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Higher Segal spaces via higher excision

1. from (1-) Segal to 2-Segal.
2. higher Segal spaces via cyclic Polytopes
3. a "manifold calculus" on Δ
4. higher Segal spaces via higher excision

Classical Segal spaces/objects

[Segal '74, Dwyer-Kan-Smith '89, Hirschowitz-Simpson '98, Rezk '01, ..]

Notation: Δ = category of finite, non-empty linearly ordered sets

(full subcat of Cat)

every object is uniquely isom. to some $[n] = \{0 < 1 < \dots < n\}$

Key observation: simplicial sets $\Delta^{op} \to Set$

can encode categories/monoids

$$[\Delta^{op}, Set] \longleftarrow Cat \; : N \quad \text{nerve}$$

$$NC := Hom_{Cat}(-, C) \longleftarrow\!\shortmid C \qquad \text{fully faithful}$$

Essential image = {Segal simplicial sets}

$X : \Delta^{op} \to Set$ **Segal** if for each $n \geq 2$

$$X_{[n]} \xrightarrow{\cong} X_{\{0,1\}} \underset{X_{\{1\}}}{\times} X_{\{1,2\}} \underset{X_{\{2\}}}{\times} \dots \underset{X_{\{n-1\}}}{\times} X_{\{n-1,n\}}$$

bijection induced by $[n] \hookleftarrow \{i-1, i\}$

- Can recover C by

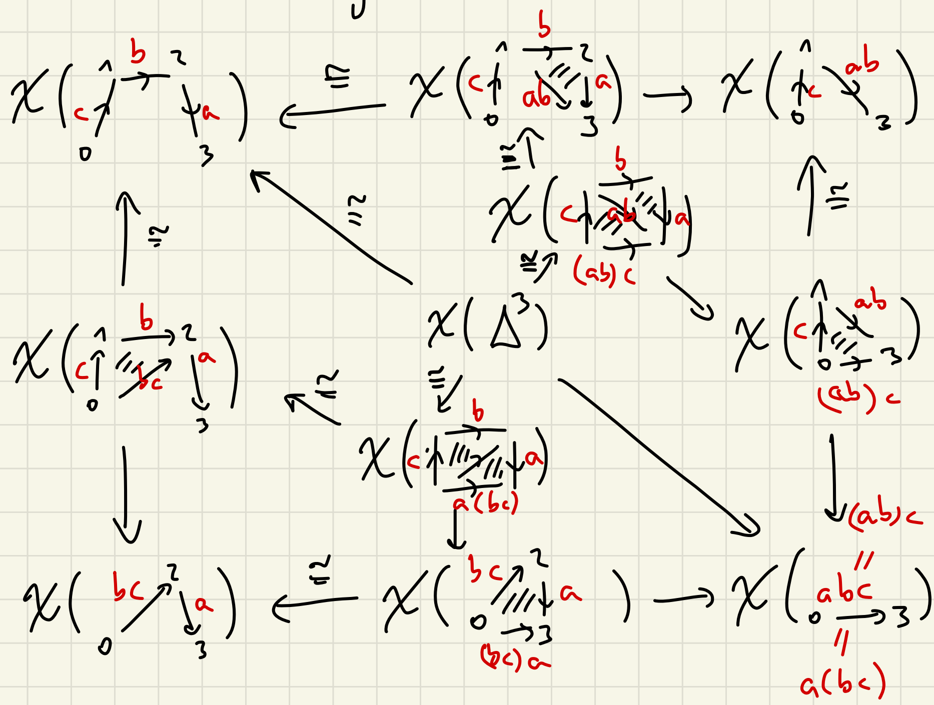
$$\{\text{objects}\} = \mathcal{X}_{[0]} \underset{\text{target}}{\overset{\text{source}}{\leftleftarrows}} \{\text{arrows}\} = \mathcal{X}_{\{0,1\}}$$

induced by $\{0\} \underset{1}{\overset{0}{\rightrightarrows}} \{0,1\}$

<u>composition</u>

$$\begin{tikzcd}
\mathcal{X}(0 \to 1 \to 2) \arrow[d, equal] & \mathcal{X}(\Delta^2) \arrow[l, "\simeq"'] \arrow[r] \arrow[d, equal] & \mathcal{X}(0 \to 2) \arrow[d, equal] \\
\mu : \mathcal{X}_{\{0,1\}} \times_{\mathcal{X}_{\{1\}}} \mathcal{X}_{\{1,2\}} & \mathcal{X}_{\{0,1,2\}} \arrow[l, "\simeq"'] \arrow[r] & \mathcal{X}_{\{0,2\}}
\end{tikzcd}$$

<u>associativity</u>



$\rightsquigarrow$ pass to homotopy theory:

replace Set by some ∞-category / model category $\mathcal{C}$

e.g. $\mathcal{C}$ = Grpds / Spaces / Spectra / ∞-cats / ...

Let $\mathcal{X} : \Delta^{op} \to \mathcal{C}$ be a simplicial object

we can evaluate $\mathcal{X}$ on any simplical complex/set

by right (homotopy) Kan extension

$$\begin{tikzcd} \Delta^{op} \arrow[r, "\mathcal{X}"] \arrow[d, hook] & \mathcal{C} \\ {[\Delta^{op}, \mathrm{Set}]} \arrow[ur, dotted, "\mathcal{X}"'] & \end{tikzcd}$$

e.g. $\mathcal{X}\left(\Delta^{\{0,1,2,3\}}_{\square}\right) := \mathcal{X}\left(\Delta^{\{0,1,2\}}\right) \underset{\mathcal{X}(\Delta^{\{0,2\}})}{\overset{(h)}{\times}} \mathcal{X}\left(\Delta^{\{0,2,3\}}\right)$

$$\mathcal{X}(K) = \mathcal{X}\Big(\operatorname*{colim}_{\Delta^i \to K} \Delta^i\Big) := (\mathrm{ho})\lim_{\Delta^i \to K} \mathcal{X}(\Delta^i)$$

$\mathcal{X} : \Delta^{op} \to \mathcal{C}$ is <u>Segal</u> if $\forall n \geq 2$:

$$\mathcal{X}_{[n]} = \mathcal{X}(\Delta^n) \xrightarrow{\simeq} \mathcal{X}\left(\overset{0}{\bullet} - \overset{1}{\bullet} - \overset{2}{\bullet} - \cdots - \overset{n-1}{\bullet} - \overset{n}{\bullet}\right)$$

$$\rotatebox{90}{=} \quad \mathcal{X}\left(\overset{0}{\bullet} - \overset{1}{\bullet}\right) \underset{\mathcal{X}(\bullet)}{\overset{(h)}{\times}} \cdots \underset{\mathcal{X}(\bullet)}{\overset{(h)}{\times}} \mathcal{X}\left(\overset{n-1}{\bullet} - \overset{n}{\bullet}\right)$$

is an equiv.

