Lax factorisation systems and categories of partial maps

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Plan of the talk

- Introduction on partial maps and **Ord**-enrichments;
- Lax weak orthogonality;
- Lax factorisation systems;
- Lax factorisation systems for $\mathcal{P}(\mathscr{C})$ generated by total maps;
- Extracting an oplax WFS for $\mathcal{P}(\mathscr{C})$ from a WFS on $\mathscr{C}$;
- Lax factorisation systems for $\mathcal{P}(\mathbf{Set})$;
- Open problems on lax factorisation for some pointed **Ord**-categories.

Ord-enriched categories

A category $\mathscr{C}$ is **Ord**-enriched if on every $\mathscr{C}(A, B)$ is defined a partial order among morphisms. This order must be compatible with compositions, i.e. for any h, k morphisms in $\mathscr{C}$:

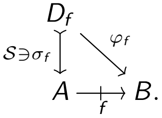
$$f \leq g \qquad \Longrightarrow \qquad f \circ h \leq g \circ h \qquad k \circ f \leq k \circ g.$$

Any **Ord**-enriched category $\mathscr{C}$ gives rise to the category $\mathscr{C}^2_{lax}$, whose objects are $\mathrm{Arr}(\mathscr{C})$ and morphisms are lax commutative squares as:

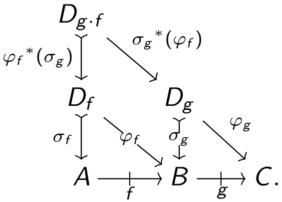
$$(u, v) : f \longrightarrow g \qquad\qquad \begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \nearrow\!\!/ & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D. \end{array}$$

Partial maps

Let $\mathscr{C}$ be a category with pullbacks and $\mathcal{S}$ a class of monomorphisms closed under pullbacks and composition. A partial map is a span

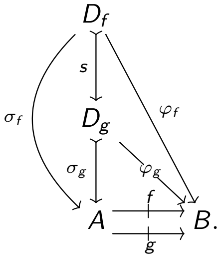


If $\sigma_f = \mathrm{id}_A$ we say that f is **total**.
They form a **category of partial maps** $\mathcal{P}(\mathscr{C})$ whose composition law is



Discrete **Ord**-enrichment

A category of partial maps comes equipped with a partial order among arrows. We define $f \preceq g$ if there exists $s \in \mathcal{S}$ such that

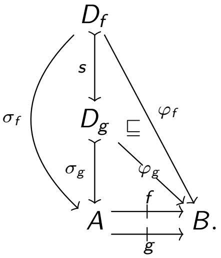


Properties

- *If f is total and $f \preceq g$ then $f = g$.*
- *If $g \cdot f$ is total, then f is total.*

General **Ord**-enrichment

If $\mathscr{C}$ has an **Ord**-enrichment denoted by $\sqsubseteq$ and morphisms in $\mathcal{S}$ are faithful, we can induce an **Ord**-enrichment on $\mathcal{P}(\mathscr{C})$.
We define $f \leq g$ if there exists $s \in \mathcal{S}$ such that



Properties

- *If f is total and $f \leq g$ then g is total.*
- *If $g \cdot f$ is total, then f is total.*

$\mathcal{P}$ (**Set**)

Set is a suitable category for this construction. In fact it has pullbacks and partial domains in **Set** are equivalence classes of monomorphisms.

Properties

- *Domains for compositions are* $D_{g \cdot f} = \varphi_f^{-1}(D_g)$.
- *The discrete* Ord*-enrichment here induces that* $f \preceq g$ *iff* f *is a restriction of* g.

Lax weak orthogonality

Two morphisms in $\mathscr{C}$ are said **laxly weakly orthogonal**, denoted by $f \pitchfork g$, if for any lax square $(u, v) : f \longrightarrow g$, there exists a morphism $d : cod(f) \longrightarrow dom(g)$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{\;u\;} & C \\
{\scriptstyle f}\downarrow & \overset{\wedge|}{} \nearrow^{d} \underset{\wedge|}{} & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\;v\;]{} & D.
\end{array}
\qquad\qquad
\begin{cases} u \leq d \circ f \\ g \circ d \leq v \end{cases}
$$

The morphism d is called **lax diagonal morphism** for (u, v).
Given two classes of morphisms $\mathcal{H}, \mathcal{H}' \subseteq \mathscr{C}^2_{lax}$ we write:

- $\mathcal{H} \pitchfork \mathcal{H}'$ if for every $h \in \mathcal{H}$ e $h' \in \mathcal{H}'$, then $h \pitchfork h'$;
- $\mathcal{H}^{\pitchfork} = \{f | h \pitchfork f \text{ for any } h \in \mathcal{H}\}$;
- ${}^{\pitchfork}\mathcal{H} = \{f | f \pitchfork h \text{ for any } h \in \mathcal{H}\}$.

