A unified framework for notions of algebraic theory

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Conceptual levels in study of algebra

1. **Algebra**
   A set (an object) equipped with an algebraic structure.
   E.g., the group $\mathfrak{S}_5$, the ring $\mathbb{Z}$.
2. **Algebraic theory**
   Specification of a type of algebras.
   E.g., the clone of groups, the operad of monoids.
3. **Notion of algebraic theory**
   Framework for a type of algebraic theories.
   E.g., $\{$clones$\}$, $\{$operads$\}$.

This talk: unified account of **notions of algebraic theory**.

Examples of notions of algebraic theory

1. **Clones/Lawvere theories** [Lawvere, 1963]
   Categorical equivalent of **universal algebra**.
2. **Symmetric operads, non-symmetric operads** [May, 1972]
   Originates in homotopy theory for **algebras-up-to-homotopy**.
3. **Clubs/generalised operads** [Burroni, 1971; Kelly, 1972]
   Classical approach to **categories with structure** [Kelly 1972].
   The 'globular operad' approach to higher categories [Batanin 1998, Leinster 2004].
4. **PROPs, PROs** [Mac Lane 1965]
   'Many-in, many-out' version of (non-)symmetric operads.
5. **Monads** [Godement, 1958; Linton, 1965; Eilenberg–Moore, 1965]
   Monads on **Set** = infinitary version of clones.

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Metatheory and theory

Definition

1. A **metatheory** is a monoidal category $\mathcal{M} = (\mathcal{M}, I, \otimes)$.
2. A **theory in** $\mathcal{M}$ is a monoid $\mathrm{T} = (T, e, m)$ in $\mathcal{M}$. That is,
   - T: an object of $\mathcal{M}$;
   - $e \colon I \longrightarrow T$;
   - $m \colon T \otimes T \longrightarrow T$;

   satisfying the associativity and unit laws.

'**Metatheory**' (technical term) formalises '**notion of algebraic theory**' (non-technical term).

Example: clones

Definition

The category **F**

- object: the sets $[n] = \{1, \ldots, n\}$ for all $n \in \mathbb{N}$;
- morphism: all functions.

Example: clones

Definition

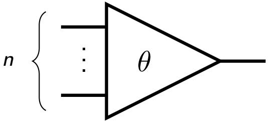
The **metatheory of clones** is the monoidal category $([\mathbf{F}, \mathbf{Set}], I, \bullet)$ where $\bullet$ is the *substitution monoidal product* [Kelly–Power 1993; Fiore–Plotkin–Turi 1999].

- $I = \mathbf{F}([1], -) \in [\mathbf{F}, \mathbf{Set}]$;
- for $X, Y \in [\mathbf{F}, \mathbf{Set}]$,

$$
\begin{aligned}
(Y \bullet X)_n &= \int^{[m] \in \mathbf{F}} Y_m \times (X_n)^m \\
&= \Big(\sum_{[m] \in \mathbf{F}} Y_m \times (X_n)^m \Big) / \text{action of } \mathbf{F} .
\end{aligned}
$$

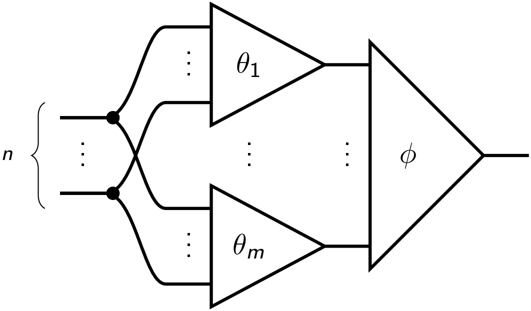
Example: clones

$\theta \in X_n$



An element of $(Y \bullet X)_n = \int^{[m]\in \mathbf{F}} Y_m \times (X_n)^m$ is:

$\phi \in Y_m, \theta_i \in X_n$



modulo action of $\mathbf{F}$.

Example: clones

Definition (classical; see e.g., [Taylor, 1993])

A **clone** C is given by

- $(C_n)_{n\in\mathbb{N}}$: a family of sets;
- $\forall n \in \mathbb{N}$, $\forall i \in \{1, \ldots, n\}$, an element $p_i^{(n)} \in C_n$;
- $\forall n, m \in \mathbb{N}$, a function

$$\circ_m^{(n)} \colon C_m \times (C_n)^m \longrightarrow C_n$$

satisfying the associativity and the unit axioms.

(In universal algebra, people sometimes omit C_0.)

Example: clones

Example

$\mathcal{C}$: category with finite products

$C \in \mathcal{C}$

The clone $\mathrm{End}(C)$ of **endo-multimorphisms on** C is defined by:

- $\mathrm{End}(C)_n = \mathcal{C}(C^n, C)$;
- $p_i^{(n)} \in \mathrm{End}(C)_n$ is the i-th projection $p_i^{(n)} \colon C^n \longrightarrow C$;
- $\circ_m^{(n)} \colon \mathrm{End}(C)_m \times (\mathrm{End}(C)_n)^m \longrightarrow \mathrm{End}(C)_n$ maps $(g, f_1, \dots, f_m)$ to $g \circ \langle f_1, \dots, f_m \rangle$:

$$C^n \xrightarrow{\langle f_1, \dots, f_m \rangle} C^m \xrightarrow{\quad g \quad} C.$$

(In fact, every clone is isomorphic to $\mathrm{End}(C)$ for some $\mathcal{C}$ and $C \in \mathcal{C}$.)

