# Spaces vs Categories, Perfect Maps vs Discrete Cofibrations

Walter Tholen

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Largely based on

WT, Leila Yeganeh:

*The comprehensive factorization of Burroni's T-functors*

arXiv:2012.08043

http://www.tac.mta.ca/tac/volumes/36/8/36-08abs.html

# Overview

- Two special classes of maps ...
- ... guiding us to Burroni's $\mathbb{T}$-categories and $\mathbb{T}$-functors
- A network of specialized and generalized $\mathbb{T}$-categories
- Two special factorization systems leading to ...
- ... the comprehensive factorization system for $\mathbb{T}$-functors
- On-going and future work

# Two instances of "nice" maps $p : E \to B$ of ...

... topological spaces:

$p$ *perfect* $\iff$ $p$ is proper and separated
$\iff$ $p$ is a pullback-stably closed map and distinct points of a fibre are sep'ed by disjoint open sets
$\iff$ $p$ is closed, has compact fibres, and is separated

... and of (small) categories:

$p$ *discrete cofibration* $\iff$ $\forall x \in E_0 : x\backslash E \longrightarrow px\backslash B$ bijective on objects

"Nice", because they share good properties, such as containing all isos, being closed under composition and stable under pullback, ...

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# Let's search for a parent!
# (Only categorists may have a chance to find one.)

$p : E \longrightarrow B$ discrete cofibration in Cat:

$$\begin{array}{ccc} x & \overset{f}{\dashrightarrow} & y \\ \Big| & & \vdots \\ a & \xrightarrow{u} & b \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{ob}E & \xleftarrow{\mathrm{dom}} & E \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{ob}B & \xleftarrow{\mathrm{dom}} & B \end{array}$$

$p : E \longrightarrow B$ perfect map in Top:

$$\begin{array}{ccc} \mathfrak{x} & \overset{\mathrm{conv}}{\dashrightarrow} & y \\ \Big| & & \vdots \\ \mathfrak{a} & \xrightarrow{\mathrm{conv}} & b \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Ult}E & \xleftarrow{\mathrm{dom}} & \mathrm{Conv}E \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathrm{Ult}B & \xleftarrow{\mathrm{dom}} & \mathrm{Conv}B \end{array}$$

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$$

# But where does the arrow $\mathfrak{x} \to y$ live?

Think (small) *multicategory A* à la Lambek 1969:

$$(x_1, ..., x_n) \longrightarrow y$$

- set $A_0$ of objects
- set $A_1$ of morphisms
- "domain" and "codomain" functions

  $$\mathrm{dom} : A_1 \longrightarrow \mathrm{L}A_0 \quad \text{and} \quad \mathrm{cod} : A_1 \longrightarrow A_0$$

  where $\mathrm{L}$ belongs to the *list* (or free-monoid) monad on Set
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# Burroni's 1971 generalization and internalization Step 1: $\mathbb{T}$- graphs

$\mathcal{C}$ category with pullbacks, $\mathbb{T} = (T, \mu, \eta)$ *any* monad on $\mathcal{C}$

Gph($\mathbb{T}$), the category of $\mathbb{T}$-*graphs* in $\mathcal{C}$:

$$\begin{array}{ccccc} TA_0 & \xleftarrow{d_0^A} & A_1 & \xrightarrow{c_0^A} & A_0 \\ \downarrow{\scriptstyle Tf_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ TB_0 & \xleftarrow{d_0^B} & B_1 & \xrightarrow{c_0^B} & B_0 \end{array} \qquad \begin{array}{c} A = (A_0, A_1, d_0^A, c_0^A) \\ f= \downarrow (f_0, f_1) \\ B = (B_o, B_1, d_0^B, c_0^B) \end{array}$$

RelGph($\mathbb{T}$), the category of *relational* (or *regular*) $\mathbb{T}$-graphs:

full subcategory of Gph($\mathbb{T}$) of those $A$ with $(d_0^A, c_0^A)$ monic

Alg($\mathbb{T}$) = full subcat of Gph($\mathbb{T}$) of those $A$ with $A_1 = TA_0$, $d_0^A = 1_{TA_0}$