A Segal objects $X : \Delta^{op} \to \mathcal{C}$ encodes a category object in $\mathcal{C}$ which is associative up to coherent homotopies.

Rezk: $\left\{ \text{(complete) Segal } \Delta^{op} \to \text{Spaces} \right\} \simeq \infty\text{-Cats}$

<u>Example</u>: $\mathcal{A}$ an additive category,

e.g. $\mathcal{A} = \mathrm{proj}(R)$

$\rightsquigarrow$ Segal object

$$[n] \longmapsto \left\{ \begin{tikzcd} A_1 \arrow[r] & A_1 \oplus A_2 \arrow[r] & A_1 \oplus A_2 \oplus A_3 & \\ & A_2 \arrow[u] \arrow[r] & A_2 \oplus A_3 \arrow[u] & \\ & & A_3 \arrow[u] & \ddots \\ & & & A_n \end{tikzcd} \right\}^{\simeq}$$

$$\begin{tikzcd} \Delta^{op} \arrow[r] \arrow[d] & \mathrm{Grpd} \\ \Gamma^{op} = \mathrm{Fin}_* \arrow[ur, dashed, "\text{by permuting the } A_i"'] & \end{tikzcd}$$

($\rightsquigarrow$ can group complete to get $K(R)$)

2-Segal objects

[Dyckerhoff–Kapranov, Gálvez-Carrillo–Kock–Tonks]

sometimes composition/addition/multiplication is not uniquely defined but multivalued

e.g. extension of modules in R-mod $= \mathcal{A}$

$$A, B \in \mathcal{A} \longmapsto E = A \cdot B \text{ for each}$$

extension/s.e.s
$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}$$

- still associative in the following sense:

given $A, B, C \in \mathcal{A}$

$$\left\{ \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[r] \arrow[d] & F \arrow[dd] \\ 0 \arrow[r] & B \arrow[d] \arrow[r, phantom, "\square"] & {} \\ & 0 \arrow[r] & C \end{tikzcd} \right\} \qquad \left\{ \begin{tikzcd} A \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & F \arrow[d] \\ 0 \arrow[r] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ & 0 \arrow[r] & C \end{tikzcd} \right\}$$

$G := F \amalg_E B \cong F/A$ $\quad\cong\quad$ 3rd iso thm

$$\left\{ \begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & F \arrow[d] \\ 0 \arrow[r] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G \arrow[d] \\ & 0 \arrow[r] & C \end{tikzcd} \right\} \cong$$

denotes pushout + pullback

$E := F \times_G B$

$:= \ker(F \to C)$

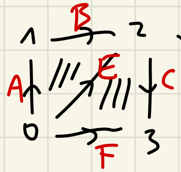
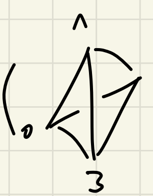
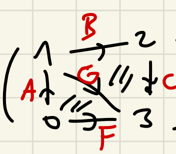
Can capture this data in the

Waldhausen $S_\bullet$-construction:

$$[n] \longmapsto \left\{ \begin{array}{ccccccccc}
A_{01} & \to & A_{02} & \to & A_{03} & \cdots & \to & A_{0n} \\
\downarrow & \square & \downarrow & \square & \downarrow & & \square & \\
0 & \to & A_{12} & \to & A_{13} & & & \vdots \\
 & & \downarrow & \square & \downarrow & & & \\
 & & 0 & \to & A_{23} & & & \vdots \\
 & & & & & \ddots & & A_{n-2,n} \\
 & & & & & \downarrow & \square & \downarrow \\
 & & & & & 0 & \to & A_{n-1,n}
\end{array} \right\}^{(\simeq)}$$

$$\Delta^{op} \longrightarrow \mathrm{Grpd} / \mathrm{Spaces} / \mathrm{Stacks} / \mathrm{Cats}$$

which satisfies <u>2-Segal conditions</u>, e.g.:

$\mathcal{X}\Big($  $\Big)$ $\xleftarrow{\ \simeq\ }$ $\mathcal{X}\Big($  $\Big)$ $\xrightarrow{\ \simeq\ }$ $\mathcal{X}\Big($  $\Big)$

2-2 Pachner move

"A, B, C, E = AB F = EC"

"A, B, C, G = BC F = AG"

"A, B, C & all their possible composites"

given a 2-Segal object $\mathcal{X} : \Delta^{op} \to \mathcal{C}$
and a suitable "linearization"/cohomology procedure

$$H : \mathcal{C} \longrightarrow \mathrm{Vect}/R\text{-Mod}/\ldots$$

can consider convolution algebra

$$H(\mathcal{X}_{[1]}) \underset{H(\mathcal{X}_{[0]})}{\otimes} H(\mathcal{X}_{[1]}) \longrightarrow H(\mathcal{X}_{[1]})$$

"cohomology classes" on $\mathcal{X}_{[1]}$

$$(f,g)(e) := \int_{\{a\cdot b = e\}} f(a)\cdot g(b)$$

fiber of $\mathcal{X}_{[2]} \longrightarrow \mathcal{X}_{[1]}$ over e

$$\left[\begin{array}{l} \mu : \quad \mathcal{X}_{[1]} \underset{\mathcal{X}_{[0]}}{\times} \mathcal{X}_{[1]} \longleftarrow \mathcal{X}_{[2]} \longrightarrow \mathcal{X}_{[1]} \\ \qquad H(\;\ldots\;) \xrightarrow[\text{back}]{\text{pull-}} H(\;..\;) \xrightarrow[\text{forward}]{\text{push}} H(\;..\;) \end{array}\right]$$

For $\mathcal{X} = S_\bullet(\mathcal{A})$ these are called Hall algebras

Comparison with other algebraic structures:

Thm [W.] $\{\text{2-Segal spaces}\} \xhookrightarrow[\text{faithful}]{\text{fully}} \left\{\begin{matrix}\text{(colored, non-symm.)} \\ \infty\text{-operads}\end{matrix}\right\}$

essential image = "invertible" ∞-operads

Thm [Stern] $\{\text{2-Segal } \Delta^{op} \to \mathcal{C}\} \overset{\simeq}{\longleftrightarrow} \mathrm{Alg}(\mathrm{Span}(\mathcal{C}))$

Higher Segal objects

[Dyckerhoff - Kapranov, Poguntke]

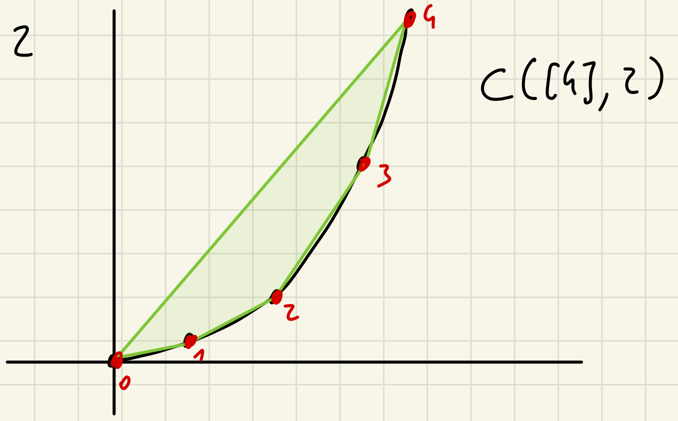
How do Segal & 2-Segal objects fit into a hierarchy?

