Topos-theoretic completeness theorems

Christian Espíndola

Brno, MUNI

May 7th, 2020

The logic of accessible categories

Definition

A κ -coherent (positive existential) formula is a formula of the form:

$$\bigvee_{i<\kappa} \exists \mathbf{x} \bigwedge_{j<\alpha_i} \phi_{ij}$$

where each $\alpha_i < \kappa$ and each ϕ_{ij} is an atomic formula.

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FACT (Rosicky 1981): Every accessible category is equivalent to the category of models of a theory axiomatized by basic sequents. (Alternatively, we will see that it arises as the category of κ -points of a κ -topos).

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Definition

The system of axioms and rules for $\kappa\text{-coherent}$ logic consists of

• Identity axiom:

 $\phi \vdash_{\mathbf{x}} \phi$

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Substitution rule:

$$\frac{\phi \vdash_{\mathsf{x}} \psi}{\phi[\mathsf{s}/\mathsf{x}] \vdash_{\mathsf{y}} \psi[\mathsf{s}/\mathsf{x}]}$$

where ${\boldsymbol y}$ is a string of variables including all variables occurring in the string of terms ${\boldsymbol s}.$

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• Cut rule:

$$\frac{\phi \vdash_{\mathbf{x}} \psi \quad \psi \vdash_{\mathbf{x}} \theta}{\phi \vdash_{\mathbf{x}} \theta}$$

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• Equality axioms:

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 $\top \vdash_x x = x$

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$(\mathbf{x} = \mathbf{y}) \land \phi \vdash_{\mathbf{z}} \phi[\mathbf{y}/\mathbf{x}]$

where **x**, **y** are contexts of the same length and type and **z** is any context containing **x**, **y** and the free variables of ϕ .

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• Conjunction axioms and rules:

$$\bigwedge_{i < \gamma} \phi_i \vdash_{\mathbf{x}} \phi_j$$

$$\frac{\{\phi \vdash_{\mathbf{x}} \psi_i\}_{i < \gamma}}{\phi \vdash_{\mathbf{x}} \bigwedge_{i < \gamma} \psi_i}$$

for each cardinal $\gamma < \kappa$.

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• Disjunction axioms and rules:

$$\phi_j \vdash_{\mathbf{x}} \bigvee_{i < \gamma} \phi_i$$

$$\frac{\{\phi_i \vdash_{\mathbf{x}} \theta\}_{i < \gamma}}{\bigvee_{i < \gamma} \phi_i \vdash_{\mathbf{x}} \theta}$$

for each cardinal γ .

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• Existential rule:

$$\frac{\phi \vdash_{\mathsf{x}\mathsf{y}} \psi}{\exists \mathsf{y}\phi \vdash_{\mathsf{x}} \psi}$$

where no variable in ${\bf y}$ is free in $\psi.$

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$$\phi \wedge \bigvee_{i < \gamma} \psi_i \vdash_{\mathbf{x}} \bigvee_{i < \gamma} \phi \wedge \psi_i$$

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• Frobenius axiom:

$$\phi \land \exists \mathbf{y} \psi \vdash_{\mathbf{x}} \exists \mathbf{y} (\phi \land \psi)$$

where no variable in \mathbf{y} is in the context \mathbf{x} .

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The logic of accessible categories

Definition

• Transfinite transitivity:

$$\frac{\phi_{f} \vdash_{\mathbf{y}_{f}} \bigvee_{g \in \gamma^{\beta+1}, g|_{\beta} = f} \exists \mathbf{x}_{g} \phi_{g} \qquad \beta < \kappa, f \in \gamma^{\beta}}{\phi_{f} \dashv_{\mathbf{y}_{f}} \bigwedge_{\alpha < \beta} \phi_{f|_{\alpha}} \qquad \beta < \kappa, \text{ limit } \beta, f \in \gamma^{\beta}} \frac{\phi_{f} \dashv_{\mathbf{y}_{f}} \bigwedge_{\beta < \beta} \phi_{f|_{\alpha}}}{\phi_{\emptyset} \vdash_{\mathbf{y}_{\emptyset}} \bigvee_{f \in B} \exists_{\beta < \delta_{f}} \mathbf{x}_{f|_{\beta+1}} \bigwedge_{\beta < \delta_{f}} \phi_{f|_{\beta+1}}}}$$

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for each cardinal γ , where \mathbf{y}_f is the canonical context of ϕ_f , provided that, for every $f \in \gamma^{\beta+1}$, $FV(\phi_f) = FV(\phi_{f|_\beta}) \cup \mathbf{x}_f$ and $\mathbf{x}_{f|_{\beta+1}} \cap FV(\phi_{f|_\beta}) = \emptyset$ for any $\beta < \gamma$, as well as $FV(\phi_f) = \bigcup_{\alpha < \beta} FV(\phi_{f|_\alpha})$ for limit β . Here $B \subseteq \gamma^{<\kappa}$ consists of the minimal elements of a given bar over the tree $\gamma^{<\kappa}$, and the δ_f are the levels of the corresponding $f \in B$. Christian Espindola (Pno, MUN) Topos-theoretic completeness theorems May 7th, 2020 10/27

Categorical semantics

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The following are κ -coherent categories: Set,

The logic above can be interpreted not just in structures in Set but in fact in any category with sufficiently similar properties.