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# Step 2: Pointed $\mathbb{T}$-graphs

PGph($\mathbb{T}$): *pointed* $\mathbb{T}$-graphs = $\mathbb{T}$-graphs equipped with $i^A : A_0 \to A_1$:

$$
\begin{array}{ccccc}
TA_0 & \xleftarrow{\eta_{A_0}} & A_0 & = & A_0 \\
\| & & \downarrow{\scriptstyle i^A} & & \| \\
TA_0 & \xleftarrow{d_0^A} & A_1 & \xrightarrow{c_0^A} & A_0
\end{array}
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A_0 & \xrightarrow{i^A} & A_1 \\
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RefGph($\mathbb{T}$): *reflexive* $\mathbb{T}$-graphs = relational pointed $\mathbb{T}$-graphs

UAlg($\mathbb{T}$): *unary* $\mathbb{T}$-algebras = reflexive $\mathbb{T}$-graphs lying in Alg($\mathbb{T}$)

$$
\begin{array}{ccc}
A_0 & = & A_0 \\
{\scriptstyle \eta_{A_0}}\downarrow & & \| \\
TA_0 & \xrightarrow{c_0^A} & A_0
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\begin{array}{ccc}
A_0 & = & A_0 \\
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TA_0 & \xrightarrow{c_0^A} & A_0
\end{array}
$$

# Step 3: $\mathbb{T}$-categories (notational preparation)

The objects $A_2 = TA_1 \times_{TA_0} A_1$ of "composable pairs" and
$A_3 = TA_2 \times_{TA_1} A_2$ of "composable triples" of a $\mathbb{T}$-graph:

$$
\begin{array}{ccc}
TA_1 & \xleftarrow{\;d_1\;} & A_2 \\
{\scriptstyle Tc_0}\downarrow & & \downarrow{\scriptstyle c_1} \\
TA_0 & \xleftarrow{\;d_0\;} & A_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
TA_2 & \xleftarrow{\;d_2\;} & A_3 \\
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Every morphism $f : A \to B$ in $\mathrm{Gph}(\mathbb{T})$ gives

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# Step 3: $\mathbb{T}$- categories (definition)

Cat($\mathbb{T}$), $\mathbb{T}$-*categories* = pointed $\mathbb{T}$-graphs equipped with $m^A : A_2 \to A_1$:

$$\begin{array}{ccccc} TA_1 & \xleftarrow{d_1} & A_2 & \xrightarrow{c_1} & A_1 \\ \downarrow{\scriptstyle \mu_{A_0}\cdot Td_0} & & \downarrow{\scriptstyle m^A} & & \downarrow{\scriptstyle c_0} \\ TA_0 & \xleftarrow{d_0} & A_1 & \xrightarrow{c_0} & A_0 \end{array} \qquad \begin{array}{ccc} A_2 & \xrightarrow{m^A} & A_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ B_2 & \xrightarrow{m^B} & B_1 \end{array}$$

subject to the associativity and neutrality laws

$$m \cdot m_1 = m \cdot m_2 \qquad \text{and} \qquad m \cdot i_1 = 1_{A_1} = m \cdot i_2,$$

where $\quad m_1 = Tm \times_{Tc_0} c_1, \quad m_2 = (\mu_{A_1} \cdot Td_1) \times_{(\mu_{A_0} \cdot Td_0)} m : \; A_3 \longrightarrow A_2$

and $\quad i_1 = Ti \times_{TA_0} 1_{TA_1}, \quad i_2 = \eta_{A_1} \times_{\eta_{A_0}} i : \; A_1 \longrightarrow A_2$

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# $\mathbb{T}$-functors = $\mathbb{T}^2$-categories