Let
$$\mu_d: \mathbb{R} \longrightarrow \mathbb{R}^d$$
$$t \longmapsto (t, t^2, \dots, t^d) \qquad \text{moment map}$$

define <u>cyclic polytopes</u>:

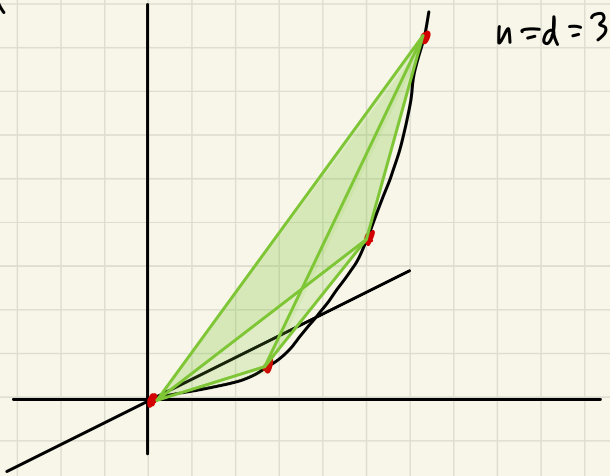
$$C([n], d) := \text{convex hull of } \mu_d(\{0, 1, \dots, n\}) \subseteq \mathbb{R}^d$$

Example: • $d = 2$



$$C([n], 2) = (n+1)\text{-gon in the plane}$$

- $C([d], d) \cong \Delta^d$



Facts: Boundary

disjoint interiors

$$\partial C([n], d+1) = \mathcal{L}([n], d) \cup \mathcal{U}([n], d)$$

where $\mathcal{U}([n], d)$ and $\mathcal{L}([n], d)$

upper boundary lower boundary

are d-dim'l simplicial complexes.

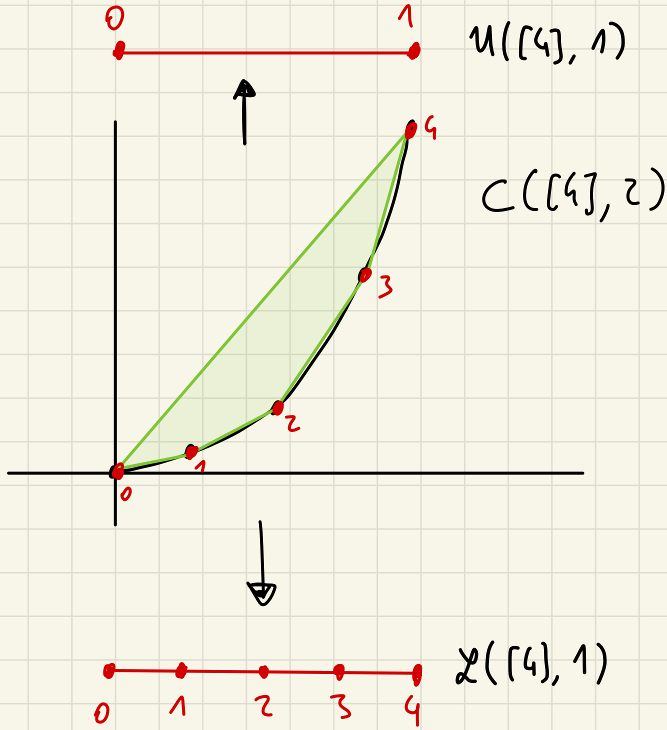
Moreover $\mathbb{R}^{d+1} \longrightarrow \mathbb{R}^d$ $(x_1, \dots, x_{d+1}) \mapsto (x_1, \dots, x_d)$