Example: clones

Proposition ([Kelly–Power, 1993; Fiore–Plotkin–Turi 1999])

There is an isomorphism of categories

$$\mathbf{Clo} \cong \mathbf{Mon}([\mathbf{F}, \mathbf{Set}], I, \bullet).$$

Example: clones

Recall again:

Definition

1. A **metatheory** is a monoidal category $\mathcal{M}$.
2. A **theory in** $\mathcal{M}$ is a monoid T in $\mathcal{M}$.

and:

Definition

The **metatheory of clones** is the monoidal category $([\mathbf{F}, \mathbf{Set}], I, \bullet)$.

Theories in $([\mathbf{F}, \mathbf{Set}], I, \bullet) =$ clones.

Example: symmetric operads

Definition

The category **P**

- object: the sets $[n] = \{1, ..., n\}$ for all $n \in \mathbb{N}$;
- morphism: all **bijections**.

Definition (cf. [Kelly 2005; Curien 2012; Hyland 2014])

The **metatheory of symmetric operads** is the monoidal category $([\mathbf{P}, \mathbf{Set}], I, \bullet)$.

Variables can be permuted, but cannot be copied nor discarded.

✓ $x_1 \cdot x_2 = x_2 \cdot x_1$; $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$.

✗ $x_1 \cdot x_1 = x_1$; $x_1 \cdot x_2 = x_1$.

Example: non-symmetric operads

Definition

The (discrete) category **N**

- object: the sets $[n] = \{1, ..., n\}$ for all $n \in \mathbb{N}$;
- morphism: all **identities**.

Definition (cf. [Kelly 2005; Curien 2012; Hyland 2014])

The **metatheory of non-symmetric operads** is the monoidal category $([\mathbf{N}, \mathbf{Set}], I, \bullet)$.

Variables cannot be permuted (nor discarded/copied).

✓ $(x_1 \cdot x_2) \cdot x_3 = x_1 \cdot (x_2 \cdot x_3)$; $\phi_m(\phi_{m'}(x_1)) = \phi_{mm'}(x_1)$.

✗ $x_1 \cdot x_2 = x_2 \cdot x_1$.

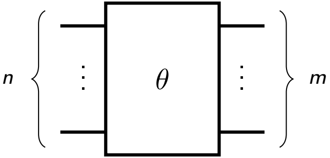
Example: PROs

Definition ([Mac Lane 1965])

A **PRO** is given by:

- a monoidal category T;
- an identity-on-objects, strict monoidal functor J from the (strict) monoidal category $\mathbf{N} = (\mathbf{N}, [0], +)$ to T.

For $n, m \in \mathbb{N}$, an element $\theta \in \mathrm{T}([n], [m])$ is depicted as



Example: PROs

Definition ([Bénabou 1973; Lawvere 1973])

$\mathcal{A}, \mathcal{B}$: (small)[1] categories
A **profunctor** (= **distributor** = **bimodule**) from $\mathcal{A}$ to $\mathcal{B}$ is a functor

$$H\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathbf{Set}.$$

Categories, profunctors and natural transformations form a bicategory.
$\Rightarrow$ For any category $\mathcal{A}$, the category $[\mathcal{A}^{\mathrm{op}} \times \mathcal{A}, \mathbf{Set}]$ of **endo-profunctors on $\mathcal{A}$** is monoidal.

[1] In this talk, I often ignore the size issues.

Example: PROs

Proposition (Folklore)

$\mathcal{A}$: *category*
To give **a monoid in** $[\mathcal{A}^{\mathrm{op}} \times \mathcal{A}, \mathbf{Set}]$ *is equivalent to giving* **a category** $\mathcal{B}$ **together with an identity-on-objects functor** $J\colon \mathcal{A} \longrightarrow \mathcal{B}$**.**

Recall:

Definition ([Mac Lane 1965])

A **PRO** is given by:

- a monoidal category T;
- an identity-on-objects, strict monoidal functor J from the (strict) monoidal category $\mathbf{N} = (\mathbf{N}, [0], +)$ to T.

Idea: use a **monoidal version of profunctors**.

Example: PROs

Definition ([Im–Kelly 1986, Gambino–Joyal 2017])

$\mathcal{M} = (\mathcal{M}, I_\mathcal{M}, \otimes_\mathcal{M}), \mathcal{N} = (\mathcal{N}, I_\mathcal{N}, \otimes_\mathcal{N})$: monoidal category
A **monoidal profunctor** from $\mathcal{M}$ to $\mathcal{N}$ is a lax monoidal functor

$$(H, h_\cdot, h) \colon \mathcal{N}^{\mathrm{op}} \times \mathcal{M} \longrightarrow (\mathbf{Set}, 1, \times).$$

That is:

- a functor $H \colon \mathcal{N}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{Set}$;
- a function $h_\cdot \colon 1 \longrightarrow H(I_\mathcal{N}, I_\mathcal{M})$;
- a natural transformation
  $h_{N,N',M,M'} \colon H(N', M') \times H(N, M) \longrightarrow H(N' \otimes_\mathcal{N} N, M' \otimes_\mathcal{M} M)$

satisfying the coherence axioms.

Example: PROs

Monoidal categories, *monoidal* profunctors and *monoidal* natural transformations form a bicategory.
$\Rightarrow$ For any monoidal category $\mathcal{M}$, the category $\mathcal{M}on\mathcal{C}at(\mathcal{M}^{\mathrm{op}} \times \mathcal{M}, \mathbf{Set})$ is monoidal.

