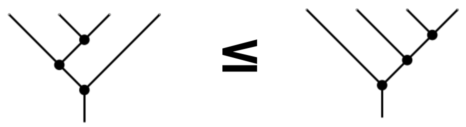
$$
\dfrac{p \longrightarrow p \qquad \dfrac{\dfrac{q \longrightarrow q \qquad \dfrac{r \longrightarrow r \qquad s \longrightarrow s}{r, s \longrightarrow r \bullet s}\,R}{q, r, s \longrightarrow q \bullet (r \bullet s)}\,R}{q \bullet r, s \longrightarrow q \bullet (r \bullet s)}\,L}{\dfrac{p, q \bullet r, s \longrightarrow p \bullet (q \bullet (r \bullet s))}{\dfrac{p \bullet (q \bullet r), s \longrightarrow p \bullet (q \bullet (r \bullet s))}{(p \bullet (q \bullet r)) \bullet s \longrightarrow p \bullet (q \bullet (r \bullet s))}\,L}\,L}\,R
$$



Skew monoidal categories and the proof-theoretic anatomy of associativity

(and unitality)

Noam Zeilberger

Masaryk University Algebra Seminar (online)

13 May 2021

based on joint work with **Tarmo Uustalu** and **Niccolò Veltri**

some papers*

(and talks)

linked in talk abstract:

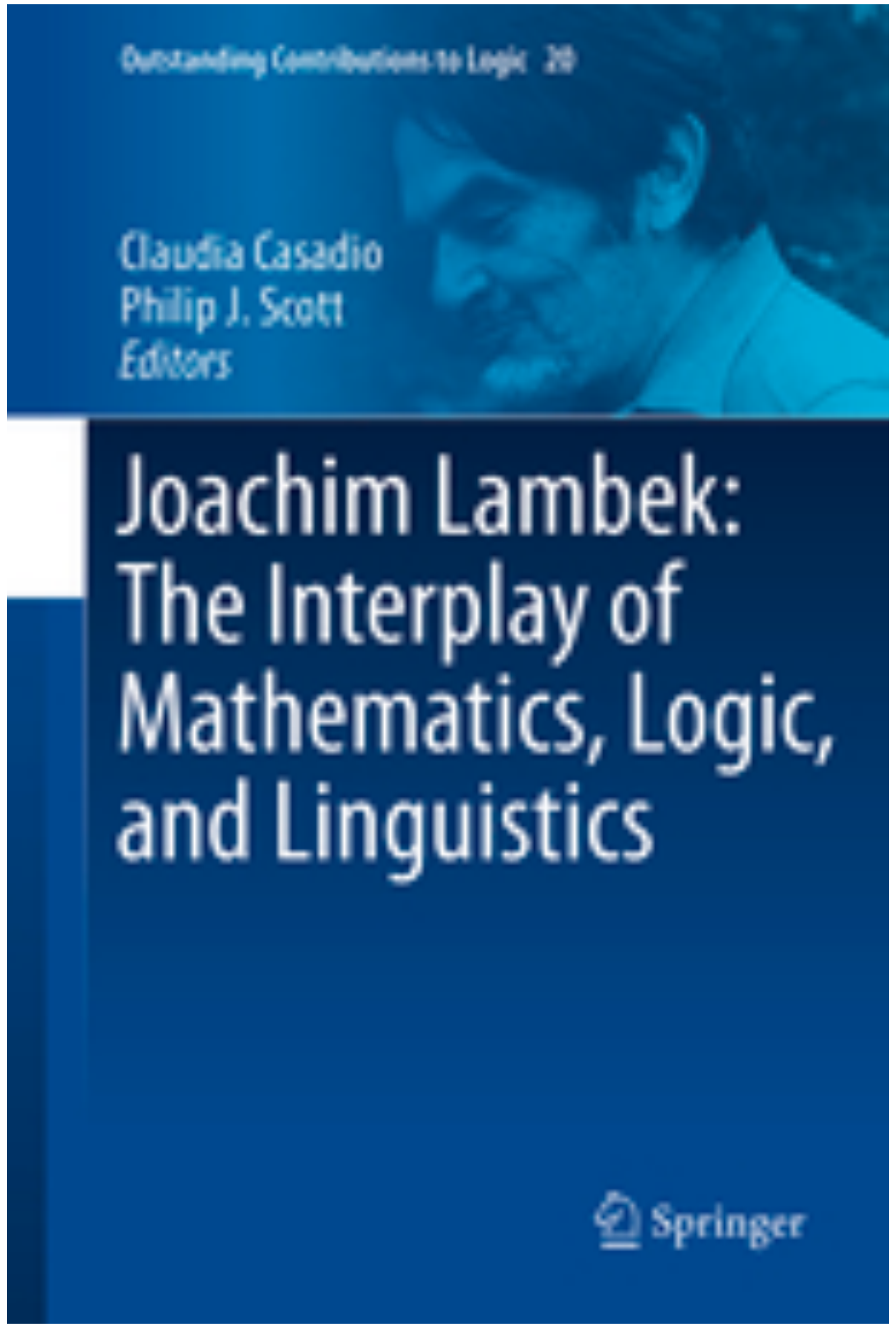
1. A sequent calculus for a semi-associative law, FSCD 2017, extended ver. LMCS 15:1 2019
2. The sequent calculus of skew monoidal categories, MFPS 2018, ext. ver. **Lambek Mem. Vol**
   Tarmo talk: https://www.youtube.com/watch?v=jg2K8L4LRt8
3. Proof theory of partially normal skew monoidal categories, ACT 2020
   Niccolò talk: https://www.youtube.com/watch?v=UdZleN5L0TA

other's in line of work:

