Covariant / Contravariant Homotopy Theories and Quillen's Theorem A

Idea: Suppose C is locally presentable cat together w/ a functorial cylinder obj.

↳ get two in general distinct model structures on C, behaving in a "covariant" or "contravariant" way wrt to the cylinder

Motivation: Homotopy theories for presheaves in higher cat. thy

↳ Covariant/contravariant model structures due to Joyal on $sSet/A$ modeling presheaves on A w/ values in ∞-groupoids

↳ (co)Cartesian model structures due to Lurie on $sSet^+/A^\#$

modeling presheaves on A w/ values in ∞-Categories.

Plan: I Model structures from a cylinder

II Cofinality

III Quillen's Thm A

I

Fix a locally presentable cat C together w/ a wfs (L, R)

Def: A cylinder on an obj. X of C is an obj $I \otimes X$ together w/ maps $\partial_0 \sqcup \partial_1 : X \sqcup X \to I \otimes X \in L$, $\sigma : I \otimes X \to X$

s.t.

$$\begin{tikzcd}
X \arrow[dr, "\partial_0"] \arrow[drr, bend left, "id"] & & \\
& I \otimes X \arrow[r, "\sigma"] & X \\
X \arrow[ur, "\partial_1"'] \arrow[urr, bend right, "id"'] & &
\end{tikzcd}$$

commutes.

A functorial on C is an endofunctor $I \otimes - : C \to C$ together w/ nat transfos $\partial_0 \sqcup \partial_0 : \mathrm{id} \sqcup \mathrm{id} \Rightarrow I \otimes -$

$\sigma : I \otimes - \Rightarrow \mathrm{id}$

s.t. evaluation at any obj. defines a cylinder on it.

<u>Notation:</u> Supp. $i : K \to L$ we get comm. sq.

$$\begin{tikzcd} K \sqcup K \arrow[r, "\partial_0 \sqcup \partial_1"] \arrow[d, "i \sqcup i"'] & I \otimes K \arrow[d, "I \otimes i"] \\ L \sqcup L \arrow[r, "\partial_0 \sqcup \partial_1"'] & I \otimes L \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "\partial_j"] \arrow[d, "i"'] & I \otimes K \arrow[d, "I \otimes i"] \\ L \arrow[r, "\partial_j"'] & I \otimes L \end{tikzcd} \quad j = 0, 1$$

Hence we get induced map

$$\partial \bar{I} \otimes i := L \sqcup L \sqcup_{K \sqcup K} I \otimes K \to I \otimes L \qquad \partial_j \otimes i := L \sqcup_K I \otimes K \to I \otimes L$$

Finally assume that the cylinder satisfies

- $I \otimes -$ commute w/ small colimits
- For each $i \in L$ the maps $\begin{matrix} \partial I \otimes i \\ \partial_j \otimes i \end{matrix} \in L$.

<u>Def:</u> Let $A \subseteq L$ be a class of morphisms. Then A is called a class of <u>right I-anodyne extensions</u>

if
- A is saturated gen by a small set of morphisms
- If $i \in L$ then $\partial_1 \otimes i \in A$
- If $i \in A$ then $\partial I \otimes i \in A$

If instead we replace $\partial_1 \otimes i$ by $\partial_0 \otimes i$ we get <u>left I-anodyne extensions</u>

Given a small set of morphisms $S \subseteq L$ it generates a class of right I-anodyne ext., denoted by $An^r(I, S)$

Def: A right (I,S)-fibration is a map having the RLP wrt $\mathrm{An}^r(I,S)$

Thm: (Cisinski, Olschok, N.) There exists a model structure on $\mathcal{C}$ s.t.

- The cofibrations are given by the class $\mathcal{L}$
- The fibrant obj. are the right (I,S)-fibrant obj.

Moreover, the fibrations between fibrant obj are the right (I,S)-fibrations.

