Independence Relations in Abstract Elementary Categories

Mark Kamsma
Supervisor: Jonathan Kirby

University of East Anglia

Masaryk University Algebra Seminar - 15 April 2021

Table of Contents

Table of Contents

Classification theory

Shelah (around 1970): classify theories by how complicated the behaviour of their definable sets is.

Classification theory

Shelah (around 1970): classify theories by how complicated the behaviour of their definable sets is.

Most well-known class is *stable theories*.

Classification theory

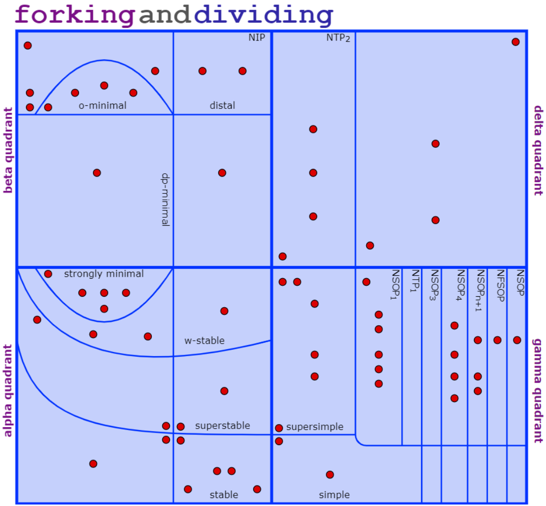
Shelah (around 1970): classify theories by how complicated the behaviour of their definable sets is.

Most well-known class is *stable theories*.

Examples of stable theories: $\mathbb{R}$-vector spaces and algebraically closed fields.

Classification theory

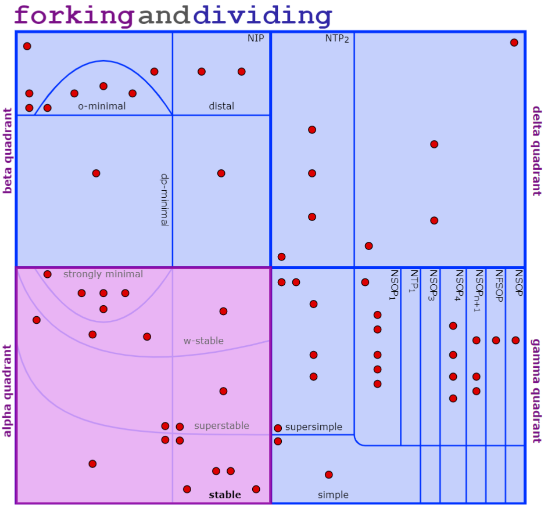
A map of the universe



Source: `https://forkinganddividing.com`

Classification theory

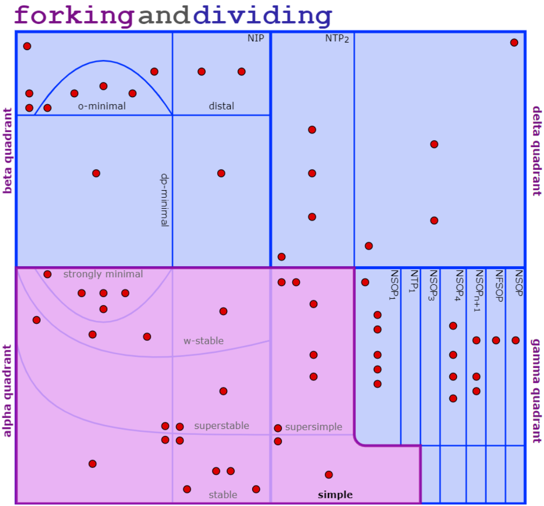
A map of the universe - Stable



Source: `https://forkinganddividing.com`

Classification theory

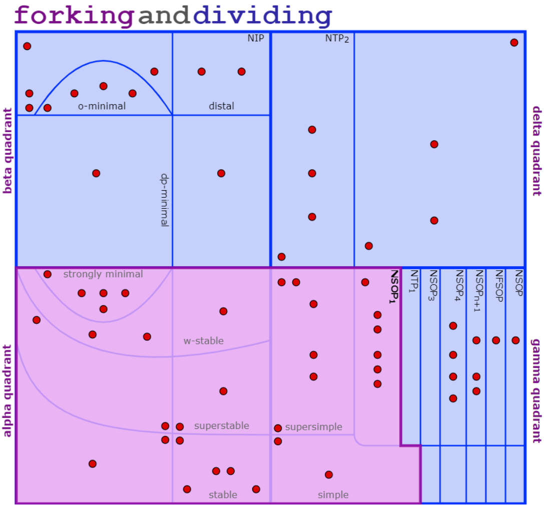
A map of the universe - Simple



Source: `https://forkinganddividing.com`

Classification theory

A map of the universe - $NSOP_1$



Source: `https://forkinganddividing.com`

Classification theory

Simple and $NSOP_1$

Example of simple non-stable theory: the random graph.