4. Eilenberg-Kelly Reloaded, MFPS 2020
   Niccolò talk: https://www.youtube.com/watch?v=pZQtKg2ACTQ
5. Deductive systems and coherence for skew prounital closed categories, LFMTP 2020

*all except (1) are by Tarmo, Niccolò, and me

Joachim Lambek: 1922-2014



THE MATHEMATICS OF SENTENCE STRUCTURE (1958)

1. Introduction. The aim of this paper is to obtain an effective rule (or algorithm) for distinguishing sentences from nonsentences, which works not only for the formal languages of interest to the mathematical logician, but also for natural languages such as English, or at least for fragments of such languages. An attempt to formulate such an algorithm is implicit in the work of Ajdukiewicz.† His method, later elaborated by Bar-Hillel [2], depends on a kind of arithmetization of the so-called *parts of speech*, here called *syntactic types*.‡

[...]

The decision problem for this system is solved affirmatively, following a procedure first proposed by Gentzen for the intuitionistic propositional calculus.††

[...]

(*TIME*	*flies*	(10,000	*copies*))	*to*	*Montreal*.
n	$n\backslash s/n^*$	n^*/n^*	n^*	$s\backslash s/n$	n

[...]

Let capitals denote sequences of types, possibly empty sequences. By "U, V" we mean the sequence obtained by juxtaposing U and V; if U is empty it means V, and if V is empty it means U. The following rules are consequences of (a) to (e), provided T, P and Q are not empty.

(1) $x \to x$

(2) *if* $T, y \to x$ *then* $T \to x/y$

(2′) *if* $y, T \to x$ *then* $T \to y\backslash x$

(3) *if* $T \to y$ *and* $U, x, V \to z$ *then* $U, x/y, T, V \to z$

(3′) *if* $T \to y$ *and* $U, x, V \to z$ *then* $U, T, y\backslash x, V \to z$

(4) *if* $U, x, y, V \to z$ *then* $U, xy, V \to z$

(5) *if* $P \to x$ *and* $Q \to y$ *then* $P, Q \to xy$

[...]

Conversely, we shall deduce rules (a) to (e) from (1) to (5), so that the two sets of rules are equivalent. For the moment we assume one additional rule, the so-called *cut*,

(6) *if* $T \to x$ *and* $U, x, V \to y$ *then* $U, T, V \to y$

It will appear later (Gentzen's theorem) that this new rule does not increase the set of theorems deducible from (1) to (5).

ON THE CALCULUS OF SYNTACTIC TYPES

(1961)

In classical physics it was possible to decide whether an equation was grammatically correct by comparing the "dimensions" of the two sides of the equation. These dimensions formed an abelian group with three generators L, M and T, admitting fractional exponents.[2]

One may ask whether it is similarly possible to assign "grammatical types" to the words of English in such a way that the grammatical correctness of a sentence can be determined by a computation on these types. As long as "John loves Jane" fails to imply "Jane loves John" one cannot expect these types to form an abelian group. Probably they should not form a group at all.

Some time ago [3] I suggested a group-like mathematical system, which I called the "syntactic calculus". For reasons that will appear later, it would have been better to call it the "associative syntactic calculus". My method was closely related to an earlier syntactic method by Bar-Hillel, which in turn goes back to the "semantic types" of Ajdukiewicz, Leśniewski and ultimately Husserl. Independent type theories were also developed by Church and Curry,[3] who calls his types "functional characters".

ON THE CALCULUS OF SYNTACTIC TYPES

(1961)

Appendix II

The associative syntactic calculus also has application in mathematics. In multiplicative ideal theory AB, A/B and $B\backslash A$ may be interpreted as the product, right residual quotient and left residual quotient of the ideals A and B respectively. More interesting perhaps is the application to bimodules worked out by G. D. Findlay and the present author in 1956. Here is a necessarily abbreviated report of our work, which has never been published.

The idea is briefly this: Each proof of a formula in the associative syntactic calculus may be used to construct a canonical mapping between functors of bimodules built up from $\otimes$ and Hom. For example the proof of the formula $C/(AB) \to (C/B)/A$ from postulates (1) to (5) gives an explicit construction for the canonical mapping of $\mathrm{Hom}_T(A \otimes {}_S B, C)$ into $\mathrm{Hom}_S(A, \mathrm{Hom}_T(B, C))$, where ${}_R A_S$, ${}_S B_T$ and ${}_R C_T$ are given bimodules. The decision procedure for the associative syntactic calculus can then be used to find all canonical mappings from one functor into another.