$\mathbb{T} = (T, \mu, \eta)$ on $\mathcal{C}$ gives a monad on $\mathbb{T}^2 = (T^2, \mu^2, \eta^2)$ on $\mathcal{C}^2$:

$$T^2 : ((u, v) : f \to g) \mapsto (Tu, Tv) : Tf \to Tg, \quad \mu^2_f := (\mu_x, \mu_y), \quad \eta^2_f := (\eta_x, \eta_y)$$

$$\begin{array}{ccc} x & \xrightarrow{u} & z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ y & \xrightarrow[v]{} & w \end{array} \longmapsto \begin{array}{ccc} Tx & \xrightarrow{Tu} & Tz \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle Tg} \\ Ty & \xrightarrow[Tv]{} & Tw \end{array} \qquad \begin{array}{ccc} TTx & \xrightarrow{\mu_x} & Tx \\ {\scriptstyle TTf}\downarrow & & \downarrow{\scriptstyle Tf} \\ TTy & \xrightarrow[\mu_y]{} & Ty \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & Tx \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ y & \xrightarrow[\eta_y]{} & Ty \end{array}$$

For $f = (f_0, f_1, d_0^f, c_0^f, i^f, m^f)$ to be a $\mathbb{T}^2$-category means to have

$$\mathbb{T}^2 f_0 \xleftarrow{d_0^f = (d_0^A, d_0^B)} f_1 \xrightarrow{c_0^f = (c_0^A, c_0^B)} f_0 \qquad f_2 \xrightarrow[(m^A, m^B)]{m^f =} f_1 \qquad f_0 \xrightarrow[(i^A, i^B)]{i^f =} f_1$$

satisfying the $\mathbb{T}^2$-category equations, which means equivalently:

$A$ and $B$ are $\mathbb{T}$-categories and $f : A \to B$ is a $\mathbb{T}$-functor.

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# $\mathbb{T}$-(pre)orders, Eilenberg-Moore algebras

Ord($\mathbb{T}$): $\mathbb{T}$-*(pre)orders* = relational $\mathbb{T}$-categories

EM($\mathbb{T}$): *Eilenberg-Moore* $\mathbb{T}$-algebras




Note: The upper and middle rectangles are pullbacks in CAT.

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$$
\begin{array}{ccccc}
\mathrm{EM}(\mathbb{T}) & \rightarrowtail & \mathrm{Ord}(\mathbb{T}) & \rightarrowtail & \mathrm{Cat}(\mathbb{T}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{UAlg}(\mathbb{T}) & \rightarrowtail & \mathrm{RefGph}(\mathbb{T}) & \rightarrowtail & \mathrm{PGph}(\mathbb{T}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Alg}(\mathbb{T}) & \rightarrowtail & \mathrm{RelGph}(\mathbb{T}) & \rightarrowtail & \mathrm{Gph}(\mathbb{T}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C} & = & \mathcal{C} & = & \mathcal{C}
\end{array}
$$

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# Special cases, examples

- $\mathbb{T} = \mathbb{I}\mathrm{d}_{\mathcal{C}}$: categories, (pre)orders, ..., *internal* to $\mathcal{C}$

  Note: while $\mathsf{EM}(\mathcal{C}) = \mathsf{UAlg}(\mathcal{C}) \cong \mathcal{C}$, generally $\mathsf{Alg}(\mathcal{C}) \not\cong \mathcal{C}$
- $\mathcal{C} = \mathsf{Set}$, $\mathbb{T} = \mathbb{L}$(ist): multicategories, multiorders, monoids, ...
- $\mathcal{C} = \mathsf{Set}$, $\mathbb{T} = \mathbb{U}$(ltrafilter), induced by

$$\hom(-,2) : \mathsf{BooA}^{\mathrm{op}} \rightleftarrows \mathsf{Set} : \hom(-,2)\,.$$

Imagined by Hausdorff 1914, and probably also by Cartan 1937, proved by Manes 1967 (compact Hausdorff case) and Barr 1970 (general case), adapted by Burroni 1971 to his setting, one has:

