$(\infty, 1)$-Comprehension Schemes

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Overview

1. Preliminary notions in $(\infty, 1)$-category theory
2. $(\infty, 1)$-Comprehension schemes:
   2.1 Motivation, definitions and some fundamental instances
   2.2 Examples and applications
3. A model categorical point of view

Preliminaries and notation

In all of the following we abbreviate $\infty := (\infty, 1)$.
Let $\mathcal{S}$ be the ∞-category of spaces.
Let Cat_∞ be the ∞-category of (small) ∞-categories.

Definition
Let $\mathcal{C}$ be an ∞-category. A $\mathcal{C}$-indexed ∞-category is a functor

$$\mathcal{I}\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty.$$

The ∞-category of $\mathcal{C}$-indexed ∞-categories is given by

$$\mathsf{IC}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty).$$

Likewise, a $\mathcal{C}$-indexed ∞-groupoid is a functor $X\colon \mathcal{C}^{op} \to \mathcal{S}$.
The ∞-category of $\mathcal{C}$-indexed ∞-groupoids is the ∞-category $\hat{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ of presheaves on $\mathcal{C}$.

Preliminaries and notation
Basic examples of indexed ∞-categories

Let $\mathcal{C}$ be an ∞-category.

- The naive indexing over Cat_∞ and over $\mathcal{S}$, respectively,

$$\mathsf{Fun}(\_\,,\mathcal{C})\colon \mathrm{Cat}_\infty^{op} \to \mathrm{Cat}_\infty,$$

$$\mathrm{Fun}(\_\,,\mathcal{C})\colon \mathcal{S}^{op} \to \mathrm{Cat}_\infty.$$

  Their underlying ∞-category is $\mathcal{C}$ itself.

Preliminaries and notation
Basic examples of indexed ∞-categories

Let $\mathcal{C}$ be an ∞-category.

- If $\mathcal{C}$ has pullbacks, the slice functor

$$\mathcal{C}_{/\_} : \mathcal{C}^{op} \to \mathrm{Cat}_\infty$$

yields the canonical indexing of $\mathcal{C}$ over itself.
Again, its underlying ∞-category is $\mathcal{C}$ whenever $\mathcal{C}$ is left exact.
When applied to $\mathcal{C} \simeq \mathcal{S}$, the two indexed ∞-categories

$$\mathrm{Fun}(\_, \mathcal{S}) \colon \mathcal{S}^{op} \to \mathrm{Cat}_\infty$$

and

$$\mathcal{S}_{/\_} : \mathcal{S}^{op} \to \mathrm{Cat}_\infty$$

are pointwise equivalent.

Let $\mathcal{C}$ be an ∞-category.

- Given a $\mathcal{C}$-indexed ∞-category $\mathcal{I}\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$, each functor $F\colon \mathcal{D} \to \mathcal{C}$ induces a change of base

$$F^*\mathcal{I}\colon \mathcal{D}^{op} \to \mathrm{Cat}_\infty.$$

- The representable presheaves $\mathcal{C}(\_\,, C)\colon \mathcal{C}^{op} \to \mathcal{S}$.

Preliminaries and notation
(Un)Straightening

Given an ∞-category $\mathcal{C}$, there are adjoint equivalences

$$\mathrm{St}\colon \mathsf{Cart}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}} \mathsf{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty)\colon \mathrm{Un},$$

$$\mathrm{St}\colon \mathsf{RFib}(\mathcal{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}} \mathsf{Fun}(\mathcal{C}^{op}, \mathcal{S})\colon \mathrm{Un}.$$

Definition
A right fibration over $\mathcal{C}$ is representable if its associated presheaf via Straightening is representable.

Up to equivalence, the representable fibrations over $\mathcal{C}$ are exactly the right fibrations of the form

$$s\colon \mathcal{C}_{/C} \twoheadrightarrow \mathcal{C}$$

for objects C in $\mathcal{C}$.