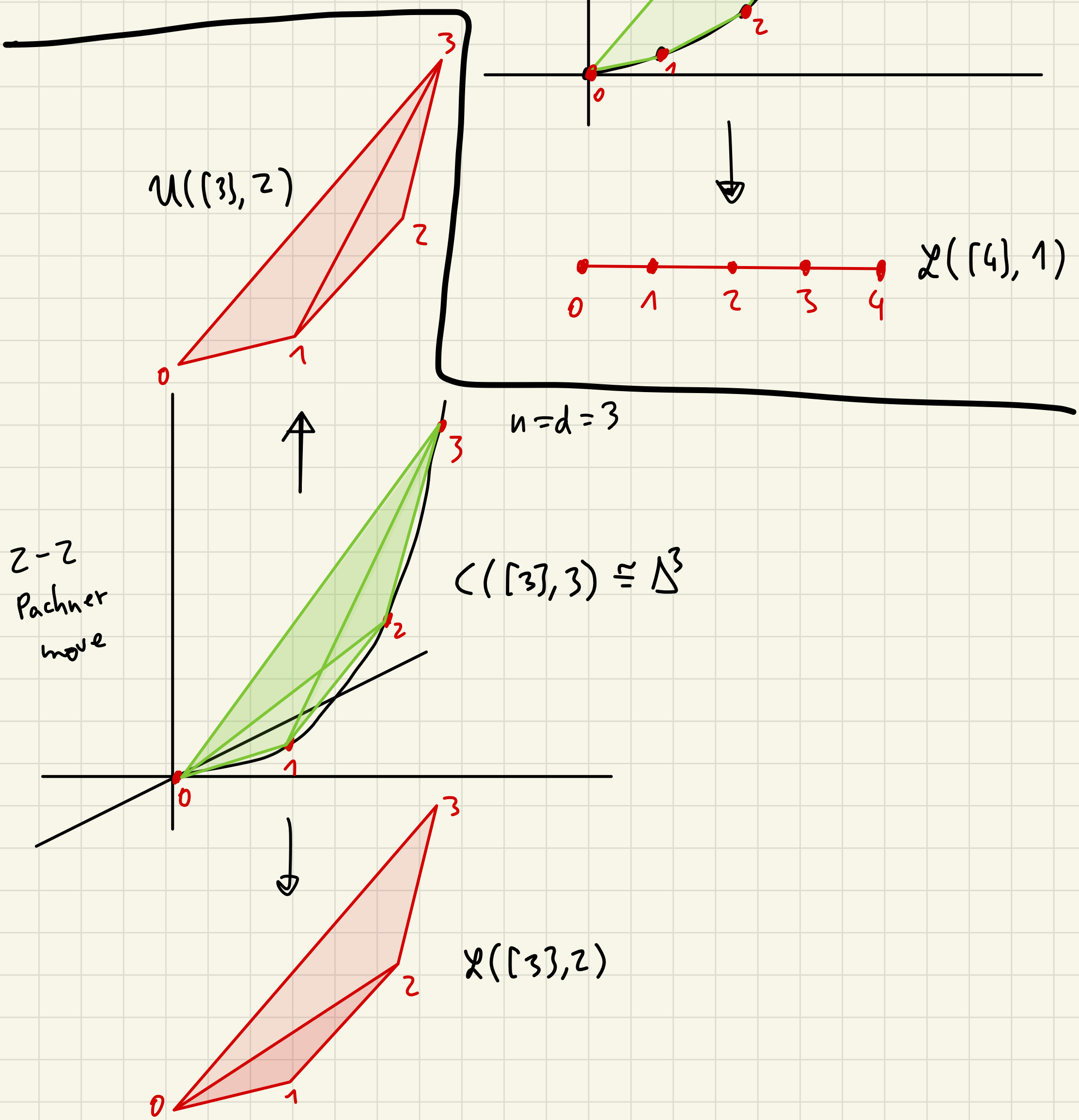
induces triangulations

$$\mathcal{U}([n], d) \xrightarrow{\cong} C([n], d) \xleftarrow{\cong} \mathcal{L}([n], d)$$

$\mathcal{U}([n],1) = \{ \overset{0}{\bullet} \text{———} \overset{1}{\bullet} \}$

$\mathcal{L}([n],1) =$
$\{ \overset{0}{\bullet} \text{—} \overset{1}{\bullet} \text{—} \overset{2}{\bullet} \cdots \text{—} \overset{n}{\bullet} \}$





Def: A simplicial object $\mathcal{X} : \Delta^{op} \to \mathcal{C}$ is called:

- <u>lower d-Segal</u> if $\forall n > d$
  $$\mathcal{X}\left(\mathcal{L}([n], d) \hookrightarrow \Delta^{[n]}\right)$$
  is an equivalence
- <u>upper d-Segal</u> if $\forall n > d$
  $$\mathcal{X}\left(\mathcal{U}([n], d) \hookrightarrow \Delta^{[n]}\right)$$
  is an equivalence
- <u>d-Segal</u> if
  $\mathcal{X}$ is upper & lower d-Segal

<u>Prop:</u> (DK, P; using combinatorics of Rambau)

$\mathcal{X} : \Delta^{op} \to \mathcal{C}$ is d-Segal $\qquad (\forall n > d)$

$\Leftrightarrow$ for <u>all</u> triangulations $\mathcal{T}$ of $C([n], d)$,

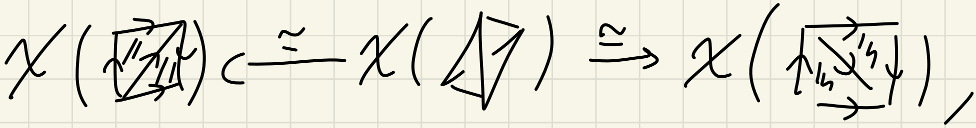
$\mathcal{X}(\mathcal{T} \longrightarrow \Delta^{[n]})$ is an equivalence

Example • X lower 1-Segal

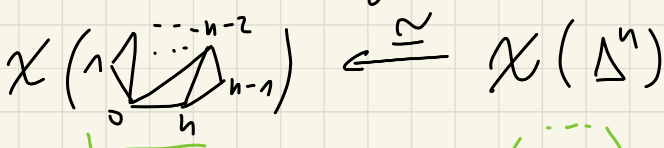
$\iff X(\Delta^{[n]}) \xrightarrow{\simeq} X(\overset{}{\underset{0}{\bullet}}\!-\!\underset{1}{\bullet}\!-\cdots-\!\underset{n-1}{\bullet}\!-\!\underset{n}{\bullet}) \quad \forall n$

$\iff X$ is Segal

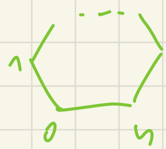
• X is 2-Segal:



more generally:

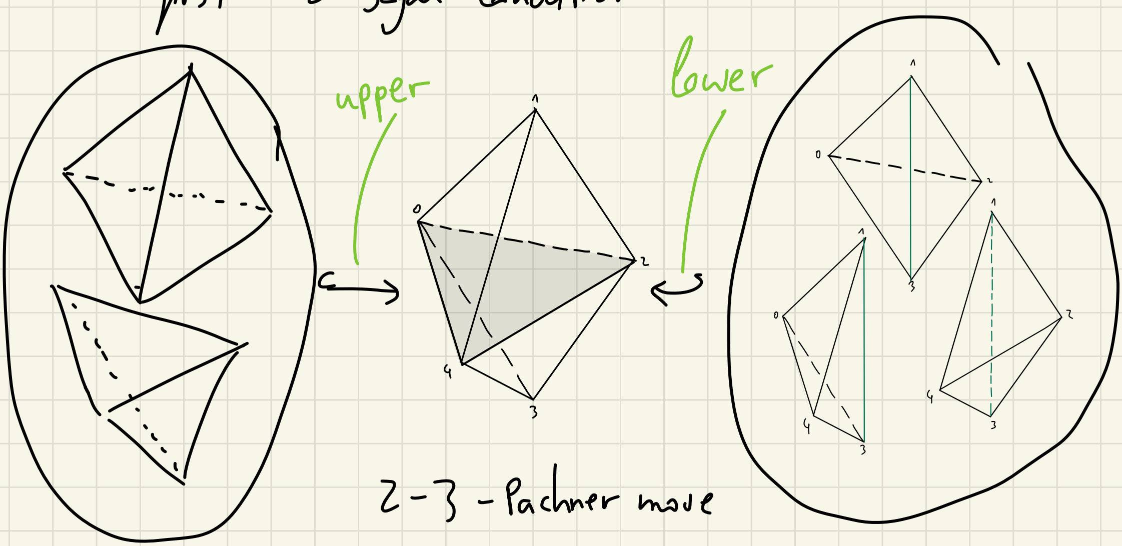


any triangulation of



(n+1)-gon

• first 3-Segal condition



2-3-Pachner move

Main example of $2k$-Segal object (Poguntke):
higher Waldhausen construction $S_\bullet^{\langle k \rangle}(\mathcal{A})$

$$\Omega^k |S_\bullet^{\langle k \rangle}(\mathcal{A})| \simeq K(\mathcal{A})$$

<u>Prop</u> (Gale's evenness criterion)

Let $d = 2k-1$ be odd.

