 and the bases of the perpendiculars dropped from O to AC and BC by B_2 and A_2, see Fig. 104. Since $A_1A_2 = 1 = B_1B_2$ (cf. Problem 3.2) and $OA_2 = OB_2$, it follows that $\triangle OA_1A_2 = \triangle OB_1B_2$, i.e., $OA_1 = OB_1$.

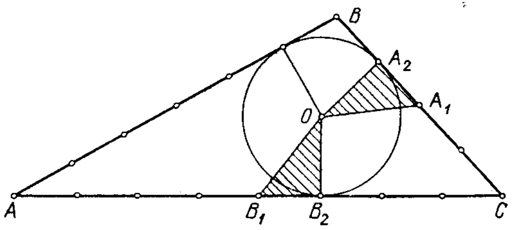


Figure 246 (Sol. 26.22)

26.23. This is possible for any $n \geq 2$. Indeed, let us inscribe into the arena a regular k-gon, where k is the number of distinct pairs that can be composed of n spotlights, i.e., $k = \frac{n(n-1)}{2}$. Then we can establish a one-to-one correspondence between the segments cut off by the sides of the k-gon and the pairs of spotlights. Let each spotlight illuminate the whole k-gon and the segments that correspond to pairs of spotlights in which it enters. (Yeah?)It is easy to verify that this illumination possesses the required properties.

Chapter 27. INDUCTION AND COMBINATORICS

§1. Induction

27.1. Prove that if the plane is divided into parts ("countries") by lines and circles, then the obtained map can be painted two colours so that the parts separated by an arc or a segment are of distinct colours.

27.2. Prove that in a convex n-gon it is impossible to select more than n diagonals so that any two of them have a common point.

27.3. Let E be the intersection point of lateral sides AD and BC of trapezoid $ABCD$, let B_{n+1} be the intersection point of lines A_nC and BD $(A_0 = A)$; let A_{n+1} be the intersection point of lines EB_{n+1} and AB. Prove that $A_nB = \frac{1}{n+1}AB$.

27.4. On a line, there are given points $A_1, \dots, A_n$ and $B_1, \dots, B_{n-1}$. Prove that

$$\sum_{i=1}^{n}\left(\frac{\prod_{1\leq k\leq n-1}\overline{A_iB_k}}{\prod_{j\neq i}\overline{A_iA_j}}\right) = 1.$$

27.5. Prove that if n points do not lie on one line, then among the lines that connect them there are not fewer than n distinct points.

See also Problems 2.12, 5.98, 22.7, 22.9–22.12, 22.20 b, 22.22, 22.23, 22.29, 23.39–23.41, 26.20.

§2. Combinatorics

27.6. Several points are marked on a circle, A is one of them. Which convex polygons with vertices in these points are more numerous: those that contain A or those that do not contain it?

27.7. On a circle, nine points are fixed. How many non-closed non-selfintersecting broken lines of nine links with vertices in these points are there?

27.8. In a convex n-gon $(n \geq 4)$ there are drawn all the diagonals and no three of them intersect at one point. Find the number of intersection points of the diagonals.

27.9. In a convex n-gon $(n \geq 4)$ all the diagonals are drawn. Into how many parts do they divide an n-gon if no three of them intersect at one point?

27.10. Given n points in plane no three of which lie on one line, prove that there exist not fewer than $\frac{\binom{n}{5}}{n-4}$ distinct convex quadrilaterals with vertices in these points.

27.11. Prove that the number of nonequal triangles with the vertices in vertices of a regular n-gon is equal to the integer nearest to $\frac{n^2}{12}$.

See also Problem 25.6.

Solutions

27.1. Let us carry out the proof by induction on the total number of lines and circles. For one line or circle the statement is obvious. Now, suppose that it is possible to paint any map given by n lines and circles in the required way and show how to paint a map given by $n+1$ lines and circles.

Let us delete one of these lines (or circles) and paint the map given by the remaining n lines and circles thanks to the inductive hypothesis. Then retain the colours of all the parts lying on one side of the deleted line (or circle) and replace the colours of all the parts lying on the other side of the deleted line (or circle) with opposite ones.

27.2. Let us prove by induction on n that in a convex n-gon it is impossible to select more than n sides and diagonals so that any two of them have a common point.

For $n = 3$ this is obvious. Suppose that the statement holds for any convex n-gon and prove it for an $(n+1)$-gon. If from every vertex of the $(n+1)$-gon there goes not more than two of the selected sides or diagonals, then the total number of selected sides or diagonals does not exceed $n+1$. Hence, let us assume that from vertex A there goes three of the selected sides or diagonals AB_1, AB_2 and AB_3, where AB_2 lies between AB_1 and AB_3. Since a diagonal or a side coming from point B_2 and distinct from AB_2 cannot simultaneously intersect AB_1 and AB_3, it is clear that only one of the chosen diagonals can go from B_2. Therefore, it is possible to delete point B_2 together with diagonal AB_2 and apply the inductive hypothesis.

27.3. Clearly, $A_0B = AB$. Let C_n be the intersection point of lines EA_n and DC, where $DC : AB = k$, $AB = a$, $A_nB = a_n$ and $A_{n+1}B = x$. Since $CC_{n+1} : A_nA_{n+1} = DC_{n+1} : BA_{n+1}$, it follows that $kx : (a_n - x) = (ka - kx) : x$, i.e., $x = \frac{aa_n}{a+a_n}$. If $a_n = \frac{a}{n+1}$, then $x = \frac{a}{n+2}$.

27.4. First, let us prove the desired statement for $n = 2$. Since $\overrightarrow{A_1B_1}+\overrightarrow{B_1A_2}+\overrightarrow{A_2A_1} = \overrightarrow{0}$, it follows that $\frac{\overline{A_1B_1}}{\overline{A_1A_2}} + \frac{\overline{A_2B_1}}{\overline{A_2A_1}} = 1$.

To prove the inductive step let us do as follows. Fix points $A_1, \dots, A_n$ and $B_1, \dots, B_{n-2}$ and consider point B_{n-1} variable. Consider the function

$$f(B_{n-1}) = \sum_{i=1}^{n} \left(\frac{\prod_{1\leq k\leq n-1} \overline{A_iB_k}}{\prod_{j\neq i} \overline{A_iA_j}} \right) = 1.$$

This function is a linear one and by the inductive hypothesis $f(B_{n-1}) = 1$ if B_{n-1} coincides with one of the points $A_1, \dots, A_n$. Therefore, this function is identically equal to 1.

27.5. Induction on n. For $n = 3$ the statement is obvious. Suppose we have proved it for $n-1$ and let us prove it then for n points. If on every line passing through two of the given points lies one more given point, then all the given points belong to one line (cf. Problem 20.13). Therefore, there exists a line on which there are exactly two given points A and B. Let us delete point A. The two cases are possible:

1) All the remaining points lie on one line l. Then there will be precisely n distinct lines: l and $n-1$ line passing through A.

2) The remaining points do not belong to one line. Then among the lines that connect them there are not fewer than $n-1$ distinct ones that connect them and all of them differ from l. Together with AB they constitute not fewer than n lines.

27.6. To any polygon, P, that does not contain point A we can assign a polygon that contains A by adding A to the vertices of P. The inverse operation, however, that is deleting of the point A, can be only performed for n-gons with $n \geq 4$. Therefore, there are more polygons that contain A than polygons without A and the difference is equal to the number of triangles with A as a vertex, i.e., $\frac{(n-1)(n-2)}{2}$.

27.7. The first point can be selected in 10 ways. Each of the following 8 points can be selected in two ways because it must be neighbouring to one of the points selected earlier (otherwise we get a self-intersecting broken line). Since the beginning and the end do not differ in this method of calculation, the result should be divided by 2. Hence, the total number of the broken lines is equal to $\frac{10\cdot 2^8}{2} = 1280$.

27.8. Any intersection point of diagonals determines two diagonals whose intersection point it serves and the endpoints of these diagonals fix a convex quadrilateral. Conversely, any four vertices of a polygon determine one intersection point of diagonals. Therefore, the total number of intersection points of diagonals is equal to the number of ways to choose 4 points of n, i.e., is equal to $\frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}$.

27.9. Let us consecutively draw diagonals. When we draw a diagonal, the number of parts into which the earlier drawn diagonals divide the polygon increases by $m+1$, where m is the number of intersection points of the new diagonals with the previously drawn ones, i.e., each new diagonal and each new intersection point of diagonals increase the number of parts by 1. Therefore, the total number of parts into which the diagonals divide an n-gon is equal to $D+P+1$, where D is the number of diagonals, P is the number of intersection points of the diagonals. It is clear that $D = \frac{n(n-3)}{2}$. By the above problem $P = \frac{n(n-1)(n-2)(n-3)}{24}$.

27.10. If we choose any five points, then there exists a convex quadrilateral with vertices in these points (Problem 22.2). It remains to notice that a quadruple of points can be complemented to a 5-tuple in $n-4$ distinct ways.

27.11. Let there be N nonequal triangles with vertices in vertices of a regular n-gon so that among them there are N_1 equailateral, N_2 non-equailateral isosceles, and N_3 scalane ones. Each equailateral triangle is equal to a triangle with fixed vertex A, a non-equailateral isosceles is equal to three triangles with vertex A and a scalane one is equal to 6 triangles. Since the total number of triangles with vertex A is equal to $\frac{(n-1)(n-2)}{2}$, it follows that $\frac{(n-1)(n-2)}{2} = N_1 + 3N_2 + 6N_3$.

Clearly, the number of nonequal equailateral triangles is equal to either 0 or 1 and the number of nonequal isosceles triangles is equal to either $\frac{n-1}{2}$ or $(\frac{n}{2}) - 1$, i.e., $N_1 = 1 - c$ and $N_1 + N_2 = \frac{n-2+d}{2}$, where c and d are equal to either 0 or 1. Therefore,

$$12N = 12(N_1 + N_2 + N_3) = 2(N_1 + 3N_2 + 6N_3) + 6(N_1 + N_2) + 4N_1 = \\ (n-1)(n-2) + 3(n-2+d) + 4(1-c) = n^2 + 3d - 4c.$$

Since $|3d - 4c| < 6$, it follows that N coincides with the nearest integer to $\frac{n^2}{12}$.

Chapter 28. INVERSION

Background

1. All the geometric transformations that we have encountered in this book so far turned lines into lines and circles into circles. The *inversion* is a transformation of another type which also preserves the class of lines and circles but can transform a line into a circle and a circle into a line. This and other remarkable properties of inversion serve as a foundation for its astounding effectiveness in solving various geometric problems.

2. In plane, consider circle S centered at O with radius R. We call the transformation that sends an arbitrary point A distinct from O into point A^* lying on ray OA at distance $OA^* = \frac{R^2}{OA}$ from O the *inversion relative* S. The inversion relative S will be also called the *inversion with center* O *and degree* R^2 and S will be called the *circle of inversion.*

3. It follows directly from the definition of inversion that it fixes points of S, moves points from inside S outside it and points from outside S inside it. If point A turns into A^* under the inversion, then the inversion sends A^* into A, i.e., $(A^*)^* = A$. The image of a line passing through the center of the inversion is this line itself.

Here we should make a reservation connected with the fact that, strictly speaking, the inversion is not a transformation of the plane because O has no image. Therefore, formally speaking, we cannot speak about the "image of the line through O" and should consider instead the union of two rays obtained from the line by deleting point O. Similar is the case with the circles containing point O. Nevertheless, we will use these loose but more graphic formulations and hope that the reader will easily rectify them when necessary.

4. Everywhere in this chapter the image of point A under an inversion is denoted by A^*.

5. Let us formulate the most important properties of inversion that are constantly used in the solution of problems.

Under an inversion with center O:

a) a line l not containing O turns into a circle passing through O (Problem 28.2);

b) a circle centered at C and passing through O turns into a line perpendicular to OC (Problem 28.3);

c) a circle not passing through O turns into a circle not passing through O (Problem 28.3);

d) the tangency of circles with lines is preserved only if the tangent point does not coincide with the center of inversion; otherwise, the circle and a line turn into a pair of parallel lines (Problem 28.4);

e) the value of the angle between two circles (or between a circle and a line, or between two lines) is preserved (Problem 28.5).

§1. Properties of inversions

28.1. Let an inversion with center O send point A to A^* and B to B^*. Prove that triangles OAB and OB^*A^* are similar.

28.2. Prove that under any inversion with center O any line l not passing through O turns into a circle passing through O.

28.3. Prove that under any inversion with center O any circle passing through O turns into a line and any circle not passing through O into a circle.

28.4. Prove that tangent circles (any circle tangent to a line) turn under any inversion into tangent circles or in a circle and a line or in a pair of parallel lines.

Let two circles intersect at point A. The *angle between the circles* is the angle between the tangents to the circles at point A. (Clearly, if the circles intersect at points A and B, then the angle between the tangents at point A is equal to the angle between the tangents at point B). The *angle between a line and a circle* is similarly defined (as the angle between the line and the tangent to the circle at one of the intersection points).

28.5. Prove that inversion preserves the angle between circles (and also between a circle and a line, and between lines).

28.6. Prove that two nonintersecting circles S_1 and S_2 (or a circle and a line) can be transported under an inversion into a pair of concentric circles.

28.7. Let S be centered in O. Through point A a line l intersecting S at points M and N and not passing through O is drawn. Let M' and N' be points symmetric to M and N, respectively, through OA and let A' be the intersection point of lines MN' and $M'N$. Prove that A' coincides with the image of A under the inversion with respect to S (and, therefore, does not depend on the choice of line l).

§2. Construction of circles

While solving problems of this section we will often say “let us perform an inversion … ” . Being translated into a more formal language this should sound as: “Let us construct with the help of a ruler and a compass the images of all the given points, lines and circles under the inversion relative to the given circle”. The possibility to perform such constructions follows from properties of inversion and Problem 28.8.

In problems on construction we often make use of the existence of inversion that sends two nonintersecting circles into concentric circles. The solution of Problem 28.6 implies that the center and radius of such an inversion (hence, the images of the circles) can be constructed by a ruler and a compass.

28.8. Construct the image of point A under the inversion relative circle S centered in O.

28.9. Construct the circle passing through two given points and tangent to the given circle (or line).

28.10. Through a given point draw the circle tangent to two given circles (or a circle and a line).

28.11. (*Apollonius' problem.*) Construct a circle tangent to the three given circles.

28.12. Through a given point draw a circle perpendicular to two given circles.

28.13. Construct a circle tangent to a given circle S and perpendicular to the two given circles (S_1 and S_2).

28.14. Through given points A and B draw a circle intersecting a given circle S under the angle of α.

§3. Constructions with the help of a compass only

According to the tradition that stems from ancient Greece, in geometry they usually consider constructions with the help of ruler and compass. But we can also make constructions with the help of other instruments, or we can, for instance, consider constructions with the help of one compass only, without a ruler. Clearly, with the help of a compass only one cannot simultaneously construct all the points of a line. Therefore, let us make a convention: we will consider a line *constructed* if two of its points are constructed.

It turns out that under such convention we can perform with the help of a compass all the constructions which can be performed with the help of a compass and a ruler. This follows from the possibility to construct using only a compass the intersection points of any line given by two points with a given circle (Problem 28.21 a)) and the intersection point of two lines (Problem 28.21 b)). Indeed, any construction with the help of ruler and compass is a sequence of determinations of the intersection points of circles and lines.

In this section we will only consider constructions with a compass only, without a ruler, i.e., the word "construct" means "construct with the help of a compass only". We will consider a segment *constructed* if its endpoints are constructed.

28.15. a) Construct a segment twice longer than a given segment.

b) Construct a segment n times longer than a given segment.

28.16. Construct the point symmetric to point A through the line passing through given points B and C.

28.17. Construct the image of point A under the inversion relative a given circle S centered in a given point O.

28.18. Construct the midpoint of the segment with given endpoints.

28.19. Construct the circle into which the given line AB turns into under the inversion relative a given circle with given center O.

28.20. Construct the circle passing through three given points.

28.21. a) Construct the intersection points of the given circle S and the line passing through given points A and B.

b) Construct the intersection point of lines A_1B_1 and A_2B_2, where A_1, B_1, A_2 and B_2 are given points.

§4. Let us perform an inversion

28.22. In a disk segment, all possible pairs of tangent circles (Fig. 105) are inscribed. Find the locus of their tangent points.

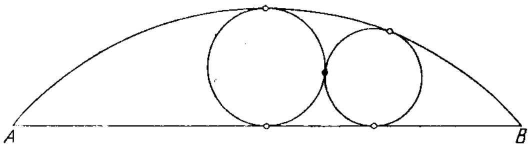


Figure 247 (28.22)

28.23. Find the set of tangent points of pairs of circles that are tangent to the legs of the given angle at given points A and B.

28.24. Prove that the inversion with the center at vertex A of an isosceles triangle ABC, where $AB = AC$, of degree AB^2 sends the base BC of the triangle into the arc $\smile BC$ of the circumscribed circle.

28.25. In a circle segment, all the possible pairs of intersecting circles are inscribed and for each pair a line is drawn through their intersection point. Prove that all these lines pass through one point, cf. Problem 3.44.

28.26. No three of the four points A, B, C, D lie on one line. Prove that the angle between the circumscribed circles of triangles ABC and ABD is equal to the angle between the circumscribed circles of triangles ACD and BCD.

28.27. Through points A and B there are drawn circles S_1 and S_2 tangent to circle S and circle S_3 perpendicular to S. Prove that S_3 forms equal angles with circles S_1 and S_2.

28.28. Two circles intersecting at point A are tangent to the circle (or line) S_1 at points B_1 and C_1 and to the circle (or line) S_2 at points B_2 and C_2 (and the tangency at B_2 and C_2 is the same as at respective points B_1 and C_1, i.e., either inner or outer). Prove that circles circumscribed about triangles AB_1C_1 and AB_2C_2 are tangent to each other.