Adjoint morphisms

As usual, in an **Ord**-enriched category, two morphisms $f : A \to B$ and $g : B \to A$ form an adjunction $f \dashv g$ if we have the following 2-cells:

$$\begin{cases} \mathrm{id}_A \leq g \cdot f \\ f \cdot g \leq \mathrm{id}_B. \end{cases}$$

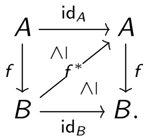
Proposition

Let f be a morphism in $\mathscr{C}$. Then the following are equivalent:

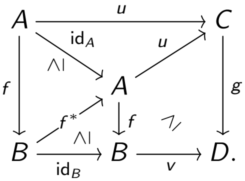
1. *f is laxly weakly orthogonal to itself;*
2. *f is a left adjoint;*
3. $f \pitchfork \mathscr{C}^2_{\mathrm{lax}}$;
4. $\mathscr{C}^2_{\mathrm{lax}} \pitchfork f$.

Sketch proof

1.⇒2. This is shown considering the square



2.⇒3.⇒1. Considering a lax square $(u, v) : f \longrightarrow g$



2.⇒4. Analogous.

3.⇒1. and 4.⇒1. are trivial.

Adjoint morphisms for partial maps

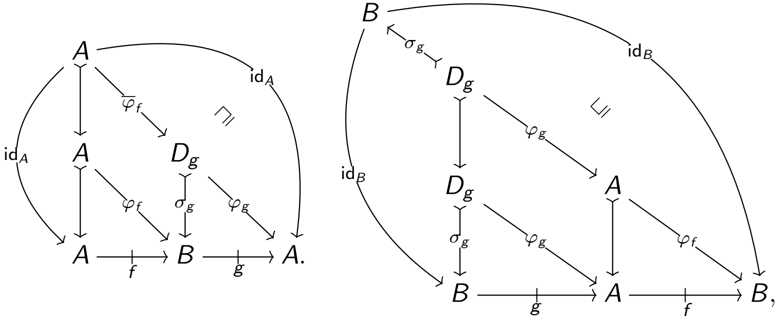
Proposition

Let $\mathscr{C}$ be an Ord*-enriched category and $\mathcal{P}(\mathscr{C})$ a category of partial maps. A pair of morphisms in $\mathcal{P}(\mathscr{C})$ constitute an adjunction $f \dashv g$ if and only if f is total and $\varphi_f = \sigma_g \cdot \overline{\varphi}_f$ such that $\overline{\varphi}_f \dashv \varphi_g$ in $\mathscr{C}$.*

Remark

In particular we deduce that if the Ord*-enrichment on $\mathscr{C}$ is discrete, then a pair of morphisms constitute an adjunction $f \dashv g$ if and only if $f = (\mathrm{id}_A, \sigma)$ and $g = (\sigma, \mathrm{id}_A)$, for some $\sigma \in \mathcal{S}$.*

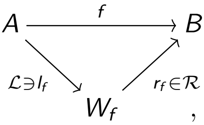
Consider the adjunction $f \dashv g$ in $\mathcal{P}(\mathscr{C})$. The 2-cell $\mathrm{id}_A \leq g \cdot f$ yields that $g \cdot f$ is total and therefore f is total, we can write



$$\begin{cases} \mathrm{id}_A \sqsubseteq \varphi_g \cdot \overline{\varphi}_f \\ \varphi_f \cdot \varphi_g = \sigma_g \cdot \overline{\varphi}_f \cdot \varphi_g \sqsubseteq \sigma_g . \end{cases}$$

This yields the existence of the adjunction $\overline{\varphi}_f \dashv \varphi_g$ in $\mathscr{C}$. The other direction is proved simply by calculation.