ON THE CALCULUS OF SYNTACTIC TYPES (1961)

The decision procedure for the associative syntactic calculus given in [**3**] can be used to find all canonical mappings (according to our recursive definition of "canonical") from one functor into another. However, this decision procedure operates not on formulas $x \to y$ but on "sequents" $x_1, x_2, \cdots, x_n \to y$. These are the associative analogues of the G-formulas of Appendix I. The suggested method for finding canonical mappings therefore does not deal with mappings $\phi : A \to B$ directly but with multilinear mappings $\emptyset : A_1 \times A_2 \times \cdots \times A_n \to B$. It has already been observed by Bourbaki[14] that linear mappings of the kind we are interested in are best defined with the help of multilinear mappings. The details are too technical to be given here, but an example may help the interested reader to reconstruct the general method.

DEDUCTIVE SYSTEMS AND CATEGORIES

II. STANDARD CONSTRUCTIONS AND CLOSED CATEGORIES

0. INTRODUCTION

We wish to explore the connection between

(1) pre-ordered sets with structure,

(2) deductive systems,

(3) categories with structure.

DEDUCTIVE SYSTEMS AND CATEGORIES

II. STANDARD CONSTRUCTIONS AND CLOSED CATEGORIES

A <u>multicategory</u> consists of a class of objects together with a class of <u>multimaps</u>

$$g\colon A_1, A_2, \ldots, A_n \longrightarrow B\ ,$$

n being any non-negative integer. Among the multimaps are the <u>identity</u> maps $1_A\colon A \longrightarrow A$. Multimaps may be composed by "substitution" as follows: Given multimaps

$$g\colon A_1, \ldots, A_n \longrightarrow \overset{A}{\not{B}},\quad f\colon \ldots, A, \ldots \longrightarrow B\ ,$$

there is a multimap

$$f(\ldots,g,\ldots)\colon \ldots, A_1, \ldots, A_n, \ldots \longrightarrow B\ .$$

Substitution, also called <u>cut</u>, must satisfy four conditions.

outline

0. Monoidal categories and representable multicategories

1. The Tamari order

2. Skew monoidal categories

3. Partially normal skew monoidal categories

monoidal categories

A category C equipped with:

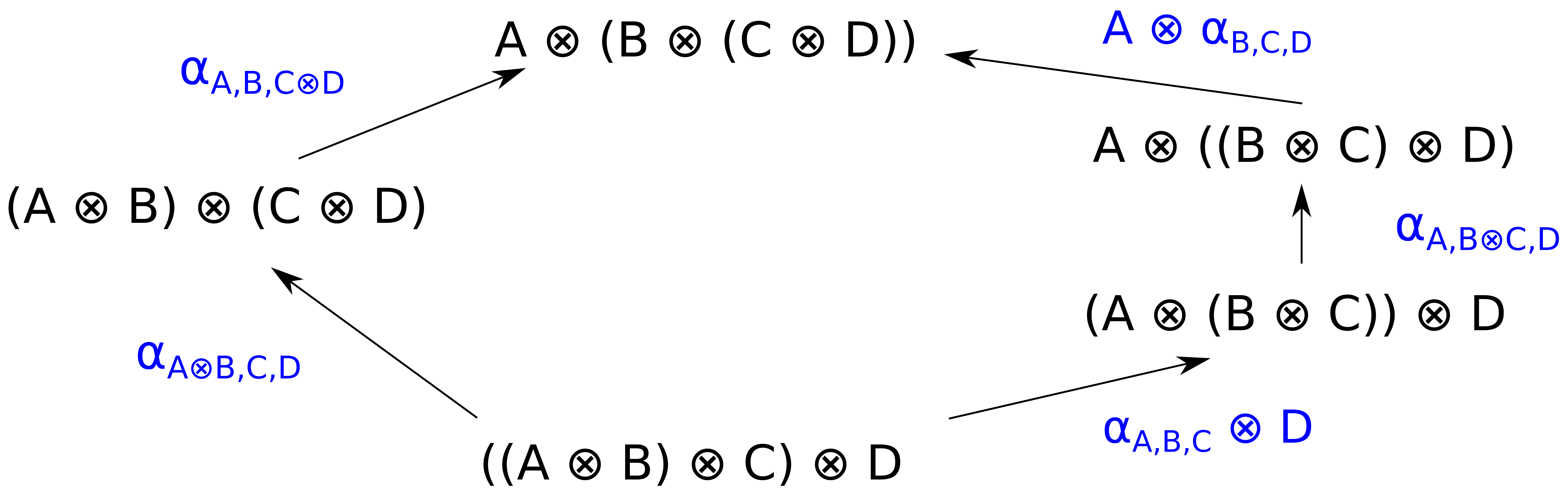
- a bifunctor $\otimes : C \times C \to C$ and an object $I \in C$
- three natural isomorphisms

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$
$$\lambda_A : I \otimes A \xrightarrow{\sim} A$$
$$\rho_A : A \otimes I \xrightarrow{\sim} A$$

- satisfying some *coherence equations*...

monoidal categories

...such as the *pentagon equation:*



Theorem (Mac Lane 1963, Kelly 1964): "given these eqns, all diagrams commute".

multicategories

Recall that a multicategory has objects, multimaps, identity maps, and composition satisfying some equations.