Definition
Let $(\cdot)^{\simeq} \colon \mathrm{Cat}_\infty \to \mathcal{S}$ be the functor assigning to each ∞-category its core, i.e. $\mathcal{C}^{\simeq}$ is the largest ∞-groupoid contained in $\mathcal{C}$.
Given a cartesian fibration $p \colon \mathcal{E} \twoheadrightarrow \mathcal{C}$, we obtain a presheaf

$$\mathcal{C}^{op} \xrightarrow{\mathrm{St}(p)} \mathrm{Cat}_\infty \xrightarrow{(\cdot)^{\simeq}} \mathcal{S}$$

whose unstraightening we denote by $p^{\times} \colon \mathcal{E}^{\times} \twoheadrightarrow \mathcal{C}$ and call the core of p.

Preliminaries and notation
Basic examples revisited

Let $\mathcal{C}$ be an ∞-category.

- The representable

$$s\colon (\mathrm{Cat}_\infty)_{/\mathcal{C}} \twoheadrightarrow \mathrm{Cat}_\infty$$

is the Unstraightening of the representable preasheaf

$$\mathrm{Cat}_\infty(\_\,,\mathcal{C}) = \mathrm{Fun}(\_\,,\mathcal{C})^{\simeq}\colon \mathrm{Cat}_\infty^{op} \to \mathcal{S}.$$

Similarly, the forgetful functor

$$s\colon \mathcal{S}_{/\mathcal{C}} \twoheadrightarrow \mathcal{S}$$

is the unstraightening of the restriction

$$\mathrm{Cat}_\infty(\_\,,\mathcal{C})\colon \mathcal{S}^{op} \to \mathcal{S}.$$

More generally . . .

Preliminaries and notation
Basic examples revisited

Let $\mathcal{C}$ be an ∞-category.

- The Unstraightening of the naive indexing

$$\mathrm{Fun}(\_\,,\mathcal{C})\colon \mathcal{S}^{op} \to \mathrm{Cat}_\infty$$

of $\mathcal{C}$, denoted in type theoretic fashion by

$$\sum_{\mathcal{D}:\mathcal{S}} \mathrm{Fun}(\mathcal{D},\mathcal{C}) \twoheadrightarrow \mathcal{S}$$

is the ∞-category of families of $\mathcal{C}$.

In the same way,

$$\mathrm{Fun}(\_\,,\mathcal{C})\colon \mathrm{Cat}_\infty^{op} \to \mathrm{Cat}_\infty$$

yields the Unstraightening

$$\sum_{\mathcal{D}:\mathrm{Cat}_\infty} \mathrm{Fun}(\mathcal{D},\mathcal{C}) \twoheadrightarrow \mathrm{Cat}_\infty.$$

Let $\mathcal{C}$ be an ∞-category.

- By construction, the core

$$\left(\sum_{\mathcal{D}:\mathrm{Cat}_\infty} \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \right)^{\times} = (\mathrm{Cat}_\infty)_{/\mathcal{C}}$$

is representable.

Let $\mathcal{C}$ be an ∞-category.

- Whenever $\mathcal{C}$ has pullbacks, the canonical indexing $\mathcal{C}_{/\_}$ of $\mathcal{C}$ over itself unstraightens to the target fibration

$$t\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \twoheadrightarrow \mathcal{C}$$

This fibration will therefore be called the canonical fibration over $\mathcal{C}$.

Preliminaries and notation
Basic examples revisited

Let $\mathcal{C}$ be an ∞-category.

- Change of base of an indexed ∞-category $\mathcal{I}\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ along a functor $F\colon \mathcal{D} \to \mathcal{C}$ unstraightens to the pullback of fibrations

$$\begin{array}{ccc} \mathsf{Un}(F^*\mathcal{I}) & \longrightarrow & \mathsf{Un}(\mathcal{I}) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D} & \xrightarrow[F]{} & \mathcal{C}. \end{array}$$

  Up to equivalence, every cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ is the change of base of the universal cartesian fibration

$$\pi^{op}\colon \mathrm{Dat}_\infty^{op} \twoheadrightarrow \mathrm{Cat}_\infty^{op}$$

  along its straightening $\mathrm{St}(p)^{op}\colon \mathcal{C} \to \mathrm{Cat}_\infty^{op}$.