A **lax weak prefactorisation system** is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$. Moreover, if every morphism f admits an $(\mathcal{L}, \mathcal{R})$-factorisation



,

then $(\mathcal{L}, \mathcal{R})$ is called **lax weak factorisation systems**.
We remark that for any lax weak prefactorisation system $(\mathcal{L}, \mathcal{R})$, $\mathcal{L} \cap \mathcal{R} = \mathbf{LA}(\mathscr{C})$.

Proposition

Given any $\mathcal{H} \subseteq \mathscr{C}^2_{lax}$, then $\left({}^{\pitchfork}\mathcal{H}, \left({}^{\pitchfork}\mathcal{H}\right)^{\pitchfork}\right)$ and $\left({}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right), \mathcal{H}^{\pitchfork}\right)$ are lax weak prefactorisation systems.

Lax functorial factorisation systems

We denote by

$$(- \circ -) : \mathscr{C}^2_{lax} \times_{\mathscr{C}} \mathscr{C}^2_{lax} \longrightarrow \mathscr{C}^2_{lax}$$

the composition functor.

Definition

A **lax functorial factorisation system** is a functor $F : \mathscr{C}^2_{lax} \longrightarrow \mathscr{C}^2_{lax} \times_{\mathscr{C}} \mathscr{C}^2_{lax}$, such that $(- \circ -)\, F = Id_{\mathscr{C}^2_{lax}}$.

Composition of F with the projections of $\mathscr{C}^2_{lax} \times_{\mathscr{C}} \mathscr{C}^2_{lax}$ yields the definition of the following functors

$$L, R : \mathscr{C}^2_{lax} \longrightarrow \mathscr{C}^2_{lax},$$
$$K : \mathscr{C}^2_{lax} \longrightarrow \mathscr{C}.$$

To avoid ambiguity, a lax functorial factorisation system will be denoted by (F, L, R, K).

Lax functorial factorisation systems

Every functorial factorisation system comes equipped with two natural transformations $\eta : \mathsf{Id}_{\mathscr{C}^2_{\text{lax}}} \Longrightarrow R$ and $\varepsilon : L \Longrightarrow \mathsf{Id}_{\mathscr{C}^2_{\text{lax}}}$. These transformations operate as follows:

$$\begin{array}{ccc} A & \xrightarrow{Lf} & Kf \\ {\scriptstyle f}\downarrow & \eta_f & \downarrow{\scriptstyle Rf} \\ B & \xrightarrow[\mathrm{id}_B]{} & B, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ {\scriptstyle Lf}\downarrow & \varepsilon_f & \downarrow{\scriptstyle f} \\ Kf & \xrightarrow[Rf]{} & B. \end{array}$$

Differently from ordinary factorisation systems, these transformations are not strict in general, but only oplax. In fact, considering η, for any lax square $(u, v) : f \longrightarrow g$ we have that

$$
\begin{array}{ccccc}
A & \xrightarrow{\;u\;} & C & \xrightarrow{\;Lg\;} & Kg \\
{\scriptstyle f}\big\downarrow & \nwarrow & \big\downarrow{\scriptstyle g} & \eta_g & \big\downarrow{\scriptstyle Rg} \\
B & \xrightarrow[\;v\;]{} & D & \xrightarrow[\;\mathrm{id}_D\;]{} & D
\end{array}
\quad \leq \quad
\begin{array}{ccccc}
A & \xrightarrow{\;Lf\;} & Kf & \xrightarrow{\;K(u,v)\;} & Kg \\
{\scriptstyle f}\big\downarrow & \eta_f & \big\downarrow{\scriptstyle Rf} & \nwarrow & \big\downarrow{\scriptstyle Rg} \\
B & \xrightarrow[\;\mathrm{id}_B\;]{} & B & \xrightarrow[\;v\;]{} & D,
\end{array}
$$

since $Lg \cdot u \leq K(u, v) \cdot Lf$ by definition of lax functorial factorisation. This amounts to have that $\eta_g \cdot (u, v) \leq R(u, v) \cdot \eta_f$. Similarly one can prove that ε is an oplax natural transformation as well.

Lax functorial weak factorisation systems

Definition

A lax weak factorisation system $(\mathcal{L}, \mathcal{R})$ is functorial if there exists a functorial factorisation system (F, L, R, K) such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$.