Composition has the type of *cut* in intuitionistic linear sequent calculus:

(cf. Lambek '58)

$$\frac{\Omega \xrightarrow{f} A \qquad \Gamma , A , \Delta \xrightarrow{g} C}{\Gamma , \Omega , \Delta \to C} \quad = \ \mathrm{cut}_{\Gamma\text{-}\Delta}(f,g)$$

$= g \circ_i f$ where $i = |\Gamma|$

multicategories

A multicategory M is said to be *representable* if for any list of objects Ω there is an object $\otimes\Omega$ equipped with a multimap

$$m_\Omega : \Omega \to \otimes\Omega$$

and a family of *bijections of multihomsets*

$$L_\Omega : M(\Gamma,\Omega,\Delta;C) \xrightarrow{\sim} M(\Gamma,\otimes\Omega,\Delta;C)$$

whose inverse is the operation of precomposing with m, i.e.,

$$cut_{\Gamma\text{-}\Delta}(m_\Omega, L_\Omega f) = f \qquad g = L_\Omega(cut_{\Gamma\text{-}\Delta}(m_\Omega, g))$$

for all $f : \Gamma , \Omega , \Delta \to C$ and $g : \Gamma , \otimes\Omega , \Delta \to C$.

multicategories

Proposition: M is representable iff it has (m_Ω , L_Ω) for Ω = A,B and $\Omega = \cdot$.
(Terminology: "M has tensors and a unit object" or "M is monoidal".)

Theorem (Lambek 1969, Hermida 2000): "monoidal categories and monoidal/representable multicategories are equivalent".

the Tamari order

Least preorder on words with a product operation s.t.:

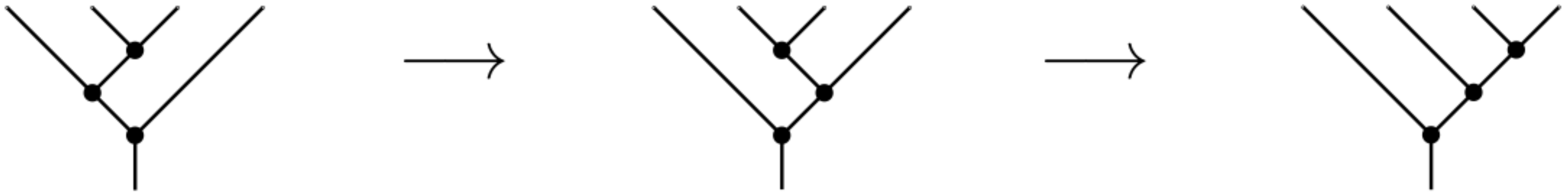
$$(A \cdot B) \cdot C \leq A \cdot (B \cdot C)$$ **semi-associativity**

$$\frac{A_1 \leq A_2 \qquad B_1 \leq B_2}{A_1 \cdot B_1 \leq A_2 \cdot B_2}$$ **monotonicity**

the Tamari order

equivalently, ordering on binary trees induced by *right rotation*, e.g.,

(A·(B·C))·D → A·((B·C)·D) → A·(B·(C·D))



the Tamari order

Let Y_n be the set of binary trees with n nodes, under the rotation order.

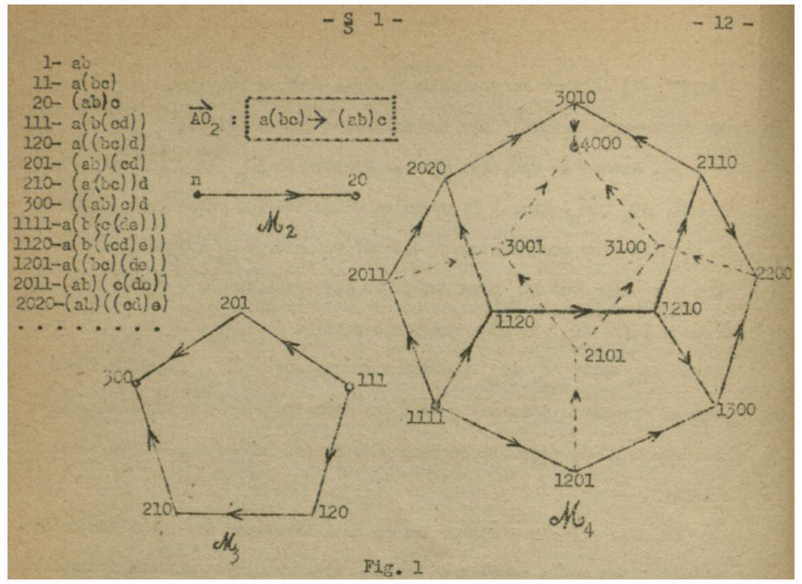


figure from Dov Tamari's PhD thesis (1951)

Three fascinating facts about Y_n:

1. it is a lattice!
   (the "Tamari lattice")

2. its Hasse diagram is the 1-skeleton of a (n-1)-dim polytope!
   (the "associahedron")

3. it contains exactly $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ intervals!
   (cf. https://oeis.org/A000260)

a sequent calculus for the Tamari order

The LMCS paper uses proof theory to explain facts #1 and #3.

