GRAY TENSOR PRODUCTS & LAX FUNCTORS OF $(\infty,2)$-CATEGORIES

INTRO (2-CATS)

$2\text{-}Cat = (Cat)\text{-}Cat$: categories enriched over $(Cat, \times, *)$

$$\begin{tikzcd} X \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd} \qquad \alpha : f \to g \text{ in } \mathcal{C}(x,y)$$

2-Functors: $F : \mathcal{C} \longrightarrow \mathcal{D}$ consist of assignments on 0,1,2 cells satisfying all the relevant coherence constraints on the nose.

Enriched category theory provides a notion of 2-natural transformation, where $\alpha : F \Rightarrow G$ consists of:

$$
\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, bend right, "Ff"'] \arrow[d, bend left, "Fg"] \arrow[d, phantom, "\overset{F\theta}{\Rightarrow}"] & GX \arrow[d, bend right, "Gf"'] \arrow[d, bend left, "Gg"] \arrow[d, phantom, "\overset{G\theta}{\Rightarrow}"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}
\qquad
\begin{aligned}
&\forall\, X, Y \in \mathcal{C}_0 \\
&\forall\, f, g : X \to Y \in \mathcal{C}_1 \\
&\forall\, \theta : f \to g \in \mathcal{C}_2
\end{aligned}
$$

Furthermore, we have a natural isomorphism of the form:

$$
2\text{-}\mathrm{Cat}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong 2\text{-}\mathrm{Cat}(\mathcal{C}, [\mathcal{D}, \mathcal{E}]_{st})
$$

- 2-functors
- (strict, 2) nat. transf.
- modifications

We can relax the notion of 2-nat transf. by using the 2-dimensional data available.

$$
\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\alpha_f}{\Uparrow}\ \text{LAX}"] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\overset{\alpha_f}{\Uparrow}\ \text{OPLAX}"] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}
$$

In $\boxtimes$, can we replace $\times$ by another monoidal structure that results in an analogous iso for $[\mathcal{D}, \mathcal{E}]_{lax}$?

The answer is yes, the correct substitute is the [(op)lax] Gray tensor product $\mathcal{C} \otimes \mathcal{D}$.

Classically, this is constructed by means of generators and relations, or exploiting density of cubes in 2-Cat (Street). A salient feature is that:

$$[1] \times [1] \qquad\qquad [1] \otimes [1]$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

(OP)LAX FUNCTORS

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

$$\mathcal{C} \ni x \longmapsto Fx \in \mathcal{D}$$

$$(f : x \to y) \longmapsto (Ff : Fx \to Fy)$$

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & y \end{tikzcd} \longmapsto \begin{tikzcd} Fx \arrow[r, bend left=50, "Ff"] \arrow[r, bend right=50, "Fg"'] \arrow[r, phantom, "\Downarrow F\alpha"] & Fy \end{tikzcd}$$

with: (I) $F(gf) \xrightarrow{\text{oplax}} F(g)\,F(f)$

lax

$\xrightarrow{f}\xrightarrow{g}$

but
$$F\left(\begin{tikzcd}[column sep=large] \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & \bullet \end{tikzcd}\right)$$
$$\|$$
$$F(\beta) *_1 F(\alpha)$$

and (II) $F(1_x) \longrightarrow 1_{Fx}$

Substituting " $\longrightarrow$ " with " $=$ " in (II) we get the normalized version.

Observe that vertically invertible 2-cells are preserved :

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "g"'] & \arrow[d, phantom, "\Downarrow\cong"] & \bullet \\ & \bullet \arrow[ur, equal] & \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[rr, "Ff"] \arrow[dr, "Fg"'] & \arrow[d, phantom, "\Downarrow\cong"] & \bullet \\ & \bullet \arrow[ur, equal] & \end{tikzcd}$$

Finally, (op)lax 2-functors need not preserve isomorphic 1-cells/ equivalences.

SCALED SIMPLICIAL SETS

Set_Δ^{sc} = Simplicial sets + scaling = subset of thin 2-simplices

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\cong"] & \bullet \\ & \bullet \arrow[ur] & \end{tikzcd}$$

Thm [Lurie] There is a model structure for $(\infty,2)$-categories on Set_Δ^{sc}, whose fibrant objects we refer to as ∞-bicategories, and whose cof's are mono's.