Let (F, L, R, K) be a lax functorial factorisation system. We define the two classes

$$\mathcal{L}_F = \{f | f \pitchfork Rf\} \qquad \mathcal{R}_F = \{f | Lf \pitchfork f\} .$$

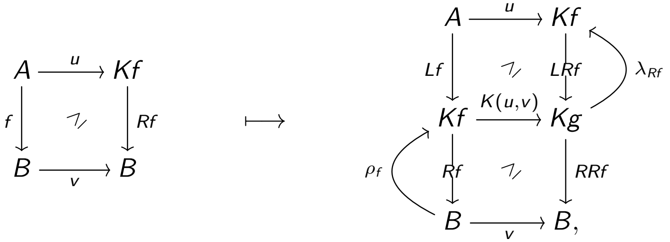
Then (F, L, R, K) is **lax predistributive** if, for every morphism f, $Lf \in \mathcal{L}_F$ and $Rf \in \mathcal{R}_F$, i.e. $Lf \pitchfork RLf$ and $LRf \pitchfork Rf$.

Let (F, L, R, K) be a predistributive lax functorial factorisation system. Then we can give the equivalent definitions

$$\mathcal{L}_F = \{f \mid \eta_f \text{ has a lax diagonal morphism}\}$$
$$\mathcal{R}_F = \{f \mid \varepsilon_f \text{ has a lax diagonal morphism}\}.$$

We will denote such lax liftings for any f by

We consider f such that ρ_f is a lax diagonal morphism for η_f and a lax square

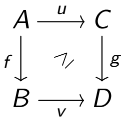


where ρ_f is a diagonal morphism of η_f existing by assumption and λ_{Rf} is a diagonal morphism of ε_{Rf} existing since $LRf \pitchfork Rf$.
This yields that $\Delta = \lambda_{Rf} \cdot K(u, v) \cdot \rho_f$ is the diagonal morphism sought and $f \pitchfork Rf$.

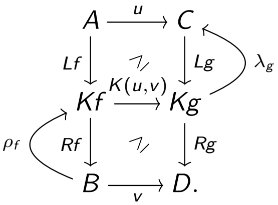
Theorem

Let (F, L, R, K) be a predistributive lax functorial factorisation system. Then $(\mathcal{L}_F, \mathcal{R}_F)$ is a lax weak factorisation system. Moreover, for any lax functorial weak factorisation system $(\mathcal{L}, \mathcal{R})$ with lax functorial factorisation (F, L, R, K), we have $(\mathcal{L}, \mathcal{R}) = (\mathcal{L}_F, \mathcal{R}_F)$.

Let $f \in \mathcal{L}_F$ and $g \in \mathcal{R}_F$. We factorise a lax square as



$\longmapsto$



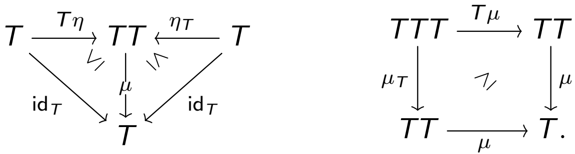
The morphism $\Delta = \lambda_g \cdot K(u, v) \cdot \rho_f$ is a lax diagonal morphism. Moreover, for any $f \pitchfork \mathcal{R}$, then $f \pitchfork Rf$, thus $f \in \mathcal{L}$, namely $\mathcal{R}^{\pitchfork} \subseteq \mathcal{L}$. Analogously $\mathcal{L}^{\pitchfork} \subseteq \mathcal{R}$.

Lax monads

For an **Ord**-enriched category $\mathscr{C}$, a **lax monad** is a triple (T, η, μ), such that