28.29. Prove that the circle passing through the midpoints of triangle's sides is tangent to its inscribed and three escribed circles. (*Feuerbach's theorem.*)

§5. Points that lie on one circle and circles passing through one point

28.30. Given four circles, S_1, S_2, S_3, S_4, where circles S_1 and S_3 intersect with both circles S_2 and S_4. Prove that if the intersection points of S_1 with S_2 and S_3 with S_4 lie on one circle or line, then the intersection points of S_1 with S_4 and S_2 with S_3 lie on one circle or line (Fig. 106).

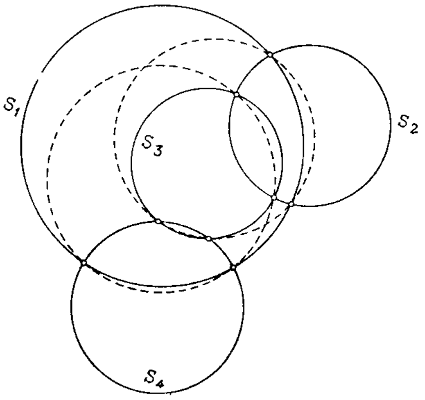


Figure 248 (28.30)

28.31. Given four circles S_1, S_2, S_3, S_4 such that S_1 and S_2 intersect at points A_1 and A_2, S_2 and S_3 at points B_1 and B_2, S_3 and S_4 at points C_1 and C_2, S_4 and S_1 at points D_1 and D_2 (Fig. 107).

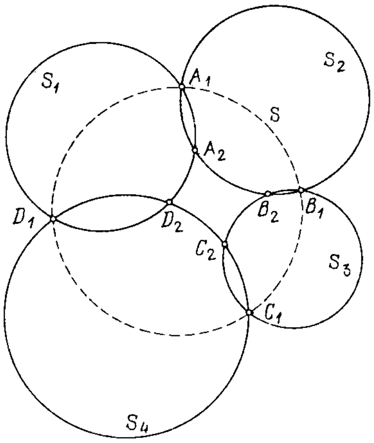


Figure 249 (28.31)

Prove that if points A_1, B_1, C_1, D_1 lie on one circle (or line) S, then points A_2, B_2, C_2, D_2 lie on one circle (or line).

28.32. The sides of convex pentagon $ABCDE$ are extended so that five-angled star $AHBKCLDMEN$ (Fig. 108) is formed. The circles are circumscribed about triangles —

the rays of the star. Prove that the five intersection points of these circles distinct from A, B, C, D, E lie on one circle.

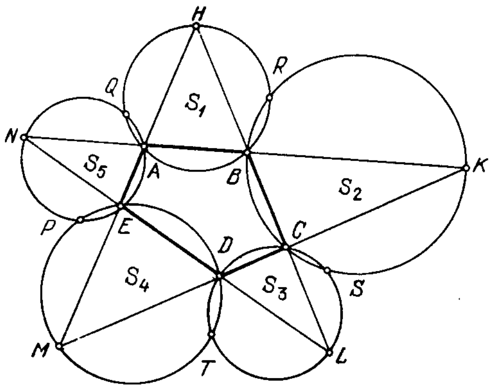


Figure 250 (28.32)

28.33. In plane, six points A_1, A_2, A_3, B_1, B_2, B_3 are fixed. Prove that if the circles circumscribed about triangles $A_1A_2B_3$, $A_1B_2A_3$ and $B_1A_2A_3$ pass through one point, then the circles circumscribed about triangles $B_1B_2A_3$, $B_1A_2B_3$ and $A_1B_2B_3$ intersect at one point.

28.34. In plane, six points A_1, A_2, B_1, B_2, C_1, C_2 are fixed. Prove that if the circles circumscribed about triangles $A_1B_1C_1$, $A_1B_2C_2$, $A_2B_1C_2$, $A_2B_2C_1$ pass through one point, then the circles circumscribed about triangles $A_2B_2C_2$, $A_2B_1C_1$, $A_1B_2C_1$, $A_1B_1C_2$ pass through one point.

28.35. In this problem we will consider tuples of n generic lines, i.e., sets of lines no two of which are parallel and no three pass through one point.

To a tuple of two generic lines assign their intersection point and to a tuple of two generic lines assign the circle passing through the three points of their pairwise intersections. If l_1, l_2, l_3, l_4 are four generic lines, then the four circles S_i corresponding to four triples of lines obtained by discarding l_i pass through one point (cf. Problem 2.83 a)) that we will assign to the foursome of lines.

This construction can be extended:

a) Let l_i, $i = 1, \dots, 5$ be five generic points. Prove that five points A_i corresponding to the foursome of lines obtained by discarding l_i lie on one circle.

b) Prove that this construction can be continued in the following way: to every tuple of n generic points assign a point if n is even or a circle if n is odd so that n circles (points) corresponding to tuples of $n-1$ lines pass through this point (belong to this circle).

28.36. On two intersecting lines l_1 and l_2, select points M_1 and M_2 not coinciding with the intersection point M of these lines. Assign to this set of lines and points the circle passing through M_1, M_2 and M.

If (l_1, M_1), (l_2, M_2), (l_3, M_3) are three generic lines with fixed points, then by Problem 2.80 a) the three circles corresponding to pairs (l_1, M_1) and (l_2, M_2), (l_2, M_2), (l_3, M_3), (l_3, M_3) and (l_1, M_1) intersect at one point that we will assign to the triple of lines with a fixed point.

a) Let l_1, l_2, l_3, l_4 be four generic lines on each of which a point is fixed so that these points lie on one circle. Prove that four points corresponding to the triples obtained by deleting one of the lines lie on one circle.

b) Prove that to every tuple of n generic lines with a point fixed on each of them so that the fixed points lie on one circle one can assign a point (if n is odd) or a circle (if n is even) so that n circles (if n is odd) or points (if n is even) corresponding to the tuples of $n-1$ lines pass through this point (resp. lie on this circle).

§6. Chains of circles

28.37. Circles S_1, S_2, … , S_n are tangent to circles R_1 and R_2 and, moreover, S_1 is tangent to S_2 at point A_1, S_2 is tangent to S_3 at point A_2, … , S_{n-1} is tangent to S_n at point A_{n-1}. Prove that points A_1, A_2, … , A_{n-1} lie on one circle.

28.38. Prove that if there exists a chain of circles S_1, S_2, … , S_n each of which is tangent to two neighbouring ones (S_n is tangent to S_{n-1} and S_1) and two given nonintersecting circles R_1 and R_2, then there are infinitely many such chains.

(?)Namely, for any circle T_1 tangent to R_1 and R_2 (in the same fashion if R_1 and R_2 do not lie inside each other, by an inner or an outer way, otherwise) there exists a similar chain of n tangent circles T_1, T_2, … , T_n. (*Steiner's porism.*)

28.39. Prove that for two nonintersecting circles R_1 and R_2 a chain of n tangent circles (cf. the preceding problem) exists if and only if the angle between the circles T_1 and T_2 tangent to R_1 and R_2 at their intersection points with the line that connects the centers of R_1 and R_2 is equal to an integer multiple of $\frac{360^\circ}{n}$ (Fig. 109).

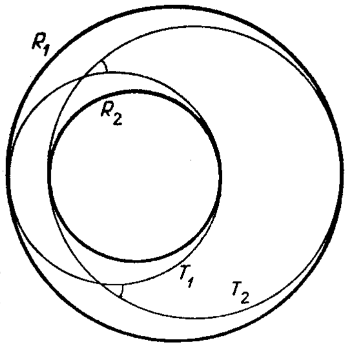


Figure 251 (28.39)

28.40. Each of six circles is tangent to four of the remaining five circles, see Fig. 110.

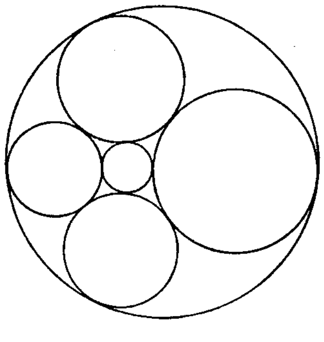


Figure 252 (28.40)

Prove that for any pair of nonintersecting circles (of these six circles) the radii and the distance between their centers are related by the formula

$$d^2 = r_1^2 + r_2^2 \pm 6r_1r_2,$$

where "plus" is taken if the circles are not inside each other and "minus" otherwise.

Solutions

28.1. Let R^2 be the degree of the inversion. Then

$$OA \cdot OA^* = OB \cdot OB^* = R^2$$

whence, $OA : OB = OB^* : OA^*$ and $\triangle OAB \sim \triangle OB^*A^*$ because $\angle AOB = \angle B^*OA^*$.

28.2. Let us drop perpendicular OC from point O to line l and take an arbitrary point M on l. Since triangles OCM and OM^*C^* are similar (Problem 28.1), $\angle OM^*C^* = \angle OCM = 90^\circ$, i.e., point M^* lies on circle S with diameter OC^*. If X is a point of S distinct from O, then it is the image under the inversion of the intersection point Y of l and OX (since the image of Y lies, on the one hand, on ray OX and, on the other hand, on circle S, as is already proved). Thus, the inversion sends line l into circle S (without point O).

28.3. The case when circle S passes through O is actually considered in the preceding problem (and formally follows from it since $(M^*)^* = M$).

Now, suppose that O does not belong to S. Let A and B be the intersection points of circle S with the line passing through O and the center of S, let M be an arbitrary point of S. Let us prove that the circle with diameter A^*B^* is the image of S. To this end it suffices to show that $\angle A^*M^*B^* = 90^\circ$. But by Problem 28.1 $\triangle OAM \sim \triangle OM^*A^*$ and $\triangle OBM \sim \triangle OM^*B^*$; hence, $\angle OMA = \angle OA^*M^*$ and $\angle OMB = \angle OB^*M^*$; more exactly, $\angle(OM, MA) = -\angle(OA^*, M^*A^*)$ and $\angle(OM, MB) = -\angle(OB^*, M^*B^*)$. (In order not to consider various cases of points' disposition we will make use of the properties of oriented angles between lines discussed in Chapter 2.) Therefore,

$$\angle(A^*M^*, M^*B^*) = \angle(A^*M^*, OA^*) + \angle(OB^*, M^*B^*) = \\ \angle(OM, MA) + \angle(MB, OM) = \angle(MB, MA) = 90^\circ.$$

28.4. If the tangent point does not coincide with the center of inversion, then after the inversion these circles (the circle and the line) will still have one common point, i.e., the tangency is preserved.

If the circles with centers A and B are tangent at point O, then under the inversion with center O they turn into a pair of lines perpendicular to AB. Finally, if line l is tangent to the circle centered at A at point O, then under the inversion with center O the line l turns into itself and the circle into a line perpendicular to OA. In each of these two cases we get a pair of parallel lines.

28.5. Let us draw tangents l_1 and l_2 through the intersection point of the circles. Since under the inversion the tangent circles or a circle and a line pass into tangent ones (cf. Problem 28.4), the angle between the images of circles is equal to the angle between the images of the tangents to them. Under the inversion centered at O line l_i turns into itself or into a circle the tangent to which at O is parallel to l_i. Therefore, the angle between the images of l_1 and l_2 under the inversion with center O is equal to the angle between these lines.

28.6. First solution. Let us draw the coordinate axis through the centers of the circles. Let a_1 and a_2 be the coordinates of the intersection points of the axes with S_1, let b_1 and b_2 be the coordinates of the intersection points of the axes with S_2. Let O be the point on the axis whose coordinate is x. Then under the inversion with center O and degree k our circles turn into the circles whose diameters lie on the axis and whose endpoints have coordinates a_1', a_2' and b_1', b_2', respectively, where

$$a_1' = x + \frac{k}{a_1 - x}, \quad a_2' = x + \frac{k}{a_2 - x}, \quad b_1' = x + \frac{k}{b_1 - x}, \quad b_2' = x + \frac{k}{b_2 - x}.$$

The obtained circles are concentric if $\frac{a_1'+a_2'}{2}=\frac{b_1'+b_2'}{2}$, i.e.,

$$\frac{1}{a_1-x}+\frac{1}{a_2-x}=\frac{1}{b_1-x}+\frac{1}{b_2-x},$$

wherefrom we have

$$(b_1+b_2-a_1-a_2)x^2+2(a_1a_2-b_1b_2)x+b_1b_2(a_1+a_2)-a_1a_2(b+b_2)=0.$$

The discriminant of this quadratic in x is equal to $4(b_1-a_1)(b_1-a_2)(b_2-a_1)(b_2-a_2)$. It is positive precisely when the circles do not intersect; this proves the existence of the required inversion.

The existence of such an inversion for the case of a circle and a line is similarly proved.

Another solution. On the line that connects centers O_1 and O_2 of the circles take point C such that the tangents drawn to the circles from C are equal. This point C can be constructed by drawing the radical axis of the circles (cf. Problem 3.53). Let l be the length of these tangents. The circle S of radius l centered in C is perpendicular to S_1 and S_2. Therefore, under the inversion with center O, where O is any of the intersection points of S with line O_1O_2, circle S turns into a line perpendicular to circles S_1^* and S_2^* and, therefore, passing through their centers. But line O_1O_2 also passes through centers of S_1^* and S_2^*; hence, circles S_1^* and S_2^* are concentric, i.e., O is the center of the desired inversion.

If S_2 is not a circle but a line, the role of line O_1O_2 is played by the perpendicular dropped from O_1 to S_2, point C is its intersection point with S_2, and l is the length of the tangent dropped from C to S_1.

28.7. Let point A lie outside S. Then A' lies inside S and we see that $\angle MA'N=\frac{1}{2}(\smile MN+\smile M'N')=\smile MN=\angle MON$, i.e., quadrilateral $MNOA'$ is an inscribed one. But under the inversion with respect to S line MN turns into the circle passing through points M, N, O (Problem 28.2). Therefore, point A^* (the image of A under the inversion) lies on the circle circumscribed about quadrilateral $MNOA'$. By the same reason points A' and A^* belong to the circle passing through M', N' and O. But these two circles cannot have other common points except O and A'. Hence, $A^*=A'$.

If A lies inside S, we can apply the already proved to line MN' and point A' (which is outside S). We get $A=(A')^*$. But then $A'=A^*$.

28.8. Let point A lie outside S. Through A, draw a line tangent to S at point M. Let MA' be a height of triangle OMA. Right triangles OMA and $OA'M$ are similar, hence, $A'O:OM=OM:OA$ and $OA'=\frac{R^2}{OA}$, i.e., point A' is the one to be found.

If A lies inside S, then we can perform the construction in the reverse order: we drop perpendicular AM to OA (point M lies on the circle). Then the tangent to S at point M intersects with ray OA at the desired point, A^*.

Proof is repeated literally.

28.9. If both given points A and B lie on the given circle (or line) S, then the problem has no solutions. Let now A not lie to S. Under the inversion with center A the circle to be found turns into the line passing through B^* and tangent to S^*. This implies the following construction. Let us perform the inversion with respect to an arbitrary circle with center A. Through B^* draw the tangent l to S^*. Perform an inversion once again. Then l turns into the circle to be constructed.

If point B^* lies on S^*, then the problem has a unique solution; if B^* lies outside S^*, then there are two solutions, and if B^* lies inside S^*, then there are no solutions.

28.10. The inversion with center at the given point sends circles S_1 and S_2 into a pair of circles S_1^* and S_2^* (or into circle S^* and line l; or into a pair of lines l_1 and l_2), respectively; the circle tangent to them turns into the common tangent to S_1^* and S_2^* (resp. into the tangent to S^* parallel to l; or into a line parallel to l_1 and l_2). Therefore, to construct the desired circle we have to construct a line tangent to S_1^* and S_2^* (resp. tangent to S^* and parallel to l; or parallel to l_1 and l_2) and perform an inversion once again.

28.11. Let us reduce this problem to Problem 28.10. Let circle S of radius r be tangent to circles S_1, S_2, S_3 of radii r_1, r_2, r_3, respectively. Since the tangency of S with each of S_i ($i = 1, 2, 3$) can be either outer or inner, there are eight possible distinct cases to consider. Let, for instance, S be tangent to S_1 and S_3 from the outside and to S_2 from the inside (Fig. 111).

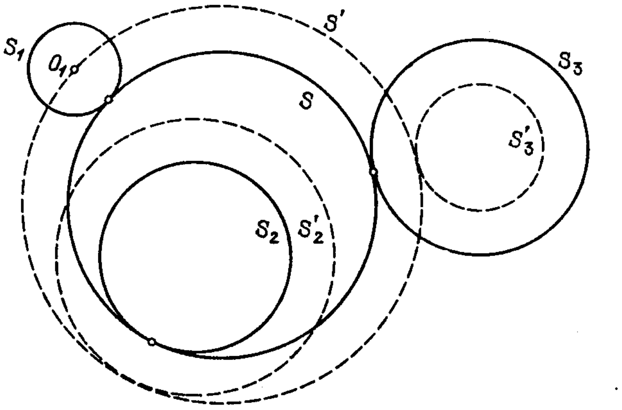


Figure 253 (Sol. 28.11)

Let us replace the circles S, S_2, S_3 with the concentric to them circles S', S_2' and S_3', respectively, so that S' is tangent to S_2' and S_3' and passes through the center O_1 of S_1. To this end it suffices that the radii of S', S_2', S_3' were equal to $r + r_1$, $r_2 + r_1$, $|r_3 - r_1|$, respectively.

Conversely, from circle S' passing through O_1 and tangent to S_2' and S_3' (from the outside if $r_3 - r_1 \geq 0$ and from the inside if $r_3 - r_1 < 0$) we can construct circle S — a solution of the problem — by diminishing the radius of S' by r_1. The construction of such a circle S' is described in the solution of Problem 28.10 (if the type of tangency is given, then the circle is uniquely constructed).

One can similarly perform the construction for the other possible types of tangency.

28.12. Under the inversion with center at the given point A the circle to be constructed turns into the line perpendicular to the images of both circles S_1 and S_2, i.e., into the line connecting the centers of S_1^* and S_2^*. Therefore, the circle circle to be constructed is the image under this inversion of an arbitrary line passing through the centers of S_1^* and S_2^*.

28.13. Let us perform an inversion that sends circles S_1 and S_2 into a pair of lines (if they have a common point) or in a pair of concentric circles (cf. Problem 28.6) with a common center A. In the latter case the circle perpendicular to both circles S_1 and S_2 turns into a line passing through A (since there are no circles perpendicular to two concentric circles); the tangent drawn from A to S^* is the image of the circle circle to be constructed under this inversion.

