Globular Multicategories with Homomorphism Types

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Funded by EPSRC

van den Berg, Garner *Types are weak ω-groupoids*

Type Theories with Id Types $\longrightarrow$ Monoidal Globular Categories $\longrightarrow$ Higher Categories

$$\vdash A : \mathrm{Type}$$

$$x, y : A \vdash \mathrm{Id}_A(x,y) : \mathrm{Type}$$

$$p, q : \mathrm{Id}_A(x,y) \vdash \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q) : \mathrm{Type}$$

$$\vdots$$

A Globular Multicategory conists of:

- A globular set of Types
- A globular set of Terms
- Terms can be composed

- A globular set of Types
  - 0-types $\qquad \bullet A \qquad\qquad \vdash A : \mathrm{Type}$
  - 1-types $\qquad A \overset{M}{\longrightarrow\!\!\!\!\!\!\!+\;\;} B \qquad\qquad x : A, y : B \vdash M(x,y) : \mathrm{Type}$
  - 2-types

$$\begin{tikzcd} A \arrow[r, bend left=50, "M", "\shortmid"{marking}] \arrow[r, bend right=50, "N"', "\shortmid"{marking}] \arrow[r, phantom, "O"] & B \end{tikzcd}$$

$$m : M(a,b), n : N(a,b) \vdash O(m,n) : \mathrm{Type}$$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

- A globular set of Terms

An n-term $f : \Gamma \to A$ is a generalized arrow

The Context Γ is a globular pasting diagram of types

A is a type

- 0-terms

$$\begin{array}{c} A \\ f\downarrow \\ B \end{array} \qquad x : A \vdash fx : B$$

- 1-terms

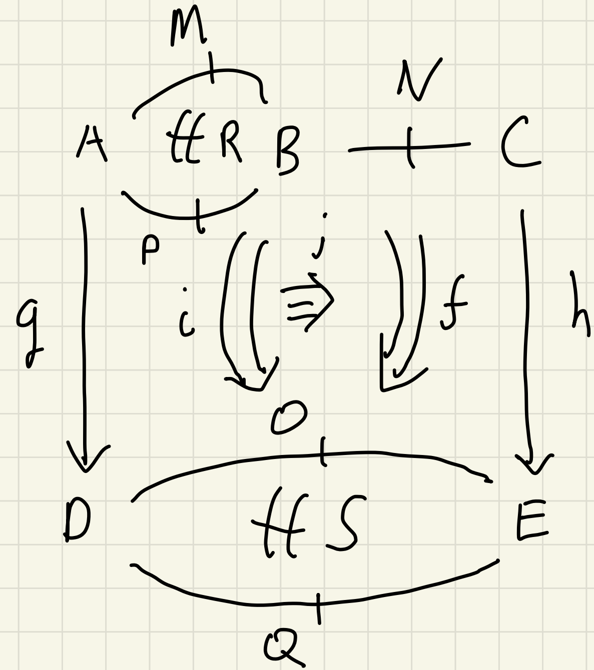
$$\begin{tikzcd} A \arrow[r, "M", "|" marking] \arrow[d, "g"'] & B \arrow[r, "N", "|" marking] \arrow[d, phantom, "\Downarrow f"] & C \arrow[d, "h"] \\ D \arrow[rr, "O"', "|" marking] & & E \end{tikzcd}$$

$$m : M(a,b),\ n : N(b,c) \vdash f(m,n) : O(ga, hc)$$

$$\begin{tikzcd} & A \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[d, phantom, "\Downarrow\phi"] & \\ D \arrow[rr, "O"', "|" marking] & {} & E \end{tikzcd}$$