The maximal simplices of $\mathcal{L}([n], d)$ correspond to subsets of the form

$$I = \dot{\bigcup_{j=1}^{k}} \{ i_j - 1, i_j \} \subset [n]$$

union is disjoint, i.e. $i_j < i_{j+1} - 1$

Example: maximal simplices of $\mathcal{L}([4], 3)$:
($k=2$)

0	1	2	3	4	$= [4]$
0	1	2	3		maximal simplices
0	1		3	4	
	1	2	3	4	

(there is a similar criterion for other cases)

Today focus on

lower $(2k-1)$ Segal & $2k$-Segal

well understood

1	$\boxed{\text{(lower 1-)Segal}}$	
2		$\boxed{\text{2-Segal}}$
3	lower 3-Segal	
4		4-Segal
5	lower 5-Segal	
6		6-Segal
7	lower 7-Segal	
	$\vdots$	$\vdots$
	"excision" on simplex category Δ	"excision" on cyclic cat Λ

Interlude: manifold calculus

Let Man be the topological/∞- cat of (smooth) mfds & embeddings

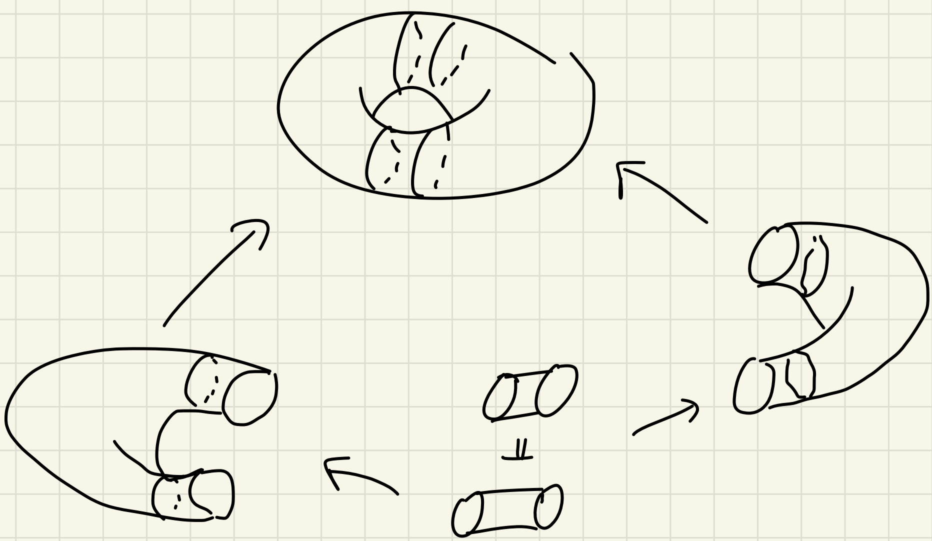
A functor $\mathcal{X} : \mathrm{Man}^{op} \longrightarrow \mathcal{C}$ is

excisive if for $M = U_0 \cup U_1$ open cover

$$\mathcal{X}\left(\begin{tikzcd} & M & \\ U_0 \arrow[ur, hook] & & U_1 \arrow[ul, hook'] \\ & U_0 \cap U_1 \arrow[ul, hook] \arrow[ur, hook'] & \end{tikzcd}\right)$$

is (htpy) pullback, i.e.

$$\mathcal{X}(M) \xrightarrow{\simeq} \mathcal{X}(U_0) \times_{\mathcal{X}(U_0 \cap U_1)} \mathcal{X}(U_1)$$



(if, for instance $\mathcal{C} = \mathcal{D}(\mathbb{R})$
$\rightsquigarrow$ yields familiar l.e.s for $H_* \mathcal{X}(-)$)

<u>Example</u>: fix $N \in Man$

- $Imm(-, N)$ is excisive
(being immersion is local)
- $Emb(-, N)$ is <u>not</u> excisive
(being embedding is global)

<u>Main idea</u> (Goodwillie-Weiss) interpolate

$$\begin{tikzcd}
Fun(Man^{op}, \mathcal{C}) \arrow[rr, bend right, "P_{\leq k}"'] & \supset \dots \supset & \left\{\begin{matrix}\text{polynomial} \\ \text{of degree} \leq k\end{matrix}\right\} \arrow[rr, bend right, "P_{\leq 1}"'] & \supset \dots \supset & \{\text{excisive}\} \arrow[d, phantom, "\shortparallel"] \\
& & & & \begin{matrix}\text{polynomial} \\ \text{of deg} \leq 1\end{matrix}
\end{tikzcd}$$

"polyn. approximation"/ adjoint

$\rightsquigarrow$ tower

$$\begin{tikzcd}
& & & & const\ \mathcal{X}(\emptyset) \arrow[d, phantom, "\shortparallel"] \\
\mathcal{X} \arrow[r] & \cdots \arrow[r] & P_{\leq k}\mathcal{X} \arrow[r] & \cdots \arrow[r] & P_{\leq 1}\mathcal{X} \arrow[r] & P_{\leq 0}\mathcal{X}
\end{tikzcd}$$

which in good cases yields information about
$\mathcal{X} : \Delta^{op} \to \mathcal{C}$.

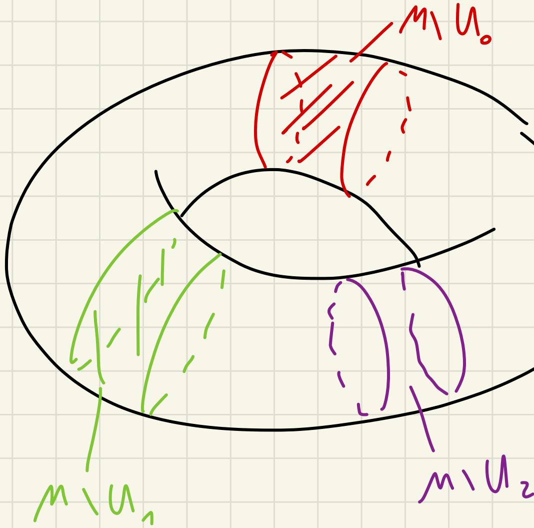
Def: A functor $\mathcal{X} : \mathrm{Man}^{op} \to \mathcal{C}$ is polynomial of degree $\leq k$ if:

whenever $M = U_0 \cup \ldots \cup U_k$ open covers with $U_i \cup U_j = M \;\forall i \neq j$