If S_1^* and S_2^* are parallel lines, then the image of the circle circle to be constructed is any of the two lines perpendicular to S_1^* and S_2^* and tangent to S^*. Finally, if S_1^* and S_2^* are lines intersecting at a point B, then the circle circle to be constructed is the image under the inversion of any of the two circles with center B and tangent to S^*.

28.14. Under the inversion with center at point A the problem reduces to the construction of a line l passing through B^* and intersecting circle S^* at an angle of α, i.e., to construction of a point X on S^* such that $\angle B^*XO = 90^\circ - \alpha$, where O is the center of S^*. This point X lies on the intersection of S^* with the arc whose points serve as the vertices of angles of $90^\circ - \alpha$ subtending segment B^*O.

28.15. a) Let AB be the given segment. Let us draw the circle with center B and radius AB. On this circle, mark chords AX, XY and YZ of the same length as AB; we get equilateral triangles ABX, XBY and YBZ. Hence, $\angle ABZ = 180^\circ$ and $AZ = 2AB$.

b) In the solution of heading a) we have described how to construct a segment BZ equal to AB on line AB. Repeating this procedure $n-1$ times we get segment AC such that $AC = nAB$.

28.16. Let us draw circles with centers B and C passing through A. Then the distinct from A intersection point of these circles is the desired one.

28.17. First, suppose that point A lies outside circle S. Let B and C be the intersection points of S and the circle of radius AO and with center A. Let us draw circles with centers B and C of radius $BO = CO$; let O and A' be their intersection points. Let us prove that A' is the desired point.

Indeed, under the symmetry through line OA the circles with centers B and C turn into each other and, therefore, point A' is fixed. Hence, A' lies on line OA. Isosceles triangles OAB and OBA' are similar because they have equal angles at the base. Therefore, $OA' : OB = OB : OA$ or $OA' = \frac{OB^2}{OA}$, as required.

Now, let point A lie inside S. With the help of the construction from Problem 28.15 a) let us construct on ray OA segments AA_2, A_2A_3, $\dots$, $A_{n-1}A_n$, $\dots$ of length OA until one of the points A_n becomes outside S. Applying to A_n the above-described construction we get a point A_n^* on OA such that $OA_n^* = \frac{R^2}{n\cdot OA} = \frac{1}{n}OA^*$. In order to construct point A^* it only remains to enlarge segment OA_n^* n times, cf. Problem 28.15 b).

28.18. Let A and B be two given points. If point C lies on ray AB and $AC = 2AB$, then under the inversion with respect to the circle of radius AB centered at A point C turns into the midpoint of segment AB. The construction is reduced to Problems 28.15 a) and 28.17.

28.19. The center of this circle is the image under an inversion of point O' symmetric to O through AB. It remains to apply Problems 28.16 and 28.17.

28.20. Let A, B, C be given points. Let us construct (Problem 28.17) the images of B and C under the inversion with center A and of arbitrary degree. Then the circle passing through A, B and C is the image of line B^*C^* under this inversion and its center can be constructed thanks to the preceding problem.

28.21. a) Making use of the preceding problem construct the center O of circle S. Next, construct points A^* and B^* — the images of A and B under the inversion with respect to S. The image of AB is circle S_1 passing through points A^*, B^* and O. Making use of Problem 28.19 we construct S_1. The desired points are the images of the intersection points of circles S and S_1, i.e., just intersection points of S and S_1.

b) Let us consider an inversion with center A_1. Line A_2B_2 turns under this inversion into the circle S passing through points A_1, A_2^* and B_2^*. We can construct S making use of Problem 28.19. Further, let us construct the intersection points of S and line A_1B_1 making use of the solution of heading a). The desired point is the image of the intersection point distinct from A_1 under the inversion considered.

28.22. Under the inversion centered in the endpoint A of the segment the configuration plotted on Fig. 105 turns into the pair of tangent circles inscribed into the angle at vertex B^*. Clearly, the set of the tangent points of such circles is the bisector of the angle and the desired locus is the image of the bisector under the inversion — the arc of the circle with endpoints A and B that divides in halves the angle between the arc of the segment and chord AB.

28.23. Let C be the vertex of the given angle. Under the inversion with center in A line CB turns into circle S; circles S_1 and S_2 turn into circle S_1^* centered in O_1 tangent to S at point B^* and line l parallel to C^*A and tangent to S_1^* at X, respectively (Fig. 112).

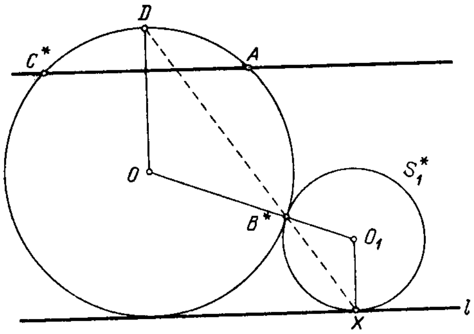


Figure 254 (Sol. 28.23)

In S, draw radius OD perpendicular to C^*A. Points O, B^* and O_1 lie on one line and $OD \parallel O_1X$. Hence,

$$\angle OB^*D = 90^\circ - \frac{\angle DOB^*}{2} = 90^\circ - \frac{\angle XO_1B^*}{2} = \angle O_1B^*X,$$

therefore, point X lies on line DB^*. Applying inversion once again we see that the desired locus of tangent points is arc $\smile AB$ of the circle passing through points A, B and D^*.

28.24. The given inversion sends line BC into the circle passing through points A, B and C so that the image of segment BC should remain inside angle $\angle BAC$.

28.25. Let S_1 and S_2 be circles inscribed into the segment; M, N their intersection points (Fig. 113). Let us show that line MN passes through point P of the circle of the segment equidistant from its endpoints A and B.

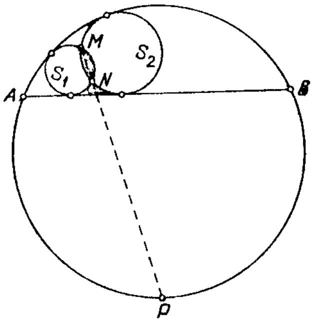


Figure 255 (Sol. 28.25)

Indeed, thanks to the preceding problem the inversion with center P and of degree PA^2 sends segment AB to arc $\smile AB$ and circles S_1 and S_2 to circles S_1^* and S_2^*, still inscribed into a segment, respectively. But the tangents to S_1 drawn from P are tangent also to S_1^*; hence, $S_1^* = S_1$ (since both these circles are similarly tangent to the three fixed points). Analogously, $S_2^* = S_2$; hence, points M and N change places under the inversion, i.e., $M^* = N$ and MN passes through the center of inversion.

28.26. Let us perform an inversion with center A. The angles of interest to us are then equal (by Problem 28.5) to the respective angles between lines B^*C^* and B^*D^* or between line C^*D^* and the circle circumscribed about triangle $B^*C^*D^*$. Both these angles are equal to a half arc $\smile C^*D^*$.

28.27. Performing an inversion with center A we get three lines passing through B: lines S_1^* and S_2^* are tangent to S^* and S_3^* is perpendicular to it. Thus, line S_3^* passes through the center of S^* and is the bisector of the angle formed by S_1^* and S_2^*. Therefore, circle S_3 divides the angle between S_1 and S_2 in halves.

28.28. The condition of the types of tangency implies that after an inversion with center A we get either two circles inscribed into the same angle or a pair of vertical angles. In either case a homothety with center A turns circles S_1^* and S_2^* into each other. This homothety sends one segment that connects tangent points into another one. Hence, lines $B_1^*C_1^*$ and $B_2^*C_2^*$ are parallel and their images under the inversion are tangent at point A.

28.29. Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB, respectively. Let us prove that, for instance, the circle circumscribed about triangle $A_1B_1C_1$ is tangent to the inscribed circle S and escribed circle S_a tangent to BC. Let points B' and C' be symmetric to B and C, respectively, through the bisector of angle $\angle A$ (i.e., $B'C'$ is the second common inner tangent to S and S_a), let P and Q be the tangent points of circles S and S_a, respectively, with side BC and let D and E be the intersection points of lines A_1B_1 and A_1C_1, respectively, with line $B'C'$.

By Problem 3.2 $BQ = CP = p - c$ and, therefore, $A_1P = A_1Q = \frac{1}{2}|b-c|$. It suffices to prove that the inversion with center A_1 and degree A_1P^2 sends points B_1 and C_1 into D and E, respectively, (this inversion sends circles S and S_a into themselves, and the circle circumscribed about triangle $A_1B_1C_1$ into line $B'C'$).

Let K be the midpoint of segment CC'. Point K lies on line A_1B_1 and

$$A_1K = \frac{BC'}{2} = \frac{|b-c|}{2} = A_1P.$$

Moreover,

$$A_1D : A_1K = BC' : BA = A_1K : A_1B_1,$$

i.e., $A_1D \cdot A_1B_1 = A_1K^2 = A_1P^2$. Similarly, $A_1E \cdot A_1C_1 = A_1P^2$.

28.30. After an inversion with center at the intersection point of S_1 and S_2 we get lines l_1, l_2 and l intersecting at one point. Line l_1 intersects circle S_4^* at points A and B, line l_2 intersects S_3^* at points C and D and line l passes through the intersection points of these circles. Hence, points A, B, C, D lie on one circle (Problem 3.9).

28.31. Let us make an inversion with center at point A_1. Then circles S_1, S_2 and S_4 turn into lines $A_2^*D_1^*$, $B_1^*A_2^*$ and $D_1^*B_1^*$; circles S_3 and S_4 into circles S_3^* and S_4^* circumscribed about triangles $B_2^*C_1^*B_1^*$ and $C_1^*D_1^*D_2^*$, respectively (Fig. 114).

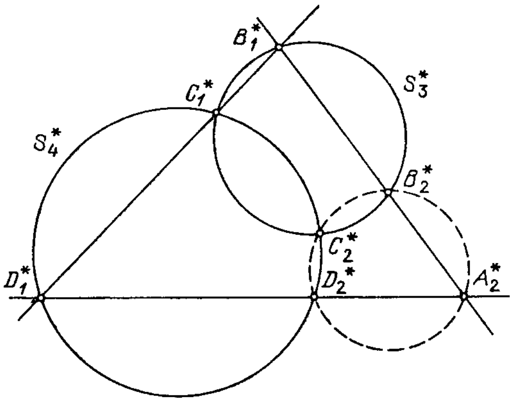


Figure 256 (Sol. 28.31)

Let us draw the circle through points B_2^*, D_2^* and A_2^*. By Problem 2.80 a) it passes through the intersection point C_2^* of circles S_3^* and S_4^*. Thus, points A_2^*, B_2^*, C_2^*, D_2^* lie on one circle. It follows, that points A_2, B_2, C_2, D_2 lie on one circle or line.

28.32. Let P, Q, R, S, T be the intersection points of circles S_1, S_2, S_3, S_4, S_5 spoken about in the formulation of the problem (cf. Fig. 108).

Let us prove, for instance, that points P, Q, R, S lie on one circle. Let us draw circle Σ circumscribed about triangle NKD. Applying the result of Problem 2.83 a) (which coincides with that of Problem 19.45) to quadrilaterals $AKDE$ and $BNDC$ we see that circles S_4, S_5 and Σ intersect at one point (namely, P) and circles S_2, S_3, Σ also intersect at one point (namely, S).

Therefore, circle Σ passes through points P and S. Now, observe that of eight intersection points of circles Σ, S_1, S_2, S_5 four, namely, N, A, B, K, lie on one line. It follows that by Problem 28.31 the remaining four points P, Q, R, S lie on one circle.

28.33. An inversion with center at the intersection point of circumscribed circles of triangles $A_1A_2B_3$, $A_1B_2A_3$ and $B_1A_2A_3$ sends these circles into lines and the statement of the problem reduces to the statement that the circles circumscribed about triangles $B_1^*B_2^*A_3^*$, $B_1^*A_2^*B_3^*$ and $A_1^*B_2^*B_3^*$ pass through one point, i.e., the statement of Problem 2.80 a).

28.34. Under an inversion with center at the intersection point of circles circumscribed about triangles $A_1B_1C_1$, $A_1B_2C_2$, $A_2B_1C_2$ and $A_2B_2C_1$ we get four lines and four circles circumscribed about triangles formed by these lines. By Problem 2.83 a) these circles pass through one point.

28.35. a) Denote by M_{ij} the intersection point of lines l_i and l_j and by S_{ij} the circle corresponding to the three remaining lines. Then point A_1 is distinct from the intersection point M_{34} of circles S_{15} and S_{12}.

Repeating this argument for each point A_i, we see that thanks to Problem 28.32 they lie on one circle.

b) Let us prove the statement of the problem by induction and considere separately the cases of even and odd n.

Let n be odd. Denote by A_i the point corresponding to the tuple of $n-1$ lines obtained by deleting line l_i and by A_{ijk} the point corresponding to the tuple of n given lines without l_i, l_j and l_k. Similarly, denote by S_{ij} and S_{ijkm} the circles corresponding to tuples of $n-2$ and $n-4$ lines obtained by deleting l_i and l_j or l_i, l_j, l_k and l_m, respectively.

In order to prove that n points A_1, A_2, $\dots$, A_n lie on the same circle, it suffices to prove that any four of them lie on one circle. Let us prove this, for instance, for points A_1, A_2, A_3 and A_4. Since points A_i and A_{ijk} lie on S_{ij}, it follows that circles S_{12} and S_{23} intersect at points A_2 and A_{123}; circles S_{23} and S_{34} intersect at points A_3 and A_{234}; circles S_{34} and S_{41} at points A_4 and A_{134}; circles S_{41} and S_{12} at points A_1 and A_{124}. But points A_{123}, A_{234}, A_{134} and A_{124} lie on one circle — circle S_{1234} — hence, by Problem 28.31 points A_1, A_2, A_3 and A_4 lie on one circle.

Let n be even. Let S_i, A_{ij}, S_{ijk}, A_{ijkm} be circles and points corresponding to tuples of $n-1$, $n-2$, $n-3$ and $n-4$ lines, respectively. In order to prove that circles S_1, S_2, $\dots$, S_n intersect at one point, let us prove that this holds for any three of them. (This suffices for $n \geq 5$, cf. Problem 26.12.) Let us prove, for instance, that S_1, S_2 and S_3 intersect at one point. By definition of points A_{ij} and circles S_i and S_{ijk}, points A_{12}, A_{13} and A_{14} lie on circle S_1; points A_{12}, A_{23} and A_{24} on S_2; points A_{13}, A_{14} and A_{34} on S_3; points A_{12}, A_{14} and A_{24} on S_{124}; points A_{13}, A_{14}, A_{34} on S_{134}; points A_{23}, A_{24}, A_{34} on S_{234}.

But the three circles S_{124}, S_{134} and S_{234} pass through point A_{1234}; hence, by Problem 28.33 circles S_1, S_2 and S_3 also intersect at one point.

28.36. a) Denote by M_{ij} the intersection point of lines l_i and l_j. Then point A_1 corresponding to the triple (l_2, l_3, l_4) is the intersection point of the circles circumscribed about triangles $M_2M_3M_{23}$ and $M_3M_4M_{34}$. By the similar arguments applied to A_2, A_3 and A_4 we see that points A_1, A_2, A_3 and A_4 lie on one circle thanks to Problem 28.31 because points M_1, M_2, M_3, M_4 lie on one circle.

b) As in Problem 28.35 b), let us prove our statement by induction; consider the cases of even and odd n separately.

Let n be even; let A_i, S_{ij}, A_{ijk} and S_{ijkm} denote points and circles corresponding to tuples of $n-1$, $n-2$, $n-3$ and $n-4$ lines, respectively. Let us prove that points A_1, A_2, A_3, A_4 lie on one circle. By definition of points A_i and A_{ijk}, circles S_{12} and S_{23} intersect at points A_2 and A_{123}; circles S_{23} and S_{34} at points A_3 and A_{234}; circles S_{34} and S_{41} at points A_4 and A_{134}; circles S_{41} and S_{12} at points A_1 and A_{124}.

Points A_{123}, A_{234}, A_{134} and A_{124} lie on circle S_{1234}; hence, by Problem 28.31 points A_1, A_2, A_3, A_4 lie on one circle. We similarly prove that any four of points A_i (hence, all of them) lie on one circle.

Proof for n odd, $n \geq 5$, literally repeats the proof of heading b) of Problem 28.35 for the case of n even.

28.37. If circles R_1 and R_2 intersect or are tangent to each other, then an inversion with the center at their intersection point sends circles $S_1, S_2, \dots, S_n$ into the circles that are tangent to a pair of straight lines and to each other at points $A_1^*, A_2^*, \dots, A_{n-1}^*$ lying on the bisector of the angle formed by lines R_1^* and R_2^* if R_1^* and R_2^* intersect, or on the line parallel to R_1^* and R_2^* if these lines do not intersect. Applying the inversion once again we see that points $A_1^*, A_2^*, \dots, A_{n-1}^*$ lie on one circle.

If circles R_1 and R_2 do not intersect, then by Problem 28.6 there is an inversion sending them into a pair of concentric circles. In this case points $A_1^*, A_2^*, \dots, A_{n-1}^*$ lie on a circle concentric with R_1^* and R_2^*; hence, points $A_1, A_2, \dots, A_{n-1}$ lie on one circle.

28.37. Let us make an inversion sending R_1 and R_2 into a pair of concentric circles. Then circles $S_1^*, S_2^*, \dots, S_n^*$ and T_1^* are equal (Fig. 115).