Classification theory

Simple and $NSOP_1$

Example of simple non-stable theory: the random graph.

Example of $NSOP_1$ non-simple theory: infinite dimensional vector spaces with a bilinear form over an algebraically closed field.

Classification theory

Simple and $NSOP_1$

Example of simple non-stable theory: the random graph.

Example of $NSOP_1$ non-simple theory: infinite dimensional vector spaces with a bilinear form over an algebraically closed field.

Existentially closed exponential fields [Haykazyan and Kirby, 2021] are also $NSOP_1$ non-simple. This is in positive logic.

Classification theory

Definition of simple

A formula $\varphi(x, y)$ has the *tree property* if there are $(a_\eta)_{\eta \in \omega^{<\omega}}$ in some model and some $k \geq 2$ such that:

1. for all $\sigma \in \omega^\omega$ the set $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent;
2. for all $\eta \in \omega^{<\omega}$ the set $\{\varphi(x, a_{\eta^\frown i}) : i < \omega\}$ is k-inconsistent.

Classification theory

Definition of simple

A formula $\varphi(x, y)$ has the *tree property* if there are $(a_\eta)_{\eta\in\omega^{<\omega}}$ in some model and some $k \geq 2$ such that:

1. for all $\sigma \in \omega^\omega$ the set $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent;
2. for all $\eta \in \omega^{<\omega}$ the set $\{\varphi(x, a_{\eta^\frown i}) : i < \omega\}$ is k-inconsistent.

We say that a theory is *simple* if no formula has the tree property.

Classification theory

Definition of simple

A formula $\varphi(x, y)$ has the *tree property* if there are $(a_\eta)_{\eta \in \omega^{<\omega}}$ in some model and some $k \geq 2$ such that:

1. for all $\sigma \in \omega^\omega$ the set $\{\varphi(x, a_{\sigma|_n}) : n < \omega\}$ is consistent;
2. for all $\eta \in \omega^{<\omega}$ the set $\{\varphi(x, a_{\eta^\frown i}) : i < \omega\}$ is k-inconsistent.

We say that a theory is *simple* if no formula has the tree property.

Hard to check...

Classification theory

Independence relations

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \underset{C}{\overset{V}{\downarrow}} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

We say that A *is independent from* B *over* C.

Classification theory

Independence relations

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \underset{C}{\overset{V}{\downarrow}} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

We say that A *is independent from* B *over* C.

We call $\downarrow$ an *independence relation*.

Classification theory

Independence relations - properties

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \mathop{\downarrow}_{C}^{V} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

For $\mathbb{R}$-vector spaces $\downarrow$ has nice properties.

Classification theory

Independence relations - properties

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \underset{C}{\overset{V}{\downarrow}} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

For $\mathbb{R}$-vector spaces $\downarrow$ has nice properties.

If $A \downarrow_C^V B$ then also $B \downarrow_C^V A$ (symmetry).

Classification theory

Independence relations - properties

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \mathop{\downarrow}_{C}^{V} B \iff \text{span}(A \cup C) \cap \text{span}(B \cup C) \subseteq \text{span}(C).$$

For $\mathbb{R}$-vector spaces $\downarrow$ has nice properties.

If $A \downarrow_C^V B$ then also $B \downarrow_C^V A$ (symmetry).

If $B' \subseteq B$ and $A \downarrow_C^V B$ then also $A \downarrow_C^V B'$ (monotonicity).

Classification theory

Independence relations - properties

Let V be an $\mathbb{R}$-vector space and let $A, B, C \subseteq V$. We define:

$$A \underset{C}{\overset{V}{\downarrow}} B \iff \operatorname{span}(A \cup C) \cap \operatorname{span}(B \cup C) \subseteq \operatorname{span}(C).$$

For $\mathbb{R}$-vector spaces $\downarrow$ has nice properties.

If $A \downarrow_C^V B$ then also $B \downarrow_C^V A$ (symmetry).

If $B' \subseteq B$ and $A \downarrow_C^V B$ then also $A \downarrow_C^V B'$ (monotonicity).

And some more...