($\Leftrightarrow$ $M \setminus U_0, \ldots, M \setminus U_k$ closed & pw. disjoint)

have

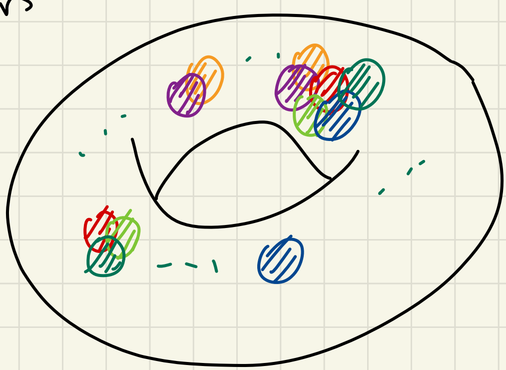
$$\mathcal{X}(M) \xrightarrow{\simeq} \lim_{I \subset [k]} \mathcal{X}\Big(\bigcap_{i \in I} U_i\Big)$$



Cover M by $(k+1)$ many mfds, each of which (and their $\cap$) can be complicated

interplay

"good k-covers"



Cover M by a large number of mfds, each of which (and their $\cap$) is very simple

Def: • A k-cover of a mfd is

$$M = \bigcup_{i \in I} U_i \qquad U_i \subseteq M \text{ open}$$

s.th $\forall$ $S \subset M$, $|S| \leq k$ finite set

$$\exists \; i \in I : \quad S \subset U_i$$

$$\left(\iff \{ U_i^{\times k} \}_{i \in I} \text{ is cover of } M^{\times k} \right)$$

• A k-cover is good if additionally for each $I_0 \subset I$ finite

$\bigcap_{i \in I_0} U_i$ is (diffeom to) disj union of $\leq k$ many disks.

Thm [Boavida-Weiss]

A functor $X : \mathrm{Man}^{op} \to \mathcal{C}$ is polynomial of degree $\leq k$ if and only if for each good k-cover $\{U_i\}_{i \in I}$ of M

$$X(M) \xrightarrow{\sim} \lim_{\substack{I_0 \subset I \\ \text{finite}}} X\left(\bigcap_{i \in I_0} U_i \right)$$

("satisfies descent/is a sheaf wrt good k-covers")

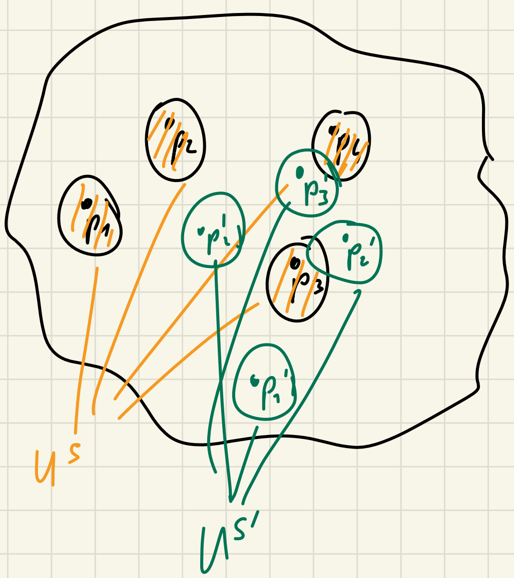
<u>Lemma</u> (basic diff. geometry)

Every manifold has a good k-covering

<u>Pf Sketch</u>:

for each $S = \{p_1, \dots, p_k\} \subset M$, $|S| = k$ choose very small convex (geodesically wrt a chosen metric) pw. disjoint balls $U_1^S \ni p_1$, ... $U_k^S \ni p_k$

$\rightsquigarrow \mathcal{U} = \{U^S \mid S \text{ as above}\}$ good k-cover of M.



Back to the simplex category

<u>Informal analogy</u> $\Delta \leftrightsquigarrow \mathrm{Man}$

We think of an object $[n] \in \Delta$
as a "manifold" (not in any real mathematical sense)

which has:
- "points" $p_i = \{i-1, i\}$, $i = 1, \dots, n$
- "open sets" $\triangleq$ subsets $I \subseteq [n]$
- where "$p_i \in I$" if $\{i-1, i\} \subseteq I$
- "open sets" may contain <u>no</u> "point",

  e.g. $I = \{0\} \subseteq [n]$

  $I, I' \subseteq [n]$
- we say that "open sets" $I, I' \subseteq [n]$
  "cover" the "manifold" $[n]$ if
  for each "point" p we have "$p \in I$" or "$p \in I'$"

<u>Note</u> it does <u>not</u> suffice to ask $[n] = I \cup I'$
(as normal subsets)

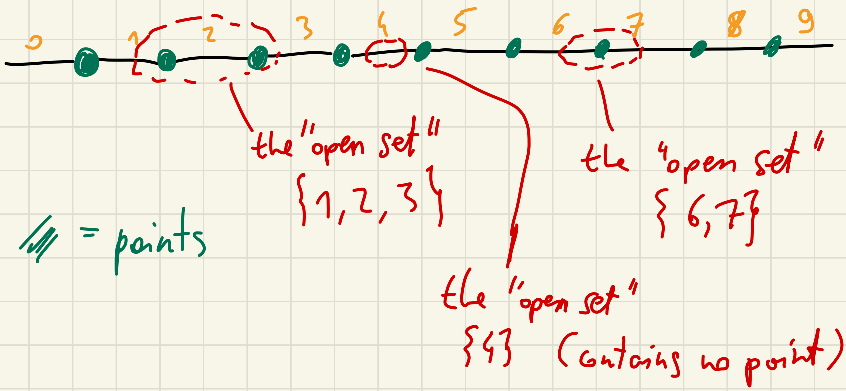
e.g. $\underline{0\,1\,2\,3\,4} = [n]$

$0\,1\,2 = I$

$3\,4 = I'$

$p = \{2,3\} \not\subseteq I$, $\not\subseteq I'$

<u>picture</u> of $[n]$ as a "manifold"



This analogy $\Delta \longleftrightarrow \mathrm{Man}$ leads to a true theorem!