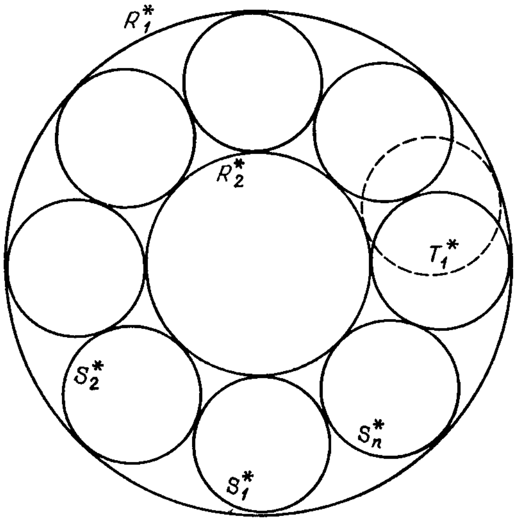


Figure 257 (Sol. 28.37)

Turning the chain of circles $S_1^*, \dots, S_n^*$ about the center of the circle R_1^* so that S_1^* becomes T_1^* and making an inversion once again we get the desired chain $T_1, T_2, \dots, T_n$.

28.39. The center of inversion that sends circles R_1 and R_2 into concentric circles lies (see the solution of Problem 28.6) on the line that connects their centers. Therefore, making this inversion and taking into account that the angle between circles, as well as the type of tangency, are preserved under an inversion, we reduce the proof to the case of concentric circles R_1 and R_2 with center O and radii r_1 and r_2, respectively.

Let us draw circle S with center P and of radius $\frac{1}{2}(r_1 - r_2)$ tangent to R_1 from the inside and to R_2 from the outside and let us draw circles S' and S'' each of radius $\frac{1}{2}(r_1 + r_2)$ with centers A and B, respectively, tangent to R_1 and R_2 at their intersection points with line OP (Fig. 116).

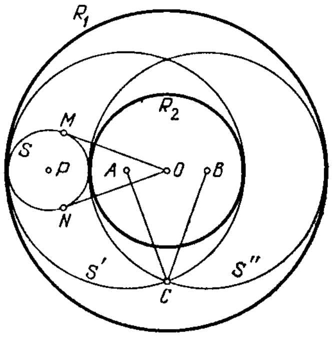


Figure 258 (Sol. 28.39)

Let OM and ON be tangent to S drawn at O. Clearly, the chain of n circles tangent to R_1 and R_2 exists if and only if $\angle MON = m\frac{360^\circ}{n}$. (In this case the circles of the chain run m times about the circle R_2.)

Therefore, it remains to prove that the angle between circles S' and S'' is equal to $\angle MON$. But the angle between S' and S'' is equal to the angle between their radii drawn to the intersection point C. Moreover, since

$$PO = r_1 - \frac{r_1 - r_2}{2} = \frac{r_1 + r_2}{2} = AC,$$

$$PN = \frac{r_1 - r_2}{2} = r_1 - \frac{r_1 + r_2}{2} = OA,$$

$$\angle PNO = \angle AOC = 90^\circ,$$

we have $\triangle ACO = \triangle PON$. Therefore,

$$\angle ACB = 2\angle ACO = 2\angle PON = \angle NOM.$$

28.40. Let R_1 and R_2 be a pair of circles without common points. The remaining four circles constitute a chain and, therefore, by the preceding problem circles S' and S'' tangent to R_1 and R_2 at the intersection points of the latter with the line connecting their centers intersect at right angle (Fig. 117). If R_2 lies inside R_1, then the radii r' and r'' of circles S' and S'' are equal to $\frac{1}{2}(r_1 + r_2 + d)$ and $\frac{1}{2}(r_1 + r_2 - d)$, respectively, and the distance between their centers is equal to $d' = 2r_1 - r_1 - r_2 = r_1 - r_2$. The angle between S' and S'' is equal to the angle between the radii drawn to the intersection point, hence, $(d')^2 = (r')^2 + (r'')^2$ or, after simplification, $d^2 = r_1^2 + r_2^2 - 6r_1r_2$.

If R_1 and R_2 are not inside one another, then the radii of S' and S'' are equal to $\frac{1}{2}(d + (r_1 - r_2))$ and $\frac{1}{2}(d - (r_1 - r_2))$, respectively, and the distance between their centers is $d' = r_1 + r_2 + d - (r_1' + r_2') = r_1 + r_2$. As a result we get $d^2 = r_1^2 + r_2^2 + 6r_1r_2$.

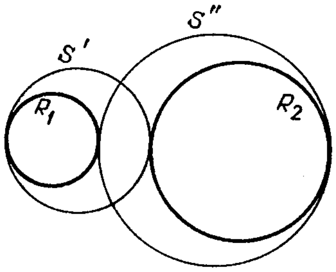


Figure 259 (Sol. 28.40)

Chapter 29. AFFINE TRANSFORMATIONS

§1. Affine transformations

A transformation of the plane is called an *affine* one if it is continuous, one-to-one, and the image of every line is a line.

Shifts and *similarity transformations* are particular cases of affine transformations.

A *dilation* of the plane relative axis l with coefficient k is a transformation of the plane under which point M turns into point M' such that $\overrightarrow{OM'} = k\overrightarrow{OM}$, where O is the projection of M to l. (A dilation with coefficient smaller than 1 is called a *contraction.*)

29.1. Prove that a dilation of the plane is an affine transformation.

29.2. Prove that under an affine transformation parallel lines turn into parallel ones.

29.3. Let A_1, B_1, C_1, D_1 be images of points A, B, C, D, respectively, under an affine transformation. Prove that if $\overrightarrow{AB} = \overrightarrow{CD}$, then $\overrightarrow{A_1B_1} = \overrightarrow{C_1D_1}$.

Problem 29.3 implies that we can define the image of vector $\overrightarrow{AB}$ under an affine transformation L as $\overrightarrow{L(A)L(B)}$ and this definition does not depend on the choice of points A and B that determine equal vectors.

29.4. Prove that if L is an affine transformation, then

a) $L(\overrightarrow{0}) = \overrightarrow{0}$;

b) $L(\mathbf{a}+\mathbf{b}) = L(\mathbf{a}) + L(\mathbf{b})$;

c) $L(k\mathbf{a}) = kL(\mathbf{a})$.

29.5. Let A', B', C' be images of points A, B, C under an affine transformation L. Prove that if C divides segment AB in the ratio $AC : CB = p : q$, then C' divides segment $A'B'$ in the same ratio.

29.6. Given two points O and O' in plane and two bases $\{\mathbf{e}_1, \mathbf{e}_2\}$ and $\{\mathbf{e}'_1, \mathbf{e}'_2\}$.

a) Prove that there exists a unique affine transformation that sends O into O' and a the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ into the basis $\{\mathbf{e}'_1, \mathbf{e}'_2\}$.

b) Given two triangles ABC and $A_1B_1C_1$ prove that there exists a unique affine transformation that sends A into A_1, B into B_1 and C into C_1.

c) Given two parallelograms, prove that there exists a unique affine transformation that sends one of them into another one.

29.7. Prove that if a non-identity affine transformation L sends each point of line l into itself, then all the lines of the form $ML(M)$, where M is an arbitrary point not on l, are parallel to each other.

29.8. Prove that any affine transformation can be represented as a composition of two dilations and an affine transformation that sends any triangle into a similar triangle.

29.9. Prove that any affine transformation can be represented as a composition of a dilation (contraction) and an affine transformation that sends any triangle into a similar triangle.

29.10. Prove that if an affine transformation sends a circle into itself, then it is either a rotation or a symmetry.

29.11. Prove that if M' and N' are the images of polygons M and N, respectively, under an affine transformation, then the ratio of areas of M and N is equal to the ratio of areas of M' and N'.

§2. How to solve problems with the help of affine transformations

29.12. Through every vertex of a triangle two lines are drawn. The lines divide the opposite side of the triangle into three equal parts. Prove that the diagonals connecting opposite vertices of the hexagon formed by these lines intersect at one point.

29.13. On sides AB, BC and CD of parallelogram $ABCD$ points K, L and M, respectively, are taken. The points divide the sides in the same ratio. Let b, c, d be lines passing through points B, C, D parallel to lines KL, KM, ML, respectively. Prove that lines b, c, d pass through one point.

29.14. Given triangle ABC, let O be the intersection point of its medians and M, N and P be points on sides AB, BC and CA, respectively, that divide these sides in the same ratio (i.e., $AM : MB = BN : NC = CP : PA = p : q$). Prove that:

a) O is the intersection point of the medians of triangle MNP;

b) O is the intersection point of the medians of the triangle formed by lines AN, BP and CM.

29.15. In trapezoid $ABCD$ with bases AD and BC, a line is drawn through point B parallel to side CD and intersecting diagonal AC at point P; through point C a line is drawn parallel to AB and intersecting diagonal BD at Q. Prove that PQ is parallel to the bases of the trapezoid.

29.16. In parallelogram $ABCD$, points A_1, B_1, C_1, D_1 lie on sides AB, BC, CD, DA, respectively. On sides A_1B_1, B_1C_1, C_1D_1, D_1A_1 of quadrilateral $A_1B_1C_1D_1$ points A_2, B_2, C_2, D_2, respectively, are taken. It is known that

$$\frac{AA_1}{BA_1} = \frac{BB_1}{CB_1} = \frac{CC_1}{DC_1} = \frac{DD_1}{AD} = \frac{AD_2}{D_1D_2} = \frac{D_1C_2}{C_1C_2} = \frac{C_1B_2}{B_1B_2} = \frac{B_1A_2}{A_1A_2}.$$

Prove that $A_2B_2C_2D_2$ is a parallelogram with sides parallel to the sides of $ABCD$.

29.17. On sides AB, BC and AC of triangle ABC, points M, N and P, respectively, are taken. Prove that:

a) if points M_1, N_1 and P_1 are symmetric to points M, N and P through the midpoints of the corresponding sides, then $S_{MNP} = S_{M_1N_1P_1}$.

b) if M_1, N_1 and P_1 are points on sides AC, BA and CB, respectively, such that $MM_1 \parallel BC$, $NN_1 \parallel CA$ and $PP_1 \parallel AB$, then $S_{MNP} = S_{M_1N_1P_1}$.

Solutions

29.1. We have to prove that if A', B', C' are images of points A, B, C under the dilation with respect to line l with coefficient k and point C lies on line AB, then point C' lies on line $A'B'$. Let $\overrightarrow{AC} = t\overrightarrow{AB}$. Denote by A_1, B_1, C_1 the projections of points A, B, C, respectively, on line l and let

$$\mathbf{a} = \overrightarrow{A_1A},\ \ \mathbf{b} = \overrightarrow{B_1B},\ \ \mathbf{c} = \overrightarrow{C_1C},$$

$$\mathbf{a}' = \overrightarrow{A_1A'},\ \ \mathbf{b}' = \overrightarrow{B_1B'},\ \ \mathbf{c}' = \overrightarrow{C_1C'},$$

$$\mathbf{x} = \overrightarrow{A_1B_1},\ \ \mathbf{y} = \overrightarrow{A_1C_1}.$$

Since the ratio of lengths of proportional vectors under the projection on line l is preserved, then $\mathbf{y} = t\mathbf{x}$ and $\mathbf{y} + (\mathbf{c} - \mathbf{a}) = t(\mathbf{y} + (\mathbf{b} - \mathbf{a}))$. By subtracting the first equality from the second one we get $(\mathbf{c}-\mathbf{a})=t(\mathbf{b}-\mathbf{a})$. By definition of a dilation $\mathbf{a}'=k\mathbf{a}$, $\mathbf{b}'=k\mathbf{b}$, $\mathbf{c}'=k\mathbf{c}$; hence,

$$\overrightarrow{A'C'}=\mathbf{y}+k(\mathbf{c}-\mathbf{a})=t\mathbf{x}+k(t(\mathbf{b}-\mathbf{a}))=t(\mathbf{x}+k(\mathbf{b}-\mathbf{a}))=t\overrightarrow{A'B'}.$$

29.2. By definition, the images of lines are lines and from the property of an affine transformation to be one-to-one it follows that the images of nonintersecting lines do not intersect.

29.3. Let $\overrightarrow{AB}=\overrightarrow{CD}$. First, consider the case when points A, B, C, D do not lie on one line. Then $ABCD$ is a parallelogram. The preceding problem implies that $A_1B_1C_1D_1$ is also a parallelogram; hence, $\overrightarrow{A_1B_1}=\overrightarrow{C_1D_1}$.

Now, let points A, B, C, D lie on one line. Take points E and F that do not lie on this line and such that $\overrightarrow{EF}=\overrightarrow{AB}$. Let E_1 and F_1 be their images. Then $\overrightarrow{A_1B_1}=\overrightarrow{E_1F_1}=\overrightarrow{C_1D_1}$.

29.4. a) $L(\overrightarrow{0})=L(\overrightarrow{AA})=\overrightarrow{L(A)L(A)}=\overrightarrow{0}$.

b)
$$L(\overrightarrow{AB}+\overrightarrow{BC})=L(\overrightarrow{AC})=\overrightarrow{L(A)L(C)}=\overrightarrow{L(A)L(B)}+\overrightarrow{L(B)L(C)}=\\ L(\overrightarrow{AB})+L(\overrightarrow{BC}).$$

c) First, suppose k is an integer. Then

$$L(k\mathbf{a})=L(\mathbf{a}+\dots+\mathbf{a})=L(\mathbf{a})+\dots+L(\mathbf{a})=kL(\mathbf{a}).$$

Now, let $k=\frac{m}{n}$ be a rational number. Then

$$nL(k\mathbf{a})=L(nk\mathbf{a})=L(m\mathbf{a})=mL(\mathbf{a});$$

hence,

$$L(k\mathbf{a})=\frac{mL(\mathbf{a})}{n}=kL(\mathbf{a}).$$

Finally, if k is an irrational number, then there always exists a sequence k_n ($n\in\mathbb{N}$) of rational numbers tending to k (for instance, the sequence of decimal approximations of k). Since L is continuous,

$$L(k\mathbf{a})=L(\lim_{n\to\infty}k_n\mathbf{a})=\lim_{n\to\infty}k_nL(\mathbf{a})=kL(\mathbf{a}).$$

29.5. By Problem 29.4 c) the condition $q\overrightarrow{AC}=p\overrightarrow{CB}$ implies that

$$q\overrightarrow{A'C'}=qL(\overrightarrow{AC})=L(q\overrightarrow{AC})=L(p\overrightarrow{CB})=pL(\overrightarrow{CB})=p\overrightarrow{C'B'}.$$

29.6. a) Define the map L as follows. Let X be an arbitrary point. Since $\mathbf{e}_1$, $\mathbf{e}_2$ is a basis, it follows that there exist the uniquely determined numbers x_1 and x_2 such that $\overrightarrow{OX}=x_1\mathbf{e}_1+x_2\mathbf{e}_2$. Assign to X point $X'=L(X)$ such that $\overrightarrow{O'X'}=x_1\mathbf{e}_1'+x_2\mathbf{e}_2'$. Since $\mathbf{e}_1'$, $\mathbf{e}_2'$ is also a basis, the obtained map is one-to-one. (The inverse map is similarly constructed.)

Let us prove that the image of any line AB under L is a line. Let $A'=L(A)$, $B'=L(B)$; let a_1, a_2, and b_1, b_2 be the coordinates of points A and B, respectively, in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, i.e., $\overrightarrow{OA}=a_1\mathbf{e}_1+a_2\mathbf{e}_2$, $\overrightarrow{OB}=b_1\mathbf{e}_1+b_2\mathbf{e}_2$. Let us consider an arbitrary point C on line AB. Then $\overrightarrow{AC}=k\overrightarrow{AB}$ for some k, i.e.,

$$\overrightarrow{OC}=\overrightarrow{OA}+k(\overrightarrow{OB}-\overrightarrow{OA})=\\ ((1-k)a_1+kb_1)\mathbf{e}_1+((1-k)a_2+kb_2)\mathbf{e}_2.$$

Hence, if $C' = L(C)$, then

$$\overrightarrow{O'C'} = ((1-k)a_1 + kb_1)\mathbf{e}'_1 + ((1-k)a_2 + kb_2)\mathbf{e}'_2 = \overrightarrow{O'A'} + k(\overrightarrow{O'B'} - \overrightarrow{O'A'}),$$

i.e., point C' lies on line $A'B'$.

The uniqueness of L follows from the result of Problem 29.4. Indeed, $L(\overrightarrow{OX}) = x_1L(\mathbf{e}_1) + x_2L(\mathbf{e}_2)$, i.e., the image of X is uniquely determined by the images of vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and point O.

b) To prove it, it suffices to make use of the previous heading setting $O = A$, $\mathbf{e}_1 = \overrightarrow{AB}$, $\mathbf{e}_2 = \overrightarrow{AC}$, $O' = A_1$, $\mathbf{e}'_1 = \overrightarrow{A_1B_1}$, $\mathbf{e}'_2 = \overrightarrow{A_1C_1}$.

c) Follows from heading b) and the fact that parallel lines turn into parallel lines.

29.7. Let M and N be arbitrary points not on line l. Denote by M_0 and N_0 their projections to l and by M' and N' the images of M and N under L. Lines M_0M and N_0N are parallel because both of them are perpendicular to l, i.e., there exists a number k such that $\overrightarrow{M_0M} = k\overrightarrow{N_0N}$. Then by Problem 29.4 c) $\overrightarrow{M_0M'} = k\overrightarrow{N_0N'}$. Hence, the image of triangle M_0MM' under the parallel translation by vector $\overrightarrow{M_0N_0}$ is homothetic with coefficient k to triangle N_0NN' and, therefore, lines MM' and NN' are parallel.

29.8. Since an affine map is uniquely determined by the images of vertices of any fixed triangle (see Problem 29.6 b)), it suffices to prove that with the help of two dilations one can get from any triangle an arbitrary triangle similar to any before given one, for instance, to an isosceles right triangle. Let us prove this.

Let ABC be an arbitrary triangle, BN the bisector of the outer angle $\angle B$ adjacent to side BC. Then under the dilation with respect to BN with coefficient $\frac{\tan 45^\circ}{\tan \angle CBN}$ we get from triangle ABC triangle $A'B'C'$ with right angle $\angle B'$. With the help of a dilation with respect to one of the legs of a right triangle one can always get from this triangle an isosceles right triangle.