Let $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration. Then we obtain a homotopy cartesian square of the form

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathrm{Dat}_\infty^{op} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi^{op}} \\ \mathcal{C} & \xrightarrow[\mathrm{St}(p)]{} & \mathrm{Cat}_\infty^{op}. \end{array}$$

- Similarly, up to equivalence, every right fibration is the change of base of the universal right fibration $\mathcal{S}_*^{op} \twoheadrightarrow \mathcal{S}^{op}$ along its straightening.

What is a "comprehension scheme" anyway?

In set theory, we have the

Axiom scheme of restricted Comprehension:

For every set X and every class $S = \{x \mid \phi(x)\}$, the class

$$X \cap S = \{x \in X \mid \phi(x)\}$$

is a set.

What is a "comprehension scheme" anyway?

Generally, given

- a background theory (some form of "meta-mathematics"),
- therein a model $\mathbf{M}$ formalizing some fragment of that mathematics,
- some logical structure S over $\mathbf{M}$ defined by external meta-mathematical means,

we say that $\mathbf{M}$ satisfies the comprehension scheme associated to S if the structure S applied to any mathematical object M in $\mathbf{M}$ can be internalized in $\mathbf{M}$.

A short dictionary of comprehension

Mathematics:	**Set Theory**	**Higher Category Theory**
Meta-mathematics:	The naive set theory of proper classes	$(\infty,1)$-Category Theory over the ∞-topos $\mathcal{S}$ of spaces
A model of a fragment of mathematics:	A set theoretic universe V	An indexed $(\infty,1)$-category $\mathcal{E}\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$
A mathematical object:	A set X in V	A diagram $X\colon I \to \mathcal{E}$ over $\mathcal{C}$
Some external structure:	A class S	A functor of $(\infty,1)$-categories $G\colon I \to J$
Some induced structure over a mathematical object:	$S \cap X$	The presheaf $G^* \downarrow X$ of J-shaped diagrams in $\mathcal{E}$ extending X in $\mathcal{E}$ over $\mathcal{C}$
Meaning of Comprehension:	V has S-comprehension if $S \cap X$ is small for all $X \in V$	$\mathcal{E}$ has G-comprehension if the presheaf $G^* \downarrow X$ is representable for all I-diagrams X in $\mathcal{E}$.

$(\infty, 1)$-Comprehension

For the rest of the talk, fix an ∞-category $\mathcal{C}$.

Definition
Let $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration. For a simplicial set $I \in \mathbf{S}$, let $\mathrm{Rect}(I, \mathcal{E})$ be given by the pullback

$$\begin{array}{ccc} \mathrm{Rect}(I,\mathcal{E}) & \hookrightarrow & (\mathcal{E}^I)^\times \\ \downarrow & \lrcorner & \downarrow \\ & & \mathcal{E}^I \\ \downarrow & & \downarrow p^I \\ \mathcal{C} & \underset{\Delta}{\hookrightarrow} & \mathcal{C}^I . \end{array}$$

Note: $\mathrm{Rect}(\Delta^0, \mathcal{E}) = \mathcal{E}^\times$.

$(\infty,1)$-Comprehension

Proposition

Let $\mathcal{I} \in \mathrm{IC}(\mathcal{C})$. Then for every simplicial set I, the straightening of the right fibration $\mathrm{Rect}(I, \mathrm{Un}(\mathcal{I})) \twoheadrightarrow \mathcal{C}$ is equivalent to the composition

$$\mathcal{C}^{op} \xrightarrow{\mathcal{I}} \mathrm{Cat}_\infty \xrightarrow{(\cdot)^I} \mathrm{Cat}_\infty \xrightarrow{(\cdot)^{\simeq}} \mathcal{S}$$

Definition

Let $G\colon I \to J$ be a map between simplicial sets. Say a cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ has G-comprehension if the restriction $G^*\colon \mathrm{Rect}(J, \mathcal{E}) \to \mathrm{Rect}(I, \mathcal{E})$ has a right adjoint.