Proposition

$\mathcal{M}$*: monoidal category*
To give **a monoid in** $\mathcal{M}on\mathcal{C}at(\mathcal{M}^{\mathrm{op}} \times \mathcal{M}, \mathbf{Set})$ *is equivalent to giving* **a monoidal category** $\mathcal{N}$ **together with an identity-on-objects strict monoidal functor** $J \colon \mathcal{M} \longrightarrow \mathcal{N}$.

Definition

The **metatheory of PROs** is the monoidal category $\mathcal{M}on\mathcal{C}at(\mathbf{N}^{\mathrm{op}} \times \mathbf{N}, \mathbf{Set})$.

Other examples

Definition

The **metatheory of PROPs** is the monoidal category $\mathcal{S}ym\mathcal{M}on\mathcal{C}at(\mathbf{P}^{\mathrm{op}} \times \mathbf{P}, \mathbf{Set})$ of symmetric monoidal endo-profunctors on $\mathbf{P}$.

Definition

$\mathcal{C}$: category with finite limits; S: cartesian monad on $\mathcal{C}$
The **metatheory of clubs over** S is the monoidal category $(\mathcal{C}/S1, \eta_1, \bullet)$.

Definition

$\mathcal{C}$: category.
The **metatheory of monads on** $\mathcal{C}$ is the monoidal category $\mathcal{E}nd(\mathcal{C}) = ([\mathcal{C}, \mathcal{C}], \mathrm{id}_{\mathcal{C}}, \circ)$.

Summary of this section

Quite often, **algebraic theories** can be seen as **monoid objects** in a suitable monoidal category.

$\Longrightarrow$ Take this observation as a starting point of the framework.

- **Metatheory** $=$ monoidal category (idea: formalise *notion of algebraic theory*).
- **Theory** in $\mathcal{M}$ $=$ monoid in $\mathcal{M}$ (idea: formalise *algebraic theory*).

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One theory, various models

Important feature of notions of algebraic theory (esp. of clones, operads, PROs, PROPs): **a single theory can have models in many categories.**

Example

A clone can have its models in **any category with finite products**. Models of the clone of groups

- in **Set**: ordinary groups;
- in **FinSet**: finite groups;
- in **Top**: topological groups;
- in **Mfd**: Lie groups;
- in **Grp**: abelian groups.

One theory, various models

Given a notion of algebraic theory, ...

1. first define a **notion of model**, i.e., what it means to be a model of a theory;
2. then consider a **model** of a theory following the notion of model.

Example

For clones, ...

1. $\mathcal{C}$: category with finite product
   a model in $\mathcal{C}$ of a clone T is an object $C \in \mathcal{C}$ together with a clone morphism $\mathrm{T} \longrightarrow \mathrm{End}(C)$;
2. find a particular model, i.e., an object $C \in \mathcal{C}$ together with a clone morphism $\mathrm{T} \longrightarrow \mathrm{End}(C)$.

One theory, various models

For **metatheories** (formalising notions of algebraic theory),
we introduce **metamodels** (formalising notions of model) later.

First we look at two simple subclasses of metamodels:

- enrichment;
- (left) oplax action.

Definitions

Definition (cf. [Campbell 2018])

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: metatheory; $\mathrm{T} = (T, e, m)$: theory in $\mathcal{M}$.

1. An **enrichment in $\mathcal{M}$** is a category $\mathcal{C}$ equipped with
   - $\langle -, - \rangle \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{M}$: a functor;
   - $j_C \colon I \longrightarrow \langle C, C \rangle$: a nat. tr.;
   - $M_{A,B,C} \colon \langle B, C \rangle \otimes \langle A, B \rangle \longrightarrow \langle A, C \rangle$: a nat. tr.

   satisfying the suitable coherence axioms.
   ($\forall C \in \mathcal{C}$, $\mathrm{End}(C) = (\langle C, C \rangle, j_C, M_{C,C,C})$: monoid in $\mathcal{M}$.)
2. A **model of T with respect to** $(\mathcal{C}, \langle -, - \rangle)$ is an object C of $\mathcal{C}$ together with a monoid morphism $\mathrm{T} \longrightarrow \mathrm{End}(C)$. That is,
   - $\chi \colon T \longrightarrow \langle C, C \rangle$: a morphism in $\mathcal{M}$

   commuting with unit and multiplication.

Definitions

$\mathcal{M}$: metatheory
T: theory in $\mathcal{M}$
$(\mathcal{C}, \langle -, - \rangle)$: enrichment in $\mathcal{M}$

We obtain the category

$$\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \langle -, - \rangle))$$

of **models** and **homomorphisms** together with a forgetful functor

$$\begin{tikzcd} \mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \langle -, - \rangle)) \arrow[d, "U"] \\ \mathcal{C} \end{tikzcd}$$

Example: clones $[\mathbf{F}, \mathbf{Set}]$

Definition

$\mathcal{C}$: category with finite products

The **standard $\mathcal{C}$-metamodel of clones** is the enrichment $\langle -,- \rangle \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow [\mathbf{F}, \mathbf{Set}]$ given by