$\mathbb{U}$-*orders* $\Longleftrightarrow$ *topologies* (on the set $A_0$ of objects), in terms of an ultrafilter convergence relation $\rightsquigarrow$, satisfying two basic axioms:

Reflexivity: $\dot{x} \rightsquigarrow x$; Transitivity: $(\mathfrak{X} \rightsquigarrow \mathfrak{y}$ and $\mathfrak{y} \rightsquigarrow z) \Longrightarrow \Sigma\mathfrak{X} \rightsquigarrow z$

Open problem: $\mathbb{U}$-categories = ultracategories (Clementino-T 2003)?

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Open problem: $\mathbb{U}$-categories $=$ ultracategories (Clementino-T 2003)?

# Special cases, examples

- $\mathbb{T} = \mathbb{I}\mathrm{d}_{\mathcal{C}}$: categories, (pre)orders, ..., *internal* to $\mathcal{C}$

  Note: while $\mathsf{EM}(\mathcal{C}) = \mathsf{UAlg}(\mathcal{C}) \cong \mathcal{C}$, generally $\mathsf{Alg}(\mathcal{C}) \not\cong \mathcal{C}$
- $\mathcal{C} = \mathsf{Set}$, $\mathbb{T} = \mathbb{L}$(ist): multicategories, multiorders, monoids, ...
- $\mathcal{C} = \mathsf{Set}$, $\mathbb{T} = \mathbb{U}$(ltrafilter), induced by

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# Some categorical properties

In the chain

$$\mathrm{EM}(\mathbb{T}) \rightarrowtail \mathrm{Ord}(\mathbb{T}) \rightarrowtail \mathrm{Cat}(\mathbb{T}) \longrightarrow \mathrm{PGph}(\mathbb{T}) \longrightarrow \mathrm{Gph}(\mathbb{T})$$

each category but $\mathrm{EM}(\mathbb{T})$ is fibred over $\mathcal{C}$, via $(-)_0$, with the arrows preserving cartesian morphisms, and all categories are (small) complete when $\mathcal{C}$ is.

If, in addition, $\mathcal{C}$ is wellpowered, then $\mathrm{Ord}(\mathbb{T})$ is topological over $\mathcal{C}$ and, in particular, (small) cocomplete when $\mathcal{C}$ is cocomplete.

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# Well-behaviour under transformations

$\mathbb{T} = (T, \mu, \eta)$ on $\mathcal{C}$, $\quad \mathbb{S} = (S, \nu, \varepsilon)$ on $\mathcal{D}$, $\quad (F, \alpha) : \mathbb{T} \to \mathbb{S}$ where

$F : \mathcal{C} \to \mathcal{D}$ preserves pullbacks, $\alpha : SF \to FT$ nat. transf. with

$$\alpha \cdot \delta F = F\eta \quad \text{and} \quad \alpha \cdot \nu F = F\mu \cdot \alpha T \cdot S\alpha$$

Then one has the induced (vertical) functors of the diagram

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Important special case: $\mathbb{S} = \mathbb{I}\mathrm{d}_{\mathcal{C}},\ \mathrm{F} = \mathrm{Id}_{\mathcal{C}},\ \alpha = \eta$

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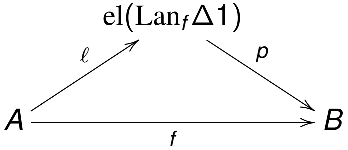
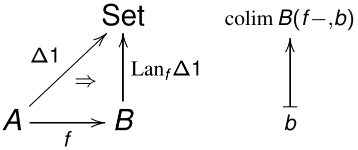
# The comprehensive factorization of Cat

Street-Walters 1973:

Every functor $f : A \to B$ of small categories factors orthogonally

(initial $=$ limit-invariant, discrete cofibration):

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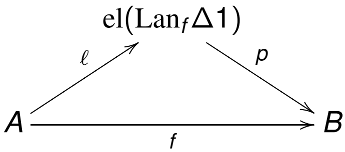