Classification theory

Kim-Pillay style theorem

We can classify theories by the existence of a nice enough independence relation.

Classification theory

Kim-Pillay style theorem

We can classify theories by the existence of a nice enough independence relation.

Theorem (Kim, Pillay, 1997)

A first-order theory T is simple if and only if there is a independence relation $\downarrow$ that satisfies a certain list of properties. In this case $\downarrow$ is given by dividing.

Classification theory

Kim-Pillay style theorem

We can classify theories by the existence of a nice enough independence relation.

Theorem (Kim, Pillay, 1997)

A first-order theory T is simple if and only if there is a independence relation $\mathop{\smile\!\!\!\!\!\!|}$ that satisfies a certain list of properties. In this case $\mathop{\smile\!\!\!\!\!\!|}$ is given by dividing.

Similar theorems exist for stable theories [Harnik, Harrington, Lascar, Shelah (1984)] and NSOP_1 theories [Chernikov, Kaplan, Kim, Ramsey (2020)].

Table of Contents

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

- It is an accessible category.

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

- It is an accessible category.
- It has directed colimits.

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

- It is an accessible category.
- It has directed colimits.
- All arrows are monomorphisms.

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

- It is an accessible category.
- It has directed colimits.
- All arrows are monomorphisms.
- It has the amalgamation property (AP):

Abstract Elementary Categories

Fix a (first-order) theory T and let $\mathrm{Mod}(T)$ be the category of its models with elementary embeddings, what nice properties does $\mathrm{Mod}(T)$ have?

- It is an accessible category.
- It has directed colimits.
- All arrows are monomorphisms.
- It has the amalgamation property (AP):

$$\begin{array}{ccc} B & \dashrightarrow & D \\ \uparrow & & \uparrow \\ A & \longrightarrow & C \end{array}$$

Abstract Elementary Categories

Category of subsets of models

Fix a (first-order) theory T. We write $\mathrm{SubMod}(T)$ for the category of subsets of models of T.

Abstract Elementary Categories

Category of subsets of models

Fix a (first-order) theory T. We write $\mathrm{SubMod}(T)$ for the category of subsets of models of T.

Objects are pairs (A, M) with $M \models T$ and $A \subseteq M$.

Abstract Elementary Categories

Category of subsets of models

Fix a (first-order) theory T. We write SubMod(T) for the category of subsets of models of T.

Objects are pairs (A, M) with $M \models T$ and $A \subseteq M$.

An arrow $f : (A, M) \to (B, N)$ is an elementary embeddings $f : A \to B$. That is, a function $f : A \to B$ such that for every formula $\varphi(\bar{x})$ and every tuple $\bar{a} \in A$ we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})).$$

Abstract Elementary Categories

Category of subsets of models

Fix a (first-order) theory T. We write SubMod(T) for the category of subsets of models of T.

Objects are pairs (A, M) with $M \models T$ and $A \subseteq M$.

An arrow $f : (A, M) \to (B, N)$ is an elementary embeddings $f : A \to B$. That is, a function $f : A \to B$ such that for every formula $\varphi(\bar{x})$ and every tuple $\bar{a} \in A$ we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})).$$

We consider Mod(T) as a full subcategory of SubMod(T) via the embedding $M \mapsto (M, M)$.

Abstract Elementary Categories

Definition

If we work in SubMod(T) we still need to keep track of which objects are the models.

Abstract Elementary Categories

Definition

If we work in SubMod(T) we still need to keep track of which objects are the models.

Definition

An *AECat*, short for *abstract elementary category*, consists of a pair $(\mathcal{C}, \mathcal{M})$ where $\mathcal{C}$ and $\mathcal{M}$ are accessible categories and $\mathcal{M}$ is a full subcategory of $\mathcal{C}$ such that:

1. $\mathcal{M}$ has directed colimits, preserved by $\mathcal{M} \hookrightarrow \mathcal{C}$;
2. all arrows in $\mathcal{C}$ (and thus in $\mathcal{M}$) are monomorphisms.

The objects in $\mathcal{M}$ are called *models*. We say that $(\mathcal{C}, \mathcal{M})$ has the *amalgamation property* (or *AP*) if $\mathcal{M}$ has the amalgamation property.

Abstract Elementary Categories

Examples

Examples of AECats.

Abstract Elementary Categories

Examples

Examples of AECats.

1. For a first-order theory T both $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.

Abstract Elementary Categories

Examples

Examples of AECats.