$(\infty,1)$-Comprehension

Given a cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$, a functor $G\colon I \to J$ and a rectangular diagram $X \in \mathrm{Rect}(J,\mathcal{E})$, consider the comma ∞-category

$$\begin{array}{ccc} G^* \downarrow X & \longrightarrow & \mathrm{Rect}(I,\mathcal{E})_{/X} \\ \downarrow & \lrcorner & \downarrow s \\ \mathrm{Rect}(J,\mathcal{E}) & \xrightarrow[G^*]{} & \mathrm{Rect}(I,\mathcal{E}) \end{array}$$

and the composition

$$(G^* \downarrow X) \to \mathcal{C}_{/p(X)}.$$

Lemma

Let $G\colon I \to J$ be a monomorphism of simplicial sets. Then a cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ has G-comprehension if and only if for every diagram $X \in \mathrm{Rect}(I,\mathcal{E})$ the presheaf $G^* \downarrow X$ of vertical J-structures extending X horizontally is representable over $\mathcal{C}_{/p(X)}$.

$(\infty, 1)$-Comprehension

Remark

G-comprehension for epimorphisms G can be interpreted in terms of definability of subfibrations.

General strategy: show that the total ∞-categories $G^* \downarrow X$ have a terminal object.

Example

If $G\colon I \to J$ is a weak categorical equivalence and $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ is any cartesian fibration, then the restriction

$$G^*\colon \mathrm{Rect}(J, \mathcal{E}) \to \mathrm{Rect}(I, \mathcal{E})$$

is a categorical equivalence as well. In particular, the fibration p satisfies G-comprehension in a trivial way.

$(\infty, 1)$-Comprehension

Definition

Let $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration.

1. Say p is small if it has $(\emptyset \to \Delta^0)$-comprehension.
   That is if and only if the core $\mathcal{E}^{\times}$ has a terminal object. Or in other words, if and only if the right fibration $p^{\times}$ is representable.
2. Say p is locally small if it has δ^1-comprehension, where $\delta^1 : \partial\Delta^1 \to \Delta^1$ is the boundary inclusion.
   That is if and only if for any two points $t_1, t_2 \in \mathcal{E}(C)$, the right fibration
   $$(\delta^1)^* \downarrow (t_1, t_2) \twoheadrightarrow \mathcal{C}_{/C}$$
   is representable.

$(\infty, 1)$-Comprehension
Examples

Example

A representable right fibration

$$\mathcal{C}_{/C} \twoheadrightarrow \mathcal{C}$$

is locally small if and only if the diagonal

$$\Delta \colon \mathcal{C}_{/C} \to \mathcal{C}_{/C} \times_{\mathcal{C}} \mathcal{C}_{/C}$$

has a right adjoint.
Thus, all representable fibrations over $\mathcal{C}$ are locally small if and only if the ∞-category $\mathcal{C}$ has all equalizers.

$(\infty, 1)$-Comprehension
Examples

Example

Let $\mathcal{C}$ be a potentially large ∞-category.
Then $\mathcal{C}$ is a small (resp. locally small) ∞-category if and only if the fibration

$$\sum_{\mathcal{D}:\mathrm{Cat}_\infty} \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \twoheadrightarrow \mathrm{Cat}_\infty$$

of families in $\mathcal{C}$ is small (resp. locally small).

$(\infty, 1)$-Comprehension
Examples

Example

The terminal map

$$\mathcal{C} \twoheadrightarrow \Delta^0$$

is a cartesian fibration, its cartesian edges are exactly the equivalences in $\mathcal{C}$.

For any simplicial set I, we obtain an equivalence

$$\mathrm{Rect}(I, \mathcal{C}) \simeq \mathrm{Fun}(I, \mathcal{C})^{\simeq}.$$

The fibration $\mathcal{C} \twoheadrightarrow \Delta^0$ is small if and only if the core $\mathcal{C}^{\simeq}$ is contractible.

It is locally small if and only if all hom-spaces of $\mathcal{C}$ are contractible.

$(\infty, 1)$-Comprehension
Examples

Proposition

Suppose $\mathcal{C}$ has pullbacks. Then the canonical fibration $t\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \twoheadrightarrow \mathcal{C}$ is locally small if and only if $\mathcal{C}$ is locally cartesian closed.