The starting point is a very simple sequent calculus:

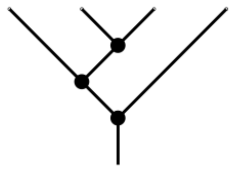
$$\frac{}{A \longrightarrow A}\ id \qquad \frac{\Theta \longrightarrow A \quad \Gamma, A, \Delta \longrightarrow B}{\Gamma, \Theta, \Delta \longrightarrow B}\ cut$$

$$\frac{A, B, \Delta \longrightarrow C}{A \bullet B, \Delta \longrightarrow C}\ \bullet L \qquad \frac{\Gamma \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma, \Delta \longrightarrow A \bullet B}\ \bullet R$$

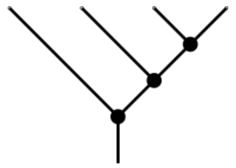
(compare with Lambek '58!)

a sequent calculus for the Tamari order

Example:



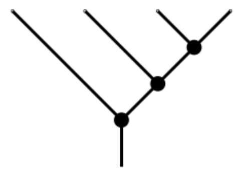
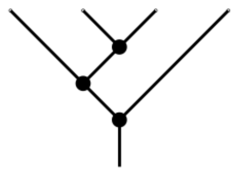
≤



$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \longrightarrow p} \qquad \dfrac{\dfrac{\dfrac{}{q \longrightarrow q} \qquad \dfrac{\dfrac{}{r \longrightarrow r} \quad \dfrac{}{s \longrightarrow s}}{r, s \longrightarrow r \bullet s}\, R}{q, r, s \longrightarrow q \bullet (r \bullet s)}\, R}{q \bullet r, s \longrightarrow q \bullet (r \bullet s)}\, L}{p, q \bullet r, s \longrightarrow p \bullet (q \bullet (r \bullet s))}\, R}{p \bullet (q \bullet r), s \longrightarrow p \bullet (q \bullet (r \bullet s))}\, L}{(p \bullet (q \bullet r)) \bullet s \longrightarrow p \bullet (q \bullet (r \bullet s))}\, L$$

a sequent calculus for the Tamari order

Counterexample:

 $\not\leq$ 

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{p \longrightarrow p} \qquad \dfrac{\overline{q \longrightarrow q} \quad \overline{r \longrightarrow r}}{q, r \longrightarrow q \bullet r}\,R}{p, q, r \longrightarrow p \bullet (q \bullet r)}\,R \qquad \overline{s \longrightarrow s}}{p, q, r, s \longrightarrow (p \bullet (q \bullet r)) \bullet s}\,R}{p, q, r \bullet s \longrightarrow (p \bullet (q \bullet r)) \bullet s}\,L^{\mathrm{amb}}}{p, q \bullet (r \bullet s) \longrightarrow (p \bullet (q \bullet r)) \bullet s}\,L^{\mathrm{amb}}}{p \bullet (q \bullet (r \bullet s)) \longrightarrow (p \bullet (q \bullet r)) \bullet s}\,L$$

left representability*

multicategorically, the restriction on the left rule corresponds to weakening the universal property of the tensor...

$$L_\Omega : M(\Gamma,\Omega,\Delta;C) \risingdotseq\!\!\rightarrow M(\Gamma,\otimes\Omega,\Delta;C)$$

to the case $\Gamma = \cdot$. (We'll get back to this later.)

*cf §4.4 of Bourke & Lack (2018b), "Skew monoidal categories and skew multicategories"

completeness

Proposition: if A ≤ B then A → B.

reflexivity = id

transitivity = cut

semi-associativity =

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \longrightarrow A} \quad \dfrac{\dfrac{}{B \longrightarrow B} \quad \dfrac{}{C \longrightarrow C}}{B, C \longrightarrow B \bullet C} R}{A, B, C \longrightarrow A \bullet (B \bullet C)} R}{A \bullet B, C \longrightarrow A \bullet (B \bullet C)} L}{(A \bullet B) \bullet C \longrightarrow A \bullet (B \bullet C)} L$$

monotonicity =

$$\dfrac{\dfrac{A_1 \longrightarrow A_2 \quad B_1 \longrightarrow B_2}{A_1, B_1 \longrightarrow A_2 \bullet B_2} R}{A_1 \bullet B_1 \longrightarrow A_2 \bullet B_2} L$$

soundness

Proposition: if $A \to B$ then $A \leq B$.