- for $A, B \in \mathcal{C}$ and $[m] \in \mathbf{F}$,

$$\langle A, B \rangle_m \quad = \quad \mathcal{C}(A^m, B) \; .$$

So a model of a theory $\mathbb{T} = (T, e, m)$ consists of

- an object $C \in \mathcal{C}$;
- $$\frac{\dfrac{\text{a nat. tr. } \chi \colon T \longrightarrow \langle C, C \rangle \text{ (w/ cond.)}}{\forall m \in \mathbb{N}, \text{ a function } \chi_m \colon T_m \longrightarrow \mathcal{C}(C^m, C) \text{ (w/ cond.)}}}{\forall m \in \mathbb{N}, \forall \theta \in T_m, \text{ a morphism } [\![\theta]\!]_\chi \colon C^m \longrightarrow C \text{ (w/ cond.).}}$$

Example: PROs $\mathcal{M}on\mathcal{C}at(\mathbf{N}^{\mathrm{op}} \times \mathbf{N}, \mathbf{Set})$

Definition

$\mathcal{C} = (\mathcal{C}, I, \otimes)$: monoidal category

The **standard $\mathcal{C}$-metamodel of PROs** is the enrichment $\langle -, - \rangle \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{M}on\mathcal{C}at(\mathbf{N}^{\mathrm{op}} \times \mathbf{N}, \mathbf{Set})$ given by

- for $A, B \in \mathcal{C}$ and $n, m \in \mathbf{N}$,

$$\langle A, B \rangle([n], [m]) \; = \; \mathcal{C}(A^{\otimes m}, B^{\otimes n}) \; .$$

There are analogous enrichments for non-symmetric operads, symmetric operads and PROPs.

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Definitions

Definition

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: metatheory; $\mathrm{T} = (T, e, m)$: theory in $\mathcal{M}$.

1. A **(left) oplax action of** $\mathcal{M}$ is a category $\mathcal{C}$ equipped with
   - $*\colon \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$: a functor;
   - $\varepsilon_C\colon I * C \longrightarrow C$: a nat. tr.;
   - $\delta_{X,Y,C}\colon (Y \otimes X) * C \longrightarrow Y * (X * C)$: a nat. tr.

   satisfying the suitable coherence axioms.
2. A **model of** T **with respect to** $(\mathcal{C}, *)$ is an object C of $\mathcal{C}$ together with a left T-action γ on C. That is,
   - $\gamma\colon T * C \longrightarrow C$: a morphism in $\mathcal{C}$

   satisfying the left unit and associativity axioms.

Definitions

$\mathcal{M}$: metatheory
T: theory in $\mathcal{M}$
$(\mathcal{C}, *)$: oplax action of $\mathcal{M}$

We obtain the category

$$\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, *))$$

of **models** and **homomorphisms** together with a forgetful functor

$$\begin{tikzcd} \mathbf{Mod}(\mathsf{T}, (\mathcal{C}, *)) \arrow[d, "U"] \\ \mathcal{C} \end{tikzcd}$$

Example: monads $[\mathcal{C}, \mathcal{C}]$

Definition

$\mathcal{C}$: category

The **standard $\mathcal{C}$-metamodel of monads on $\mathcal{C}$** is the action $\mathrm{ev}_{\mathcal{C}} \colon [\mathcal{C}, \mathcal{C}] \times \mathcal{C} \longrightarrow \mathcal{C}$ given by evaluation.

So a model of a theory $\mathsf{T} = (T, m, e)$ consists of

- an object $C \in \mathcal{C}$;
- a morphism $\gamma \colon TC \longrightarrow C$ in $\mathcal{C}$

satisfying the left unit and associativity axioms.That is, an **Eilenberg–Moore algebra** of T.

$$\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \mathrm{ev}_{\mathcal{C}})) \cong \mathcal{C}^{\mathsf{T}}$$

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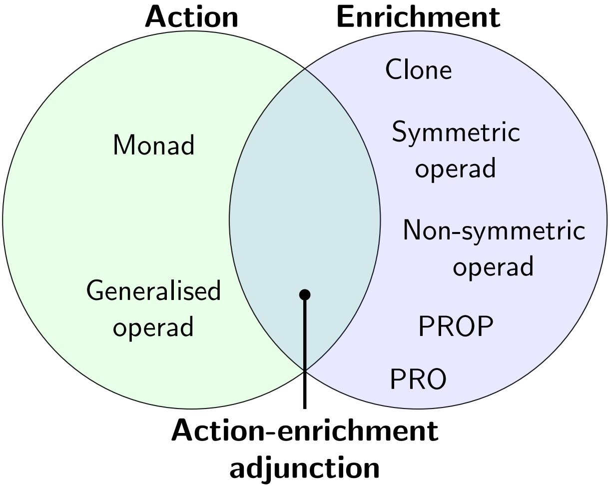
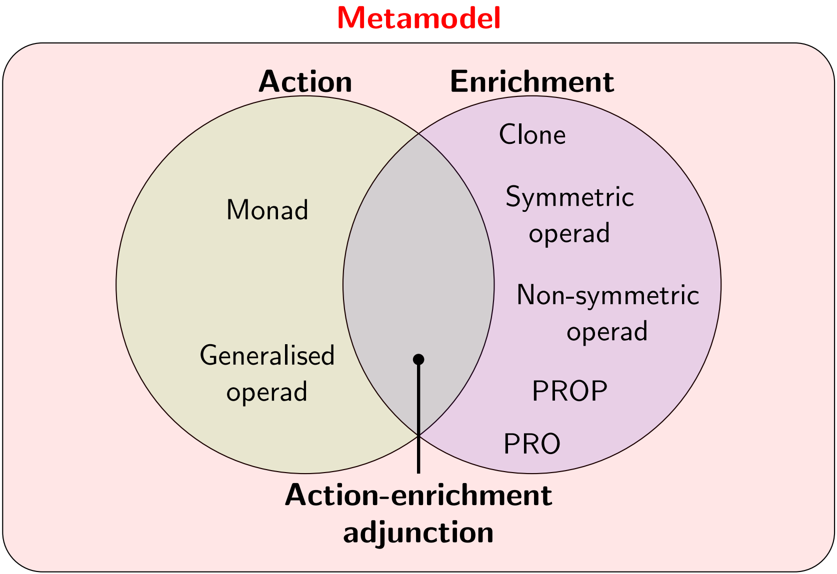
Unifying the two approaches