1. For a first-order theory T both $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.
2. For a positive theory T we let $\mathrm{Mod}(T)$ be the category of e.c. models and immersions and define $\mathrm{SubMod}(T)$ similarly, then again $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.

Abstract Elementary Categories

Examples

Examples of AECats.

1. For a first-order theory T both $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.
2. For a positive theory T we let $\mathrm{Mod}(T)$ be the category of e.c. models and immersions and define $\mathrm{SubMod}(T)$ similarly, then again $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.
3. For any continuous theory T we can form $(\mathrm{SubMetMod}(T), \mathrm{MetMod}(T))$ and $(\mathrm{MetMod}(T), \mathrm{MetMod}(T))$, which are AECats with AP.

Abstract Elementary Categories

Examples

Examples of AECats.

1. For a first-order theory T both $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.
2. For a positive theory T we let $\mathrm{Mod}(T)$ be the category of e.c. models and immersions and define $\mathrm{SubMod}(T)$ similarly, then again $(\mathrm{SubMod}(T), \mathrm{Mod}(T))$ and $(\mathrm{Mod}(T), \mathrm{Mod}(T))$ are AECats with AP.
3. For any continuous theory T we can form $(\mathrm{SubMetMod}(T), \mathrm{MetMod}(T))$ and $(\mathrm{MetMod}(T), \mathrm{MetMod}(T))$, which are AECats with AP.
4. Any Abstract Elementary Class (AEC) $\mathcal{K}$ yields AECats $(\mathrm{SubSet}(\mathcal{K}), \mathcal{K})$ and $(\mathcal{K}, \mathcal{K})$, which have AP precisely when $\mathcal{K}$ has AP.

Abstract Elementary Categories

Types

Model theory, and definitely classification theory, is all about types.

Abstract Elementary Categories

Types

Model theory, and definitely classification theory, is all about types.

For $\bar{a} \in M$ the *type* of $\bar{a}$ is

$$\mathrm{tp}(\bar{a}; M) = \{\varphi(\bar{x}) : M \models \varphi(\bar{a})\}.$$

Abstract Elementary Categories

Types

Model theory, and definitely classification theory, is all about types.

For $\bar{a} \in M$ the *type* of $\bar{a}$ is

$$\mathrm{tp}(\bar{a}; M) = \{\varphi(\bar{x}) : M \models \varphi(\bar{a})\}.$$

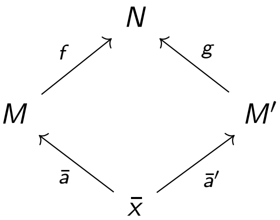
Proposition

Let $\bar{a} \in M$ and $\bar{a}' \in M'$ be tuples of matching lengths. Then $\mathrm{tp}(\bar{a}; M) = \mathrm{tp}(\bar{a}'; M')$ if and only if there are elementary embeddings $M \xrightarrow{f} N \xleftarrow{g} M'$ such that $f(\bar{a}) = g(\bar{a}')$.

Abstract Elementary Categories

Galois types

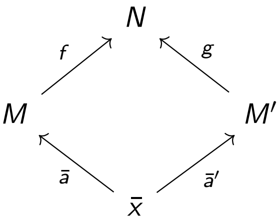
In a picture, we have $\mathrm{tp}(\bar{a}; M) = \mathrm{tp}(\bar{a}'; M')$ iff



Abstract Elementary Categories

Galois types

In a picture, we have $\text{tp}(\bar{a}; M) = \text{tp}(\bar{a}'; M')$ iff



This inspires the definition of Galois types. In AECs Galois types are widely used, but are still phrased in terms of elements. We replace those elements by arrows.

Abstract Elementary Categories

Galois types

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. We write $((a_i)_{i \in I}; M)$ to mean that the a_i are arrows into M and that M is a model.

Abstract Elementary Categories

Galois types

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. We write $((a_i)_{i\in I}; M)$ to mean that the a_i are arrows into M and that M is a model. Two tuples $((a_i)_{i\in I}; M)$ and $((a'_i)_{i\in I}; M')$ are said to have the same *Galois type*, written as

$$\mathrm{gtp}((a_i)_{i\in I}; M) = \mathrm{gtp}((a'_i)_{i\in I}; M'),$$

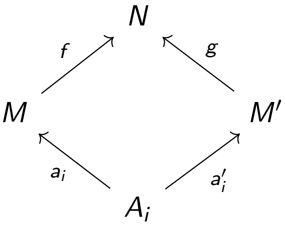
Abstract Elementary Categories

Galois types

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP. We write $((a_i)_{i\in I}; M)$ to mean that the a_i are arrows into M and that M is a model. Two tuples $((a_i)_{i\in I}; M)$ and $((a'_i)_{i\in I}; M')$ are said to have the same *Galois type*, written as

$$\mathrm{gtp}((a_i)_{i\in I}; M) = \mathrm{gtp}((a'_i)_{i\in I}; M'),$$

if $\mathrm{dom}(a_i) = \mathrm{dom}(a'_i)$ for all $i \in I$, and there are $M \xrightarrow{f} N \xleftarrow{g} M'$, such that the following commutes for all $i \in I$:



