Topos-theoretic completeness theorems

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The logic of accessible categories

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Definition

A κ-coherent (positive existential) formula is a formula of the form:

$$\bigvee_{i<\kappa} \exists \mathbf{x} \bigwedge_{j<\alpha_i} \phi_{ij}$$

where each $\alpha_i < \kappa$ and each ϕ_{ij} is an atomic formula.

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whose intended meaning is the sentence $\forall \mathbf{x}(\phi \to \psi)$.

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FACT (Rosicky 1981): Every accessible category is equivalent to the category of models of a theory axiomatized by basic sequents. (Alternatively, we will see that it arises as the category of κ-points of a κ-topos).

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Definition

The system of axioms and rules for κ-coherent logic consists of

- Identity axiom:

$$\phi \vdash_{\mathbf{x}} \phi$$

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Definition

- Substitution rule:

$$\frac{\phi \vdash_{\mathbf{x}} \psi}{\phi[\mathbf{s}/\mathbf{x}] \vdash_{\mathbf{y}} \psi[\mathbf{s}/\mathbf{x}]}$$

  where **y** is a string of variables including all variables occurring in the string of terms **s**.

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  2. $$(\mathbf{x} = \mathbf{y}) \wedge \phi \vdash_{\mathbf{z}} \phi[\mathbf{y}/\mathbf{x}]$$

     where **x**, **y** are contexts of the same length and type and **z** is any context containing **x**, **y** and the free variables of ϕ.

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Definition

- Conjunction axioms and rules:

$$\bigwedge_{i<\gamma} \phi_i \vdash_{\mathbf{x}} \phi_j$$

$$\frac{\{\phi \vdash_{\mathbf{x}} \psi_i\}_{i<\gamma}}{\phi \vdash_{\mathbf{x}} \bigwedge_{i<\gamma} \psi_i}$$

for each cardinal $\gamma < \kappa$.

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Definition

- Disjunction axioms and rules:

$$\phi_j \vdash_{\mathbf{x}} \bigvee_{i<\gamma} \phi_i$$

$$\frac{\{\phi_i \vdash_{\mathbf{x}} \theta\}_{i<\gamma}}{\bigvee_{i<\gamma} \phi_i \vdash_{\mathbf{x}} \theta}$$

for each cardinal γ.

The logic of accessible categories

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- Existential rule:

$$\frac{\phi \vdash_{\mathbf{xy}} \psi}{\exists \mathbf{y} \phi \vdash_{\mathbf{x}} \psi}$$

where no variable in **y** is free in ψ.

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Definition

- Transfinite transitivity:

$$\frac{\phi_f \vdash_{\mathbf{y}_f} \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} \exists \mathbf{x}_g \phi_g \qquad \beta < \kappa, f \in \gamma^\beta \qquad \phi_f \dashv\vdash_{\mathbf{y}_f} \bigwedge_{\alpha < \beta} \phi_{f|_\alpha} \qquad \beta < \kappa, \text{ limit } \beta, f \in \gamma^\beta}{\phi_\emptyset \vdash_{\mathbf{y}_\emptyset} \bigvee_{f \in B} \exists_{\beta < \delta_f} \mathbf{x}_{f|_{\beta+1}} \bigwedge_{\beta < \delta_f} \phi_{f|_{\beta+1}}}$$

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for each cardinal γ, where $\mathbf{y}_f$ is the canonical context of ϕ_f, provided that, for every $f \in \gamma^{\beta+1}$, $FV(\phi_f) = FV(\phi_{f|_{\beta}}) \cup \mathbf{x}_f$ and $\mathbf{x}_{f|_{\beta+1}} \cap FV(\phi_{f|_{\beta}}) = \emptyset$ for any $\beta < \gamma$, as well as $FV(\phi_f) = \bigcup_{\alpha<\beta} FV(\phi_{f|_{\alpha}})$ for limit β. Here $B \subseteq \gamma^{<\kappa}$ consists of the minimal elements of a given bar over the tree $\gamma^{<\kappa}$, and the δ_f are the levels of the corresponding $f \in B$.

Categorical semantics

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Definition

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Terms are interpreted as arrows, and composition of terms as arrow composition:

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Terms are interpreted as arrows, and composition of terms as arrow composition:

$$\begin{tikzcd} \Pi_{i<\alpha} A_i \arrow[r, "{(g_0(x_0,x_1,\dots),g_1(x_0,x_1,\dots),\dots)}"] \arrow[rr, bend right, "{f(g_0(x_0,x_1,\dots),g_1(x_0,x_1,\dots),\dots))}"'] & \Pi_{j<\beta} B_j \arrow[r, "{f(y_0,y_1,\dots)}"] & C \end{tikzcd}$$

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Atomic formulas:

$$\begin{tikzcd}
{[[R(t_0, t_1, ...)]]} \arrow[rr] \arrow[d, tail] & & R \arrow[d, tail] \\
\Pi_{j<\beta} B_j \arrow[rr, "{(t_0, t_1, ...)}"] & & \Pi_{i<\alpha} A_i
\end{tikzcd}$$

$$\begin{tikzcd}
{[[s = t]]} \arrow[r, tail] & \Pi_{j<\beta} B_j \arrow[rr, bend left, "{(s_0, s_1, ...)}"] \arrow[rr, bend right, "{(t_0, t_1, ...)}"'] & & \Pi_{i<\alpha} A_i
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The connectives $\bigvee$ and $\bigwedge$ are interpreted as join and meet in the subobject lattices.