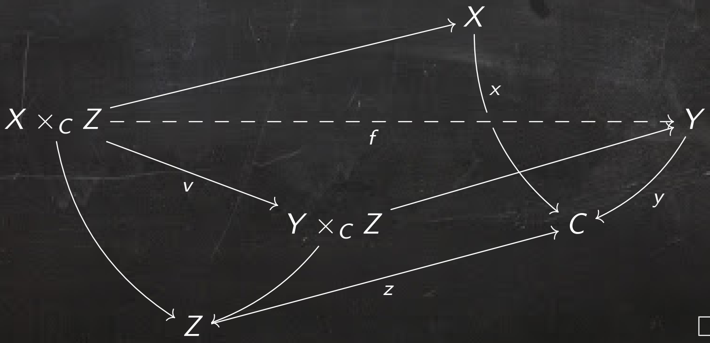
Recall that an ∞-category $\mathcal{C}$ is locally cartesian closed if and only if the presheaf

$$\mathcal{C}_{/C}(x \times_C \_, y)\colon (\mathcal{C}_{/C})^{op} \to \mathcal{S}$$

is representable for every pair $x, y \in \mathcal{C}_{/C}$.

$(\infty, 1)$-Comprehension

Proof.
We have to show that the right fibrations $\mathrm{Un}(\mathcal{C}_{/C}(x \times_C \_, y))$ and $(\delta^1)^* \downarrow (x, y) \twoheadrightarrow \mathcal{C}_{/C}$ are equivalent.
We can do so by using that the class of squares which are vertical up to equivalence and the class of cartesian squares yield a factorization system on $\mathrm{Fun}(\Delta^1, \mathcal{C})$.



$\square$

$(\infty, 1)$-Comprehension
Closure properties

Consider the class

$$\mathrm{Comp}_p := \{G \colon I \to J \mid p \text{ satisfies } G\text{-comprehension}\}.$$

Lemma

Let $p \colon \mathcal{E} \to \mathcal{C}$ be a cartesian fibration.

1. *The class Comp_p contains all weak categorical equivalences. It is closed under composition and under pushouts along monomorphisms.*
2. *The class of monomorphisms in Comp_p is closed under compositions and arbitrary pushouts.*
3. *If $\mathcal{C}$ has pullbacks, then the class of monomorphisms in Comp_p furthermore has the right cancellation property.*

$(\infty, 1)$-Comprehension

Definition

Following [1], we say that a cartesian fibration $p \colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ satisfies

- definability of invertibility (for arrows) if it has inv-comprehension, where $\mathrm{inv} \colon \Delta^1 \to I\Delta^1$ is the embedding of Δ^1 into the free groupoid generated by it.
- definability of equivalence (of parallel morphisms) if it has ∇_1-comprehension, where the map $\nabla_1 \colon (\Delta^1 \cup_{\partial\Delta^1} \Delta^1) \to \Delta^1$ identifies the two parallel non- degenerate arrows.
- definability of equivalence of parallel n-cells if it has ∇_n-comprehension, where the map $\nabla_n \colon (\Delta^n \cup_{\partial\Delta^n} \Delta^n) \to \Delta^n$ is the codiagonal identifying the two parallel non-degenerate n-cells.

[1] Johnstone - Sketches of an Elephant, B1.3.15

$(\infty, 1)$-Comprehension

Proposition

Let $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration.

- *Definability of equivalence of parallel n-cells implies δ^{n+1}-comprehension.*
- *Suppose $\mathcal{C}$ has all equalizers.*
  - *δ^n-comprehension implies definability of equivalence of parallel n-cells.*
  - *If p is locally small, it has δ^n-comprehension for all $n \geq 1$.*
  - *If p is locally small, it has definability of invertibility.*

Corollary

A cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ is small and locally small if and only if it has j-comprehension for every monomorphism j between finite simplicial sets.

Internal $(\infty, 1)$-categories

In the following, let $\mathcal{C}$ be a fixed left exact ∞-category and let

$$s\mathcal{C} := \mathrm{Fun}(N(\Delta)^{op}, \mathcal{C})$$

be the ∞-category of simplicial objects in $\mathcal{C}$.