Unifying the two approaches via metamodels



Metamodels and models

Definition (cf. [Campbell 2018])

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: metatheory; $\mathrm{T} = (T, e, m)$: theory in $\mathcal{M}$.

1. A **metamodel of** $\mathcal{M}$ is a category $\mathcal{C}$ together with:
   - $$\begin{array}{rcl} \Phi\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \longrightarrow & \mathbf{Set}\text{: a functor;} \\ (X, A, B) & \longmapsto & \Phi_X(A, B) \end{array}$$
   - $(\phi_\cdot)_C \colon 1 \longrightarrow \Phi_I(C, C)$: a nat. tr.;
   - $(\phi_{X,Y})_{A,B,C} \colon \Phi_Y(B, C) \times \Phi_X(A, B) \longrightarrow \Phi_{Y \otimes X}(A, C)$: nat. tr.

   satisfying the suitable coherence axioms.
2. A **model of** T **with respect to** $(\mathcal{C}, \Phi)$ is (C, ξ) where
   - $C \in \mathcal{C}$;
   - $\xi \in \Phi_T(C, C)$;

   satisfying the suitable coherence axioms.

Incorporating enrichments

Given an enrichment

$$\langle -, - \rangle \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{M},$$

define a metamodel

$$\Phi \colon \mathcal{M}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$$

by

$$\Phi_X(A, B) \;=\; \mathcal{M}(X, \langle A, B \rangle).$$

For any theory $\mathsf{T} = (T, e, m)$ in $\mathcal{M}$, we have

$$\frac{\text{a model } (C, \chi \colon T \longrightarrow \langle C, C \rangle) \quad \text{(via enrichment)}}{\text{a model } (C, \xi \in \Phi_T(C, C)) \qquad \text{(via metamodel).}}$$

Incorporating oplax actions

Given an oplax action

$$*\colon \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C},$$

define a metamodel

$$\Phi\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathbf{Set}$$

by

$$\Phi_X(A, B) = \mathcal{C}(X * A, B).$$

For any theory $\mathsf{T} = (T, e, m)$ in $\mathcal{M}$, we have

$$\frac{\text{a model } (C, \gamma\colon T * C \longrightarrow C) \quad \text{(via oplax action)}}{\text{a model } (C, \xi \in \Phi_T(C, C)) \qquad \text{(via metamodel).}}$$

Categories of models as hom-categories

$\mathcal{M}$: metatheory

- Metamodels of $\mathcal{M}$ form a 2-category $\mathcal{MMod}(\mathcal{M})$.
- A theory $\mathsf{T} = (T, e, m)$ in $\mathcal{M}$ can be considered as a metamodel $\Phi^{(\mathsf{T})}$ of $\mathcal{M}$ in the terminal category 1:

$$\begin{aligned} \Phi^{(\mathsf{T})} \colon \mathcal{M}^{\mathrm{op}} \times 1^{\mathrm{op}} \times 1 &\longrightarrow \mathbf{Set} \\ (X, *, *) \quad &\longmapsto \mathcal{M}(X, T). \end{aligned}$$

- For any theory T in $\mathcal{M}$ and a metamodel $(\mathcal{C}, \Phi)$ of $\mathcal{M}$, the category of models $\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \Phi))$ is isomorphic to the hom-category

$$\mathcal{MMod}(\mathcal{M})((1, \Phi^{(\mathsf{T})}), (\mathcal{C}, \Phi)).$$

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Morphisms of metatheories

Motivation: uniform method to relate different notions of algebraic theory.

$\Rightarrow$ We want a notion of **morphism of metatheories**, which suitably acts on metamodels.

Morphisms of metatheories

Definition (cf. [Im–Kelly 1986, Gambino–Joyal 2017])

$\mathcal{M} = (\mathcal{M}, I_{\mathcal{M}}, \otimes_{\mathcal{M}}), \mathcal{N} = (\mathcal{N}, I_{\mathcal{N}}, \otimes_{\mathcal{N}})$: metatheories
A **morphism of metatheories** from $\mathcal{M}$ to $\mathcal{N}$, written as

$$H = (H, h_., h) \colon \mathcal{M} \looparrowright \mathcal{N},$$

is a monoidal profunctor from $\mathcal{M}$ to $\mathcal{N}$, i.e., a lax monoidal functor

$$(H, h_., h) \colon \mathcal{N}^{\mathrm{op}} \times \mathcal{M} \longrightarrow (\mathbf{Set}, 1, \times).$$

Specifically:

- a functor $H \colon \mathcal{N}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{Set}$;
- a function $h_. \colon 1 \longrightarrow H(I_{\mathcal{N}}, I_{\mathcal{M}})$;
- a natural transformation
  $h_{N,N',M,M'} \colon H(N', M') \times H(N, M) \longrightarrow H(N' \otimes_{\mathcal{N}} N, M' \otimes_{\mathcal{M}} M)$

satisfying the coherence axioms.