29.9. Let L be a given affine transformation, O an arbitrary point, T the shift by vector $\overrightarrow{L(O)O}$ and $L_1 = T \circ L$. Then O is a fixed point of L_1. Among the points of the unit circle with center O, select a point A for which the vector $L(\overrightarrow{OA})$ is the longest. Let H be a rotational homothety with center O that sends point $L_1(A)$ into A and let $L_2 = H \circ L_1 = H \circ T \circ L$. Then L_2 is an affine transformation that preserves points O and A; hence, by Problem 29.4 c) it preserves all the other points of line OA and thanks to the choice of point A for all points M we have $|\overrightarrow{OM}| \geq |L(\overrightarrow{OM})|$.

Let us prove (which will imply the statement of the problem) that L_2 is a contraction with respect to line OA. If L_2 is the identity transformation, then it is a contraction with coefficient 1, so let us assume that L_2 is not the identity.

By Problem 29.9 all the lines of the form $\overrightarrow{ML_2(M)}$, where M is an arbitrary point not on OA, are parallel to each other. Let $\overrightarrow{OB}$ be the unit vector perpendicular to all these lines. Then B is a fixed point of L_2 because otherwise we would have had

$$|\overrightarrow{OL_2(B)}| = \sqrt{OB^2 + BL_2(B)^2} > |OB|.$$

If B does not lie on line OA, then by Problem 29.6 b) transformation L_2 is the identity. If B lies on OA, then all the lines of the form $ML_2(M)$ are perpendicular to the fixed line of transformation L_2. With the help of Problem 29.4 c) it is not difficult to show that the map with such a property is either a dilation or a contraction.

29.10. First, let us prove that an affine transformation L that sends a given circle into itself sends diametrically opposite points into diametrically opposite ones. To this end let us notice that the tangent to the circle at point A turns into the line that, thanks to the property of L to be one-to-one, intersects with the circle at a (uniquely determined) point $L(A)$, i.e., is the tangent at point $L(A)$. Therefore, if the tangents at points A and B are parallel to each other (i.e., AB is a diameter), then the tangents at points $L(A)$ and $L(B)$ are also parallel, i.e., $L(A)L(B)$ is also a diameter.

Fix a diameter AB of the given circle. Since $L(A)L(B)$ is also a diameter, there exists a movement P of the plane which is either a rotation or a symmetry that sends A and B into $L(A)$ and $L(B)$, respectively, and each of the arcs α and β into which points A and B divide the given circle into the image of these arcs under L.

Let us prove that the map $F = P^{-1} \circ L$ is the identity. Indeed, $F(A) = A$ and $F(B) = B$; hence, all points of line AB are fixed. Hence, if X is an arbitrary point of the circle, then the tangent at X intersects line AB at the same place where the tangent at point $X' = F(X)$ does because the intersection point is fixed. Since X and X' lie on one and the same of the two arcs α or β, it follows that X coincides with X'. Thus, $P^{-1} \circ L = E$, i.e., $L = P$.

29.11. Let a_1 and a_2 be two perpendicular lines. Since an affine transformation preserves the ratio of the lengths of (the segments of the) parallel lines, the lengths of all the segments parallel to one line are multiplied by the same coefficient. Denote by k_1 and k_2 these coefficients for lines a_1 and a_2. Let φ be the angle between the images of these lines. Let us prove that the given affine transformation multiplies the areas of all polygons by k, where $k = k_1k_2 \sin\varphi$.

For rectangles with sides parallel to a_1 and a_2 and also for a right triangle with legs parallel to a_1 and a_2 the statement is obvious. Any other triangle can be obtained by cuttting off the rectangle with sides parallel to a_1 and a_2 several right triangles with legs parallel to a_1 and a_2 as shown on Fig. 118 and, finally, by Problem 22.22 any polygon can be cut into triangles.

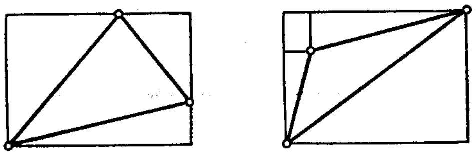


Figure 260 (Sol. 29.11)

29.12. Since an affine transformation sends an arbitrary triangle into an equilateral one (Problem 29.6 b)), the ratio of lengths of parallel segments are preserved (Problem 29.5). It suffices to prove the statement of the problem for an equilateral triangle ABC. Let points A_1, A_2, B_1, B_2, C_1, C_2 divide the sides of the triangle into equal parts and A', B', C' be the midpoints of the sides (Fig. 119). Under the symmetry through AA' line BB_1 turns into CC_2 and BB_2 into CC_1. Since symmetric lines intersect on the axis of symmetry, AA' contains a diagonal of the considered hexagon. Similarly, the remaining diagonals lie on BB' and CC'. It is clear that the medians AA', BB', CC' intersect at one point.

29.13. Problem 29.6 b) implies that an affine transformation sends an arbitrary parallelogram into a square. Since this preserves the ratio of lengths of parallel segments (Problem 29.5), it suffices to prove the statement of the problem for the case when $ABCD$ is a square. Denote by P the intersection point of lines b and d. It suffices to prove that $PC \parallel MK$. Segment KL turns under the rotation through the angle of 90° about the center of square $ABCD$ into LM, hence, lines b and d which are parallel to these respective segments are perpendicular; hence, P lies on the circle circumscribed about $ABCD$. Then $\angle CPD = \angle CBD = 45°$.

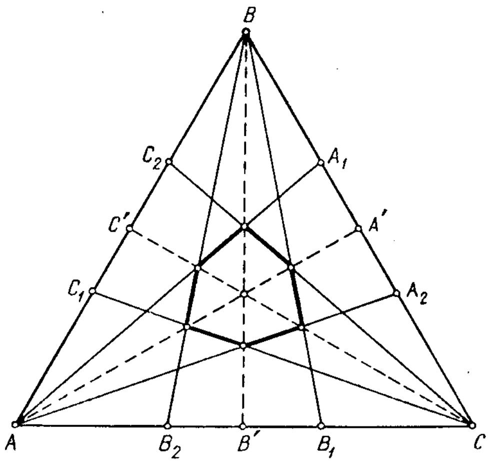


Figure 161 (Sol. 29.12)

Therefore, the angle between lines CP and b is equal to 45° but the angle between lines MK and KL is also equal to 45° and $b \parallel KL$ implying $CP \parallel MK$.

29.14. a) Let us consider an affine transformation that sends triangle ABC into a equilateral triangle $A'B'C'$. Let O', M', N', P' be the images of points O, M, N, P. Under the rotation through the angle of 120° about point O' triangle $M'N'P'$ turns into itself and, therefore, this triangle is a equilateral one and O' is the intersection point of its medians. Since under an affine transformation any median turns into a median, O is the intersection point of the medians of triangle MNP.

b) Solution is similar to the solution of heading a).

29.15. Let us consider an affine transformation that sends $ABCD$ into an isosceles trapezoid $A'B'C'D'$. For such a transformation one can take the affine transformation that sends triangle ADE, where E is the intersection point of AB and CD, into an isosceles triangle. Then the symmetry through the midperpendicular to $A'D'$ sends point P' into point Q', i.e., lines $P'Q'$ and $A'D'$ are parallel.

29.16. Any parallelogram $ABCD$ can be translated by an affine transformation into a square (for this we only have to transform triangle ABC into an isosceles right triangle). Since the problem only deals with parallel lines and ratios of segments that lie on one line, we may assume that $ABCD$ is a square. Let us consider a rotation through an angle of 90° sending $ABCD$ into itself. This rotation sends quadrilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ into themselves; hence, the quadrilaterals are also squares. We also have

$$\tan\angle BA_1B_1 = BB_1 : BA_1 = A_1D_2 : A_1A_2 = \tan\angle A_1A_2D_2,$$

i.e., $AB \parallel A_2D_2$ (Fig. 120).

29.17. a) Since an affine transformation sends any triangle into a equilateral one, the midpoints of the sides into the midpoints, the centrally symmetric points into centrally symmetric and triangles of the same area into triangles of the same area (Problem 29.11), it follows that we can assume that triangle ABC is an equilateral one with side a. Denote the lengths of segments AM, BN, CP by p, q, r, respectively. Then

$$S_{ABC} - S_{MNP} = S_{AMP} + S_{BMN} + S_{CNP} =$$
$$\frac{1}{2}\sin 60^\circ \cdot (p(a-r) + q(a-p) + r(a-q)) = \frac{1}{2}\sin 60^\circ \cdot (a(p+q+r) - (pq+qr+rp)).$$

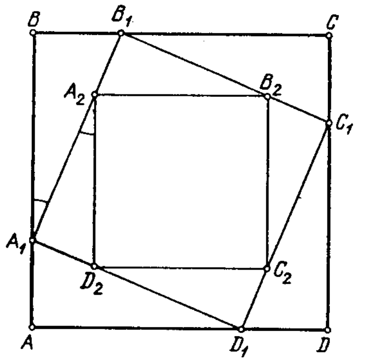


Figure 262 (Sol. 29.16)

Similarly,

$$S_{ABC} - S_{M_1N_1P_1} = \frac{1}{2}\sin 60^\circ \cdot (r(a-p) + p(a-q) + q(a-r)) =$$
$$\frac{1}{2}\sin 60^\circ \cdot (a(p+q+r) - (pq+qr+rp)).$$

b) By the same reasons as in heading a) let us assume that ABC is an equilateral triangle. Let $M_2N_2P_2$ be the image of triangle $M_1N_1P_1$ under the rotation about the center of triangle ABC through the angle of 120° in the direction from A to B (Fig. 121).

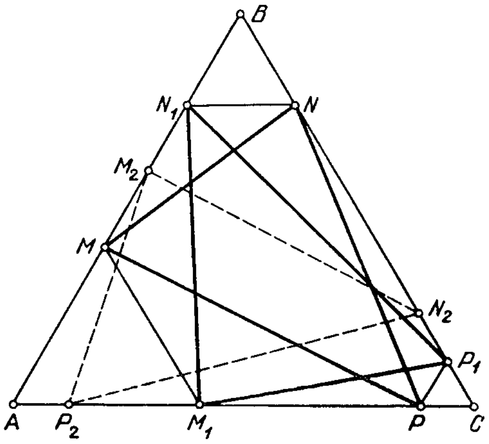


Figure 263 (Sol. 29.17)

Then $AM_2 = CM_1 = BM$. Similarly, $BN_2 = CN$ and $CP_2 = AP$, i.e., points M_2, N_2, P_2 are symmetric to points M, N, P through the midpoints of the corresponding sides. Therefore, this heading is reduced to heading a).

Chapter 30. PROJECTIVE TRANSFORMATIONS

§1. Projective transformations of the line

1. Let l_1 and l_2 be two lines on the plane, O a point that does not lie on any of these lines. The *central projection* of line l_1 to line l_2 with center O is the map that to point A_1 on line l_1 assigns the intersection point of lines OA_1 and l_2.

2. Let l_1 and l_2 be two lines on the plane, l a line not parallel to either of the lines. The *parallel projection* of l_1 to l_2 along l is the map that to point A_1 of line l_1 assigns the intersection point of l_2 with the line passing through A_1 parallel to l.

3. A map P of line a to line b is called a *projective* one if it is the composition of central or parallel projections, i.e., if there exist lines $a_0 = a$, $a_1, \dots, a_n = b$ and maps P_i of the line a_i to a_{i+1} each of which is either a central or a parallel projection and P is the composition of the maps P_i in some order. If b coincides with a, then P is called a *projective transformation* of line a.

30.1. Prove that there exists a projective transformation that sends three given points on one line into three given points on another line.

The *cross ratio* of a quadruple of points A, B, C, D lying on one line is the number

$$(ABCD) = \frac{c-a}{c-b} : \frac{d-a}{d-b},$$

where a, b, c, d are the coordinates of points A, B, C, D, respectively. It is easy to verify that the cross ratio does not depend on the choice of the coordinate system on the line. We will also write

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD}$$

in the sence that $\frac{AC}{BC}$ (resp. $\frac{AD}{BD}$) denotes the ratio of the lengths of these segments, if vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ (resp. $\overrightarrow{AD}$ and $\overrightarrow{BD}$) are similarly directed or the ratio of the lengths of these segments taken with minus sign, if these vectors are pointed in the opposite directions.

The *double ratio* of the quadruple of lines a, b, c, d passing through one point is the number

$$(abcd) = \pm\frac{\sin(a,c)}{\sin(b,c)} : \frac{\sin(a,d)}{\sin(b,d)}$$

whose sign is determined as follows: if one of the angles formed by lines a and b does not intersect with one of the lines c or d (in this case we say that *the pair of lines a and b does not divide the pair of lines c and d*) then $(abcd) > 0$; otherwise $(abcd) < 0$.

30.2. a) Given lines a, b, c, d passing through one point and line l that does not pass through this point. Let A, B, C, D be intersection points of l with lines a, b, c, d, respectively. Prove that $(abcd) = (ABCD)$.

b) Prove that the double ratio of the quadruple of points is preserved under projective transformations.

30.3. Prove that if $(ABCX) = (ABCY)$, then $X = Y$ (all points are assumed to be pairwise distinct except, perhaps, points X and Y, and lie on one line).

30.4. Prove that any projective transformation of the line is uniquely determined by the image of three arbitrary points.

30.5. Prove that any non-identity projective transformation of the line has not more than two fixed points.

30.6. A map sends line a into line b and preserves the double ratio of any quadruple of points. Prove that this map is a projective one.

30.7. Prove that transformation P of the real line is projective if and only if it can be represented in the form

$$P(x) = \frac{ax+b}{cx+d},$$

where a, b, c, d are numbers such that $ad-bc \neq 0$. (Such maps are called *fractionally-linear* ones.)

30.8. Points A, B, C, D lie on one line. Prove that if $(ABCD)=1$, then either $A=B$ or $C=D$.

30.9. Given line l, a circle and points M, N that lie on the circle and do not lie on l. Consider map P of line l to itself,; let P be the composition of the projection of l to the given circle from point M and the projection of the circle to l from point N. (If point X lies on line l, then $P(X)$ is the intersection of line NY with line l, where Y is the distinct from M intersection point of line MX with the given circle.) Prove that P is a projective transformation.

30.10. Given line l, a circle and point M that lies on the circle and does not lie on l, let P_M be the projection map of l to the given circle from point M (point X of line l is mapped into the distinct from M intersection point of line XM with the circle), R the movement of the plane that preserves the given circle (i.e., a rotation of the plane about the center of the circle or the symmetry through a diameter). Prove that the composition $P_M^{-1} \circ R \circ P_M$ is a projective transformation.

Remark. If we assume that the given circle is identified with line l via a projection map from point M, then the statement of the problem can be reformulated as follows: the map of a circle to itself with the help of a movement of the plane is a projective transformation of the line.

§2. Projective transformations of the plane

Let α_1 and α_2 be two planes in space, O a point that does not belong to any of these planes. The *central projection map* of α_1 to α_2 with center O is the map that to point A_1 of plane α_1 assigns the intersection point of OA_1 with plane α_2.

30.11. Prove that if planes α_1 and α_2 intersect, then the central projection map of α_1 to α_2 with center O determines a one-to-one correspondence of plane α_1 with deleted line l_1 onto plane α_2 with deleted line l_2, where l_1 and l_2 are the intersection lines of planes α_1 and α_2, respectively, with planes passing through O and parallel to α_1 and α_2. On l_1, the map is not defined.

A line on which the central projection map is not defined is called the singular line of the given projection map.

30.12. Prove that under a central projection a nonsingular line is projected to a line.

In order to define a central projection everywhere it is convenient to assume that in addition to ordinary points every line has one more so-called infinite point sometimes denoted by ∞. If two points are parallel, then we assume that their infinite points coincide; in other words, parallel lines intersect at their infinite point.

We will also assume that on every plane in addition to ordinary lines there is one more, infinite line, which hosts all the infinite points of the lines of the plane. The infinite line intersects with every ordinary line l lying in the same plane in the infinite point of l.

If we introduce infinite points and lines, then the central projection map of plane α_1 to plane α_2 with center at point O is defined through(?) points of α_1 and the singular line is mapped into the infinite line of α_2, namely, the image of point M of the singular line is the infinite point of line OM; this is the point at which the lines of plane α_2 parallel to OM intersect.

30.13. Prove that if together with the usual (finite) points and lines we consider infinite ones, then

a) through any two points only one line passes;

b) any two lines lying in one plane intersect at one point;

c) a central projection map of one plane to another one is a one-to-one correspondence.

A map P of plane α to plane β is called a projective *one if it is the composition of central projections and affine transformations, i.e., if there exist planes $\alpha_0 = \alpha$, $\alpha_1, \dots, \alpha_n = \beta$ and maps P_i of plane α_i to α_{i+1} each of which is either a central projection or an affine transformation and P is the composition of the P_i. If plane α coincides with β, map P is called a* projective transformation *of α. The preimage of the infinite line will be called the* singular line *of the given projective transformation.*

30.14. a) Prove that a projective transformation P of the plane sending the infinite line into the infinite line is an affine transformation.

b) Prove that if points A, B, C, D lie on a line parallel to the singular line of a projective transformation P of plane α, then $P(A)P(B) : P(C)P(D) = AB : CD$.

c) Prove that if a projective transformation P sends parallel lines l_1 and l_2 into parallel lines, then either P is affine or its singular line is parallel to l_1 and l_2.

d) Let P be a one-to-one transformation of the set of all finite and infinite points of the plane, let P send every line into a line. Prove that P is a projective map.

30.15. Given points A, B, C, D no three of which lie on one line and points A_1, B_1, C_1, D_1 with the same property.

a) Prove that there exists a projective transformation sending points A, B, C, D to points A_1, B_1, C_1, D_1, respectively.

b) Prove that the transformation from heading a) is unique, i.e., any projective transformation of the plane is determined by the images of four generic points (cf. Problem 30.4).

c) Prove statement of heading a) if points A, B, C lie on one line l and points A_1, B_1, C_1, D_1 on one line l_1.

d) Is transformation from heading c) unique?