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Existential quantification is interpreted through image factorization:

$$\begin{tikzcd}
{[[\phi(\mathbf{x}, y)]]} \arrow[r] \arrow[d, two heads] & \Pi_{i<\alpha} A_i \times B \arrow[d] \\
{[[\exists y \phi(\mathbf{x}, y)]]} \arrow[r, tail] & \Pi_{i<\alpha} A_i
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- Objects are α-equivalence classes of κ-coherent formulas in context $[\mathbf{x}, \phi]$, where $\mathbf{x}$ contains (but is not necessarily equal to) the free variables of ϕ.
- Morphisms $[\mathbf{x}, \phi] \to [\mathbf{y}, \psi]$ are equivalence classes $[\mathbf{xy}, \theta]$ of provably functional formulas.

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There is a canonical structure in which $[x, \top]$ is the interpretation of the sort corresponding to the variable x, while relations $R \rightarrowtail \Pi_{i<\alpha} A_i$ and functions $f_j : \Pi_{i<\alpha} A_i \to B$ are interpreted respectively as $[x_0 x_1 ..., R(x_0, x_1, ...)]$ and $[x_0 x_1 ... y, f(x_0, x_1, ...) = y]$.

Syntactic categories

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Theorem

The syntactic category $\mathcal{C}_{\mathbb{T}}$ of a κ-coherent theory $\mathbb{T}$ is a κ-coherent category and its canonical structure is the initial model M_0 of $\mathbb{T}$. Moreover, models of $\mathbb{T}$ in any other κ-coherent category $\mathcal{D}$ are precisely the images of M_0 through κ-coherent functors $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$.

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Theorem

The κ-classifying topos of $\mathbb{T}$ is the topos of sheaves $\mathcal{S}h(\mathcal{C}_{\mathbb{T}}, \tau)$ with the topology given by jointly epic families of arrows.

$$
\begin{tikzcd}
\mathcal{C}_{\mathbb{T}} \arrow[rr, "Y"] \arrow[dr] & & \mathcal{S}h(\mathcal{C}_{\mathbb{T}}, \tau) \arrow[dl, bend left, "f^* \text{ (preserves } \kappa\text{-small limits)}"] \\
& \mathcal{E} \arrow[ur, "f_*"] &
\end{tikzcd}
$$

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Theorem

(Espindola 2017) Assume $\kappa^{<\kappa} = \kappa$. The κ-classifying topos of a κ-coherent theory $\mathbb{T}$ with at most κ many axioms (in canonical form) has enough κ-points

Axiomatizing saturated models

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$$
\begin{tikzcd}
\mathcal{C}_{\mathbb{T}_\kappa} \arrow[r, "ev"] \arrow[d] & \mathcal{S}et^{\mathcal{K}_\kappa} \arrow[d, bend right, "f^*"'] \\
\mathcal{C}_{\mathbb{T}_{\kappa^+}} \arrow[r] \arrow[d] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[u, bend right, "f_*"'] \arrow[d, bend right] \\
\mathcal{C}_{\mathbb{T}^{sat}_{\kappa^+}} \arrow[r] & \mathcal{S}h(K_\kappa^{op}, \tau_D) \arrow[u, bend right] \arrow[uu, bend right=60, "g_*"']
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\end{tikzcd}
$$

Here g_* is the double negation subtopos, i.e., the subtopos corresponding to the dense topology:

$$
\begin{tikzcd}
C \arrow[r, dashed] \arrow[d, dashed] & A_i \arrow[dr] & \dots & A_j \arrow[dl] \\
B \arrow[rr] & & A &
\end{tikzcd}
$$

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If M is a saturated model of size κ^+, for every $p : N \to N'$ in $\mathcal{K}_\kappa$, each $N \to M$ extends to some $p' : N' \to M$. This is the same as saying that $M : \mathcal{S}et^{\mathcal{K}_\kappa} \to \mathcal{S}et$ maps $p^* : [N', -] \to [N, -]$ to an epimorphism, since:

$$\varinjlim_i ev_{N_i}([N, -]) = \varinjlim_i [N, N_i] \cong [N, \varinjlim_i N_i] \cong [N, M]$$

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$$\varinjlim_i ev_{N_i}([N, -]) = \varinjlim_i [N, N_i] \cong [N, \varinjlim_i N_i] \cong [N, M]$$

It follows that M factors through $a : \mathcal{S}et^{\mathcal{K}_\kappa} \to \mathcal{S}h(K_\kappa^{op}, \tau_D)$ and that $\mathcal{S}h(K_\kappa^{op}, \tau_D)$ is the κ^+-classifying topos of $\mathbb{T}_{\kappa^+}^{sat}$.