- $T : \mathscr{C} \longrightarrow \mathscr{C}$ is a locally monotone functor;
- $\eta : \mathrm{Id} \Longrightarrow T$ is an oplax natural transformation;
- $\mu : TT \Longrightarrow T$ is an oplax natural transformation;

and such that the following lax monads laws are satisfied



Lax algebraic weak factorisation systems

A **lax algebraic weak factorisation system** is a functorial factorisation system (F, L, R, K) such that (R, η) is part of a lax monad (R, η, Θ), (L, ε) is part of a lax comonad (L, ε, Ω), and there exists a distributive law $\Delta : LR \Longrightarrow RL$ of the comonad over the monad such that the following diagram commutes

$$
\begin{array}{ccccc}
LRR & \xrightarrow{\Delta_R} & RLR & \xrightarrow{R\Delta} & RRL \\
{\scriptstyle L\Theta}\downarrow & & & & \downarrow{\scriptstyle \Theta_L} \\
LR & & \xrightarrow{\qquad\quad\Delta\qquad\quad} & & RL \\
{\scriptstyle \Omega_R}\downarrow & & & & \downarrow{\scriptstyle R\Omega} \\
LLR & \xrightarrow{L\Delta} & LRL & \xrightarrow{\Delta_L} & RLL.
\end{array}
$$

Proposition

A lax algebraic weak factorisation system (F, L, R, K) is a predistributive lax functorial factorisation system.

Oplax case

Dual arguments can be discussed for $\mathscr{C}^2_{oplax}$. Morphisms here are oplax squares as

$$(u, v) : f \longrightarrow g \qquad\qquad \begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \swarrow & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D. \end{array}$$

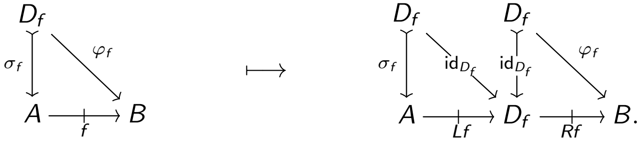
Here the oplax weak orthogonality relation $f \pitchfork g$ requires that, for any oplax square $(u, v) : f \longrightarrow g$, there exists a morphism $d : cod(f) \longrightarrow dom(g)$ such that

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \overset{\vee\!|}{} \nearrow^{d} \underset{\vee\!|}{} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array} \qquad\qquad \begin{cases} d \circ f \leq u \\ v \leq g \circ d. \end{cases}$$

The main difference here will be that self orthogonal morphisms are right adjoint morphisms.

Total maps

Let us consider $\mathscr{C}$ **Ord**-enriched category and $\mathcal{P}(\mathscr{C})$ a category of partial maps. Any morphism may be factorised as follows



This factorisation is functorial and lax predistributive.
The underlying lax weak factorisation system is given by

$$\mathcal{L}_F = \mathbf{PLA} = \{f | \varphi_f = \sigma \cdot \overline{\varphi}_f \text{ with } \sigma \in \mathcal{S} \text{ and } \overline{\varphi}_f \in \mathbf{LA}(\mathscr{C})\}, \quad \mathcal{R}_F = \mathbf{Tot}.$$

This lax functorial factorisation system is actually a **lax algebraic weak factorisation system**.

Sketch proof

- One proves functoriality and defines the lax (co)monad naturally.
- If $f \in \mathbf{PLA}$, then Rf is a left adjoint morphism and therefore $f \pitchfork Rf$, i.e. $f \in \mathcal{L}_F$. On the other hand if $f \in \mathcal{L}_F$, one can prove that the existence of a lax diagonal morphism for the square η_f yields that

$$\varphi_f = \sigma_\delta \cdot \overline{\varphi}_f \qquad \textit{and} \qquad \overline{\varphi}_f \dashv \varphi_\delta$$

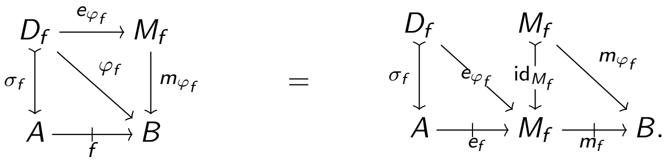
  Hence we have that $\mathcal{L}_F = \mathbf{PLA}$.
- If f is total, then Lf is an identity, therefore $Lf \pitchfork f$, i.e. $f \in \mathcal{R}_F$. If $f \in \mathcal{R}_F$ and δ' a lax diagonal morphism for ε_f, then $\mathrm{id}_A \leq \delta \cdot Lf$. Hence $Lf = \mathrm{id}_A$.