In space, consider the unit sphere with center in the origin. Let $N(0, 0, 1)$ be the sphere's north pole. The *stereographic projection* of the sphere to the plane is the map that to every point M of the sphere assigns distinct from N intersection point of line MN with plane Oxy. It is known (see, for example, Solid Problem 16.19 b)) that the stereographic projection sends a circle on the sphere into a circle in plane. Make use of this fact while solving the following two problems:

30.16. Givin a circle and a point inside it.

a) Prove that there exists a projective transformation that sends the given circle into a circle and the given point into the center of the given circle's image.

b) Prove that if a projective transformation sends the given circle into a circle and point M into the center of the given circle's image, then the singular line of this transformation is perpendicular to a diameter through M.

30.17. In plane, there are given a circle and a line that does not intersect the circle. Prove that there exists a projective transformation sending the given circle into a circle and the given line into the infinite line.

30.18. Given a circle and a chord in it. Prove that there exists a projective transformation that sends the given circle into a circle and the given chord into the diameter of the given circle's image.

30.19. Given circle S and point O inside it, consider all the projective maps that send S into a circle and O into the center of the image of S. Prove that all such transformations map one and the same line into the infinite line.

The preimage of the infinite line under the above transformations is called *the polar line* of point O relative circle S.

30.20. A projective transformation sends a circle into itself so that its center is fixed. Prove that this transformation is either a rotation or a symmetry.

30.21. Given point O and two parallel lines a and b. For every point M we perform the following construction. Through M draw a line l not passing through O and intersecting lines a and b. Denote the intersection points pf l with a and b by A and B, respectively, and let M' be the intersection point of OM with the line parallel to OB and passing through A.

a) Prove that point M' does not depend on the choice of line l.

b) Prove that the transformation of the plane sending M into M' is a projective one.

30.22. Prove that the transformation of the coordinate plane that every point with coordinates (x, y) sends into the point with coordinates $(\frac{1}{x}, \frac{y}{x})$ is a projective one.

(?)**30.23.** Let O be the center of a lens, π a plane passing through the optic axis a of the lens, a and f the intersection lines of π with the plane of the lens and the focal plane, respectively, ($a \parallel f$). In the school course of physics it is shown that if we neglect the lens, then the image M' of point M that lies in plane π is constructed as follows, see Fig. 122.

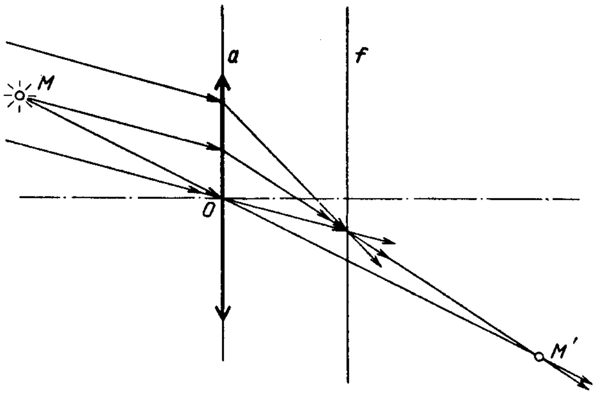


Figure 264 (30.23)

Through point M draw an arbitrary line l; let A be the intersection point of lines a and l, let B be the intersection point of f with the line passing through O parallel to l. Then M' is defined as the intersection point of lines AB and OM.

Prove that the transformation of plane π assigning to every of its points its image is a projective one.

Thus, through a magnifying glass we can see the image of our world thanks to projective transformations.

§3. Let us transform the given line into the infinite one

30.24. Prove that the locus of the intersection points of quadrilaterals $ABCD$ whose sides AB and CD belong to two given lines l_1 and l_2 and sides BC and AD intersect at a given point P is a line passing through the intersection point Q of lines l_1 and l_2.

30.25. Let O be the intersection point of the diagonals of quadrilateral $ABCD$; let E (resp. F) be the intersection point of the continuations of sides AB and CD (resp. BC and AD). Line EO intersects sides AD and BC at points K and L, respectively, and line FO intersects sides AB and CD at points M and N, respectively. Prove that the intersection point X of lines KN and LM lies on line EF.

30.26. Lines a, b, c intersect at one point O. In triangles $A_1B_1C_1$ and $A_2B_2C_2$, vertices A_1 and A_2 lie on line a; B_1 and B_2 lie on line b; C_1 and C_2 lie on line c. Let A, B, C be the intersection points of lines B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, A_1B_1 and A_2B_2, respectively. Prove that points A, B, C lie on one line (*Desargue's theorem.*)

30.27. Points A, B, C lie on line l and points A_1, B_1, C_1 on line l_1. Prove that the intersection points of lines AB_1 and BA_1, BC_1 and CB_1, CA_1 and AC_1 lie on one line (*Pappus's theorem.*)

30.28. Given convex quadrilateral $ABCD$. Let P, Q be the intersection points of the continuations of the opposite sides AB and CD, AD and BC, respectively, R an arbitrary point inside the quadrilateral. Let K, L, M be the intersection point of lines BC and PR, AB and QR, AK and DR, respectively. Prove that points L, M and C lie on one line.

30.29. Given two triangles ABC and $A_1B_1C_1$ so that lines AA_1, BB_1 and CC_1 intersect at one point O and lines AB_1, BC_1 and CA_1 intersect at one point O_1. Prove that lines AC_1, BA_1 and CB_1 also intersect at one point O_2. (*Theorem on doubly perspective triangles.*)

30.30. Given two triangles ABC and $A_1B_1C_1$ so that lines AA_1, BB_1 and CC_1 intersect at one point O, lines AA_1, BC_1 and CB_1 intersect at one point O_1 and lines AC_1, BB_1 and CA_1 intersect at one point O_2, prove that lines AB_1, BA_1 and CC_1 also intersect at one point O_3. (*Theorem on triply perspective triangles.*)

30.31. Prove that the orthocenters of four triangles formed by four lines lie on one line.

30.32. Given quadrilateral $ABCD$ and line l. Denote by P, Q, R the intersection points of lines AB and CD, AC and BD, BC and AD, respectively. Denote by P_1, Q_1, R_1 the midpoints of the segments which these pairs of lines cut off line l. Prove that lines PP_1, QQ_1 and RR_1 intersect at one point.

30.33. Given triangle ABC and line l. Denote by A_1, B_1, C_1 the midpoints of the segments cut off line l by angles $\angle A$, $\angle B$, $\angle C$ and by A_2, B_2, C_2 the intersection points of lines AA_1 and BC, BB_1 and AC, CC_1 and AB, respectively. Prove that points A_2, B_2, C_2 lie on one line.

30.34. (*Theorem on a complete quadrilateral.*) Given four points A, B, C, D and the intersection points P, Q, R of lines AB and CD, AD and BC, AC and BD, respectively; the intersection points K and L of line QR with lines AB and CD, respectively. Prove that $(QRKL) = -1$.

30.35. Is it possible to paint 1991 points of the plane red and 1991 points blue so that any line passing through two points of distinct colour contains one more of coloured points? (We assume that coloured points are distinct and do not belong to one line.)

§4. Application of projective maps that preserve a circle

The main tools in the solution of problems of this section are the results of Problems 30.16 and 30.17.

30.36. Prove that the lines that connect the opposite tangent points of a circumscribed quadrilateral pass through the intersection point of the diagonals of this quadrilateral.

30.37. Consider a triangle and the inscribed circle. Prove that the lines that connect the triangle's vertices with the tangent points of the opposite sides intersect at one point.

30.38. a) Through point P all secants of circle S are drawn. Find the locus of the intersection points of the tangents to S drawn through the two intersection points of S with every secant.

b) Through point P the secants AB and CD of circle S are drawn, where A, B, C, D are the intersection points of the secants with the circle. Find the locus of the intersection points of AC and BD.

30.39. Given circle S, line l, point M on S and not on l and point O not on S. Consider a map P of line l which is the composition of the projection map of l to S from M, of S to itself from O and S to l from M, i.e., for any point A point $P(A)$ is the intersection point of lines l and MC, where C is the distinct from B intersection point of S with line OB and B is the distinct from A intersection point of S with line MA. Prove that P is a projective map.

Remark. If we assume that a projection map from point M identifies circle S with line l, then the statement of the problem can be reformulated as follows: every central projection of a circle to itself is a projective transformation.

30.40. Consider disk S, point P outside S and line l passing through P and intersecting the circle at points A and B. Denote the intersection point of the tangents to the disk at points A and B by K.

a) Consider all the lines passing through P and intersecting AK and BK at points M and N, respectively. Prove that the locus of the tangents to S drawn through M and N and distinct from AK and BK is a line passing through K and having the empty intersection with the interior of S.

b) Let us select various points R on the circle and draw the line that connects the distinct from R intersection points of lines RK and RP with S. Prove that all the obtained lines pass through one point and this point belongs to l.

30.41. An escribed circle of triangle ABC is tangent to side BC at point D and to the extensions of sides AB and AC at points E and F, respectively. Let T be the intersection point of lines BF and CE. Prove that points A, D and T lie on one line.

30.42. Let $ABCDEF$ be a circumscribed hexagon. Prove that its diagonals AD, BE and CF intersect at one point. (*Brianchon's theorem.*)

30.43. Hexagon $ABCDEF$ is inscribed in circle S. Prove that the intersection points of lines AB and DE, BC and EF, CD and FA lie on one line. (*Pascal's theorem.*)

30.44. Let O be the midpoint of chord AB of circle S, let MN and PQ be arbitrary chords through O such that points P and N lie on one side of AB; let E and F be the intersection points of chord AB with chords MP and NQ, respectively. Prove that O is the midpoint of segment EF. (*The butterfly problem.*)

30.45. Points A, B, C and D lie on a circle, SA and SD are tangents to this circle, P and Q are the intersection points of lines AB and CD, AC and BD, respectively. Prove that points P, Q and S lie on line line.

§5. Application of projective transformations of the line

30.46. On side AB of quadrilateral $ABCD$ point M_1 is taken. Let M_2 be the projection of M_1 to line BC from D, let M_3 be the projection of M_2 to CD from A, M_4 the projection of M_3 on DA from B, M_5 the projection of M_4 to AB from C, etc. Prove that $M_{13} = M_1$ (hence, $M_{14} = M_2$, $M_{15} = M_3$, etc.).

30.47. Making use of projective transformations of the line prove the theorem on a complete quadrilateral (Problem 30.34).

30.48. Making use of projective transformations of the line prove Pappus's theorem (Problem 30.27).

30.49. Making use of projective transformations of the line prove the butterfly problem (Problem 30.44).

30.50. Points A, B, C, D, E, F lie on one circle. Prove that the intersection points of lines AB and DE, BC and EF, CD and FA lie on one line. (*Pascal's theorem.*)

30.51. Given triangle ABC and point T, let P and Q be the bases of perpendiculars dropped from point T to lines AB and AC, respectively; let R and S be the bases of perpendiculars dropped from point A to lines TC and TB, respectively. Prove that the intersection point X of lines PR and QS lies on line BC.

§6. Application of projective transformations of the line in problems on construction

30.52. Given a circle, a line, and points A, A', B, B', C, C', M on this line. By Problems 30.1 and 30.3 there exists a unique projective transformation of the given line to itself that maps points A, B, C into A', B', C', respectively. Denote this transformation by P. Construct with the help of a ruler only a) point $P(M)$; b) fixed points of map P. (*J. Steiner's problem.*)

The problem of constructing fixed points of a projective transformation is the key one for this section in the sense that all the other problems can be reduced to it, cf. also remarks after Problems 30.10 and 30.39.

30.53. Given two lines l_1 and l_2, two points A and B not on these lines, and point E of line l_2. Construct with a ruler and compass point X on l_1 such that lines AX and BX intercept on line l_2 a segment a) of given length a; b) divisible in halves by E.

30.54. Points A and B lie on lines a and b, respectively, and point P does not lie on any of these lines. With the help of a ruler and compass draw through P a line that intersects lines a and b at points X and Y, respectively, so that the lengths of segments AX and BY a) are of given ratio; b) have a given product.

30.55. With the help of a ruler and compass draw through a given point a line on which three given lines intercept equal segments.

30.56. Consider a circle S, two chords AB and CD on it, and point E of chord CD. Construct with a ruler and compass point X on S so that lines AX and BX intercept on CD a segment a) of given length a; b) divided in halves by E.

30.57. a) Given line l, point P outside it, a given length, and a given angle α. Construct with a ruler and compass segment XY on l of the given length and subtending an angle of value α and with vertex in P.

b) Given two lines l_1 and l_2, points P and Q outside them, and given angles α and β. Construct with the help of a ruler and compass point X on l_1 and point Y on l_2 such that segment XY subtends an angle of value α with vertex in P and another angle equal to β with vertex in Q.

30.58. a) Given a circle, n points and n lines. Construct with the help of a ruler only an n-gon whose sides pass through the given points and whose vertices lie on the given lines.

b) With the help of ruler only inscribe in the given circle an n-gon whose sides pass through n given points.

c) With the help of a ruler and compass inscribe in a given circle a polygon certain sides of which pass through the given points, certain other sides are parallel to the given lines and the remaining sides are of prescribed lengths (about each side we have an information of one of the above three types).

§7. Impossibility of construction with the help of a ruler only

30.59. Prove that with the help of a ruler only it is impossible to divide a given segment in halves.

30.60. Given a circle on the plane, prove that its center is impossible to construct with the help of a ruler only.

Solutions

30.1. Denote the given lines by l_0 and l, the given points on l_0 by A_0, B_0, C_0 and the given points on l by A, B, C. Let l_1 be an arbitrary line not passing through A. Take an arbitrary point O_0 not on lines l_0 and l_1. Denote by P_0 the central projection map of l_0 to l_1 with center at O_0 and by A_1, B_1, C_1 the projections of points A_0, B_0, C_0, respectively, under P_0.

Let l_2 be an arbitrary line through point A not coinciding with l and not passing through A_1. Take point O_1 on line AA_1 and consider the central projection map P_1 of l_1 to l_2 with center at O_1. Denote by A_2, B_2, C_2 the projections of points A_1, B_1, C_1, respectively, under P_1. Clearly, A_2 coincides with A.

Finally, let P_2 be the projection map of l_2 to l which in the case when lines BB_2 and CC_2 are not parallel is the central projection with center at the intersection point of these lines; if lines BB_2 and CC_2 are parallel this is the parallel projection along either of these lines.

The composition $P_2 \circ P_1 \circ P_0$ is the required projective transformation.

30.2. a) Denote the intersection point of the four given lines by O; let H be the projection of H on l and $h = OH$. Then

$$\begin{aligned} 2S_{OAC} &= OA \cdot OC \sin(a,c) = h \cdot AC, \\ 2S_{OBC} &= OB \cdot OC \sin(b,c) = h \cdot BC, \\ 2S_{OAD} &= OA \cdot OD \sin(a,d) = h \cdot AD, \\ 2S_{OBD} &= OB \cdot OD \sin(b,d) = h \cdot BD. \end{aligned}$$

Dividing the first equality by the second one and the third one by the fourth one we get

$$\frac{OA\sin(a,c)}{OB\sin(b,c)} = \frac{AC}{BC}, \qquad \frac{OA\sin(a,d)}{OB\sin(b,d)} = \frac{AD}{BD}.$$

Dividing the first of the obtained equalities by the second one we get $|(ABCD)| = |(abcd)|$. To prove that the numbers $(ABCD)$ and $(abcd)$ are of the same sign, we can, for example, write down all the possible ways to arrange points on the line (24 ways altogether) and verify case by case that $(ABCD)$ is positive if and only if the pair of lines a, b does not separate the pair of lines c, d.

b) follows immediately from heading a).

30.3. Let a, b, c, x, y be the coordinates of points A, B, C, X, Y, respectively. Then

$$\frac{x-a}{x-b} : \frac{c-a}{c-b} = \frac{y-a}{y-b} : \frac{c-a}{c-b}.$$

Therefore, since all the points are distinct, $(x-a)(y-b) = (x-b)(y-a)$. By simplifying we get $ax - bx = ay - by$. Dividing this equality by $a - b$ we get $x = y$.

30.4. Let the image of each of the three given points under one projective transformation coincide with the image of this point under another projective transformation. Let us prove then that the images of any other point under these transformations coincide. Let us denote the images of the given points by A, B, C. Take an arbitrary point and denote by X and Y its images under the given projective transformations. Then by Problem 30.2 $(ABCX) = (ABCY)$ and, therefore, $X = Y$ by Problem 30.3.

30.5. This problem is a corollary of the preceding one.

30.6. On line a, fix three distinct points. By Problem 30.1 there exists a projective map P which maps these points in the same way as the given map. But in the solution of Problem 30.4 we actually proved that any map that preserves the cross ratio is uniquely determined by the images of three points. Therefore, the given map coincides with P.

30.7. First, let us show that the fractionally linear transformation

$$P(x) = \frac{ax+b}{cx+d}, \quad ad - bc \neq 0$$

preserves the cross ratio. Indeed, let x_1, x_2, x_3, x_4 be arbitrary numbers and $y_i = P(x_i)$. Then

$$y_i - y_j = \frac{ax_i + b}{cx_i + d} - \frac{ax_j + b}{cx_j + d} = \frac{(ad-bc)(x_i - x_j)}{(cx_i + d)(cx_j + d)};$$

hence, $(y_1y_2y_3y_4) = (x_1x_2x_3x_4)$.

In the solution of Problem 30.4 we have actually proved that if a transformation of the line preserves the cross ratio, then it is uniquely determined by the images of three arbitrary distinct points. By Problem 30.2 b) projective transformations preserve the cross ratio. It remains to prove that for any two triples of pairwise distinct points x_1, x_2, x_3 and y_1, y_2, y_3 there exists a fractionally linear transformation P such that $P(x_i) = y_i$.

For this, in turn, it suffices to prove that for any three pairwise distinct points there exists a fractionally linear transformation that sends them into points $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$.

Indeed, if P_1 and P_2 be fractionally linear transformations such that $P_1(x_i) = z_i$ and $P_2(y_i) = z_i$, then $P_2^{-1}(P_1(x_i)) = y_i$. The inverse to a fractionally linear transformation is a fractionally linear transformation itself because if $y = \frac{ax+c}{cx+d}$, then $x = \frac{dy-b}{-cy+a}$; the verification of the fact that the composition of fractionally linear transformations is a fractionally linear transformation is left for the reader.