Relation to lax/oplax monoidal functors

- A **lax** monoidal functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ induces a morphism $F_*\colon \mathcal{M} \longrightarrow\!\!\!\!\!\!\mapsto \mathcal{N}$ defined as

$$\begin{aligned} F_*\colon \mathcal{N}^{\mathrm{op}} \times \mathcal{M} &\longrightarrow \mathbf{Set} \\ (N, M) &\longmapsto \mathcal{N}(N, FM). \end{aligned}$$

- An **oplax** monoidal functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ induces a morphism $F^*\colon \mathcal{N} \longrightarrow\!\!\!\!\!\!\mapsto \mathcal{M}$ defined as

$$\begin{aligned} F^*\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{N} &\longrightarrow \mathbf{Set} \\ (M, N) &\longmapsto \mathcal{N}(FM, N). \end{aligned}$$

- A **strong** monoidal functor $F\colon \mathcal{M} \longrightarrow \mathcal{N}$ induces both F_* and F^*, and they form an adjunction (in the bicategory of metatheories)

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "F_*" description] \arrow[r, phantom, "\perp"] & \mathcal{N}. \arrow[l, bend left, "F^*" description] \end{tikzcd}$$

Morphisms of metatheories act on metamodels

$\mathcal{M}, \mathcal{N}$: metatheory
$H = (H, h_{\cdot}, h)\colon \mathcal{M} \longrightarrow\!\!\!\!\!\!\mapsto \mathcal{N}$: morphism of metatheories
$(\mathcal{C}, \Phi)$: metamodel of $\mathcal{M}$

$\Rightarrow$ We have a metamodel $(\mathcal{C}, H\Phi)$ of $\mathcal{N}$ defined as:

$$\begin{aligned} H\Phi\colon \mathcal{N}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\longrightarrow \mathbf{Set} \\ (N, A, B) &\longmapsto \int^{M \in \mathcal{M}} H(N, M) \times \Phi_M(A, B). \end{aligned}$$

$\mathcal{MM}od(-)$ extends to a pseudofunctor from the bicategory of metatheories to the 2-category of 2-categories 2-$\mathcal{C}at$.

Isomorphisms between categories of models

$\mathcal{M}, \mathcal{N}$: metatheory
$F\colon \mathcal{M} \longrightarrow \mathcal{N}$: **strong monoidal functor**
T: theory in $\mathcal{M}$
$(\mathcal{C}, \Phi)$: metamodel of $\mathcal{N}$

We can take ...

- the category of models $\mathbf{Mod}(F_*\mathrm{T}, (\mathcal{C}, \Phi))$ (using $\mathcal{N}$);
- the category of models $\mathbf{Mod}(\mathrm{T}, (\mathcal{C}, F^*\Phi))$ (using $\mathcal{M}$).

By the 2-adjunction

$$\begin{tikzcd} \mathcal{MM}od(\mathcal{M}) \arrow[rr, bend left, "\mathcal{MM}od(F_*)"] \arrow[rr, phantom, "\perp"] & & \mathcal{MM}od(\mathcal{N}), \arrow[ll, bend left, "\mathcal{MM}od(F^*)"] \end{tikzcd}$$

these two categories of models are canonically isomorphic.

Isomorphisms between categories of models

Example

$[\mathbf{F}, \mathbf{Set}]$: the metatheory of clones
$[\mathbf{Set}, \mathbf{Set}]$: the metatheory of monads on **Set**

Using the inclusion functor $J\colon \mathbf{F} \longrightarrow \mathbf{Set}$, we obtain a strong monoidal functor $\mathrm{Lan}_J\colon [\mathbf{F}, \mathbf{Set}] \longrightarrow [\mathbf{Set}, \mathbf{Set}]$.

T: clone = theory in $[\mathbf{F}, \mathbf{Set}]$
$(\mathbf{Set}, \Phi)$: the standard **Set**-metamodel of $[\mathbf{Set}, \mathbf{Set}]$

We have:

- $\mathrm{Lan}_{J*}\mathsf{T}$: the finitary monad corresponding to T;
- $(\mathbf{Set}, \mathrm{Lan}_J{}^*\Phi)$: the standard **Set**-metamodel of $[\mathbf{F}, \mathbf{Set}]$.

$\Rightarrow$ The classical result on compatibility of semantics of clones (= Lawvere theories) and monads on **Set** [Linton, 1965].

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The structure-semantics adjunctions

$\mathcal{M}$: metatheory
$(\mathcal{C}, \Phi)$: metamodel of $\mathcal{M}$

We have a functor

$$\mathrm{Sem} \colon \mathbf{Mon}(\mathcal{M})^{\mathrm{op}} \longrightarrow \mathbf{CAT}/\mathcal{C}$$

mapping T to $U \colon \mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \Phi)) \longrightarrow \mathcal{C}$.

Q. Does Sem have a left adjoint?
$\Longrightarrow$ If so, the left adjoint is called the **structure functor** [Lawvere 1963, Linton 1966, Linton 1969, Dubuc, 1970, Street 1972, Avery 2017].