more generally, if $\Gamma \to B$ then $\otimes\Gamma \leq B$, where

$$\otimes(A_0,A_1,\ldots,A_n) := (A_0 \bullet A_1)\cdots\bullet A_n$$

proof by induction on sequent calculus derivations.

key lemma ("oplaxity"): $\otimes(\Gamma,\Delta) \leq \otimes\Gamma \bullet \otimes\Delta$

coherence theorem

A derivation is *focused* if it stays in the following subsystem:

$$\frac{A, B, \Delta \longrightarrow C}{A \bullet B, \Delta \longrightarrow C} \bullet L \qquad \frac{\Gamma^{\mathrm{irr}} \longrightarrow A \quad \Delta \longrightarrow B}{\Gamma^{\mathrm{irr}}, \Delta \longrightarrow A \bullet B} \bullet R^{\mathrm{foc}} \qquad \frac{}{p \longrightarrow p}\, id^{\mathrm{atm}}$$

(Γ *irreducible* if atomic leftmost formula; no cut allowed.)

Theorem: every valid sequent has a unique focused derivation.

applications

1. new proof that Y_n is a lattice — Friedman & Tamari 1967, Huang & Tamari 1971

key idea: prove in mutual induction w/lattice structure on contexts

2. new proof that # intervals in Y_n is $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ — Chapoton 2006

NB: "interval" = valid entailment $A \leq B$

key idea: count focused derivations! Easy using generating functions...

See LMCS paper for details.

skew monoidal categories

Szlachányi 2012

A category C equipped with:

- a bifunctor $\otimes : C \times C \to C$ and an object $I \in C$
- three natural transformations

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$

$$\lambda_A : I \otimes A \to A$$

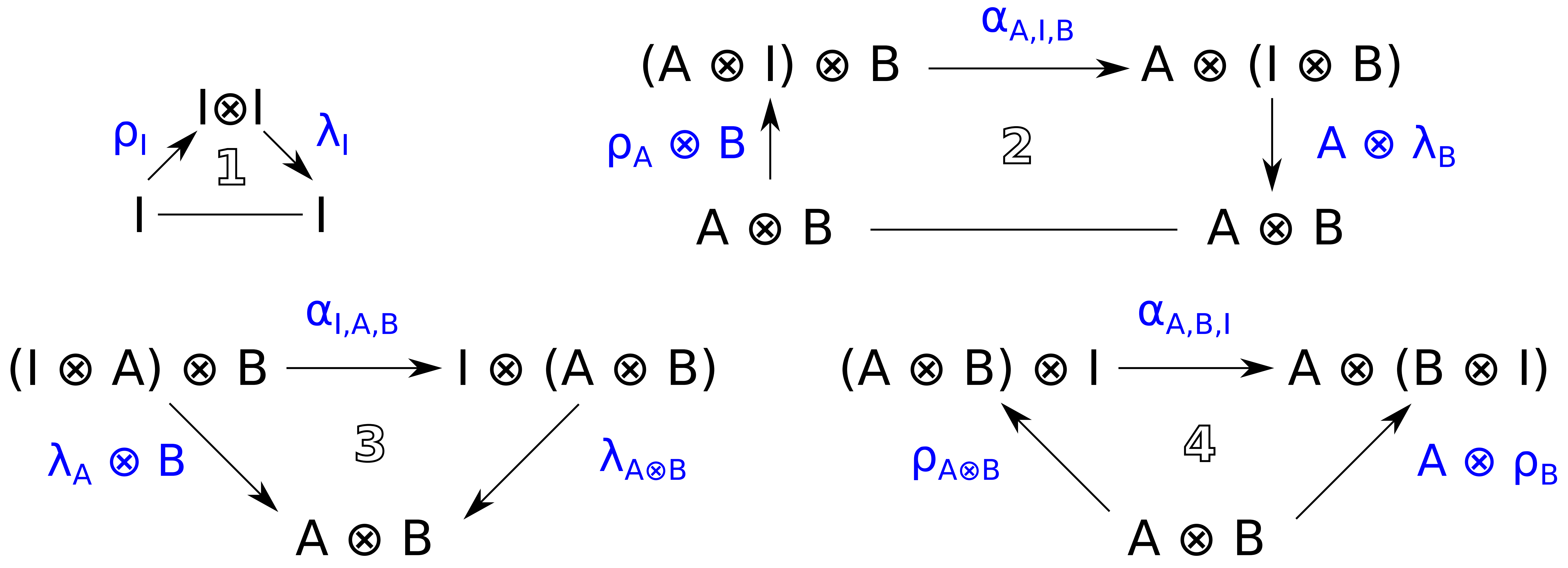
$$\rho_A : A \to A \otimes I$$

- satisfying five coherence equations...

skew monoidal categories

Szlachányi 2012

...including the pentagon eqn + four more:



No longer true that "all diagrams commute"!

skew monoidal categories

Some examples:

- $(\mathbb{N},\leq)$ with "skewed addition" $x \otimes^n y := (x \dot{-} n) + y$

  truncated subtraction

  note: only λ iso in this example

- the category of pointed sets with $(A,x) \otimes (B,y) := (A+B,\text{inl } x)$

  note: only α iso in this example
  compare with $(A,x) + (B,y) := (A+B,[\text{inl } x])/\text{inl } x\sim\text{inr } y$

- $A \otimes^D B := A \otimes D(B)$ where D is a lax monoidal comonad

- the functor category $[\mathbb{J},\mathbb{C}]$ with $F \otimes^J G := \text{Lan}(J,F) \circ G$ defined by left kan extension along a functor $J : \mathbb{J} \to \mathbb{C}$.