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Example

The following are κ -coherent categories: Set, presheaf categories $Set^{\mathcal{C}}$, $Sh(\mathcal{C}, \tau)$ where \mathcal{C} has κ -small limits and τ satisfies the transfinite transitivity property, the syntactic category of any theory in κ -coherent logic.

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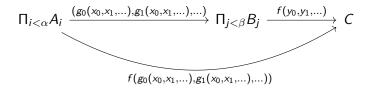
A structure on a κ -coherent category is a collection of objects A_i (representing the sorts), morphisms $f_j : \prod_{i < \alpha} A_i \to B$ (representing function symbols) and monomorphisms $R \to \prod_{i < \alpha} A_i$ (representing relation symbols).

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Terms are interpreted as arrows, and composition of terms as arrow composition:

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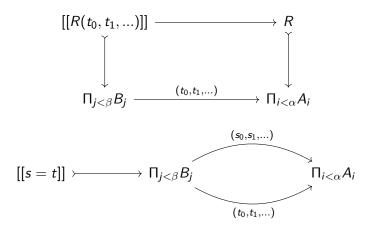


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Formulas $\phi(x_0, x_1, ...)$ are interpreted as subobjects $[[\phi]] \rightarrow \prod_{i < \alpha} A_i$ of the corresponding product of sorts:

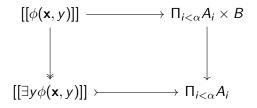
Formulas $\phi(x_0, x_1, ...)$ are interpreted as subobjects $[[\phi]] \rightarrow \prod_{i < \alpha} A_i$ of the corresponding product of sorts: Atomic formulas:



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Existential quantification is interpreted through image factorization:



Syntactic categories

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Objects are α-equivalence classes of κ-coherent formulas in context [x, φ], where x contains (but is not necessarily equal to) the free variables of φ.

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There is a canonical structure in which $[x, \top]$ is the interpretation of the sort corresponding to the variable x, while relations $R \rightarrow \prod_{i < \alpha} A_i$ and functions $f_j : \prod_{i < \alpha} A_i \rightarrow B$ are interpreted respectively as $[x_0x_1..., R(x_0, x_1, ...)]$ and $[x_0x_1...y, f(x_0, x_1, ...) = y]$.

Syntactic categories

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Theorem

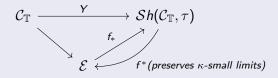
The syntactic category $C_{\mathbb{T}}$ of a κ -coherent theory \mathbb{T} is a κ -coherent category and its canonical structure is the initial model M_0 of \mathbb{T} . Moreover, models of \mathbb{T} in any other κ -coherent category \mathcal{D} are precisely the images of M_0 through κ -coherent functors $F : C_{\mathbb{T}} \to \mathcal{D}$.

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Theorem

The κ -classifying topos of \mathbb{T} is the topos of sheaves $Sh(\mathcal{C}_{\mathbb{T}}, \tau)$ with the topology given by jointly epic families of arrows.



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$\kappa\text{-}{\rm classifying}$ toposes

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κ -classifying toposes

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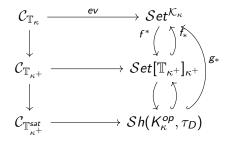
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Theorem

(Espindola 2017) Assume $\kappa^{<\kappa} = \kappa$. The κ -classifying topos of a κ -coherent theory \mathbb{T} with at most κ many axioms (in canonical form) has enough κ -points

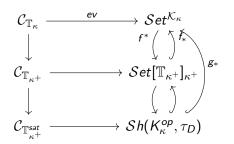
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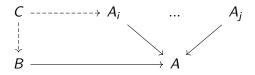


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Here g_* is the double negation subtopos, i.e., the subtopos corresponding to the dense topology:



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If M is a saturated model of size κ^+ , for every $p: N \to N'$ in \mathcal{K}_{κ} , each $N \to M$ extends to some $p': N' \to M$. This is the same as saying that $M: Set^{\mathcal{K}_{\kappa}} \to Set$ maps $p^*: [N', -] \to [N, -]$ to an epimorphism, since:

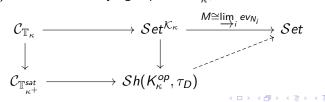
$$\varinjlim_{i} ev_{N_i}([N,-]) = \varinjlim_{i} [N, N_i] \cong [N, \varinjlim_{i} N_i] \cong [N, M]$$

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$$\lim_{i} ev_{N_i}([N,-]) = \lim_{i} [N, N_i] \cong [N, \lim_{i} N_i] \cong [N, M]$$

It follows that M factors through $a : Set^{\mathcal{K}_{\kappa}} \to Sh(\mathcal{K}_{\kappa}^{op}, \tau_D)$ and that $Sh(\mathcal{K}_{\kappa}^{op}, \tau_D)$ is the κ^+ -classifying topos of $\mathbb{T}_{\kappa^+}^{sat}$.



Interpreting universal quantification

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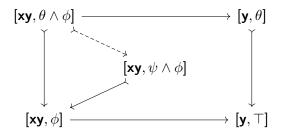
Although the inverse image of a κ -geometric morphism $f^* : \mathcal{E} \to \mathcal{F}$ between κ -toposes preserves the sequent $\phi \vdash_{\mathbf{x}} \psi$, the sentence $\forall \mathbf{x} (\phi \to \psi)$ corresponding to the sequent above should be interpreted as a subobject $S \to 1$ and this interpretation might not be preserved. The interpretation of $\forall \mathbf{x} (\phi \to \psi)$ should reflect the syntactic rule for universal quantification: Although the inverse image of a κ -geometric morphism $f^* : \mathcal{E} \to \mathcal{F}$ between κ -toposes preserves the sequent $\phi \vdash_{\mathbf{x}} \psi$, the sentence $\forall \mathbf{x} (\phi \to \psi)$ corresponding to the sequent above should be interpreted as a subobject $S \mapsto 1$ and this interpretation might not be preserved. The interpretation of $\forall \mathbf{x} (\phi \to \psi)$ should reflect the syntactic rule for universal quantification:

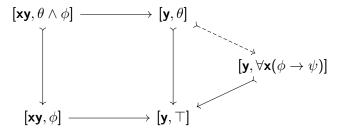
$$\frac{\theta \land \phi \vdash_{\mathsf{x}\mathsf{y}} \psi}{\theta \vdash_{\mathsf{y}} \forall \mathsf{x}(\phi \to \psi)}$$

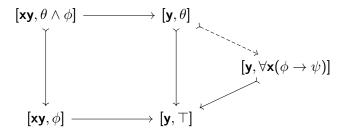
where no variable in \mathbf{x} is free in θ .

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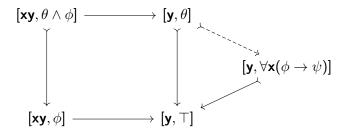
The categorical interpretation of the rule above is exactly that of an adjunction between subobject lattices:







It follows that universal quantification can be interpreted as a right adjoint to the pullback functor between subobject lattices.



It follows that universal quantification can be interpreted as a right adjoint to the pullback functor between subobject lattices. Hence, the interpretation of the sentence $\forall \mathbf{x}(\phi \rightarrow \psi)$ is preserved by a functor F if the functor preserves right adjoint to pullbacks: $F(\forall_f A) = \forall_{F(f)}F(A)$.

Kripke-Joyal semantics

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Theorem

(Infinitary Joyal) The evaluation functor:

$$ev: \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+}
ightarrow \mathcal{S}et^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda}$$

preserves the interpretation of the sentence $\forall \mathbf{x}(\phi \rightarrow \psi)$.

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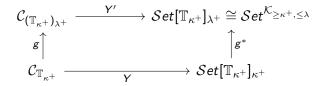
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Proof.



Kripke-Joyal semantics

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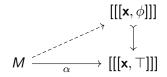
In a sheaf topos, for a morphism $\phi : E \to F$, the right adjoint to the pullback functor $\phi^{-1} : Sub(F) \to Sub(E)$ is the sheaf $\forall_{\phi} : Sub(E) \to Sub(F)$ defined as:

 $y \in \forall_{\phi}(A)(C) \iff \text{for all } f: D \to C \ \phi_D^{-1}(F(f)(y)) \subseteq A(D)$

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In a presheaf topos like $Set^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$, the subobject above is easier to compute thanks to the Kripke-Joyal semantics: $M \Vdash \phi(\alpha)$ if and only if:



Kripke-Joyal semantics

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It follows that $\phi = \top$ if $M \Vdash \phi(\alpha)$ for every $\alpha : M \to [[[\mathbf{x}, \top]]]$.