$(\infty,2)$-Cats $=$ $(\infty,1)$-$(\infty\text{-Cat})$ $=$ $\mathcal{C}(x,y) \in \mathrm{Cat}_\infty$

$\mathrm{Cat}(f,g) \in$ Spaces $\simeq$ Kan

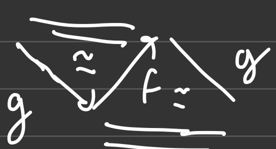
$\cong$

Gpd_∞

Let's define

$$\mathrm{Map}_{\text{oplax}}\big((\mathcal{C}, T_\mathcal{C}), (\mathcal{E}, T_\mathcal{E})\big) = \mathrm{Map}\big((\mathcal{C}, L_c), (\mathcal{E}, T_\mathcal{E})\big)$$

where $L_c = \{ \alpha \in T_\mathcal{C} : \alpha|_{01}$ or $\alpha|_{12}$ is invertible in $\mathcal{C}\}$



Lemma. Let $(\mathcal{C}, T_\mathcal{C}) \to (\mathcal{D}, T_\mathcal{D})$ be a bicategorical equivalence. Then $(\mathcal{C}, L_\mathcal{C}) \to (\mathcal{D}, L_\mathcal{D})$ is again a bicategorical equivalence.

In particular, it induces an equivalence of the form

$$\mathrm{Map}_{\mathrm{oplax}}(\mathcal{D}, \mathcal{E}) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{oplax}}(\mathcal{C}, \mathcal{E})$$

NOT TRUE FOR ORDINARY VERSION!

Why is it a good notion?

$$\begin{tikzcd} \cdot \arrow[r, "gf"] \arrow[d, "f"'] \arrow[dr, phantom, "\simeq"] & \cdot \\ \cdot \arrow[ur, "g"'] & \end{tikzcd} \longmapsto \begin{tikzcd} \cdot \arrow[r, "F(gf)"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\Downarrow"] & \cdot \\ \cdot \arrow[ur, "F(g)"'] & \end{tikzcd}$$

Let's define:

$$\mathrm{Map}(X \otimes Y, Z) \overset{?}{\simeq} \widetilde{\mathrm{Map}}_{\mathrm{oplax}}(X \times Y, Z)$$

subspace spanned by $X \times Y \rightsquigarrow Y$ oplax s.t.

i) $\forall x \in \mathcal{C}, y \in \mathcal{D}$ $\phi_{|\{x\} \times Y}, \phi_{|X \times \{y\}}$ are ordinary maps of ∞-bicats

ii) $\forall\, f : x \to x'$ in X

$\forall\, g : y \to y'$ in Y

the 2-simplex

$$\begin{tikzcd}
(x,y) \arrow[r, "{(f,y)}"] \arrow[dr, "{(f,g)}"'] & (x',y) \arrow[d, "{(x',g)}"] \\
 & x',y'
\end{tikzcd}$$

in $X \times Y$ is mapped to a thin one in Z.

Reason:

$$\begin{tikzcd}
{} \arrow[r, "{(f,y)}"] \arrow[d, "{(x,g)}"'] \arrow[dr] \arrow[dr, phantom, "\Downarrow\ \ \Uparrow \simeq"] & {} \arrow[d, "{(x',g)}"] \\
{} \arrow[r, "{(f,y')}"'] & {}
\end{tikzcd}
\simeq
\begin{tikzcd}
{} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & {} \arrow[d] \\
{} \arrow[r] & {}
\end{tikzcd}$$

Def. Given $(X, T_X), (Y, T_Y) \in Set^{\Delta}_{sc}$, define

$$X \otimes Y = \left(X \times Y, \{ (\alpha, \beta) : \alpha \in T_X, \beta \in T_Y, \alpha_{12} \text{ deg in } X \text{ or } \beta_{01} \text{ deg in } Y \} \right)$$

Thm.

$$Set_{\Delta}^{sc} \times Set_{\Delta}^{sc} \xrightarrow{\otimes} Set_{\Delta}^{sc}$$

replace deg with invertible

is a Quillen bifunctor.

We have inclusions

$$\begin{tikzcd} (X \times Y, L_{X \times Y}) \arrow[r, hook] & (X \times Y, T_{\simeq}) \\ & X \otimes Y \arrow[u, hook, "\simeq"'] \end{tikzcd}$$

that induce a restriction functor:

$$\mathrm{Map}(X \otimes Y, Z) \xrightarrow{\theta} \mathrm{Map}_{oplax}(X \times Y, Z)$$

Thm. The map f identifies the domain with $\widetilde{\mathrm{Map}}_{\mathrm{oplax}}(X \times Y, Z) \subset \mathrm{Map}_{\mathrm{oplax}}(X \times Y, Z)$.