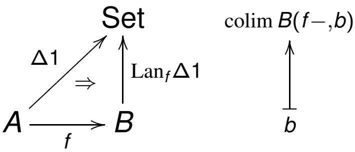


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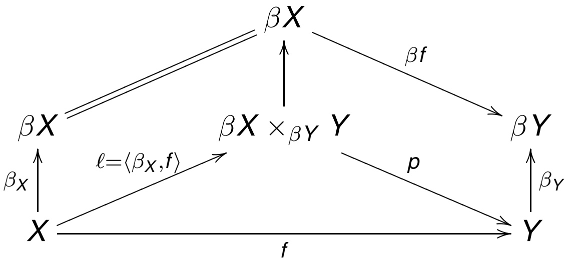


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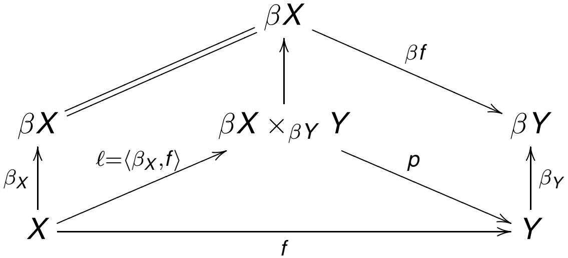


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# Goal: Establish a “comprehensive factorization” in Cat($\mathbb{T}$)!

$\mathcal{C}$ category with pullbacks, $\mathbb{T}$ monad on $\mathcal{C}$

Call $f : A \to B$ in Cat($\mathbb{T}$) *perfect* or a *discrete cofibration* if

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# In a cat $\mathcal{A}$ with pbs: When does an arbitrary class $\mathcal{M}$ belong to an orth. factorization system $(\mathcal{E}, \mathcal{M})$?

That's the case if, and only if,

- $\mathcal{M}$ is closed under composition;
- $\mathcal{M}$ is stable under pullback;
- (Ehrbar-Wyler 1968) for every object $B \in \mathcal{A}$, the full subcategory $\mathcal{M}/B$ is reflective in the comma category $\mathcal{A}/B$.

# Pullback stability of $\mathcal{P}$

The class $\mathcal{P}$ is trivially closed under composition in Cat($\mathbb{T}$):
it's just the pasting of pullback diagrams!

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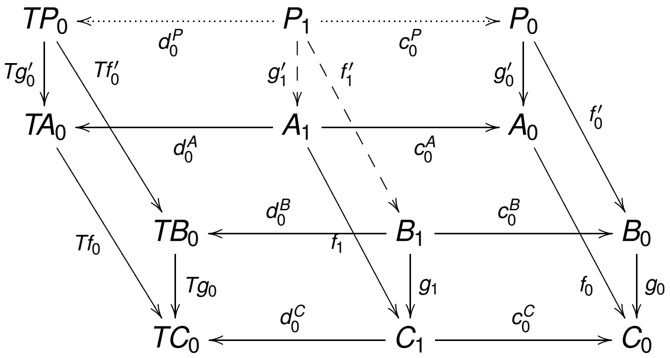


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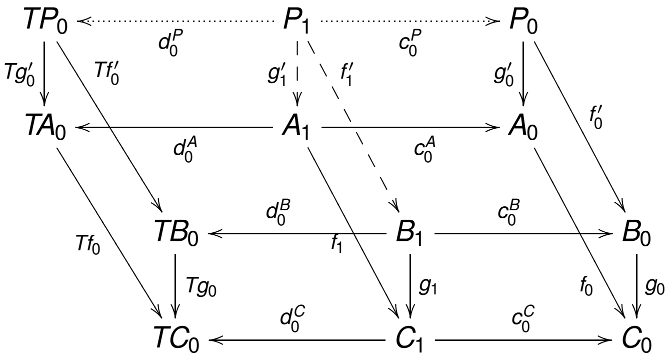