Abstract Elementary Categories

Galois types

Having the same Galois type is clearly a symmetric and reflexive relation and by assuming AP it is also transitive. So a "Galois type" is an equivalence class of arrows.

Abstract Elementary Categories

Galois types

Having the same Galois type is clearly a symmetric and reflexive relation and by assuming AP it is also transitive. So a "Galois type" is an equivalence class of arrows.

Galois types correspond to the usual syntactic types from a first-order theory in the sense that they induce the same equivalence classes. This is also true for AECats obtained from positive logic and continuous logic.

Table of Contents

Independence relations

Independence as commuting squares

In [Lieberman et al., 2019] an independence relation on an accessible category is a collection of commuting squares.

$$\begin{array}{ccc} M_1 & \longrightarrow & N \\ \uparrow & \downarrow & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array} \iff M_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$$

Independence relations

Independence as commuting squares

In [Lieberman et al., 2019] an independence relation on an accessible category is a collection of commuting squares.

$$\begin{array}{ccc} M_1 & \longrightarrow & N \\ \uparrow & \downarrow & \uparrow \\ M_0 & \longrightarrow & M_2 \end{array} \qquad \Longleftrightarrow \qquad M_1 \underset{M_0}{\overset{N}{\downarrow}} M_2$$

They prove, among other things, canonicity of stable-like independence relations.

Independence relations

Independence in AECats

Definition

In an AECat with AP, an *independence relation* is a relation on triples of subobjects of models. If such a triple (A, B, C) of a model M is in the relation, we call it *independent* and denote this by:

$$A \underset{C}{\overset{M}{\downarrow}} B.$$

Independence relations

Independence in AECats

Definition

In an AECat with AP, an *independence relation* is a relation on triples of subobjects of models. If such a triple (A, B, C) of a model M is in the relation, we call it *independent* and denote this by:

$$A \underset{C}{\overset{M}{\perp}} B.$$

In an AECat of the form $(\mathcal{M}, \mathcal{M})$ this is essentially the same as the notion in [Lieberman et al., 2019]. That is, modulo some basic properties, we can recover one from the other.

Independence relations

Independence in AECats

Definition

In an AECat with AP, an *independence relation* is a relation on triples of subobjects of models. If such a triple (A, B, C) of a model M is in the relation, we call it *independent* and denote this by:

$$A \underset{C}{\overset{M}{\perp}} B.$$

In an AECat of the form $(\mathcal{M}, \mathcal{M})$ this is essentially the same as the notion in [Lieberman et al., 2019]. That is, modulo some basic properties, we can recover one from the other.

Advantage: more intuitive independence calculus.

Independence relations

Independence in AECats

Definition

In an AECat with AP, an *independence relation* is a relation on triples of subobjects of models. If such a triple (A, B, C) of a model M is in the relation, we call it *independent* and denote this by:

$$A \underset{C}{\overset{M}{\downarrow}} B.$$

In an AECat of the form $(\mathcal{M}, \mathcal{M})$ this is essentially the same as the notion in [Lieberman et al., 2019]. That is, modulo some basic properties, we can recover one from the other.

Advantage: more intuitive independence calculus.
Disadvantage: we lose the nice way of viewing the independence relation itself as a category.

Independence relations

Independence in AECats - basic independence relation

We call $\downarrow$ a *basic independence relation* if it satisfies:

Independence relations

Independence in AECats - basic independence relation

We call $\mathop{\smile\!\!\!\!\!\!|}$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \mathop{\smile\!\!\!\!\!\!|}_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \mathop{\smile\!\!\!\!\!\!|}_{C'}^{M'} B'$.

Independence relations

Independence in AECats - basic independence relation

We call $\mathop{\smile\!\!\!\!\!\!|}\,$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \mathop{\smile\!\!\!\!\!\!|}_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \mathop{\smile\!\!\!\!\!\!|}_{C'}^{M'} B'$.