  This implies that $\mathcal{R}_F = \mathbf{Tot}$.
- Predistributivity follows trivially since for every f

$$Lf \in \mathbf{PLA} = \mathcal{L}_F \qquad \textit{and} \qquad Rf \in \mathbf{Tot} = \mathcal{R}_F.$$

Extracting an oplax WFS for $\mathcal{P}(\mathscr{C})$ from a WFS on $\mathscr{C}$

We consider $\mathscr{C}$ a category with a stable WFS $(\mathcal{E}, \mathcal{M})$ and $\mathcal{P}(\mathscr{C})$ category of partial maps. Then for any partial morphism f we have the factorisation



We define

$$\overline{\mathcal{E}} = \{f | \varphi_f \in \mathcal{E}\}, \qquad \overline{\mathcal{M}} = \{f | \varphi_f \in \mathcal{M}\}.$$

Stability under pullback of the class $\mathcal{E}$ enables us to build appropriate lax diagonal morphisms for oplax squares $e \to m$ and prove that $\overline{\mathcal{E}} \pitchfork \overline{\mathcal{M}}$.

Moreover we have that if $(\mathcal{E}, \mathcal{M})$ is functorial on $\mathscr{C}$ through (F, L, R, K), then the functors can be extended to $\mathcal{P}(\mathscr{C})$ as

$$\overline{L}f = (\sigma_f, L\varphi_f) \qquad \textit{and} \qquad \overline{R}f = (\mathrm{id}_{K\varphi_f}, R\varphi_f)\,.$$

These are functors on $\mathcal{P}(\mathscr{C})^2_{\text{oplax}}$.

Proposition

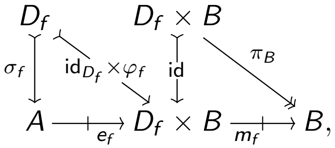
The pair $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is an oplax weak factorisation system. If $(\mathcal{E}, \mathcal{M})$ is functorial underlying (F, L, R, K), then $(\overline{\mathcal{E}}, \overline{\mathcal{M}})$ is functorial underlying $(\overline{F}, \overline{L}, \overline{R}, \overline{K})$. Moreover if (F, L, R, K) is an algebraic weak factorisation system, then $(\overline{F}, \overline{L}, \overline{R}, \overline{K})$ is an oplax weak algebraic factorisation system.

What happens for $\mathcal{P}(\mathbf{Set})$

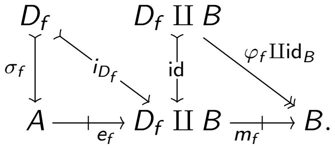
We recall that for $\mathcal{P}(\mathbf{Set})$ left adjoint partial morphisms are injective total maps and right adjoint partial morphisms are partial bijective maps.
Then on $\mathcal{P}(\mathbf{Set})$ we have the following factorisation systems.

- First we have lax functorial weak factorisation system $(\overline{\mathbf{Mono}}, \mathbf{Tot})$ as described in general. We remark that $\overline{\mathbf{Mono}}$ is here given by f such that φ_f is monic.

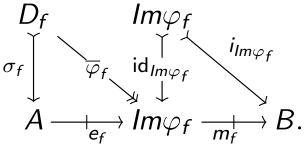
- Since we know that $(\mathbf{Mono}, \mathbf{Epi})$ is a weak factorisation system in **Set** and **Mono** is stable under pullback, then $(\overline{\mathbf{Mono}}, \overline{\mathbf{Epi}})$ is an oplax weak factorisation system in $\mathcal{P}(\mathbf{Set})$. Each morphism f may be factorised as



but also as

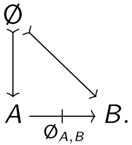


- Finally $(\mathbf{Epi}, \mathbf{Mono})$ is a stable orthogonal factorisation system. Then $(\overline{\mathbf{Epi}}, \overline{\mathbf{Mono}})$ is an oplax algebraic weak factorisation system. Each morphism f is factorised as



Pointed categories

Consider $\mathcal{P}(\mathbf{Set})$. In this category $\emptyset$ is a zero object, in fact for every pair of sets A, B we have the zero map



We define the class of morphisms

$$\mathcal{O} = \{\emptyset_{A,B} | A, B \in \mathbf{Set}\}.$$

We want to apply to $\mathcal{O}$ the proposition and its oplax version seen before:

Proposition

Given any $\mathcal{H} \subseteq \mathscr{C}^2_{lax}$, then $\left({}^{\pitchfork}\mathcal{H}, \left({}^{\pitchfork}\mathcal{H}\right)^{\pitchfork}\right)$ and $\left({}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right), \mathcal{H}^{\pitchfork}\right)$ are lax weak prefactorisation systems.

