

Covariant / Contravariant Homotopy Theories and Quillen's Theorem A

Idea: Suppose C is locally presentable cat together w/ a functorial cylinder obj.

↳ get two in general distinct model structures on C , behaving in a "covariant" or "contravariant" way w.r.t. to the cylinder

Motivation: Homotopy theories for presheaves in higher cat. thy

↳ Covariant / contravariant model structures due to Joyal on $\mathcal{S}Set / \mathcal{A}$ modeling presheaves

on \mathcal{A} w/ values in ∞ -groupoids

↳ (co)Cartesian model structures due to Lurie on $\mathcal{S}Set^+ / \mathcal{A}^\#$

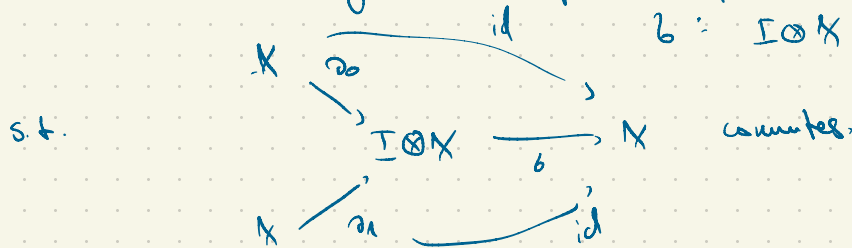
modelly preserves m & w/ values
in ∞ -Categories.

- Plan:
- I Model structures from a cylinder
 - II Cohirality
 - III Quillen's Theorem

(I)

Fix a locally presentable cat \mathcal{C} together w/ a
wfs (L, R)

Def: A cylinder on an obj. X of \mathcal{C} is an obj
 $I \otimes X$ together w/ maps $\partial_0, \partial_1: X \cup X \rightarrow I \otimes X \in L$
 $\iota: I \otimes X \rightarrow X$



A functional on C is an endofunctor $I \otimes - : C \rightarrow C$
 together w/ unit maps $\partial_0 \cup \partial_0 : id \cup id \Rightarrow I \otimes -$
 $\partial : I \otimes - \Rightarrow id$

s.t. evaluation at any obj. defines a cylinder on it.

Notation: Supp. $i: k \rightarrow L$ we get comm. sq.

$$\begin{array}{ccc}
 k \cup k \xrightarrow{\partial_0 \cup \partial_0} I \otimes k & & k \xrightarrow{\partial_j} I \otimes k \\
 i \cup i \downarrow & & \downarrow I \otimes i \quad j=0,1 \\
 L \cup L \xrightarrow{\partial_0 \cup \partial_0} I \otimes L & & L \xrightarrow{\partial_j} I \otimes L
 \end{array}$$

hence we get induced map

$$\begin{array}{c}
 L \cup L \cup_{k \cup k} I \otimes k \rightarrow I \otimes L \\
 \partial \otimes i :=
 \end{array}$$

$$\begin{array}{c}
 L \cup_k I \otimes k \rightarrow I \otimes L \\
 \partial \otimes i :=
 \end{array}$$

Finally assume that the cylinder satisfies

- $I \otimes$ - commute w/ small colimits.
- For each $i \in L$ the maps $\partial I \otimes i \in L$.
 $\partial_j \otimes i$

Def: let $A \subseteq L$ be a class of morphisms. Then A is called a class of right I-acyclic extensions

- if
- A is saturated gen by a small set of morphisms
 - If $i \in L$ then $\partial_1 \otimes i \in A$
 - If $i \in A$ then $\partial I \otimes i \in A$

If instead we replace $\partial_1 \otimes i$ by $\partial_0 \otimes i$ we get
left I-acyclic extensions

Given a small set of morphisms $S \subseteq L$ it generates a class of right I-acyclic ext., denoted by $A_n^v(I, S)$

Def: A right (I, S) -fibration is a map having the RLP
wrt to $An^{\vee}(I, S)$

Thm: (Cisinski, Osobko, N.) There exists a model
structure on \mathcal{C} st.

- The cofibrations are given by the class \mathcal{L}
- The fibrant obj. are the right (I, S) -fibrant obj.

Moreover, the fibrations between fibrant obj. are the
right (I, S) -fibrations.

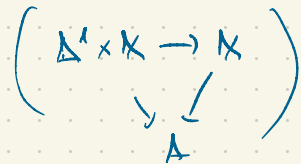
Example:

- $\mathcal{C} = s\text{Set}$
- $(L, R) = (\text{Mono}, \text{Triv}) \Rightarrow$ Kan-Quillen model
structure.
- $I \otimes - = \mathcal{S}^1 \times -$
- $S = \emptyset$

- $C = \text{sSet}/A$

- $(L, R) = (\text{Mono}, \text{Triv})$

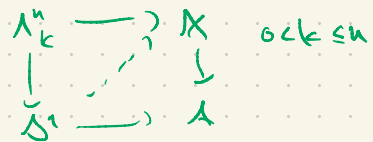
- $\Gamma \otimes - = \Delta^1 \times -$



\Rightarrow Joyal's Convariant model structure on sSet/A .