<u>Def.</u> A collection of subsets $I_0, \dots, I_k \subseteq [n]$ is called <u>compatible</u> if for each $i = 1, \dots, n$ there is at most one $j \in [k]$ s.th

$$\{i-1, i\} \not\subseteq I_j$$

($\triangleq$ each pair $I_j, I_{j'}$ $(j \neq j)$ form an "open cover" of the "manifold" $[n]$)

Thm (W.) Fix $k \geq 1$. Let $\mathcal{X} : \Delta^{op} \to \mathcal{C}$. TFAE:

- $\mathcal{X}$ is polynomial of degree $\leq k$, i.e

$$\mathcal{X}_{[n]} \xrightarrow{\simeq} \lim_{J \subseteq [k]} \mathcal{X}\Big(\bigcap_{i \in J} I_i\Big)$$

  for each compatible $I_0, \dots, I_k \subseteq [n]$

- $\mathcal{X}$ is lower $2k-1$ Segal.

recall construction of good k-covers:

$\triangleq$ given k "points" $p_1 = \{i_1 - 1, i_1\}, \dots, p_k = \{i_k - 1, i_k\}$

need "very small open balls around p_j"

$\rightsquigarrow$ just take $U_{p_j} = \{i_j - 1, i_j\}$

$\rightsquigarrow$ get canonical "good k-cover" of $[n]$

$$\left\{ \bigcup_{j=1}^{k} \{i_j - 1, i_j\} \;\middle|\; \begin{array}{l} 0 < i_1 < i_2 - 1 < i_2 < \dots \\ \dots < i_k \leq n \end{array} \right\}$$

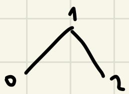
are exactly the maximal simplices of

$\mathcal{L}([n], 2k-1)$

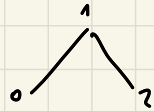
Example: $k=1$

compatible "k-covers"	"good k-covers"

$n=2$



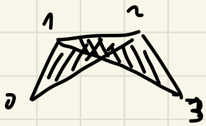
$$\begin{array}{ccc} 0&1&2\\ \hline 0&1&\\ &1&2 \end{array} \qquad\qquad \begin{array}{ccc} 0&1&2\\ \hline 0&1&\\ &1&2 \end{array}$$



$n=3$

$$\begin{array}{cccc} 0&1&2&3\\ \hline 0&1&&\\ &1&2&3 \end{array} \qquad\qquad \begin{array}{cccc} 0&1&2&3\\ \hline 0&1&&\\ &1&2&\\ &&2&3 \end{array}$$

$$\begin{array}{cccc} 0&1&2&3\\ \hline 0&1&2&\\ &&2&3 \end{array}$$



$$\begin{array}{cccc} 0&1&2&3\\ \hline 0&1&2&\\ &1&2&3 \end{array}$$

$n=4$

$$\begin{array}{ccccc} 0&1&2&3&4\\ \hline 0&1&2&3&\\ &&2&3&4 \end{array} \qquad\qquad \begin{array}{ccccc} 0&1&2&3&4\\ \hline 0&1&&&\\ &1&2&&\\ &&2&3&\\ &&&3&4 \end{array}$$

$$\begin{array}{ccccc} 0&1&2&3&4\\ \hline 0&1&&3&4\\ &1&2&3& \end{array}$$

$\vdots$

opens = constant

size of opens increases

size of opens = constant

opens increases

What is the intrinsic meaning of compatible collections $I_0, \ldots, I_k \subseteq [n]$?

$\underline{k=1}$ consider the square

$$\begin{tikzcd}
 & {[n]} & \\
I_0 \arrow[ur, hook] & (*) & I_1 \arrow[ul, hook'] \\
 & I_0 \cap I_1 \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

in Δ

<u>Lemma</u>: $I_0, I_1 \subseteq [n]$ are compatible

$\iff$ $(*)$ is a pushout square

<u>Note</u>: $(*)$ is always a pullback square.

In Δ there are other pushout squares, e.g.

$$\begin{tikzcd}
\{0,0'\} \arrow[r, hook] \arrow[d, two heads, "s"] \arrow[dr, phantom, "(1)"] & \{0,0',1\} \arrow[d, two heads, "{0,0' \mapsto 0}"] \\
\{0\} \arrow[r, hook] & \{0,1\}
\end{tikzcd}$$

is also pullback

$$\begin{tikzcd} \{0,0'\} \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "(2)"] & \{0\} \arrow[d] \\ \{0\} \arrow[r] & \{0\} \end{tikzcd}$$

is not pullback

If $\mathcal{X} : \Delta^{op} \to \mathcal{C}$ is Segal

then $\mathcal{X}(1)$ is pullback

$$\left[\left\{ \begin{tikzcd} & 0' \arrow[dr, "f"] \arrow[d, phantom, "\equiv"] & \\ 0 \arrow[ur, "e"] \arrow[rr, "g"'] & & 1 \end{tikzcd} \;,\; e \simeq s^*(id_0) \right\} \simeq \left\{ 0 \xrightarrow{g} 1 \right\} \right.$$

$$\begin{tikzcd} & 0 \arrow[dr, "g"] \arrow[d, phantom, "\equiv"] & \\ 0 \arrow[ur, "id"] \arrow[rr, "g"'] & & 1 \end{tikzcd} \longleftarrow\!\shortmid\; g$$

"unitality" of encoded category $\Big]$

but $\mathcal{X}(2)$ is not necessarily

By explicitly characterizing the squares in Δ which are pushout & pullback (= bicartesian) one sees that

$X : \Delta^{op} \to \mathcal{C}$ is (lower 1-) Segal

$\iff$ X sends bicartesian squares to pullbacks

More generally:

powerset poset of set S

A cube $C : \mathcal{P}(S) \to \Delta$ is called **strongly bicartesian** if every 2-dimensional face

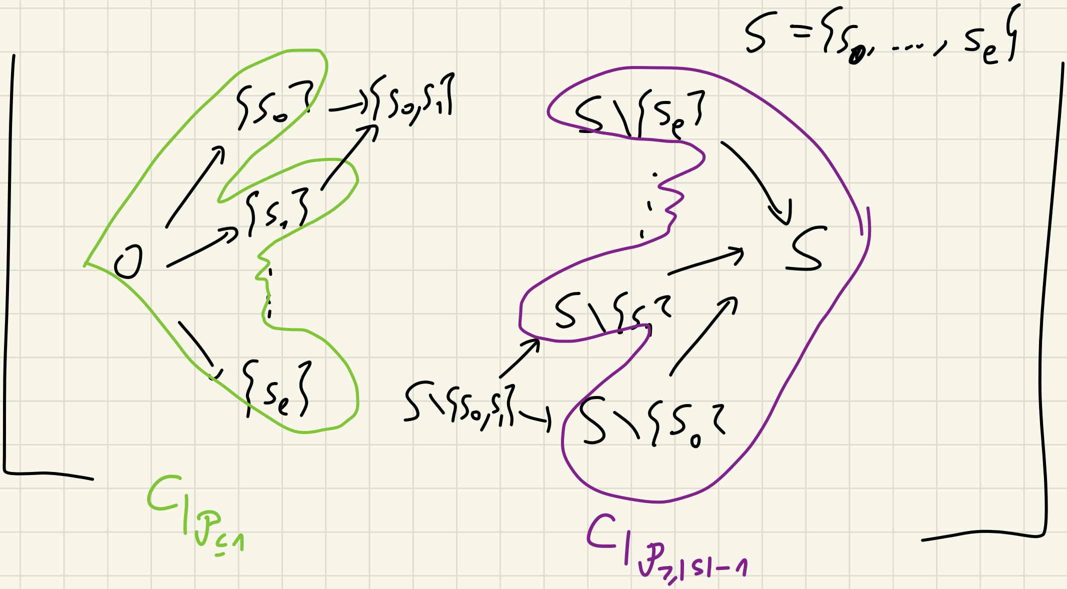
$$\begin{tikzcd} C(T \cap T') \arrow[r] \arrow[d] & C(T') \arrow[d] \\ C(T) \arrow[r] & C(T \cup T') \end{tikzcd} \quad , \; T, T' \subseteq S$$

of $\mathcal{P}(S)$ is bicartesian.