  Altenkirch, Chapman, Uustalu, "Monads need not be endofunctors"

a skew sequent calculus

Warning: units may be subtler than they appear!

Extend the sequent calculus with an explicit "stoup" on the left, which may be empty or contain a formula.

$$\frac{}{A \mid \ \longrightarrow A}\ \textsf{id} \qquad \frac{A \mid \Gamma \longrightarrow C}{- \mid A, \Gamma \longrightarrow C}\ \textsf{shift}$$

$$\frac{S \mid \Gamma \longrightarrow A \quad A \mid \Delta \longrightarrow C}{S \mid \Gamma, \Delta \longrightarrow C}\ \textsf{scut} \qquad \frac{- \mid \Gamma \longrightarrow A \quad S \mid \Delta_0, A, \Delta_1 \longrightarrow C}{S \mid \Delta_0, \Gamma, \Delta_1 \longrightarrow C}\ \textsf{ccut}$$

$$\frac{- \mid \Gamma \longrightarrow C}{\mathsf{I} \mid \Gamma \longrightarrow C}\ \textsf{IL} \qquad \frac{}{- \mid \ \longrightarrow \mathsf{I}}\ \textsf{IR} \qquad \frac{A \mid B, \Gamma \longrightarrow C}{A \otimes B \mid \Gamma \longrightarrow C}\ \otimes\textsf{L} \qquad \frac{S \mid \Gamma \longrightarrow A \quad - \mid \Delta \longrightarrow B}{S \mid \Gamma, \Delta \longrightarrow A \otimes B}\ \otimes\textsf{R}$$

a skew sequent calculus

Warning: units may be subtler than they appear!

example derivations and non-derivations:

$$\rho = \quad \dfrac{\dfrac{}{X \mid\ \longrightarrow X}\,\mathsf{id} \qquad \dfrac{}{- \mid\ \longrightarrow \mathsf{I}}\,\mathsf{IR}}{X \mid\ \longrightarrow X \otimes \mathsf{I}}\,\otimes\mathsf{R} \qquad\qquad \dfrac{X \mid \mathsf{I} \overset{??}{\longrightarrow} X}{X \otimes \mathsf{I} \mid\ \longrightarrow X}\,\otimes\mathsf{L} \quad = \rho^{-1}??$$

$$\lambda = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{X \mid\ \longrightarrow X}\,\mathsf{id}}{- \mid X \longrightarrow X}\,\mathsf{shift}}{\mathsf{I} \mid X \longrightarrow X}\,\mathsf{IL}}{\mathsf{I} \otimes X \mid\ \longrightarrow X}\,\otimes\mathsf{L} \qquad\qquad \dfrac{X \mid\ \overset{??}{\longrightarrow} \mathsf{I} \qquad - \mid\ \overset{??}{\longrightarrow} X}{X \mid \longrightarrow \mathsf{I} \otimes X}\,\otimes\mathsf{R} \quad = \lambda^{-1}??$$

completeness + soundness

Let $\mathrm{Fsk}_{\mathrm{At}}$ be the free skew monoidal category over a set of atoms.

(we give an explicit construction of $\mathrm{Fsk}_{\mathrm{At}}$ by generators and relations)

For any $f : A \to B \in \mathrm{Fsk}_{\mathrm{At}}$ there is a derivation $\mathrm{cmplt}(f) : A \mid \to B$

For any derivation $g : S \mid \Gamma \to B$ there is $\mathrm{sound}(g) : [\![S \mid \Gamma]\!] \to B \in \mathrm{Fsk}_{\mathrm{At}}$

where $[\![A_0 | A_1,\ldots,A_n]\!] := (A_0 \bullet A_1) \cdots \bullet A_n$

$[\![- | A_1,\ldots,A_n]\!] := (I \bullet A_1) \cdots \bullet A_n$

Moreover, cmplt and sound respect equality, if we impose a suitable equivalence relation $\doteq$ on derivations. (See paper for details.)

