Towards a 2-dimensional spectral construction

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Many prominent dualities in mathematics are instances of a common construction centered on the notion of *spectral functor*. Roughly stated, one starts with a locally finitely presentable category, equipped with a subcategory of distinguished *local objects* encoding point-like data and a factorization system (*Etale maps*, *Local maps*) where the etale maps behave as duals of distinguished continuous maps. Several manners of axiomatizing the correct relation between those ingredients have been proposed, either through topos theoretic methods by "localizing" local objects with a Grothendieck topology generated by etale maps, or in an alternative (though tightly related) way based on the notion of *local right ajoint* (or equivalently *stable* functor). Then the *spectrum* of a given object is constructed as a topos classifying etale maps under this given object toward local objects, equipped with a *structural sheaf* playing the role of the "free local object" under it. This defines a spectral functor from the ambient locally finitely presentable category to a category of *locally structured toposes*, forming an adjunction with a corresponding *global section* functor.

This construction provide a convenient template for several prominent 1-categorical examples, as dualities for rings in algebraic geometry, or also Stone-like dualities for different classes of propositional algebras. The strong analogy between those dualities and their corresponding first order syntax-semantics dualities suggests the later could be understood as instances of a convenient 2-dimensional spectral construction. In this talk we will expose the ongoing work devoted to concretize this intuition.

After recalling the 1-dimensional version of the construction and the details of some prominent Stone-like examples, we introduces a notion of stable 2-functor and provide a method to construct an associated notion of spectral 2-sites, defining the spectrum as the associated Grothendieck 2-topos equiped with a distinguished structural stack. In particular we give a special interest in determining the local objects and the factorization system associated to doctrines corresponding to fragments of first order logics, as Lex, Reg, or Coh; in those situations, the construction simplifies as the spectral site happens to be 1-truncated so that one recover the corresponding 1-dimensional notion of *classifying topos* of a theory as the spectrum, and the geometry of the spectrum actually arises from the geometric properties of local toposes and etale geometric morphisms.