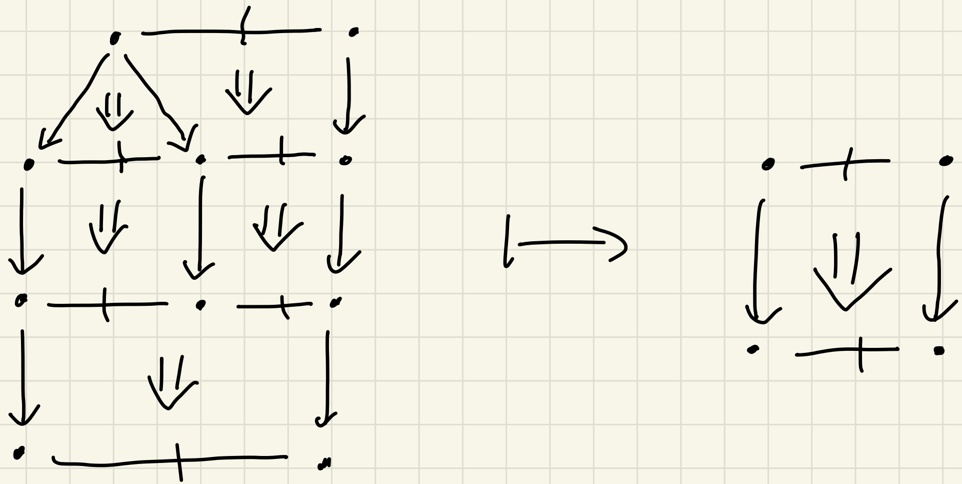
$$a : A \vdash \phi(a) : O(ga, hc)$$

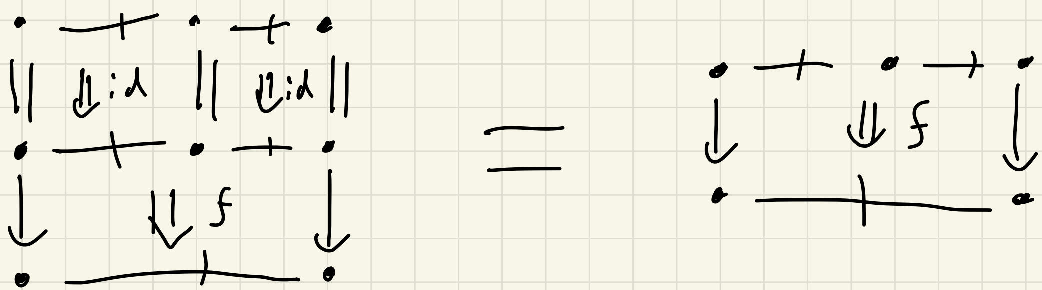
- 2-terms



$$r : R(m,p),\ n : N(b,c) \ \vdash\ j(r,n) : S\big(i(p,n),\ f(m,n)\big)$$

Terms have an associative and unital composition





Example

Every dependent type theory induces a globular multicategory.

Types $\longleftrightarrow$ Types + Dependent Types

Terms $\longleftrightarrow$ Terms + Terms in dependent Contexts

Example: Spans

Let $\mathcal{C}$ be a category with pullbacks.
There is a globular multicategory $\mathrm{Span}\,\mathcal{C}$:

- 0-types are objects in $\mathcal{C}$
- 1-types are spans

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet & & \bullet \end{tikzcd}$$

- 2-types are spans between spans

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[d] \arrow[drr] & & \bullet \arrow[d] \arrow[dll] \\ \bullet & & \bullet \end{tikzcd}$$

$\vdots$

- terms are transformations between spans

We could also restrict the class of spans that we allow

Typically we want a type $f_!$ be some sort of

2-sided fibration.

e.g. van den Berg and Garner

Virtual Double Categories are 1-dimensional Globular Multicategories

e.g. Every pseudo double category

Typical Examples:

- 0-types are "category-like"
- 1-types are profunctors
- 0-terms are functors
- 1-terms are transformations

$\vdots$

We think of objects in globular multicategories as "higher category-like"

First example:

- 0-types are strict ω-categories
- 1-types are profunctors
- 2-types are profunctors between profunctors

  $\vdots$

- 0-terms are functors
- higher terms are transformations

Type theories with Id types

Type theories with Hom-types

Spans

Globular Multicategories

ω-categories

collections of "category-like" objects

A globular multicategory has **homomorphism types** when:

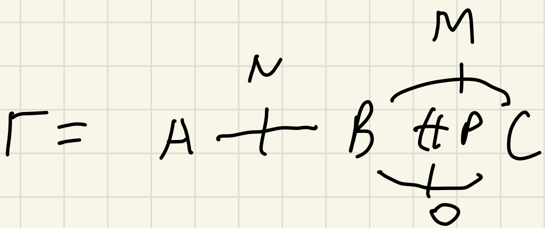
For each n-type A, there is an $(n+1)$-type