Definition

A Segal object in $\mathcal{C}$ is a simplicial object $X \in s\mathcal{C}$ such that its associated Segal maps

$$\xi_n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are equivalences in $\mathcal{C}$. A Segal object X in $\mathcal{C}$ is complete if the degeneracy

$$s_0 \colon X_0 \to \mathrm{Equiv}(X)$$

is an equivalence in $\mathcal{C}$.

We obtain full ∞-subcategories $\mathrm{CS}(\mathcal{C}) \subseteq \mathrm{S}(\mathcal{C}) \subseteq s\mathcal{C}$ of Segal objects and complete Segal objects in $\mathcal{C}$, respectively.

Internal $(\infty, 1)$-categories

Definition

A Segal groupoid in $\mathcal{C}$ is a Segal object X in $\mathcal{C}$ such that the natural map $\mathrm{Equiv}(X) \to X_1$ is an equivalence in $\mathcal{C}$.
The full ∞-subcategory of Segal groupoids in $\mathrm{S}(\mathcal{C})$ is denoted by $\mathrm{SGpd}(\mathcal{C})$.

The pushfoward of the Yoneda embedding

$$y_*\colon s\mathcal{C} \to s\hat{\mathcal{C}}$$

and the equivalence $s\hat{\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}^{op}, s\mathcal{S})$ yield a functor of the form

$$\mathrm{CS}(\mathcal{C}) \xrightarrow{y_*} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{CS}(\mathcal{S})) \xrightarrow{(p_h)_*} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$$

which we call the externalization functor associated to $\mathcal{C}$ and which we denote by

$$\mathrm{Ext}\colon \mathrm{CS}(\mathcal{C}) \to \mathrm{IC}(\mathcal{C}).$$

Remark

- *For a complete Segal object X in $\mathcal{C}$, think*

$$\mathrm{Ext}(X) \sim \mathcal{C}(\_\,, X(\cdot)).$$

- *If $\mathcal{C}$ is a 1-category, then $\mathrm{CS}(\mathcal{C})$ is the 1-category of internal categories in $\mathcal{C}$ and $\mathrm{Ext}\colon \mathrm{CS}(\mathcal{C}) \to \mathrm{IC}(\mathcal{C})$ is exactly the 1-categorical externalization studied in [2] and [3].*
- *A Segal object X in $\mathcal{C}$ is a Segal groupoid if and only if its externalization is an indexed ∞-groupoid.*
- *The cartesian fibrations corresponding to an externalization $\mathrm{Ext}(X)\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ of a complete Segal object X in $\mathcal{C}$ are exactly the "representable" Segal cartesian fibrations $\mathcal{C}_{/X}$ in [4].*

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[2] Johnstone - Sketches of an Elephant, B2.3

Question: Given a cartesian fibration $p \colon \mathcal{E} \twoheadrightarrow \mathcal{C}$, how can we check whether there is a complete Segal object X such that $p \simeq \mathrm{Ext}(X)$?

Theorem

Let $p \colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration and suppose $\mathcal{C}$ has pullbacks.

1. *The fibration p is both small and locally small if and only if there is a complete Segal object X in $\mathcal{C}$ such that $\mathrm{Ext}(X)$ is equivalent to $\mathcal{E}$ over $\mathcal{C}$.*
2. *If p is a right fibration, then p is small (and hence the unstraightening of a representable presheaf) if and only if there is a complete Segal groupoid X in $\mathcal{C}$ such that $\mathrm{Ext}(X)$ is equivalent to $\mathcal{E}$ over $\mathcal{C}$.*

Internal $(\infty, 1)$-categories
Externalization

Corollary

Suppose $\mathcal{C}$ has pullbacks and let $\mathrm{ic}(\mathcal{C}) \subset \mathrm{IC}(\mathcal{C})$ be the full ∞-subcategory of small and locally small indexed ∞-categories over $\mathcal{C}$. Then the externalization construction

$$\mathrm{Ext}\colon \mathrm{CS}(\mathcal{C}) \to \mathrm{ic}(\mathcal{C})$$

is an equivalence of ∞-categories.