MONOTONICITY $A \mathop{\smile\!\!\!\!\!\!|}_C^M B$ and $A' \leq A$ implies $A' \mathop{\smile\!\!\!\!\!\!|}_C^M B$.

Independence relations

Independence in AECats - basic independence relation

We call $\downarrow$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \downarrow_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \downarrow_{C'}^{M'} B'$.

MONOTONICITY $A \downarrow_C^M B$ and $A' \leq A$ implies $A' \downarrow_C^M B$.

TRANSITIVITY $A \downarrow_B^M C$, $A \downarrow_C^M D$ with $B \leq C$ implies $A \downarrow_B^M D$.

Independence relations

Independence in AECats - basic independence relation

We call $\mathop{\smile\hskip-0.9em\raise0.3ex\hbox{$\vert$}}$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \downarrow_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \downarrow_{C'}^{M'} B'$.

MONOTONICITY $A \downarrow_C^M B$ and $A' \leq A$ implies $A' \downarrow_C^M B$.

TRANSITIVITY $A \downarrow_B^M C$, $A \downarrow_C^M D$ with $B \leq C$ implies $A \downarrow_B^M D$.

SYMMETRY $A \downarrow_C^M B$ implies $B \downarrow_C^M A$.

Independence relations

Independence in AECats - basic independence relation

We call $\downfall$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \downfall_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \downfall_{C'}^{M'} B'$.

MONOTONICITY $A \downfall_C^M B$ and $A' \leq A$ implies $A' \downfall_C^M B$.

TRANSITIVITY $A \downfall_B^M C$, $A \downfall_C^M D$ with $B \leq C$ implies $A \downfall_B^M D$.

SYMMETRY $A \downfall_C^M B$ implies $B \downfall_C^M A$.

EXISTENCE $A \downfall_C^M C$ for all $A, C \leq M$.

Independence relations

Independence in AECats - basic independence relation

We call $\downfree$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \downfree^M_C B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \downfree^{M'}_{C'} B'$.

MONOTONICITY $A \downfree^M_C B$ and $A' \leq A$ implies $A' \downfree^M_C B$.

TRANSITIVITY $A \downfree^M_B C$, $A \downfree^M_C D$ with $B \leq C$ implies $A \downfree^M_B D$.

SYMMETRY $A \downfree^M_C B$ implies $B \downfree^M_C A$.

EXISTENCE $A \downfree^M_C C$ for all $A, C \leq M$.

EXTENSION For $(a, b, c; M)$ such that $A \downfree^M_C B$ and $B \leq B' \leq M$ then there is an extension $M \to M'$ with some $a' : A \to M'$ such that $\mathrm{gtp}(a', b, c; M') = \mathrm{gtp}(a, b, c; M')$ and $A' \downfree^{M'}_C B'$.

Independence relations

Independence in AECats - basic independence relation

We call $\downarrow$ a *basic independence relation* if it satisfies:

INVARIANCE If $A \downarrow_C^M B$ and $\mathrm{gtp}(A, B, C; M) = \mathrm{gtp}(A', B', C'; M')$ then we also have $A' \downarrow_{C'}^{M'} B'$.

MONOTONICITY $A \downarrow_C^M B$ and $A' \leq A$ implies $A' \downarrow_C^M B$.

TRANSITIVITY $A \downarrow_B^M C$, $A \downarrow_C^M D$ with $B \leq C$ implies $A \downarrow_B^M D$.

SYMMETRY $A \downarrow_C^M B$ implies $B \downarrow_C^M A$.

EXISTENCE $A \downarrow_C^M C$ for all $A, C \leq M$.

EXTENSION For $(a, b, c; M)$ such that $A \downarrow_C^M B$ and $B \leq B' \leq M$ then there is an extension $M \to M'$ with some $a' : A \to M'$ such that $\mathrm{gtp}(a', b, c; M') = \mathrm{gtp}(a, b, c; M')$ and $A' \downarrow_C^{M'} B'$.

UNION Let $(B_i)_{i \in I}$ be a directed system with a cocone into some model M, and suppose $B = \mathrm{colim}_{i \in I} B_i$ exists. Then if $A \downarrow_C^M B_i$ for all $i \in I$, we have $A \downarrow_C^M B$.

Independence relations

Independence in AECats - advanced properties

There are also the following advanced properties.

Independence relations

Independence in AECats - advanced properties

There are also the following advanced properties.

Base-Monotonicity $A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C} B$ and $C \leq C' \leq B$ implies $A \mathop{\smile\!\!\!\!\!\!|}^{M}_{C'} B$.