In $\mathcal{P}(\mathbf{Set})$ we obtain:

- $\mathcal{O}^{\pitchfork} = \mathbf{Tot} \quad \Rightarrow \quad \left(\overline{\mathbf{Mono}}, \mathbf{Tot}\right)$;
- ${}^{\pitchfork}\mathcal{O} = \mathbf{LA} \quad \Rightarrow \quad (\mathbf{LA}, \mathbf{All})$;
- $\mathcal{O}^{\pitchfork} = \overline{\mathbf{Epi}} \quad \Rightarrow \quad \left(\overline{\mathbf{Mono}}, \overline{\mathbf{Epi}}\right)$;
- ${}^{\pitchfork}\mathcal{O} = \overline{\mathbf{Epi}} \quad \Rightarrow \quad \left(\overline{\mathbf{Epi}}, \overline{\mathbf{Mono}}\right)$.

Can this behaviour be generalized?

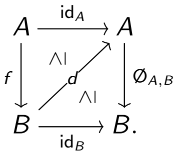
Consider an **Ord**-category $\mathscr{C}$ that has an **initial object** I and such that all initial morphisms $i_X \in \mathcal{S}$ and they are minimal. Then I is still an initial object in $\mathcal{P}(\mathscr{C})$, moreover if I is either a zero object or a strict initial object, then I is a zero object in $\mathcal{P}(\mathscr{C})$. We consider

$$\mathcal{O} = \{\emptyset_{A,B} = (i_A, i_B) : A \longrightarrow B | A; B \in \mathbf{Ob}(\mathscr{C})\}.$$

Lemma

For any $\mathcal{P}(\mathscr{C})$, a minimal map f such that $D_f = I$ is right absorbent. Whenever I is actually a zero-object, then it is both left and right absorbent.

We consider $f \in {}^{\pitchfork}\mathcal{O}$. We have



Then ${}^{\pitchfork}\mathcal{O} = \mathcal{U} = \{f \mid \mathrm{id}_A \leq d \cdot f \text{ for some } d\}$.

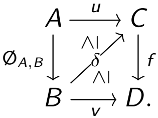
Remark

If the partial order on $\mathscr{C}$ is discrete, then $\mathrm{id}_A \leq d \cdot f$ yields that f is a split monomorphism and a left adjoint morphism.

Conjecture

The complement $\mathcal{U}^{\pitchfork} = \{f | f \cdot d' \leq \mathrm{id}_B \text{ for some } d'\}$, the intersection $\mathcal{U}^{\pitchfork} \cap \mathcal{U}$ being exactly the left adjoint morphisms.

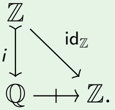
If $f \in \mathcal{O}^{\pitchfork}$, then there exists a lax diagonal morphism δ as in



Notice that a necessary and sufficient condition for the existence of such δ is that $u = \emptyset$. Then $\mathcal{O}^{\pitchfork} = \mathcal{DD} = \{f | f \cdot u = \emptyset \implies u = \emptyset\}$.
Open problem: What is ${}^{\pitchfork}\mathcal{DD}$ in general?

Remark

In Set *we have* $\mathcal{DD} = \mathrm{Tot}$. *In general it is only true that* $\mathcal{DD} \supseteq \mathrm{Tot}$. *Counterexample in* Ab*:*



Similarly in the oplax case we have:

- $\mathcal{DI} = {}^{\pitchfork}\mathcal{O} = \{f | v \cdot f = \emptyset \implies v = \emptyset\}$.
  Open problem: What is $\mathcal{DI}^{\pitchfork}$ in general?
- $\mathcal{V} = \mathcal{O}^{\pitchfork} = \{f | \mathrm{id}_B \leq f \cdot d \text{ for some } d\}$.

Conjecture

The complement ${}^{\pitchfork}\mathcal{V} = \{f | f \cdot d' \leq \mathrm{id}_A \text{ for some } d'\}$, *the intersection being exactly the right adjoint morphisms.*

Thanks for the attention

Essential bibliography

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