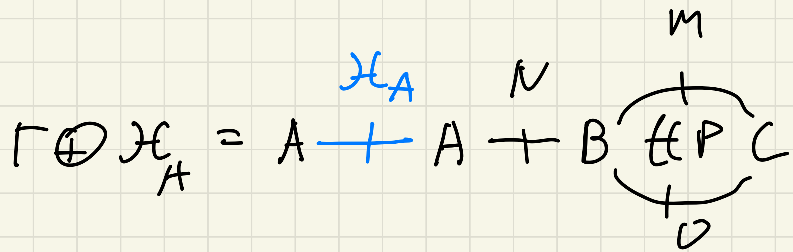
$$
\begin{tikzcd}
A \arrow[r, "\mathcal{H}_A", "|"{marking}] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, bend left=50, "M", "|"{marking}, ""{name=U, below}] \arrow[r, bend right=50, "M"', "|"{marking}, ""{name=D, above}] & C
\arrow[from=D, to=U, Rightarrow, "\mathcal{H}_M"]
\end{tikzcd}
$$

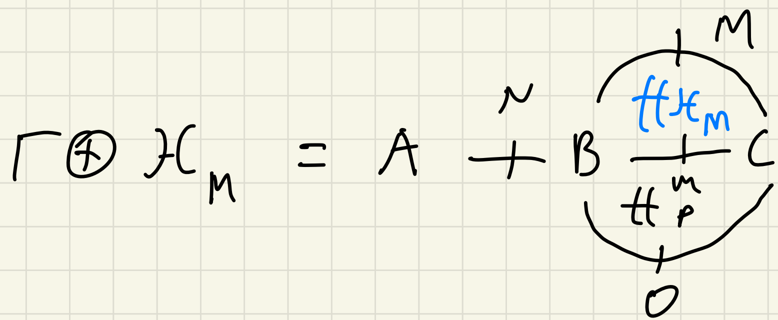
an $(n+1)$-term $r_A : A \longrightarrow \mathcal{H}_A$

$$
\begin{tikzcd}
& A \arrow[dl, equal] \arrow[dr, equal] \arrow[d, phantom, "\Downarrow r_A"] & \\
A \arrow[rr, "\mathcal{H}_A"', "|"{marking}] & {} & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "M", "|"{marking}] \arrow[d, equal] \arrow[dr, phantom, "((\Rrightarrow r_M))"] & C \arrow[d, equal] \\
B \arrow[r, bend left=50, "M", "|"{marking}, ""{name=U, below}] \arrow[r, bend right=50, "M"', "|"{marking}, ""{name=D, above}] & C
\arrow[from=D, to=U, Rightarrow, "\mathcal{H}_A"]
\end{tikzcd}
$$

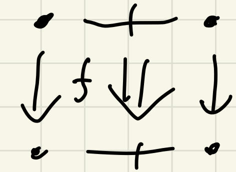
For each k-type (variable) $x : A$, $0 \leq k < n$, in an n-context Γ, there is a context $\Gamma \oplus \mathcal{H}_x$ "with a hom type at x added"





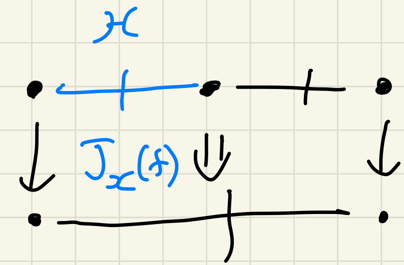


Let $f : \Gamma \to A$ be an n-term



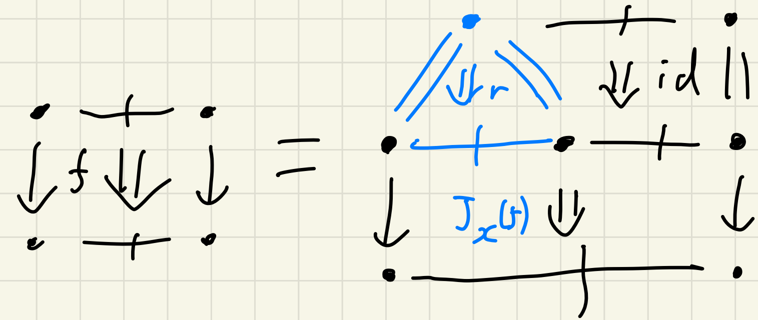
For each $(n-1)$-variable $x : A$ in Γ

there is an n-term



$$J_x(f) : \Gamma \oplus \mathcal{H}_x \longrightarrow A$$

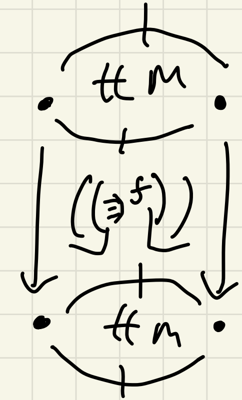
s.t. $\Gamma_x \,;\, J_x(f) = f$



For each $k < n$, k-variable x and

$$\dot{j}_s : s\Gamma \oplus \mathcal{H}_x \to sA \quad \text{s.t.} \quad r_x ; \dot{j}_s = sf,$$