Corollary

Suppose $\mathcal{C}$ has pullbacks. Then a cartesian fibration $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ has G-comprehension for all finite monomorphisms $G\colon I \to J$ if and only if it is the externalization of a complete Segal object X in $\mathcal{C}$.

Recall the universal cocartesian fibration $\pi\colon \mathrm{Dat}_\infty \twoheadrightarrow \mathrm{Cat}_\infty$ and the universal left fibration $\mathcal{S}_* \twoheadrightarrow \mathcal{S}$.

Proposition

The universal cartesian fibration $\pi^{op}\colon \mathrm{Dat}_\infty^{op} \twoheadrightarrow \mathrm{Cat}_\infty^{op}$ has $(\emptyset \to I)$-comprehension for every simplicial set I. Given a simplicial set I, the right fibration

$$\mathrm{Rect}(I, \mathrm{Dat}_\infty^{op}) \twoheadrightarrow \mathrm{Cat}_\infty^{op}$$

is represented by a fibrant replacement of I^{op} in the Joyal model structure.

Internal $(\infty, 1)$-categories
The universal fibrations

In particular, for every $n \geq 0$ we have an equivalence

$$\mathrm{Rect}(\Delta^n, \mathrm{Dat}_\infty^{op}) \simeq (\mathrm{Cat}_\infty^{op})_{/\Delta^n}). \quad (1)$$

Corollary

The universal cartesian fibration is small and locally small.

One can in fact show that the equivalences in (1) imply that

$$\pi^{op} \colon \mathrm{Dat}_\infty^{op} \twoheadrightarrow \mathrm{Cat}_\infty^{op}$$

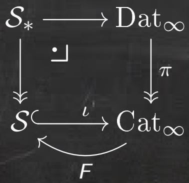
is the externalization of the complete Segal object $\Delta^\bullet \in \mathrm{Cat}_\infty^{op}$.

Transfer of comprehension schemes

Lemma

Let $p\colon \mathcal{E} \twoheadrightarrow \mathcal{C}$ be a cartesian fibration and suppose $\mathcal{C}$, $\mathcal{D}$ are ∞-category with pullbacks. If $F\colon \mathcal{D} \to \mathcal{C}$ is a functor with a right adjoint, then $\mathrm{Comp}_{F^*p} \subseteq \mathrm{Comp}_p$.

Example



The inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ has a left adjoint $F\colon \mathrm{Cat}_\infty \to \mathcal{S}$ which assigns to a ∞-category the free ∞-groupoid generated by it. It follows that the universal right fibration satisfies all comprehension schemes satisfied by the universal cartesian fibration.

Transfer of comprehension schemes

Lemma

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor between ∞-categories with pullbacks. Suppose it has a right adjoint R. Then for every complete Segal object X in $\mathcal{C}$, there is a homotopy cartesian square of the form

$$\begin{array}{ccc} \mathrm{Ext}(R_* X) & \longrightarrow & \mathrm{Ext}(X) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathcal{D} & \xrightarrow[F]{} & \mathcal{C}. \end{array}$$

Corollary

The universal right fibration $\mathcal{S}_*^{op} \twoheadrightarrow \mathcal{S}^{op}$ is the externalization of the complete Segal object $F\Delta^\bullet$ in $\mathcal{S}$.

This recovers the fact that the universal right fibration is represented by the terminal object in $\mathcal{S}$ (Lurie - Higher Topos Theory, 3.3.2).

Transfer of comprehension schemes

Applied to the universal cartesian fibration, the Lemma yields a tool to compute the straightening of indexed ∞-categories with a left adjoint.
In fact, we have the following characterization whenever $\mathcal{C}$ is complete.

Proposition

Suppose $\mathcal{C}$ is complete. Then a $\mathcal{C}$-indexed ∞-category $\mathcal{I}\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ is equivalent to the externalization of a complete Segal object X in $\mathcal{C}$ if and only if it has a left adjoint.

In particular, if $\mathcal{C}$ is a presentable ∞-category, a cartesian fibration over $\mathcal{C}$ is the externalization of a complete Segal object if and only if its Straightening preserves limits.
For right fibrations this is exactly the basic fact that limit preserving presheaves are representable.