Example:

- $\mathcal{C} = sSet$
- $(L,R) = (\mathrm{Mono}, \mathrm{Triv}) \Rightarrow$ Kan-Quillen model structure.
- $I \otimes - = \Delta^1 \times -$
- $S = \emptyset$

- $C = sSet/_A$ $\Rightarrow$ Joyal's contravariant model structure on $sSet/_A$.
- $(L, R) = (\text{Mono}, \text{Triv})$
- $I \otimes - = \Delta^1 \times -$

$$\left(\begin{tikzcd} \Delta^1 \times X \arrow[r] \arrow[dr] & X \arrow[d] \\ & A \end{tikzcd}\right)$$

- $S = \emptyset$

Fibrations are right fibrations w/ target A

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & A \end{tikzcd} \quad 0 < k \leq n$$

This models presheaves on A w/ values in ∞-grpds.

<u>Ex:</u> Consider $sSet^+$: This has obj. pairs (X, E_X)

where $X \in sSet$

$E_X \subseteq X_1$ containing all degen. ones

If $A \in sSet$ get
- $A^\flat$ where only degen. 1-simplices are marked
- $A^\#$ where all — " —

- $C = sSet^+/(A, E_A)$
- $(L, R) = (\mathrm{Mono}, \mathrm{Triv})$
- $I \otimes - = (\Delta^1)^\# \times -$
- $S = \left| \begin{array}{l} (\Lambda^n_k)^\flat \to (\Delta^n)^\flat \quad n \geq 2, 0 < k < n \\ J^\flat \to J^\# \end{array} \right.$

Nerve of the free walking isom.

Get the Cartesian model structure on $sSet^+/_{(A, E_A)}$

The fibrant obj. are maps $(X, E_X) \to (A, E_A)$

where $X \to A$ is an inner fibration

and E_X is the set of cartesian arrows over E_A.

In case $(A, E_A) = A^\#$ then the fibrant obj

Cartesian fibrations over A modeling presheaves on A

w/ values in ∞-Cats.

II Cofinality

Def: A map $A \to B$ is called <u>cofinal</u> if for all obj X and all map $B \to X$ the induced map

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & X & \end{tikzcd}$$

is a covariant equiv. in $\mathcal{C}/_X$

Ex: The canonical examples are

- left I-anodyne extensions
- The class R

In fact: A morphism is cofinal $\iff$ it can be factored as

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "\text{left I-an. ext}"'] & & B \\ & C \arrow[ur, "\in R"'] & \end{tikzcd}$$

Def: Let $X \to Y$ be a map. Supp. we have cart. sq

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] & B \arrow[r] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ A' \arrow[r, "j"'] & B' \arrow[r] & Y \end{tikzcd}$$

Then $X \to Y$ is called smooth if for all such diagrams whenever j is cofinal so is i.

Remark: One can show that right I-fibrations always satisfy the smoothness condition for a nice class of left I-anodyne ext.

Example: In the covariant / contravariant model structures on sSet/A we recover the usual notions of (co)final functors.

Right fibrations and Cartesian fibrations are smooth.

Example: Consider the contravariant model structure associated to

- $\Delta^1 \times -$
- $S = \{ \partial\Delta^{n+2} \to \Delta^{n+2} \}$.

The fibrant obj. are right fibrations w/ n-truncated fibers.

Here the notion of (co)finality can be used to define weaker notions of (co)limits.

Define an n-initial obj in an ∞-Cat C to be a n-cofinal map $\Delta^0 \longrightarrow C$.

For ex. a 1-initial is just an initial obj in hoC.

Thm: (N., Raptis, Schrade)

Let $F: C \to D$ be a functor between ∞-Cats. Assume C has fin. limits and F preserves them.

Then F has a left adj. $\Leftrightarrow$ $C_{d/}$ has a 1-initial obj

$\forall d \in D$.

Corollary: (NRS) Let $F: C \to D$ as above

Then F has a left adjoint $\Leftrightarrow$ hoF

$$\begin{tikzcd} C/d \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D/d \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

III Quillen's Thm A.

<u>Thm:</u> (Quillen) Let $F: C \to D$ be a functor between 1-cats. Then F induces a htpy equiv of classifying spaces if $C/_d$ is contractible $\forall d \in D$

This is a consequence of

<u>Thm:</u> (Joyal, Lurie)

A functor $F: C \to D$ is cofinal iff $C/_d$ is contractible.

<u>Thm:</u> (L.) A functor is n-cofinal iff $C/_d$ is n-connected.

<u>Corollary:</u> A functor between 1-cat $F: C \to D$ induces an n-conn. map of classifying spaces if $C/_d$ is n-conn.