Example: clones

$[\mathbf{F}, \mathbf{Set}]$: the metatheory of clones
$(\mathbf{Set}, \Phi)$: the standard metamodel of $[\mathbf{F}, \mathbf{Set}]$ in $\mathbf{Set}$

One can try

$$\text{“Str”} : \mathbf{CAT}/\mathbf{Set} \longrightarrow \mathbf{Mon}([\mathbf{F}, \mathbf{Set}])^{\mathrm{op}}$$

mapping $(V \colon \mathcal{A} \longrightarrow \mathbf{Set})$ to $\mathrm{End}(V)$. (So $\mathrm{End}(V)_n = [\mathcal{A}, \mathbf{Set}]((-)^n \circ V, V)$.)

$\Longrightarrow$ Works *except for the size issue.* Call $V \in \mathbf{CAT}/\mathbf{Set}$ **tractable** if $[\mathcal{A}, \mathbf{Set}]((-)^n \circ V, V)$ is small for all n. Then [Linton 1966]:

$$\mathbf{Mon}([\mathbf{F}, \mathbf{Set}])^{\mathrm{op}} \underset{\mathrm{Sem}}{\overset{\mathrm{Str}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} (\mathbf{CAT}/\mathbf{Set})_{\mathrm{tr}}$$

Example: monads

$[\mathcal{C},\mathcal{C}]$: the metatheory of monads on $\mathcal{C}$
$(\mathcal{C},\Phi)$: the standard metamodel of $[\mathcal{C},\mathcal{C}]$ in $\mathcal{C}$

One can try

$$\text{“Str”} : \mathbf{CAT}/\mathcal{C} \longrightarrow \mathbf{Mon}([\mathcal{C},\mathcal{C}])^{\mathrm{op}}$$

mapping $(V : \mathcal{A} \longrightarrow \mathcal{C})$ to $\mathrm{Ran}_V V$ (the **codensity monad** of V).

$\Longrightarrow$ Works *provided* $\mathrm{Ran}_V V$ *exists.* Call $V \in \mathbf{CAT}/\mathcal{C}$ **tractable** if $\mathrm{Ran}_V V$ exists. Then [Dubuc 1970]:

$$\begin{tikzcd} \mathbf{Mon}([\mathcal{C},\mathcal{C}])^{\mathrm{op}} \arrow[r, shift right, phantom, "\perp"] & (\mathbf{CAT}/\mathcal{C})_{\mathrm{tr}} \arrow[l, shift right, "\mathrm{Str}"'] \end{tikzcd} \quad \mathbf{Mon}([\mathcal{C},\mathcal{C}])^{\mathrm{op}} \xrightarrow[\mathrm{Sem}]{} (\mathbf{CAT}/\mathcal{C})_{\mathrm{tr}}$$

Observation

It seems difficult to find a suitable "tractability" condition for each $\mathcal{M}$ and $(\mathcal{C}, \Phi)$ so that the restricted adjunction exists.

$\Longrightarrow$ What if we try to seek an "extended" adjunction rather than "restricted" one? (Cf. [Linton 1969, Avery 2017])

The general structure-semantics adjunctions

Theorem

$\mathcal{M}$: metatheory
$(\mathcal{C}, \Phi)$: metamodel of $\mathcal{M}$

Consider $\widehat{\mathcal{M}} = [\mathcal{M}^{\mathrm{op}}, \mathbf{SET}]$ with the convolution monoidal structure [Day 1970]. Then we have an adjunction

$$\mathbf{Mon}(\widehat{\mathcal{M}})^{\mathrm{op}} \underset{\mathrm{Sem}}{\overset{\mathrm{Str}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{CAT}/\mathcal{C}.$$

This restricts to the classical structure-semantics adjunctions for clones and monads.

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Conclusion

- Unified account of various **notions of algebraic theory** and their semantics.
- Morphism of metatheories as a uniform method to compare different notions of algebraic theory.
  - Strong monoidal functor $\mapsto$ adjoint pair of morphisms $\mapsto$ isomorphisms of categories of models.

Future work:

- Clearer understanding of the scope of our framework.
  - In particular, intrinsic characterisation of the forgetful functors $U\colon \mathbf{Mod}(\mathsf{T},(\mathcal{C},\Phi)) \longrightarrow \mathcal{C}$ arising in our framework (a Beck type theorem).
- Incorporate various constructions on algebraic theories: sums, distributive laws, tensor products, ...

The relation between action and enrichment

According to a categorical folklore [Kelly, Gordon–Power, ...]:

Proposition

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: monoidal category (metatheory); $\mathcal{C}$: category

1. $*\colon \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C}$*: oplax left action s.t. for each* $C \in \mathcal{C}$
$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "(-) * C"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, dashed, "\exists \langle C, - \rangle"] \end{tikzcd} .$$
Then $\langle -, - \rangle$ defines an enrichment.

2. $\langle -, - \rangle \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{M}$*: enrichment s.t. for each* $C \in \mathcal{C}$
$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, dashed, "\exists (-) * C"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\langle C, - \rangle"] \end{tikzcd} .$$
Then $*$ defines an oplax left action.