$$\dot{j}_t : t\Gamma \oplus \mathcal{H}_x \to tA \quad \text{s.t.} \quad r_x ; \dot{j}_t = tf,$$



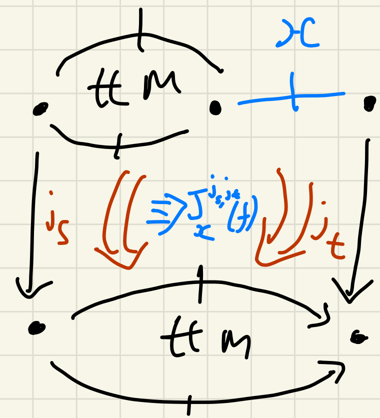
a term

$$J_x^{\dot{j}_s, \dot{j}_t}(f) : \Gamma \oplus \mathcal{H}_x \to A \quad \text{s.t.}$$

$$r_x ; J_x^{\dot{j}_s, \dot{j}_t}(f) = f,$$

$$s J_x^{\dot{j}_s, \dot{j}_t}(f) = \dot{j}_s$$

$$t J_x^{\dot{j}_s, \dot{j}_t}(f) = \dot{j}_t$$



Type theories with Id types

Spans

ω-categories

Globular Multicategories with Hom types

collections of "category-like" objects

Globular Multicategories with Hom types $\longrightarrow$ Higher Categories

- Internally
- Externally

Internally

Observation: The theory of Globular Multicategories with Hom types is essentially algebraic

$$\begin{tikzcd}[column sep=huge] \mathrm{GlobMult}_{\mathcal{H}} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] & \mathrm{GlobMultGraph} \arrow[l, bend left, "F_{\mathcal{H}}"] \end{tikzcd}$$

Presheaves on the Shapes that I have been drawing.

$\mathcal{O}$-types

By Yoneda a $\mathcal{O}$-type in a globular multicategory is an arrow from the generic $\mathcal{O}$-type $\bullet \longrightarrow X$

When we have Hom types, by adjointness, this is a Hom type preserving arrow $F_{\mathcal{H}}\bullet \longrightarrow X$

$F_{\mathcal{H}}\bullet$ is the "theory of $\mathcal{O}$-types with Hom-types"

Claim: $F_{\mathcal{H}}^{\bullet}$ is a normalized contractible globular operad.

So every $\mathcal{O}$-type has the structure of a weak ω-category.

Def: We say that Hom types are strict when there is a unique J-term $J_x(f)$ for each variable x.

$\Longleftrightarrow$ Extensionality in type theory

$$r_x ; f = r_x ; f' \Longrightarrow f = f'$$

We think of objects in this case as "strict ω-category-like".

$$\begin{tikzcd}
\mathrm{GlobMult}_{\mathcal{H}} \arrow[r, bend left, "\text{forgetful}"] \arrow[r, phantom, "\top"] \arrow[d, bend left, "F_{str}"] \arrow[d, phantom, "\vdash"] & \mathrm{GlobMultGraph} \arrow[l, bend left, "F_{\mathcal{H}}"] \\
\mathrm{GlobMult}_{\mathcal{H}_{str}} \arrow[u, bend left, "\text{forgetful}"]
\end{tikzcd}$$

Theorem:

Let η be the unit of the strictification adjunction

Let X be a globular multigraph

Then
$$\begin{tikzcd} F_{\mathcal{H}} X \arrow[d, two heads, "\sim"'] \\ U S F_{\mathcal{H}} X \end{tikzcd}$$
is an acyclic fibration.

Example: $F_{\mathcal{K}_{str}}\, \bullet = T$ the terminal globular operad.