[Equivalently

C is right Kan extension of $C|_{\mathcal{P}_{\geq |S|-1}(S)}$

and left Kan extension of $C|_{\mathcal{P}_{\leq 1}(S)}$]



<u>Observation</u> for $I_0, \dots, I_k \subseteq [n]$, the cube

$$\underbrace{\begin{array}{ccccc} \mathcal{P}([k]) & \cong & \mathcal{P}^{op}([k]) & \longrightarrow & \Delta \\ [k]\setminus J & \longleftrightarrow & J & \longmapsto & \bigcap_{i \in J} I_i \end{array}}_{\text{intersection cube}}$$

(with $\bigcap \emptyset = [n]$)

is strongly bicartesian

$\iff$ the collection $I_0, \dots, I_k \subseteq [n]$ is compatible

[$:\iff$ each $\{i-1, i\}$ belongs to all but at most one I_j]

Theorem (W.) Fix $k \geq 1$. Let $X : \Delta^{op} \to \mathcal{C}$.

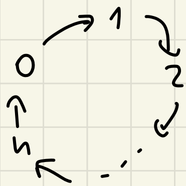
TFAE:
- X is lower $(2k-1)$-Segal
- X is polynomial of degree $\leq k$ (sends strongly bicartesian intersection $(k+1)$-cubes to limits)
- X sends all strongly bicartesian $(k+1)$-cubes to limit diagrams

I call such X weakly excisive.

compare to Goodwillie's functor calculus, where X is k-excisive $\iff$ X sends strongly cocartesian $(k+1)$-cubes to limit diagrams

What about 2k-Segal?

Λ = category of cyclically ordered sets

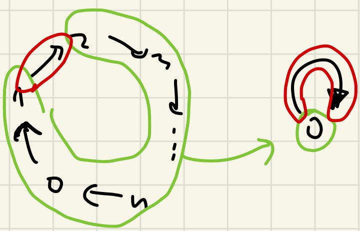
$\langle n \rangle =$ 

morphisms

$$\Lambda(\langle n\rangle,\langle m\rangle) \cong \left\{ \begin{array}{c}\text{monotone degree 1-maps}\\ (S^1, \mathbb{Z}/_{n+1}) \longrightarrow (S^1, \mathbb{Z}/_{m+1})\end{array}\right\} \Big/ \text{htpy}$$

e.g.

$$\Lambda(\langle n\rangle,\langle 0\rangle) = \{\text{linear orders on } \langle n\rangle\}$$



$$\rightsquigarrow \quad \Delta = \Lambda/_{\langle 0\rangle} \longrightarrow \Lambda$$

$$[n] \longmapsto (\langle n\rangle \longrightarrow \langle 0\rangle)$$

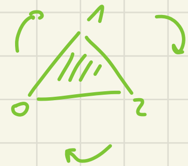
collapse all arrows except $n \to 0$

$$\mathrm{Aut}(\langle n\rangle) = \mathbb{Z}/_{n+1}$$

$\rightsquigarrow$ cyclic objects $X : \Lambda^{op} \to \mathcal{C}$

$=$ simplicial object with cyclic symmetries

$$\begin{tikzcd} X_0 & X_1 \arrow[l, shift left] \arrow[l, shift right] \arrow[loop below, "\mathbb{Z}/2\mathbb{Z}"] & X_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[loop below, "\mathbb{Z}/3\mathbb{Z}"] & \cdots \arrow[l, shift left=3] \arrow[l, shift left] \arrow[l, shift right] \arrow[l, shift right=3] \end{tikzcd}$$



Theorem (W.) Fix $k \geq 1$. Let $X : \Delta^{op} \to \mathcal{C}$.

TFAE:
- X is $2k$-Segal
- X is weakly k-excisive w.r.t. $\Delta \to \Lambda$,

  i.e. X sends to limit diagrams in $\mathcal{C}$ those $(k+1)$-cubes in Δ that are strongly bicartesian (in Δ and remain so) in Λ.

Corollary: A cyclic object $Y : \Lambda^{op} \to \mathcal{C}$ is $2k$-Segal (i.e. $\Delta^{op} \to \Lambda^{op} \to \mathcal{C}$ is)

$\iff$ Y is weakly k-excisive

(strongly bicart. cubes $\mapsto$ limit cubes)

[DK]: cyclic 2-Segal objects $Y : \Lambda^{op} \to \mathcal{C}$
yield invariants of marked surfaces
$\rightsquigarrow$ topological Fukaya categories

Further work:

- the situation becomes much simpler if $\mathcal{C}$ is additive ∞-cat: $\left(\text{\& weakly idemp. complete}\right)$

has products/coproducts & additive homotopy cat.

Thm [W.] (∞-categorical Dold-Kan correspondence)

$$\begin{array}{ccc}
\mathrm{Fun}(\Delta^{op}, \mathcal{C}) & \overset{\simeq}{\longleftrightarrow} & Ch_{\geq 0}(\mathcal{C}) \\
\cup & & \cup \\
\{\text{lower } (2k-1)\text{-Segal}\} & \simeq & \left\{ \begin{array}{c} k\text{-truncated complexes} \\ X_0 \leftarrow \cdots \leftarrow X_k \leftarrow 0 \leftarrow \cdots \end{array} \right\} \\
\parallel & & \\
\{2k\text{-Segal}\} & \simeq &
\end{array}$$

∞-cat of coherent connective chain complexes in $\mathcal{C}$

- If target $\mathcal{C}$ is presentable & stable (e.g. $\mathcal{C}$ = Spectra) can relate

$$\{2k\text{-Segal } \Gamma^{op} \to \mathcal{C}\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{array}{c} \text{(finitary) } k\text{-excisive functors} \\ \mathrm{Spaces}_* \to \mathcal{C} \end{array} \right\}$$

in the sense of Goodwillie