Thus, we have to prove that if x_1, x_2, x_3 are arbitrary distinct numbers, then there exist numbers a, b, c, d such that $ad - bc \neq 0$ and

$$ax_1 + b = 0, \quad ax_2 + b = cx_2 + d, \quad cx_3 + d = 0.$$

Find b and d from the first and third equations and substitute the result into the third one; we get

$$a(x_2 - x_1) = c(x_2 - x_3)$$

wherefrom we find the solution: $a = (x_2 - x_3)$, $b = x_1(x_3 - x_2)$, $c = (x_2 - x_1)$, $d = x_3(x_1 - x_2)$. We, clearly, have $ad - bc = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \neq 0$.

30.8. First solution. Let a, b, c, d be the coordinates of the given points. Then by the hypothesis $(c-a)(d-b) = (c-b)(d-a)$. After simplification we get $cb + ad = ca + bd$.

Transfer everything to the left-hand side and factorize; we get $(d-c)(b-a)=0$, i.e., either $a=b$ or $c=d$.

Second solution. Suppose that $C \neq D$, let us prove that in this case $A=B$. Consider the central projection map of the given line to another line, let the projection send point D into ∞. Let A', B', C' be the projections of points A, B, C, respectively. By Problem 30.2 $(ABCD)=(A'B'C'\infty)=1$, i.e., $\overrightarrow{AC}=\overrightarrow{BC}$. But this means that $A=B$.

30.9. By Problem 30.6 it suffices to prove that the map P preserves the cross ratio. Let A, B, C, D be arbitrary points on line l. Denote by A', B', C', D' their respective images under P and by a, b, c, d and a', b', c', d' the lines MA, MB, MC, MD and NA', NB', NC', ND', respectively. Then by Problem 30.2 a) we have $(ABCD)=(abcd)$ and $(A'B'C'D')=(a'b'c'd')$ and by the theorem on an inscribed angle $\angle(a,c)=\angle(a',c')$, $\angle(b,c)=\angle(b',c')$, etc.; hence, $(abcd)=(a'b'c'd')$.

30.10. Let $N=R^{-1}(M)$, $m=R(l)$, P_N be the projection map of l to the circle from point N, Q the projection map of line m to l from point M. Then $P_M^{-1}\circ R\circ P_M = Q\circ R\circ P_N^{-1}\circ P_M$. But by the preceding problem the map $P_N^{-1}\circ P_M$ is a projective one.

30.11. Lines passing through O and parallel to plane α_1 (resp. α_2) intersect plane α_2 (resp. α_1) at points of line l_2 (resp. l_1). Therefore, if a point lies on one of the planes α_1, α_2 and does not lie on lines l_1, l_2, then its projection to another plane is well-defined. Clearly, the distinct points have distinct images.

30.12. The central projection to plane α_2 with center O sends line l into the intersection of the plane passing through O and l with α_2.

30.13. This problem is a direct corollary of the axioms of geometry and the definition of infinite lines and points.

30.14. a) Problem 30.13 c) implies that if together with the ordinary (finite) points we consider infinite ones, then P is a one-to-one correspondence. Under such an assumption the infinite line is mapped to the infinite line. Therefore, the set of finite points is also mapped one-to-one to the set of finite points. Since P sends lines into lines, P is an affine map.

b) Denote by l the line on which points A, B, C, D lie and by l_0 the singular line of map P. Take an arbitrary point O outside plane α and consider plane β that passes through line l and is parallel to the plane passing through line l_0 and point O. Let Q be the composition of the central projection of α on β with center O with the subsequent rotation of the space about axis l that sends β into α. The singular line of map Q is l_0.

Therefore, the projective transformation $R=P\circ Q^{-1}$ of α sends the infinite line into the infinite line and by heading a) is an affine transformation, in particular, it preserves the ratio of segments that lie on line l. It only remains to notice that transformation Q preserves the points of line l.

c) The fact that the images of parallel lines l_1 and l_2 are parallel lines means that the infinite point A of these lines turns into an infinite point, i.e., A lies on the preimage l of the infinite line. Therefore, either l is the infinite line and then by heading a) P is an affine transformation or l is parallel to lines l_1 and l_2.

d) Denote by l_∞ the infinite line. If $P(l_\infty)=l_\infty$, then P determines a one-to-one transformation of the plane that sends every line into a line and, therefore, by definition is an affine one.

Otherwise denote $P(l_\infty)$ by a and consider an arbitrary projective transformation Q for which a is the singular line. Denote $Q\circ P$ by R. Then $R(l_\infty)=l_\infty$ and, therefore, as was shown above, R is an affine map. Hence, $P=Q^{-1}\circ R$ is a projective map.

30.15. a) It suffices to prove that points A, B, C, D can be transformed by a projective transformation into vertices of a square. Let E and F be (perhaps, infinite) intersection points of line AB with line CD and BC with AD, respectively. If line EF is not infinite, then there exists a central projection of plane ABC to a plane α for which EF is the singular line. For the center of projection one may take an arbitrary point O outside plane ABC and for plane α an arbitrary plane parallel to plane OEF and not coinciding with it. This projection maps points A, B, C, D into the vertices of a parallelogram which can be now transformed into a square with the help of an affine transformation.

If line EF is an infinite one, then $ABCD$ is already a parallelogram.

b) Thanks to heading a) it suffices to consider the case when $ABCD$ and $A_1B_1C_1D_1$ is one and the same parallelogram. In this case its vertices are fixed and, therefore, two points on an infinite line in which the extensions of the opposite sides of the parallelogram intersect are also fixed. Hence, by Problem 30.14 a) the map should be an affine one and, therefore, by Problem 20.6 the identity one.

c) Since with the help of a projection we can send lines l and l_1 into the infinit line (see the solution of heading a)), it suffices to prove that there exists an affine transformation that maps every point O into a given point O_1 and lines parallel to given lines a, b, c into lines parallel to given lines a_1, b_1, c_1, respectively.

We may assume that lines a, b, c pass through O and lines a_1, b_1, c_1 pass through O_1. On c and c_1, select arbitrary points C and C_1, respectively, and draw through each of them two lines a', b' and a_1', b_1' parallel to lines a, b and a_1, b_1, respectively. Then the affine transformation that sends the parallelogram bounded by lines a, a', b, b' into the parallelogram bounded by lines a_1, a_1', b_1, b_1' (see Problem 29.6 c)) is the desired one.

d) Not necessarily. The transformation from Problem 30.21 (as well as the identity transformation) preserves point O and line a.

30.16. a) On the coordinate plane Oxz consider points $O(0,0)$, $N(0,1)$, $E(1,0)$. For an arbitrary point M that lies on arc $\smile NE$ of the unit circle (see Fig. 123), denote by P the midpoint of segment EM and by M^* and P^* the intersection points of lines NM and NP, respectively, with line OE.

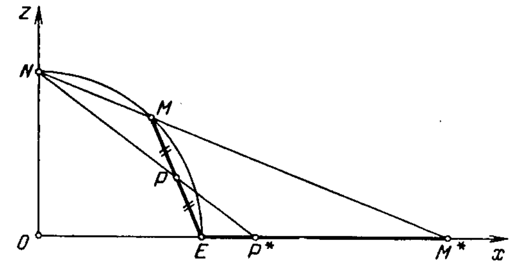


Figure 265 (Sol. 30.16)

Let us prove that for an arbitrary number $k > 2$ we can select point M soy that $M^*E : P^*E = k$. Let $A(a, b)$ be an arbitrary point on the plane, $A^*(t, 0)$ the intersection point of lines NA and OE, $B(0, b)$ the projection of point A to line ON. Then

$$t = \frac{A^*O}{ON} = \frac{AB}{BN} = \frac{a}{1-b}.$$

Therefore, if (x, z) are coordinates of point M, then points P, M^*, P^* have coordinates

$$P\left(\frac{x+1}{2}, \frac{z}{2}\right), \quad M^*\left(\frac{x}{1-z}, 0\right), \quad P^*\left(\frac{(x+1)/2}{1-(z/2)}, 0\right),$$

respectively, and, therefore,

$$M^*E : P^*E = \left(\frac{x}{1-z} - 1\right) : \left(\frac{x+1}{2-z} - 1\right) = \frac{x+z-1}{1-z} : \frac{x+z-1}{2-z} = \frac{2-z}{1-z}.$$

Clearly, the solution of the equation $\frac{2-z}{1-z} = k$ is $z = \frac{k-2}{k-1}$ and, if $k > 2$, then $0 < z < 1$ and, therefore, point $M(\sqrt{1-z^2}, z)$ is the desired one.

Now, let us prove the main statement of the problem. Denote the given circle and point inside it, respectively, by S and C. If point C is the center of S, then the identity transformation is the desired projective transformation.

Therefore, let us assume that C is not the center. Denote by AB the diameter that contains point C. Let, for definiteness, $BC > CA$. Set $k = BA : AC$. Then $k > 2$ and, therefore, as was proved, we can place point M on the unit circle in plane Oxz so that $M^*E : P^*E = k = BA : CA$. Therefore, by a similarity transformation we can translate circle S into a circle S_1 constructed in plane Oxy with segment EM^* as a diameter so that the images of points A, B, C are E, M^*, P^*, respectively.

The stereographic projection maps S_1 into circle S_2 on the unit sphere symmetric through plane Oxz; hence, through line EM as well. Thus, EM is a diameter of S_2 and the midpoint P of EM is the center of S_2.

Let α be the plane containing circle S_2. Clearly, the central projection of plane Oxy to plane α from the north pole of the unit sphere sends S_1 into S_2 and point P^* into the center P of S_2.

b) The diameter AB passing through M turns into a diameter. Therefore, the tangents at points A and B turn into tangents. But if the parallel lines pass into parallel lines, then the singular line is parallel to them (see Problem 30.14 c)).

30.17. On the coordinate plane Oxz consider points $O(0,0)$, $N(0,1)$, $E(1,0)$. For an arbitrary point M on arc $\smile NE$ of the unit circle denote by P the intersection of segment EM with line $z = 1$. Clearly, by moving point M along arc NE we can make the ratio $EM : MP$ equal to an arbitrary number. Therefore, a similarity transformation can send the given circle S into circle S_1 constructed on segment EM as on diameter in plane α perpendicular to Oxz so that the given line l turns into the line passing through P perpendicularly to Oxz. Circle S_1 lies on the unit sphere with the center at the origin and, therefore, the stereographic projection sends S_1 to circle S_2 in plane Oxy. Thus, the central projection of plane α to plane Oxy from N sends S_1 to S_2 and line l into the infinite line.

30.18. Let M be an arbitrary point on the given chord. By Problem 30.16 there exists a projective transformation that sends the given circle into a circle S and point M into the center of S. Since under a projective transformation a line turns into a line, the given chord will turn into a diameter.

30.19. Let us pass through point O two arbitrary chords AC and BD. Let P and Q be the intersection points of the extensions of opposite sides of quadrilateral $ABCD$. Consider an arbitrary projective transformation that maps S into a circle, S_1, and O into the center of S_1. It is clear that this transformation sends quadrilateral $ABCD$ into a rectangle and, therefore, it sends line PQ into the infinite line.

30.20. A projective transformation sends any line into a line and since the center is fixed, every diameter turns into a diameter. Therefore, every infinite point — the intersection point of the lines tangent to the circle in diametrically opposite points — turns into an infinite point. Therefore, by Problem 30.14 a) the given transformation is an affine one and by Problem 29.12 it is either a rotation or a symmetry.

30.21. a) Point M' lies on line OM and, therefore, its position is uniquely determined by the ratio $MO : OM'$. But since triangles MBO and MAM' are similar, $MO : OM' =$

$MB : BA$ and the latter relation does not depend on the choice of line l due to Thales' theorem.

b) **First solution.** If we extend the given transformation (let us denote it by P) by defining it at point O setting $P(O) = O$, then, as is easy to verify, P determines a one-to-one transformation of the set of all finite and infinite points of the plane into itself. (In order to construct point M from point M' we have to take an arbitrary point A on line a and draw lines AM', OB so that it is parallel to AM', and AB.) It is clear that every line passing through O turns into itself. Every line l not passing through O turns into the line parallel to OB and passing through M. Now, it only remains to make use of Problem 30.14 c).

Second solution (sketch). Denote the given plane by π and let $\pi' = R(\pi)$, where R is a rotation of the space about axis a. Denote $R(O)$ by O' and let P be the projection map of plane π to plane π' from the intersection point of line OO' with the plane passing through b parallel to π'. Then $R^{-1} \circ P$ coincides (prove it on your own) with the transformation mentioned in the formulation of the problem.

30.22. First solution. Denote the given transformation by P. Let us extend it to points of the line $x = 0$ and infinite points by setting $P(0, k) = M_k$, $P(M_k) = (0, k)$, where M_k is an infinite point on the line $y = kx$. It is easy to see that the map P extended in this way is a one-to-one correspondence.

Let us prove that under P every line turns into a line. Indeed, the line $x = 0$ and the infinite line turn into each other. Let $ax + by + c = 0$ be an arbitrary other line (i.e., either b or c is nonzero). Since $P \circ P = E$, the image of any line coincides with its preimage. Clearly, point $P(x, y)$ lies on the considered line if and only if $\frac{a}{x} + \frac{by}{x} + c = 0$, i.e., $cx + by + a = 0$. It remains to make use of Problem 30.14 d).

Second solution (sketch). Denote lines $x = 1$ and $x = 0$ by a and b, respectively, and point $(-1, 0)$ by O. Then the given transformation coincides with the transformation from the preceding problem.

30.23. If we denote line f by b, then the transformation mentioned in this problem is the inverse to the transformation of Problem 30.21.

30.24. Consider a projective transformation for which line PQ is the singular one. The images l_1' and l_2' of lines l_1 and l_2 under this transformation are parallel and the images of the considered quadrilaterals are parallelograms two sides of which lie on lines l_1' and l_2' and the other two sides are parallel to a fixed line (the infinite point of this line is the image of point P). It is clear that the locus of the intersection points of the diagonals of such parallelograms is the line equidistant from l_1' and l_2'.

30.25. Let us make a projective transformation whose singular line is EF. Then quadrilateral $ABCD$ turns into a parallelogram and lines KL and MN into lines parallel to the sides of the parallelogram and passing through the intersection point of its diagonals, i.e., into the midlines. Therefore, the images of points K, L, M, N are the midpoints of the parallelogram and, therefore, the images of lines KN and LM are parallel, i.e., point X turns into an infinite point and, therefore, X lies on the singular line EF.

30.26. Let us make the projective transformation with singular line AB. The images of points under this transformation will be denoted by primed letters. Let us consider a homothety with center at point O' (or a parallel translation if O' is an infinite point) that sends C_1' to C_2'. Under this homothety segment $B_1'C_1'$ turns into segment $B_2'C_2'$ because $B_1'C_1' \parallel B_2'C_2'$. Similarly, $C_1'A_1'$ turns to $C_2'A_2'$. Therefore, the corresponding sides of triangles $A_1'B_1'C_1'$ and $A_2'B_2'C_2'$ are parallel, i.e., all three points A', B', C' lie on the infinite line.

30.27. Let us consider the projective transformation whose singular line passes through the intersection points of lines AB_1 and BA_1, BC_1 and CB_1 and denote by A', B', ... the images of points A, B, ... Then $A'B'_1 \parallel B'A'_1$, $B'C'_1 \parallel C'B'_1$ and we have to prove that $C'A'_1 \parallel A'C'_1$ (see Problem 1.12 a)).

30.28. As a result of the projective transformation with singular line PQ the problem is reduced to Problem 4.54.

30.29. This problem is a reformulation of the preceding one. Indeed, suppose that the pair of lines OO_1 and OB separates the pair of lines OA and OC and the pair of lines OO_1 and O_1B separates the pair of lines O_1A and O_1C (consider on your own in a similar way the remaining ways of disposition of these lines). Therefore, if we redenote points A_1, B, B_1, C_1, O, O_1 and the intersection point of lines AB_1 and CC_1 by D, R, L, K, Q, P and B, respectively, then the preceding problem implies that the needed lines pass through point M.

30.30. Let us consider the projective transformation with singular line O_1O_2 and denote by A', B', ... the images of points A, B, ... Then $A'C'_1 \parallel C'_1A'_1 \parallel B'B'_1$, $B'C'_1 \parallel C'B'_1 \parallel A'A'_1$. Let us, for definiteness sake, assume that point C lies inside angle $\angle A'O'B'$ (the remaining cases can be reduced to this one after a renotation). Making, if necessary, an affine transformation we can assume that the parallelogram $O'A'C'_1B'$ is a square and, therefore, $O'A'_1C'B'_1$ is also a square and the diagonals $O'C'_1$ and $O'C'$ of these squares lie on one line. It remains to make use of the symmetry through this line.

30.31. It suffices to prove that the orthocenters of each triple of triangles formed by the given lines lie on one line. Select some three triangles. It is easy to see that one of the given lines (denoted by l) is such that one of the sides of each of the chosen triangles lies on l. Denote the remaining lines by a, b, c and let A, B, C, respectively, be their intersection points with l.

Denote by l_1 the infinite line and by A_1 (resp. B_1, C_1) the infinite points of the lines perpendicular to a (resp. b, c). Then the fact that the orthocenters of the three selected triangles lie on one line is a direct corollary of Pappus's theorem (Problem 30.27).

30.32. Perform a projective transformation with singular line parallel to l and passing through the intersection point of lines PP_1 and QQ_1; next, perform an affine transformation that makes the images of lines l and PP_1 perpendicular to each other. We may assume that lines PP_1 and QQ_1 are perpendicular to line l and our problem is to prove that line RR_1 is also perpendicular to l (points P_1, Q_1, R_1 are the midpoints of the corresponding segments because these segments are parallel to the singular line; see Problem 30.14 b)). Segment PP_1 is both a median and a hight, hence, a bisector in the triangle formed by lines l, AB and CD.