a focused subsystem

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{L}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{shift} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} C}{T \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{switch} \qquad \frac{}{X \mid \quad \longrightarrow_{\mathsf{R}} X}\ \mathsf{id}^{\mathrm{foc}}$$

$$\frac{- \mid \Gamma \longrightarrow_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{IL} \qquad \frac{}{- \mid \quad \longrightarrow_{\mathsf{R}} \mathsf{I}}\ \mathsf{IR}^{\mathrm{foc}}$$

$$\frac{A \mid B, \Gamma \longrightarrow_{\mathsf{L}} C}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} A \quad - \mid \Delta \longrightarrow_{\mathsf{L}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{R}} A \otimes B}\ \otimes\mathsf{R}^{\mathrm{foc}}$$

Theorem: for any derivation $f : S \mid \Gamma \to B$ there is a focused derivation $\mathrm{focus}(f) : S \mid \Gamma \to_{\mathsf{L}} B$. Moreover, $f \doteq g$ iff $\mathrm{focus}(f) = \mathrm{focus}(g)$.

coherence theorem(s)

With soundness + completeness, focusing gives us a two-part coherence theorem for skew monoidal categories.

Coherence (equality): two maps $f,g : A \to B \in \mathrm{Fsk}_{\mathrm{At}}$ are equal iff focus(cmplt(f)) = focus(cmplt(g)).

Coherence (enumeration): the homsets of $\mathrm{Fsk}_{\mathrm{At}}$ can be enumerated without duplicates as $\mathrm{Fsk}_{\mathrm{At}}(A,B) = \{ \mathrm{sound}(\mathrm{emb}_{L}(f)) \mid f : A \mid\!\to_{L} B \}$.

notes

Lack and Street (2014) also proved a coherence theorem for Fsk of the form *coherence (equality),* building on Huang & Tamari (1972). Bourke and Lack (2018a) refined this with a more explicit description of the morphisms of Fsk.

Bourke and Lack (2018b) defined *skew multicategories,* and proved an equivalence between skew monoidal cats and *left representable* skew multicats.

We give a light reformulation of B&L(b)'s definitions inspired by the sequent calculus in the Lambek Volume paper. Our focused sequent calc can be seen as a *canonical* construction of the free left representable skew multicat (and hence Fsk).

The development in our paper has been formalized in Agda, see Niccolò's webpage (http://cs.ioc.ee/~niccolo/skewmonseqcalc/).

partial skewness/normality

A skew monoidal category is said to be *left/right/associative normal* if the corresponding transformation λ/ρ/α is invertible.

"A monoidal category is just a fully normal skew monoidal category."

The ACT2020 paper explains how to adapt the skew sequent calculus to reflect the three normality conditions (eight possible combinations).

Agda: https://github.com/niccoloveltri/skewmoncats-normal

associative normality (focused)

Idea: introduce a judgment S | Ω ⁝ Γ → C with an "anteroom" Ω for formulae

$$\frac{S \mid \Omega, A, B \,\vdots\, \Gamma \longrightarrow_{\mathsf{C}} C}{S \mid \Omega, A \otimes B \,\vdots\, \Gamma \longrightarrow_{\mathsf{C}} C}\ \otimes\mathsf{C} \qquad \frac{S \mid \Omega \,\vdots\, D, \Gamma \longrightarrow_{\mathsf{C}} C \quad D \neq A \otimes B}{S \mid \Omega, D \,\vdots\, \Gamma \longrightarrow_{\mathsf{C}} C}\ \mathsf{move} \qquad \frac{S \mid \Gamma \longrightarrow_{\mathsf{L}} C}{S \mid \ \vdots\, \Gamma \longrightarrow_{\mathsf{C}} C}\ \mathsf{switch_{LC}}$$

$$\frac{A \mid \Gamma \longrightarrow_{\mathsf{L}} C}{- \mid A, \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{pass} \qquad \frac{- \mid \Gamma \longrightarrow_{\mathsf{L}} C}{\mathsf{I} \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{IL} \qquad \frac{A \mid B \,\vdots\, \Gamma \longrightarrow_{\mathsf{C}} C}{A \otimes B \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \otimes\mathsf{L} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} C}{T \mid \Gamma \longrightarrow_{\mathsf{L}} C}\ \mathsf{switch_{RL}}$$

$$\frac{}{X \mid \ \longrightarrow_{\mathsf{R}} X}\ \mathsf{ax} \qquad \frac{}{- \mid \ \longrightarrow_{\mathsf{R}} \mathsf{I}}\ \mathsf{IR} \qquad \frac{T \mid \Gamma \longrightarrow_{\mathsf{R}} A \quad - \mid \Delta \longrightarrow_{\mathsf{L}} B}{T \mid \Gamma, \Delta \longrightarrow_{\mathsf{R}} A \otimes B}\ \otimes\mathsf{R}$$

Remark: Lack and Street (2014) observed that the free associative-normal skewmoncat on one gen is iso to $\Delta_\perp$, and proved that Fsk → $\Delta_\perp$ is faithful. Can we prove this directly?

concluding thoughts

proof theory and category theory are extremely closely related, as emphasized by Lambek.

skew monoidal categories have a very interesting proof theory! (as do skew closed categories.)

a more conceptual understanding of left representability would be desirable.

can we can find other applications of proof theory to combinatorics and vice versa?