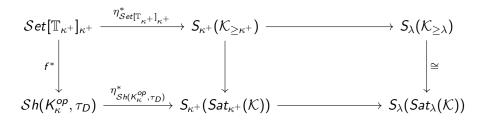
It follows that $\phi = \top$ if $M \Vdash \phi(\alpha)$ for every $\alpha : M \to [[[\mathbf{x}, \top]]]$. Moreover, since the functor $ev : Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to Set^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$ is given by evaluation, we have $[[[\mathbf{x}, \phi]]] = ev_{\phi}$ for every κ -coherent ϕ , and thus, by Yoneda, $M \Vdash \phi(\alpha)$ if and only if $M \models \phi(\alpha)$. It follows that $\phi = \top$ if $M \Vdash \phi(\alpha)$ for every $\alpha : M \to [[[\mathbf{x}, \top]]]$. Moreover, since the functor $ev : Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to Set^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$ is given by evaluation, we have $[[[\mathbf{x}, \phi]]] = ev_{\phi}$ for every κ -coherent ϕ , and thus, by Yoneda, $M \Vdash \phi(\alpha)$ if and only if $M \models \phi(\alpha)$. If ϕ is not κ -coherent, the following properties are useful:

- $M \Vdash (\phi \to \psi)(\alpha)$ if and only if for every $f : M \to N$, $N \Vdash \phi(f(\alpha)) \implies N \Vdash \psi(f(\alpha))$
- $M \Vdash \neg \phi(\alpha)$ if and only if for every $f : M \to N, N \nvDash \phi(f(\alpha))$
- $M \Vdash \forall \mathbf{x} \phi(\mathbf{x})$ if and only if for every $f : M \to N$ and $\alpha \in N$, $M \Vdash \phi(\alpha)$

Wrapping up

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$$\begin{array}{cccc} \mathcal{S}et[\mathbb{T}_{\kappa^{+}}]_{\kappa^{+}} & \xrightarrow{\eta^{*}_{\mathcal{S}et[\mathbb{T}_{\kappa^{+}}]_{\kappa^{+}}}} & S_{\kappa^{+}}(\mathcal{K}_{\geq \kappa^{+}}) & \longrightarrow & S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\ & & \downarrow & & \downarrow \\ & & \downarrow & & \downarrow \\ \mathcal{S}h(\mathcal{K}^{op}_{\kappa}, \tau_{D}) & \xrightarrow{\eta^{*}_{\mathcal{S}h(\mathcal{K}^{op}_{\kappa}, \tau_{D})}} & S_{\kappa^{+}}(\mathcal{S}at_{\kappa^{+}}(\mathcal{K})) & \longrightarrow & S_{\lambda}(\mathcal{S}at_{\lambda}(\mathcal{K})) \end{array}$$

We conclude that $M_{\lambda} \nvDash \neg \forall \mathbf{x} (\phi \rightarrow \psi)$.

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$$\begin{array}{c|c} \mathcal{S}et[\mathbb{T}_{\kappa^{+}}]_{\kappa^{+}} & \xrightarrow{\eta^{*}_{\mathcal{S}et[\mathbb{T}_{\kappa^{+}}]_{\kappa^{+}}}} & S_{\kappa^{+}}(\mathcal{K}_{\geq \kappa^{+}}) & \longrightarrow & S_{\lambda}(\mathcal{K}_{\geq \lambda}) \\ f^{*} & \downarrow & \downarrow & \downarrow \\ f^{*} & \downarrow & \downarrow & \downarrow \\ \mathcal{S}h(\mathcal{K}^{op}_{\kappa}, \tau_{D}) & \xrightarrow{\eta^{*}_{\mathcal{S}h(\mathcal{K}^{op}_{\kappa}, \tau_{D})}} & S_{\kappa^{+}}(Sat_{\kappa^{+}}(\mathcal{K})) & \longrightarrow & S_{\lambda}(Sat_{\lambda}(\mathcal{K})) \end{array}$$

We conclude that $M_{\lambda} \nvDash \neg \forall \mathbf{x} (\phi \rightarrow \psi)$. Since $ev : Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow Set^{\mathcal{K}_{\geq \kappa^+}, \leq \lambda}$ preserves the sentence in question, $[[\forall \mathbf{x} (\phi \rightarrow \psi)]]$ is not 0 in $Set[\mathbb{T}_{\kappa^+}]_{\kappa^+}$.

We conclude that $M_{\lambda} \nvDash \neg \forall \mathbf{x} (\phi \rightarrow \psi)$. Since $ev : Set[\mathbb{T}_{\kappa^+}]_{\kappa^+} \rightarrow Set^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}}$ preserves the sentence in question, $[[\forall \mathbf{x} (\phi \rightarrow \psi)]]$ is not 0 in $Set[\mathbb{T}_{\kappa^+}]_{\kappa^+}$. Since this latter is two-valued, $[[\forall \mathbf{x} (\phi \rightarrow \psi)]]$ must be 1.

Thank you!

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