- $S = \emptyset$

Fibrations are right fibrations w/ target A



This model presheaves on A w/ values in ω -grps.

Ex: Consider $s\text{Set}^+$. This has obj. pairs (X, E_X)
 where $X \in s\text{Set}$
 $E_X \subseteq X_1$ containing all degen. ones

If $A \in s\text{Set}$ get $\cdot A^b$ where only degen. 1-simplices are marked
 $\cdot A^\#$ where all ---

$\cdot C = s\text{Set}^+ / (A, \bar{E}_A)$

$\cdot (L, R) = (\text{Mono}, \text{Inv})$

$\cdot I \otimes - = (\Delta^1)^\# \times -$

$\cdot S = \left| \begin{array}{ccc} (\Lambda_{\leftarrow}^n)^b & \rightarrow & (\Delta^n)^b \\ \downarrow & & \downarrow \\ \downarrow & \rightarrow & \downarrow^\# \end{array} \right| \quad n \geq 2, 0 < k < n$

nerve of the free walking isom.

Set the Cartesian model structure on $\mathcal{S}\text{Set}^{\downarrow}_{(A, E_A)}$

The fibrant obj. are maps $(X, E_X) \rightarrow (A, E_A)$

where $X \rightarrow A$ is an inner fibration

and E_X is the set of cartesian arrows over E_A .

In case $(A, E_A) = A^{\#}$ then the fibrant obj.

Cartesian fibrations over A modeling presheaves on A

w/ values in $\infty\text{-Cats}$.

II continuity

Def: A map $A \rightarrow B$ is called cofinal if for all obj X and all map $B \rightarrow X$

the induced map $A \rightarrow B$ is a covariant equiv. in \mathcal{C}/X

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ X & & X \end{array}$$

EX: The canonical examples are

- left I-acyclic extensions
- The class \mathcal{R}

In fact: A morphism is cofinal \Leftrightarrow it can be factored as

$$\begin{array}{ccc} A & \longrightarrow & B \\ \text{left-I-acyc.} & \searrow & \nearrow \\ \text{ext} & C & \in \mathcal{R} \end{array}$$

Def: Let $X \rightarrow Y$ be a map. Supp. we have com. sq

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \rightarrow & X \\ | & & \downarrow L & & | \\ A' & \xrightarrow{j} & B' & \rightarrow & Y \end{array}$$

The $X \rightarrow Y$ is called smooth if for all such diagrams whenever j is cofiber so is i .

Remark: One can show that right I -fibrations always satisfy the smoothness condition for a nice class of left I -cofibrant ext.

Example: In the covariant/covariant model structure on \mathbf{sSet}/A
we recover the usual notion of (co)fibrations.

Right fibrations and Cartesian fibrations are smooth.

Example: Consider the covariant model structure associated to

$$\bullet \quad \Delta^1 \times -$$

$$\bullet \quad S = \left| \partial \Delta^{n+2} \rightarrow \Delta^{n+2} \right|.$$

The fibrant obj. are right fibrations w/ untruncated fibers.

Hence the notion of (co)fibrality can be used to define
weaker notions of (co)limits.

Define an u -initial obj in an ∞ -Cat \mathcal{C} to be
a u -colimit map $\mathcal{D}^{\circ} \rightarrow \mathcal{C}$.

For ex. a 1 -initial is just an initial obj in $\text{ho}\mathcal{C}$.

Thm: (N., Raptis, Schwede)

Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between ∞ -Cats. Assume \mathcal{C}
has fin. limits and F preserves them.

Then F has a left adj. $\Leftrightarrow \mathcal{C}_d /$ has a 1 -initial obj

$\forall d \in \mathcal{D}$.

Corollary: (NRS) Let $F: \mathcal{C} \rightarrow \mathcal{D}$ as above

Then F has a left adjoint $\Leftrightarrow \text{ho} F$

III Quillen's Theorem A.

$$\begin{array}{ccc} C/d & \rightarrow & D/d \\ \downarrow & & \downarrow \\ C & \rightarrow & D \end{array}$$

Theorem (Quillen) Let $F: C \rightarrow D$ be a functor between \mathcal{A} -cats.

Then F induces a homotopy equiv. of classifying spaces if C/d is contractible $\forall d \in D$.

This is a consequence of

Theorem (Joyal, Lurie)

A functor $F: C \rightarrow D$ is cofibrant iff C/d is contractible.

Theorem (h.) A functor is n -cofibrant iff C/d is n -connected.

Corollary: A functor between \mathcal{A} -cats $F: C \rightarrow D$ induces an n -conn. map of classifying spaces if C/d is n -conn.