Similarly, QQ_1 is a bisector in the triangle formed by lines l, AC and BD. This and the fact that $PP_1 \parallel QQ_1$ imply that $\angle BAC = \angle BDC$. It follows that quadrilateral $ABCD$ is an inscribed one and $\angle ADB = \angle ACB$. Denote the points at which l intersects lines AC and BD by M and N, respectively (Fig. 124). Then the angle between l and AD is equal to $\angle ADB - \angle QNM = \angle ACB - \angle QMN$, i.e., it is equal to the angle between l and BC. It follows that the triangle bounded by lines l, AD and BC is an isosceles one and segment RR_1 which is its median is also its hight, i.e., it is perpendicular to line l, as required.

30.33. Perform a projective transformation with singular line parallel to l and passing through point A. We may assume that point A is infinite, i.e., lines AB and AC are parallel. Then by Problem 30.14 b) points A_1, B_1, C_1 are, as earlier, the midpoints of the corresponding segments because these segments lie on the line parallel to the singular one. Two triangles formed by lines l, AB, BC and l, AC, BC are homothetic and, therefore, lines BB_1 and CC_1, which are medians of these triangles, are parallel. Therefore, quadrilateral

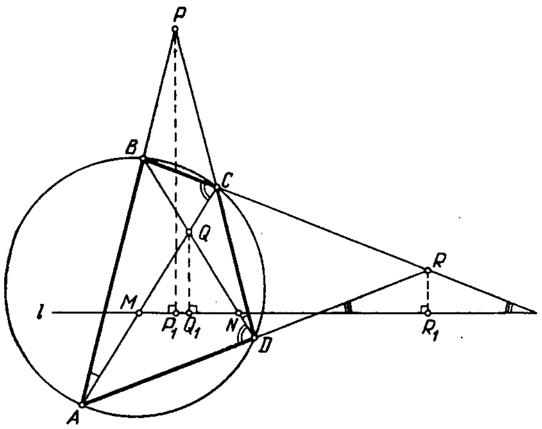


Figure 266 (Sol. 30.32)

BB_2CC_2 is a parallelogram because its opposite sides are parallel. It remains to notice that point A_2 is the midpoint of diagonal BC of this parallelogram and, therefore, it is also the midpoint of diagonal B_2C_2.

30.34. Let us make the projective transformation whose singular line is line PQ. Denote by A', B', ... the images of points A, B, ... Then $A'B'C'D'$ is a parallelogram, R' the intersection point of its diagonals, Q' is the infinite point of line $Q'R'$, K' and L' the intersection points of the sides of the parallelogram on line $Q'R'$. Clearly, points K' and L' are symmetric through point R'. Hence,

$$(Q'R'K'L') = \frac{Q'K'}{Q'L'} : \frac{R'K'}{R'L'} = 1 : \frac{R'K'}{R'L'} = -1.$$

It remains to notice that $(QRKL) = (Q'R'K'L')$ by Problem 30.2 b).

30.35. Answer: It is possible. Indeed, consider the vertices of a regular 1991-gon (red points) and points at which the extensions of the sides of this polygon intersect the infinite line (blue points). This set of points has the required properties. Indeed, for any regular n-gon, where n is odd, the line passing through its vertex and parallel to one of the sides passes through one more vertex. Any given finite set of points can be transformed by a projective transformation into a set of finite (i.e., not infinite) points.

30.36. Let us make a projective transformation that sends the circle inscribed into the quadrilateral into a circle S and the intersection point of the lines connecting the opposite tangent points into the center of S, cf. Problem 30.16 a). The statement of the problem now follows from the fact that the obtained quadrilateral is symmetric with respect to the center of S.

30.37. Let us make a projective transformation that sends the inscribed circle into a circle S and the intersection point of two of the three lines under consideration into the center of S, cf. Problem 30.16 a). Then the images of these two lines are simultaneously bisectors and hights of the image of the given triangle and, therefore, this triangle is an equilateral one. For an equilateral triangle the statement of the problem is obvious.

30.38. Let us consider, separately, the following two cases.

1) Point P lies outside S. Let us make the projective transformation that sends circle S into circle S_1 and point P into ∞ (see Problem 30.17), i.e., the images of all lines passing through P are parallel to each other. Then in heading b) the image of the locus to be found is line l, their common perpendicular passing through the center of S_1, and in heading a) the line l with the diameter of S_1 deleted.

To prove this, we have to make use of the symmetry through line l. Therefore, the locus itself is: in heading b), the line passing through the tangent points of S with the lines drawn through point P and in heading a), the part of this line lying outside S.

2) Point P lies inside S. Let us make a projective transformation that sends circle S into circle S_1 and point P into its center, cf. Problem 30.16 a). Then the image of the locus to be found in both headings is the infinite line. Therefore, the locus itself is a line.

The obtained line coincides for both headings with the polar line of point P relative to S, cf. Problem 30.19.

30.39. Denote by m the line which is the locus to be found in Problem 30.38 b) and by N the distinct from M intersection point of S with line OM. Denote by Q the composition of the projection of l to S from M and S to M from N. By Problem 30.9 this composition is a projective map.

Let us prove that P is the composition of Q with the projection of m to l from M. Let A be an arbitrary point on l, B its projection to S from M, C the projection of B to S from O, D the intersection point of lines BN and CM. By Problem 30.38 b) point D lies on line m, i.e., $D = Q(A)$. Clearly, $P(A)$ is the projection of D to l from M.

30.40. Both headings of the problem become obvious after a projective transformation that sends circle S into a circle and line KP into the infinite line, cf. Problem 30.17. The answer is as follows:

a) The locus to be found lies on the line equidistant from the images of lines AK and BK.

b) The point to be found is the center of the image of S.

30.41. Let A', B', ... be the images of points A, B, ... under the projective transformation that sends an escribed circle of triangle ABC into circle S, and chord EF into a diameter of S (see Problem 30.18). Then A' is the infinite point of lines perpendicular to diameter $E'F'$ and we have to prove that line $D'T'$ contains this point, i.e., is also perpendicular to $E'F'$.

Since $\triangle T'B'E' \sim \triangle T'F'C'$, it follows that $C'T' : T'E' = C'F' : B'E'$. But $C'D' = C'F'$ and $B'D' = B'E'$ as tangents drawn from one point; hence, $C'T' : T'E' = C'D' : D'B'$, i.e., $D'T' \parallel B'E'$.

30.42. By Problem 30.16 a) it suffices to consider the case when diagonals AD and BE pass through the center of the circle. It remains to make use of the result of Problem 6.83 for $n = 3$.

30.43. Consider the projective transformation that sends circle S into a circle and the intersection points of lines AB and DE, BC and EF into infinite points (see Problem 29.17). Our problem is reduced to Problem 2.11.

30.44. Consider a projective transformation that sends circle S into circle S_1 and point O into the center O' of S_1, cf. Problem 30.16 a). Let A', B', ... be the images of points A, B, ... Then $A'B'$, $M'N'$ and $P'Q'$ are diameters. Therefore, the central symmetry through O' sends point E' into F', i.e., O' is the midpoint of segment $E'F'$. Since chord AB is perpendicular to the diameter passing through O, Problem 30.16 b) implies that AB is parallel to the singular line. Therefore, by Problem 30.14 b) the ratio of the lengths of the segments that lie on line AB is preserved and, therefore, O is the midpoint of segment EF.

30.45. Let us consider the projective transformation that maps the given circle into circle S' and segment AD into a diameter of S' (see Problem 30.18). Let A', B', ... be the images of A, B, ... Then S turns into the infinite point S' of lines perpendicular to line $A'D'$. But $A'C'$ and $B'D'$ are hights in $\triangle A'D'P'$ and, therefore, Q' is the orthocenter of this triangle. Therefore, line $P'Q'$ is also a hight; hence, it passes through point S'.

30.46. By Problem 30.15 it suffices to consider only the case when $ABCD$ is a square. We have to prove that the composition of projections described in the formulation of the problem is the identity transformation. By Problem 30.4 a projective transformation is the identity if it has three distinct fixed points. It is not difficult to verify that points A, B and the infinite point of line AB are fixed for this composition.

30.47. Under the projection of line QR from point A to line CD points Q, R, K, L are mapped into points D, C, P, L, respectively. Therefore, by Problem 30.2 b) $(QRKL) = (DCPL)$. Similarly, by projecting line CD to line QR from point B we get $(DCPL) = (RQKL)$; hence, $(QRKL) = (RQKL)$. On the other hand,

$$(RQKL) = \frac{RK}{RL} : \frac{QK}{QL} = \left(\frac{QK}{QL} : \frac{RK}{RL}\right)^{-1} = (QRKL)^{-1}.$$

These two equalities imply that $(QRKL)^2 = 1$, i.e., either $(QRKL) = 1$ or $(QRKL) = -1$. But by Problem 30.8 the cross ratio of distinct points cannot be equal to one.

30.48. Denote the intersection points of lines AB_1 and BA_1, BC_1 and CB_1, CA_1 and AC_1 by P, Q, R, respectively, and the intersection point of lines PQ and CA_1 by R_1. We have to prove that points R and R_1 coincide. Let D be the intersection point of AB_1 and CA_1. Let us consider the composition of projections: of line CA_1 to line l_1 from point A, of l_1 to CB_1 from B, and of CB_1 to CA_1 from P. It is easy to see that the obtained projective transformation of line CA_1 fixes points C, D and A_1 and sends R into R_1. But by Problem 30.5 a projective transformation with three distinct fixed points is the identity one. Hence, $R_1 = R$.

30.49. Let F' be the point symmetric to F through O. We have to prove that $F' = F$. By Problem 30.9 the composition of the projection of line AB to circle S from point M followed by the projection of S back to AB from Q is a projective transformation of line AB. Consider the composition of this transformation with the symmetry through point O. This composition sends points A, B, O, E to B, A, F', O, respectively. Therefore, by Problem 30.2 b)

$$(ABOE) = (BAF'O).$$

On the other hand, it is clear that

$$(BAF'O) = \frac{BF'}{AF'} : \frac{BO}{AO} = \frac{AO}{BO} : \frac{AF'}{BF'} = (ABOF')$$

i.e., $(ABOE) = (ABOF')$; hence, by Problem 30.3, $E = F'$.

(?)**30.50.** Denote the intersection points of lines AB and DE, BC and EF, CD and FA by P, Q, R, respectively, and the intersection point of lines PQ and CD by R'. We have to prove that points R and R' coincide. Let G be the intersection point of AB and CD. Denote the composition of the projection of line CD on the given circle from point A with the projection of circle \$ to line BC from point E.

By Problem 30.9 this composition is a projective map. It is easy to see that its composition with the projection of BC to CD from point P fixes points C, D and G and sends point R to R'. But by Problem 30.5 a projective transformation with three fixed points is the identity one. Hence, $R' = R$.

30.51. Since angles $\angle APT$, $\angle ART$, $\angle AST$ and $\angle AQT$ are right ones, points A, P, R, T, S, Q lie on the circle constructed on segment AT as on diameter. Hence, by Pascal's theorem (Problem 30.50) points B, C and X lie on one line.

30.52. Denote the given line and circle by l and S, respectively. Let O be an arbitrary point of the given circle and let A_1, A_1', B_1, B_1', C_1, C_1' be the images of points A, A', B, B',

C, C' under the projection map of l to S from point O, i.e., A_1 (resp. A'_1, B_1, ...) is the distinct from O intersection point of line AO (resp. $A'O$, BO, ...) with circle S.

Denote by B_2 the intersection point of lines A'_1B_1 and $A_1B'_1$ and by C_2 the intersection point of lines A'_1C_1 and $A_1C'_1$. Let P_1 be the composition of the projection of line l to circle S from point O with the projection of S to line B_2C_2 from point A'_1; let P_2 be the composition of the projection of B_2C_2 to S from point A_1 with the projection of S to l from point O. Then by Problem 30.9 transformations P_1 and P_2 are projective ones and their composition sends points A, B, C to A', B', C', respectively.

It is clear that all the considered points can be constructed with the help of a ruler (in the same order as they were introduced).

a) Let M_1 be the distinct from O intersection point of line MO with circle S; $M_2 = P_1(M)$ the intersection point of lines A'_1M_1 and B_2C_2; M_3 the distinct from A_1 intersection point of line M_2A_1 with circle S; $P(M) = P_2(P_1(M))$ the intersection point of lines l and OM_3.

b) Let M_1 and N_1 be the intersection points of circle S with line B_2C_2. Then the fixed points of transformation P are the intersection points of lines OM_1 and ON_1 with line l.

30.53. a) The point X to be found is the fixed point of the composition of the projection of l_1 to l_2 from point A, the translation along line l_2 at distance a and the projection of l_2 to l_1 from point B. The fixed point of this projective map is constructed in Problem 30.52.

b) Replace the shift from the solution of heading a) with the central symmetry with respect to E.

30.54. a) Denote by k the number to which the ratio $\frac{AX}{BY}$ should be equal to. Consider the projective transformation of line a which is the composition of the projection of a to line b from point P, the movement of the plane that sends b to a and B to A and, finally, the homothety with center A and coefficient k. The required point X is the fixed point of this transformation. The construction of point Y is obvious.

b) Denote by k the number to which the product $AX \cdot BY$ should equal to and by Q the intersection point of the lines passing through points A and B parallel to lines b and a, respectively; let $p = AQ \cdot BQ$. Consider the projective transformation of line a which is the composition of the projection of a to line b from point P, projection of b to a from Q and the homothety with center A and coefficient $\frac{k}{p}$.

Let X be the fixed point of this transformation, Y its image under the first projection and X_1 the image of Y under the second projection. Let us prove that line XY is the desired one. Indeed, since $\triangle AQX_1 \sim \triangle BYQ$, it follows that

$$AX_1 \cdot BY = AQ \cdot BQ = p$$

and, therefore,

$$AX \cdot BY = \frac{k}{p} \cdot AX_1 \cdot BY = k.$$

30.55. Let P be the given point; A, B, C the points of pairwise intersections of the given lines a, b, c; let X, Y, Z be the intersection points of the given lines with line l to be found (Fig. 125).

By the hypothesis $XZ = ZY$. Let T be the intersection point of line c with the line passing through X parallel to b. Clearly, $XT = AY$. Since $\triangle XTB \sim \triangle CAB$, it follows that $XB : XT = CB : CA$ which implies $BX : YA = CB : CA$, i.e., the ratio $BX : YA$ is known. Thus, our problem is reduced to Problem 30.54 a).

30.56. a) By Problem 30.9 the composition of the projection of CD on S from A with the projection of S on CD from B is a projective transformation of line CD. Let M be a fixed point of the composition of this transformation with the shift along line CD by

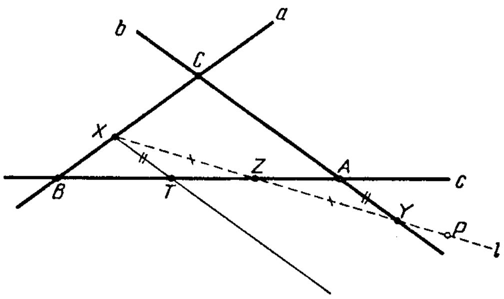


Figure 267 (Sol. 30.55)

distance a. Then the projection of M on S from A is the desired point. The fixed point of any projective transformation is constructed in Problem 30.52.

b) In the solution of heading a) replace the shift by the central symmetry through E.

30.57. a) Let us draw an arbitrary circle S through point P. By Problem 30.10 the composition of the projection of l to S from P, the rotation about the center of S through an angle of 2α and the projection of S to l from P is a projective transformation of line l. Then (by the theorem on an escribed angle) the fixed point of the composition of this transformation with the shift along line CD by given distance XY is the desired point. The fixed point of any projective transformation is constructed in Problem 30.52.

b) Let us construct arbitrary circles S_1 and S_2 passing through points P and Q, respectively. Consider the composition of projection of l_1 to S_1 from P, the rotation about the center of S_1 through an angle of 2α and the projection of S_1 to l_2 from P. By Problem 30.10 this composition is a projective map. Similarly, the composition of the projection of l_2 to S_2 from Q, the rotation about the center of S_2 through an angle of 2β and projection of S_2 to l_1 from Q is also a projective map. By the theorem on an escribed angle the fixed point of the composition of these maps is the desired point X and in order to construct it we can make use of Problem 30.52.

30.58. a) Denote the given points by $M_1, \dots, M_n$ and the given lines by $l_1, \dots, l_n$. A vertex of the polygon to be found is the fixed point of the projective transformation of line l_1 which is the composition of projections of l_1 to l_2 from M_1, l_2 to l_3 from M_2, $\dots$, l_n to l_1 from M_n. The fixed point of a projective transformation is constructed in Problem 30.52.

b) Select an arbitrary point on a given circle and with the help of projection from the given point let us identify the given circle with line l. By Problem 30.39 the central projecting of the circle to itself is a projective transformation of line l under this identification. Clearly, a vertex of the desired polygon is the fixed point of the composition of consecutive projections of the given circle to itself from given points. The fixed point of a projective transformation is constructed in Problem 30.52.

c) In the solution of heading b) certain central projections should be replaced by either rotations about the center of the circle if the corresponding side is of the given length or by symmetries if the corresponding side has the prescribed direction (the axis of the symmetry should be the diameter perpendicular to the given direction).

30.59. Suppose that we managed to find the required construction, i.e., to write an instruction the result of fulfilment of which is always the midpoint of the given segment. Let us perform this construction and consider the projective transformation that fixes the endpoints of the given segment and sends the midpoint to some other point. We can select this transformation so that the singular line would not pass through neither of the points obtained in the course of intermediate constructions.

Let us perform our imaginary procedure once again but now every time that we will encounter in the instruction words “take an arbitrary point (resp. line)” we shall take the image of the point (resp. line) that was taken in the course of the first construction.

Since a projective transformation sends any line into a line and the intersection of lines into the intersection of their images and due to the choice of the projective transformation this intersection is always a finite point, it follows that at each step of the second construction we obtain the image of the result of the first construction and, therefore, we will finally get not the midpoint of the interval but its image instead. Contradiction.

REMARK. We have, actually, proved the following statement: *if there exists a projective transformation that sends each of the objects $A_1, \dots, A_n$ into themselves and does not send an object B into itself, then it is impossible to construct object B from objects $A_1, \dots, A_n$ with the help of a ruler only.*

30.60. The statement of the problem follows directly from Remark 30.59 above and from Problem 30.16 a).

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