Transfer of comprehension schemes

Remark

If $\mathcal{C}$ is complete, it follows that every complete Segal object X in $\mathcal{C}$ yields a pullback square of the form

$$\begin{array}{ccc} \mathrm{Ext}(X) & \longrightarrow & \mathrm{Dat}_\infty^{op} \\ \downarrow & \lrcorner & \downarrow \pi^{op} \\ \mathcal{C} & \xrightarrow[\mathrm{Ext}(X)]{} & \mathrm{Cat}_\infty^{op} \end{array}$$

where the bottom map has a right adjoint.
Thus, it follows that all comprehension schemes satisfied by the universal cartesian fibration are transferred to all internal ∞-categories in $\mathcal{C}$ whenever $\mathcal{C}$ is complete.

Univalence

Definition

Suppose $\mathcal{C}$ has pullbacks. A (full) comprehension ∞-category of $\mathcal{C}$ is a (full) subfibration of the canonical fibration $t\colon \mathrm{Fun}(\Delta^1, \mathcal{C}) \twoheadrightarrow \mathcal{C}$.

Example

Suppose $\mathcal{C}$ is a locally cartesian closed ∞-category. For every map $q\colon E \to B$ in $\mathcal{C}$ we can associate the full comprehension ∞-category

$$F_q \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$$

where $F_q(C) \subseteq \mathcal{C}_{/C}$ consists of those maps with codomain C which arise as pullback of q along some map $C \to B$.

In the case that $\mathcal{C}$ is a presentable ∞-category, Gepner and Kock have characterized univalence of maps q in $\mathcal{C}$ in terms of a sheaf property of the indexed ∞-category F_q.[5]

[5] Gepner, Kock - Univalence in Locally Cartesian Closed ∞-Categories, Proposition 3.8

Univalence

Corollary

Suppose $\mathcal{C}$ is locally cartesian closed and complete and q is a map in $\mathcal{C}$. Then the following are equivalent.

- *q is univalent.*
- *the internal nerve $\mathcal{N}(q)$ is a complete Segal object in $\mathcal{C}$.*
- *the indexed ∞-category $F_q\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ has a left adjoint.*

In particular, the functor $F_q\colon \mathcal{C}^{op} \to \mathrm{Cat}_\infty$ preserves limits whenever q is univalent.

Proof.

By the proposition above, using $\mathrm{Ext}(\mathcal{N}(q)) \simeq F_q$ (see Rasekh - Complete Segal objects). $\square$

Univalence

Proposition

Suppose $\mathcal{C}$ is locally cartesian closed. A full and replete comprehension ∞-category $J \subseteq \mathrm{Fun}(\Delta^1, \mathcal{C})$ is small if and only if it is the externalization $\mathrm{Ext}(\mathcal{N}(q))$ for some univalent map $q \in \mathcal{C}$. That is, if and only if there is a univalent map $q \in \mathcal{C}$ such that $J \simeq F_q$.
In either case, the map q is a terminal object in $J^\times$.

In other words: a map q in $\mathcal{C}$ is univalent if and only if it is terminal in $(F_q)^\times$.
That means, if and only if it is a (-1)-truncated object in $\mathrm{Fun}(\Delta^1, \mathcal{C})^\times$: for all $f \in \mathrm{Fun}(\Delta^1, \mathcal{C})$, the mapping space $F_q^\times(f, q)$ is (-1)-truncated.

$$\mathrm{Fun}(\Delta^1, \mathcal{C})^\times(f, q) \simeq \begin{cases} F_q^\times(f, q) \simeq * & \text{if } f \in F_q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Univalence

This motivates to define an object $x \in \mathcal{E}$ in a general cartesian fibration $\mathcal{E} \twoheadrightarrow \mathcal{C}$ to be univalent if it is (-1)-truncated in the core $\mathcal{E}^{\times}$.

Accordingly, one can define the univalent completion of an object $x \in \mathcal{E}$ to be its (-1)-truncation in $\mathcal{E}^{\times}$ whenever it exists.

Thank you.