Independence relations

Independence in AECats - advanced properties

There are also the following advanced properties.

Base-Monotonicity $A \mathop{\smile\!\!\!\!\!\mid}^{M}_{C} B$ and $C \leq C' \leq B$ implies $A \mathop{\smile\!\!\!\!\!\mid}^{M}_{C'} B$.

Club Local Character 'For any $A, M \leq N$ there are many small $M_0 \leq M$ such that $A \mathop{\smile\!\!\!\!\!\mid}^{N}_{M_0} M$.'

Independence relations

Independence in AECats - advanced properties

There are also the following advanced properties.

Base-Monotonicity $A \downarrow_C^M B$ and $C \leq C' \leq B$ implies $A \downarrow_{C'}^M B$.

Club Local Character 'For any $A, M \leq N$ there are many small $M_0 \leq M$ such that $A \downarrow_{M_0}^N M$.'

Stationarity If $\mathrm{gtp}(a, m; N) = \mathrm{gtp}(a', m; N)$, $A \downarrow_M^N B$ and $A' \downarrow_M^N B$ then $\mathrm{gtp}(a, b, m; N) = \mathrm{gtp}(a', b, m; N)$.

Independence relations

Independence in AECats - advanced properties

There are also the following advanced properties.

BASE-MONOTONICITY $A \mathop{\smile\!\!\!\!\!\!|}\nolimits_{C}^{M} B$ and $C \leq C' \leq B$ implies $A \mathop{\smile\!\!\!\!\!\!|}\nolimits_{C'}^{M} B$.

CLUB LOCAL CHARACTER 'For any $A, M \leq N$ there are many small $M_0 \leq M$ such that $A \mathop{\smile\!\!\!\!\!\!|}\nolimits_{M_0}^{N} M$.'

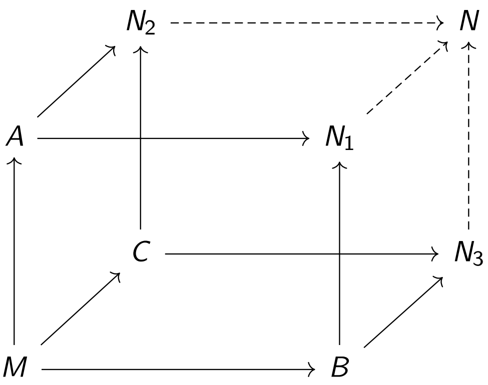
STATIONARITY If $\mathrm{gtp}(a, m; N) = \mathrm{gtp}(a', m; N)$, $A \mathop{\smile\!\!\!\!\!\!|}\nolimits_{M}^{N} B$ and $A' \mathop{\smile\!\!\!\!\!\!|}\nolimits_{M}^{N} B$ then $\mathrm{gtp}(a, b, m; N) = \mathrm{gtp}(a', b, m; N)$.

3-AMALGAMATION Picture...

Independence relations

Independence in AECats - 3-amalgamation

3-AMALGAMATION in a picture:



$$A \mathop{\downarrow}_{M}^{N_1} B$$

$$A \mathop{\downarrow}_{M}^{N_2} C$$

$$B \mathop{\downarrow}_{M}^{N_3} C$$

$$A \mathop{\downarrow}_{M}^{N} N_3$$

Independence relations

Independence in AECats - stable / simple / NSOP_1

Let $\mathop{\smile\!\!\!\!\!\!|}$ be a basic independence relation. We call $\mathop{\smile\!\!\!\!\!\!|}$ ...

Independence relations

Independence in AECats - stable / simple / NSOP$_1$

Let $\mathop{\smile}\limits^{\mid}$ be a basic independence relation. We call $\mathop{\smile}\limits^{\mid}$ ...

1. ... *NSOP$_1$-like* if it also satisfies Club Local Character and 3-amalgamation;

Independence relations

Independence in AECats - stable / simple / NSOP$_1$

Let $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}$ be a basic independence relation. We call $\mathrel{\raise0.2ex\hbox{\ooalign{\hidewidth$\vert$\hidewidth\cr\raise-0.9ex\hbox{$\smile$}}}}$ ...

1. ... *NSOP$_1$-like* if it also satisfies Club Local Character and 3-amalgamation;
2. ... *simple* if it also satisfies Club Local Character, 3-amalgamation and Base-monotonicity;

Independence relations

Independence in AECats - stable / simple / $NSOP_1$

Let $\downarrow$ be a basic independence relation. We call $\downarrow$ ...