The relation between action and enrichment

Proposition

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: *metatheory;* $\quad \mathsf{T} = (T, e, m)$: *theory in* $\mathcal{M}$
$(\mathcal{C}, *\colon \mathcal{M} \times \mathcal{C} \longrightarrow \mathcal{C})$: *oplax action*
$(\mathcal{C}, \langle -,- \rangle\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{M})$: *enrichment*

If for each $C \in \mathcal{C}$

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "(-) * C"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\langle C, - \rangle"] \end{tikzcd}$$

(compatible with structure morphisms $\delta, \varepsilon, M, j$*) then*

$$\frac{\text{a model } \gamma\colon\ T * C \longrightarrow C \quad \text{(via oplax action)}}{\text{a model } \chi\colon\ T \longrightarrow \langle C, C \rangle \quad \text{(via enrichment).}}$$

So $\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, *)) \cong \mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \langle -,- \rangle))$.

The relation between action and enrichment

Example

$([\mathbf{F}, \mathbf{Set}], I, \bullet)$: the metatheory of clones

For each $S \in \mathbf{Set}$

$$\begin{tikzcd} {[\mathbf{F}, \mathbf{Set}]} \arrow[r, bend left, "(-) * S"] \arrow[r, phantom, "\perp"] & \mathbf{Set} \arrow[l, bend left, "\langle S, - \rangle"] \end{tikzcd}$$

where

$$X * S \quad = \quad \int^{[m] \in \mathbf{F}} X_m \times S^m$$

and

$$\langle S, R \rangle_m \quad = \quad \mathbf{Set}(S^m, R) \ .$$

The category of models

$\mathcal{M} = (\mathcal{M}, I, \otimes)$: metatheory; $\mathsf{T} = (T, e, m)$: theory in $\mathcal{M}$;
$(\mathcal{C}, \Phi)$: metamodel of $\mathcal{M}$

We obtain a category

$$\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \Phi)) \quad \text{(or, } \mathbf{Mod}(\mathsf{T}, \mathcal{C}) \text{ for short)},$$

a functor

$$\begin{tikzcd} \mathbf{Mod}(\mathsf{T}, \mathcal{C}) \arrow[d, "U"] \\ \mathcal{C} \end{tikzcd}$$

and a natural transformation

$$\begin{tikzcd} \mathbf{Mod}(\mathsf{T}, \mathcal{C}) \arrow[r, "\mathrm{Hom}_{\mathbf{Mod}(\mathsf{T},\mathcal{C})}", "\shortmid" marking] \arrow[d, "U"'] \arrow[dr, phantom, "\Downarrow u"] & \mathbf{Mod}(\mathsf{T}, \mathcal{C}) \arrow[d, "U"] \\ \mathcal{C} \arrow[r, "\Phi_T"', "\shortmid" marking] & \mathcal{C} \end{tikzcd}$$

The category of models

$$
\begin{tikzcd}
\mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "U"'] \arrow[dr, phantom, "\Downarrow u"] & \mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "\Phi_T", "|" marking] \arrow[r, bend right=50, "\Phi_{T\otimes T}"', "|" marking] \arrow[r, phantom, shift right=3ex, "\Downarrow\Phi_m"] & \mathcal{C}
\end{tikzcd}
$$

$$
=
\begin{tikzcd}
\mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "U"'] \arrow[dr, phantom, "\Downarrow u"] & \mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "U"] \arrow[dr, phantom, "\Downarrow u"] & \mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "\Phi_T", "|" marking] \arrow[rr, bend right=30, "\Phi_{T\otimes T}"', "|" marking] & \mathcal{C} \arrow[r, "\Phi_T", "|" marking] \arrow[d, phantom, "\Downarrow\phi_{T,T}"] & \mathcal{C} \\
 & {} &
\end{tikzcd}
$$

The category of models

$$\begin{tikzcd}
\mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "U"'] \arrow[dr, phantom, "\Downarrow u"] & \mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "\Phi_T", "|" marking] \arrow[r, bend right, "\Phi_I"', "|" marking] \arrow[r, phantom, shift right=4, "\Downarrow\Phi_e"] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[d, "U"'] \arrow[dr, phantom, "\Downarrow U"] & \mathbf{Mod}(\mathsf{T},\mathcal{C}) \arrow[d, "U"] \\
\mathcal{C} \arrow[r, "\mathrm{Hom}", "|" marking] \arrow[r, bend right, "\Phi_I"', "|" marking] \arrow[r, phantom, shift right=4, "\Downarrow\phi_\cdot"] & \mathcal{C}
\end{tikzcd}$$

In fact, $(\mathbf{Mod}(\mathsf{T},\mathcal{C}), U, u)$ is the **universal** one as such.

$\Longrightarrow$ What is a suitable language to express this universality?

Categories of models as double limits

Definition ([Grandis–Paré 1999])

The pseudo double category $\mathbb{P}\mathbf{rof}$

- object: category;
- vertical 1-cell: functor; $(G \circ H) \circ K = G \circ (H \circ K)$
- horizontal 1-cell: profunctor; $(X \circ Y) \circ Z \cong X \circ (Y \circ Z)$
- square: natural transformation.

Monoidal category $\mathcal{M}$ defines a vertically trivial (one object, one vertical 1-cell) pseudo double category $\mathbb{H}\Sigma\mathcal{M}$.
$(\mathbf{Mod}(\mathsf{T}, (\mathcal{C}, \Phi)), U, u)$ is the **double limit** [Grandis–Paré 1999] of the lax double functor

$$\mathbb{H}\Sigma(\Delta^{\mathrm{op}}) \xrightarrow{\mathsf{T}^{\mathrm{op}}} \mathbb{H}\Sigma(\mathcal{M}^{\mathrm{op}}) \xrightarrow{\Phi} \mathbb{P}\mathbf{rof} .$$