Example: Let $I = \begin{smallmatrix} \bullet \\ \downarrow \\ \bullet \end{smallmatrix}$ be the generic O-term

- $F_{\mathcal{H}_{str}} I$ is the "theory of strict ω-functors"

$$\begin{tikzcd} F_{\mathcal{H}} I \arrow[d, two heads, "\sim"'] \arrow[dr] & \\ F_{\mathcal{H}_{str}} I & X \end{tikzcd}$$

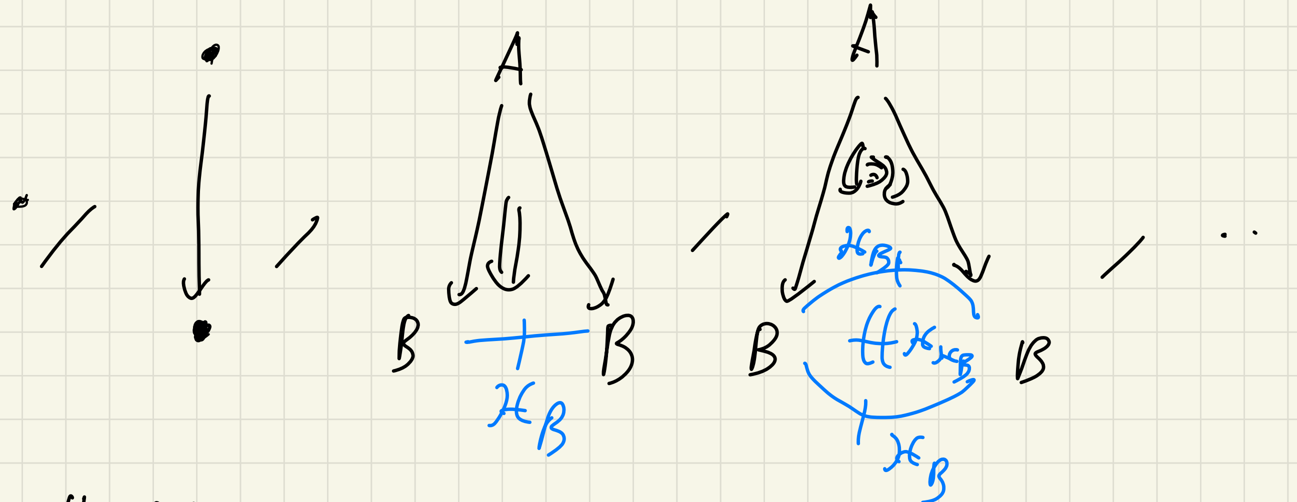
- Every O-term in X is a weak ω-functor.

In general:

- n-types are weak higher profunctors
- n-terms are transformations between them

Externally:

$\mathbb{I} := \mathcal{O}, \ I_\mathcal{O}, \ I_{I_\mathcal{O}}, \ I_{I_\mathcal{O}}, \ \ldots$



The collection of $\mathcal{O}$-types, $\mathcal{O}$-terms, transformations between $\mathcal{O}$-terms, transformations between transformations ..., should be a weak ω-category.

$\mathbb{I}$ is a globular object in $\mathrm{GlobMult}^{op}$

We have globular operads $\mathrm{End}(F_{\mathcal{H}}^{op}\mathbb{I})$ in $\mathrm{GlobMult}_{\mathcal{H}}$

$\mathrm{End}(F_{\mathcal{H}_{str}}^{op}\mathbb{I})$ in $\mathrm{GlobMult}_{\mathcal{H}_{str}}$

Theorem: $\mathrm{End}(F^{op}_{\mathcal{H}_{Str}}\mathbb{I})$ is the terminal globular operad

i.e. The collection of strict ω-categories is a strict ω-category.

Theorem:

$$\begin{tikzcd} \mathrm{End}(F^{op}_{\mathcal{H}}\mathbb{I}) \arrow[d, two heads, "\sim"] \\ \mathrm{End}(F^{op}_{\mathcal{H}_{SI}}\mathbb{I}) \end{tikzcd}$$

The homomorphism induced by strictification is an acyclic fibration

<u>Corollary</u>: $\mathrm{End}(F_{\mathcal{X}}\mathbb{I})$ is a normalized contractible globular operad.

Whenever $\mathcal{X}$ is a globular multicategory with Hom types, the collection of 0-types and terms in $\mathcal{X}$ is a globular ω-category.

Collections of "higher category like" objects $\longrightarrow$ Globular Multicategories with Hom types $\longrightarrow$ Higher Categories

Fin