1. ... *$NSOP_1$-like* if it also satisfies CLUB LOCAL CHARACTER and 3-AMALGAMATION;
2. ... *simple* if it also satisfies CLUB LOCAL CHARACTER, 3-AMALGAMATION and BASE-MONOTONICITY;
3. ... *stable* if it also satisfies CLUB LOCAL CHARACTER, 3-AMALGAMATION, BASE-MONOTONICITY and STATIONARITY.

Independence relations

Independence in AECats - canonicity

Theorem ([Kamsma, 2020])

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is a simple independence relation. Then $\mathop{\smile\!\!\!\!\!\!|}$ is given by isi-dividing.

Independence relations

Independence in AECats - canonicity

Theorem ([Kamsma, 2020])

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and suppose that $\mathop{\smile\!\!\!\!\!\!\vert}$ is a simple independence relation. Then $\mathop{\smile\!\!\!\!\!\!\vert}$ is given by isi-dividing.

Every stable independence relation is simple, so the above applies also to stable independence relations.

Independence relations

Independence in AECats - canonicity

Theorem ([Kamsma, 2020])

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is a simple independence relation. Then $\mathop{\smile\!\!\!\!\!\!|}$ is given by isi-dividing.

Every stable independence relation is simple, so the above applies also to stable independence relations.

Theorem (K (WIP))

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP satisfying the existence axiom and suppose that $\mathop{\smile\!\!\!\!\!\!|}$ is an $NSOP_1$-like independence relation. Then $\mathop{\smile\!\!\!\!\!\!|}$ is given by long Kim-dividing.

Independence relations

Independence in AECats - canonicity

Theorem ([Kamsma, 2020])

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP and suppose that $\mathop{\smile\hskip-0.9em\raise0.1ex\hbox{$\mid$}}$ is a simple independence relation. Then $\mathop{\smile\hskip-0.9em\raise0.1ex\hbox{$\mid$}}$ is given by isi-dividing.

Every stable independence relation is simple, so the above applies also to stable independence relations.

Theorem (K (WIP))

Let $(\mathcal{C}, \mathcal{M})$ be an AECat with AP satisfying the existence axiom and suppose that $\mathop{\smile\hskip-0.9em\raise0.1ex\hbox{$\mid$}}$ is an $NSOP_1$-like independence relation. Then $\mathop{\smile\hskip-0.9em\raise0.1ex\hbox{$\mid$}}$ is given by long Kim-dividing.

The 'existence axiom' is satisfied whenever we have a simple independence relation, but also for example in $(\mathrm{Mod}(T), \mathrm{Mod}(T))$.

Independence relations

Independence in AECats - classifying AECats

We recover part of the classification hierarchy based on independence relations: stable $\implies$ simple $\implies$ NSOP_1.

Independence relations

Independence in AECats - classifying AECats

We recover part of the classification hierarchy based on independence relations: stable $\implies$ simple $\implies$ NSOP_1.

Canonicity is useful. For example, suppose we have an NSOP_1-like independence relation $\downarrow$ on $(\mathcal{C}, \mathcal{M})$, and suppose that we have a counterexample to BASE-MONOTONICITY. Then $(\mathcal{C}, \mathcal{M})$ can never have a simple independence relation.

Independence relations

Independence in AECats - classifying AECats

We recover part of the classification hierarchy based on independence relations: stable $\implies$ simple $\implies$ NSOP$_1$.

Canonicity is useful. For example, suppose we have an NSOP$_1$-like independence relation $\mathop{\smile\!\!\!\!\!\!|}$ on $(\mathcal{C}, \mathcal{M})$, and suppose that we have a counterexample to $\textsc{Base-Monotonicity}$. Then $(\mathcal{C}, \mathcal{M})$ can never have a simple independence relation.

Because any simple $\mathop{\smile\!\!\!\!\!\!|}'$ on $(\mathcal{C}, \mathcal{M})$ would mean that $(\mathcal{C}, \mathcal{M})$ satisfies the existence axiom, and so by canonicity $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}'$. This cannot happen because $\mathop{\smile\!\!\!\!\!\!|}'$ satisfies $\textsc{Base-Monotonicity}$.

References

Levon Haykazyan and Jonathan Kirby. Existentially closed exponential fields. *Israel Journal of Mathematics*, January 2021.

Mark Kamsma. The Kim-Pillay theorem for Abstract Elementary Categories. *The Journal of Symbolic Logic*, 2020.

Michael Lieberman, Jiří Rosický, and Sebastien Vasey. Forking independence from the categorical point of view. *Advances in Mathematics*, 346:719–772, April 2019.