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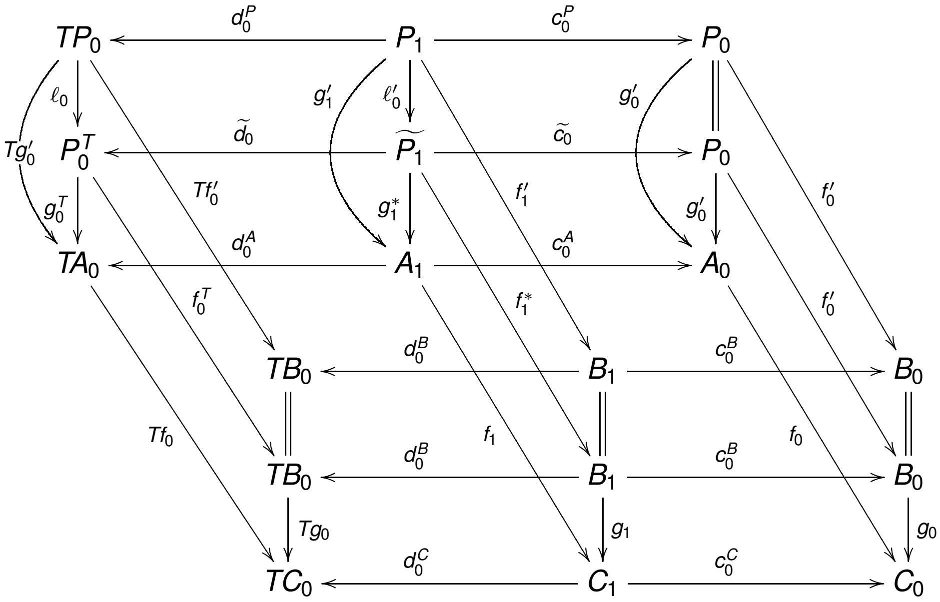
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$f_0'$
$d_0^B$
$c_0^B$
$TB_0$
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# The main Theorem

So: $\mathcal{P}$ **is** pullback stable (without assuming pb-preservation by $T$ !), but our proof is lengthy (some 9 pages in Yeganeh-T 2020).

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**Theorem**

*Assume that the category $\mathcal{C}$ with pullbacks also admits coequalizers of reflexive pairs and that these be stable under pullback and be preserved by $T$. Then, for every $\mathbb{T}$-category $B$, the full subcategory $\mathcal{P}/B$ is reflective in $\mathsf{Cat}(\mathbb{T})/B$.*

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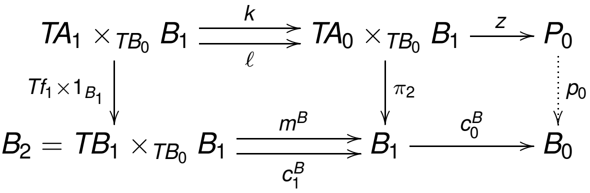

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B_2 = TB_1 \times_{TB_0} B_1 & \overset{m^B}{\underset{c_1^B}{\rightrightarrows}} & B_1 & \xrightarrow{c_0^B} & B_0
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D' & \overset{\bar{k}}{\underset{\bar{\ell}}{\rightrightarrows}} & D & \xrightarrow{\bar{z}} & P_1 \\
{\scriptstyle t'} \downarrow & & \downarrow {\scriptstyle t} & & \vdots\, {\scriptstyle c_0^P} \\
C' & \overset{k}{\underset{\ell}{\rightrightarrows}} & C & \xrightarrow{z} & P_0
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# Applications ...

- Cat($\mathcal{C}$) has a comprehensive factorization system, for any category $\mathcal{C}$ with pullbacks and pullback-stable reflexive coequalizers (Johnstone 2002) .
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- Establish the comprehensive factorization in that subcategory!
- Exponentiability in Cat($\mathbb{T}$)?
- What about the 2-categorical context?