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{T}_\kappa} \arrow[r] \arrow[d] & \mathcal{S}et^{\mathcal{K}_\kappa} \arrow[r, "M \cong \varinjlim_i ev_{N_i}"] \arrow[d] & \mathcal{S}et \\
\mathcal{C}_{\mathbb{T}_{\kappa^+}^{sat}} \arrow[r] & \mathcal{S}h(K_\kappa^{op}, \tau_D) \arrow[ur, dashed] &
\end{tikzcd}$$

Interpreting universal quantification

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Although the inverse image of a κ-geometric morphism $f^* : \mathcal{E} \to \mathcal{F}$ between κ-toposes preserves the sequent $\phi \vdash_{\mathbf{x}} \psi$, the sentence $\forall \mathbf{x}(\phi \to \psi)$ corresponding to the sequent above should be interpreted as a subobject $S \rightarrowtail 1$ and this interpretation might not be preserved.
The interpretation of $\forall \mathbf{x}(\phi \to \psi)$ should reflect the syntactic rule for universal quantification:

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The interpretation of $\forall \mathbf{x}(\phi \to \psi)$ should reflect the syntactic rule for universal quantification:

$$\frac{\theta \wedge \phi \vdash_{\mathbf{xy}} \psi}{\theta \vdash_{\mathbf{y}} \forall \mathbf{x}(\phi \to \psi)}$$

where no variable in $\mathbf{x}$ is free in θ.

Interpreting universal quantification

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The categorical interpretation of the rule above is exactly that of an adjunction between subobject lattices:

$$\begin{tikzcd}
{[\mathbf{xy}, \theta \wedge \phi]} \arrow[rr] \arrow[dd, tail] \arrow[dr, tail, dashed] & & {[\mathbf{y}, \theta]} \arrow[dd, tail] \\
& {[\mathbf{xy}, \psi \wedge \phi]} \arrow[dl, tail] & \\
{[\mathbf{xy}, \phi]} \arrow[rr] & & {[\mathbf{y}, \top]}
\end{tikzcd}$$

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{[\mathbf{xy}, \phi]} \arrow[r] & {[\mathbf{y}, \top]} &
\end{tikzcd}$$

It follows that universal quantification can be interpreted as a right adjoint to the pullback functor between subobject lattices.

Interpreting universal quantification

$$\begin{tikzcd}
{[\mathbf{xy}, \theta \wedge \phi]} \arrow[rr] \arrow[dd, tail] & & {[\mathbf{y}, \theta]} \arrow[dd, tail] \arrow[dr, dashed, tail] & \\
& & & {[\mathbf{y}, \forall \mathbf{x}(\phi \to \psi)]} \arrow[dl, tail] \\
{[\mathbf{xy}, \phi]} \arrow[rr] & & {[\mathbf{y}, \top]} &
\end{tikzcd}$$

It follows that universal quantification can be interpreted as a right adjoint to the pullback functor between subobject lattices.Hence, the interpretation of the sentence $\forall \mathbf{x}(\phi \to \psi)$ is preserved by a functor F if the functor preserves right adjoint to pullbacks: $F(\forall_f A) = \forall_{F(f)} F(A)$.

Kripke-Joyal semantics

Kripke-Joyal semantics

Theorem

(Infinitary Joyal) The evaluation functor:

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preserves the interpretation of the sentence $\forall \mathbf{x}(\phi \to \psi)$.

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preserves the interpretation of the sentence $\forall \mathbf{x}(\phi \to \psi)$.

Proof.

$$\begin{tikzcd}
\mathcal{C}_{(\mathbb{T}_{\kappa^+})_{\lambda^+}} \arrow[r, "Y'"] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\lambda^+} \cong \mathcal{S}et^{\mathcal{K}_{\geq \kappa^+, \leq \lambda}} \\
\mathcal{C}_{\mathbb{T}_{\kappa^+}} \arrow[u, "g"] \arrow[r, "Y"'] & \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \arrow[u, "g^*"']
\end{tikzcd}$$

Kripke-Joyal semantics

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In a sheaf topos, for a morphism $\phi : E \to F$, the right adjoint to the pullback functor $\phi^{-1} : \mathcal{S}ub(F) \to \mathcal{S}ub(E)$ is the sheaf $\forall_\phi : \mathcal{S}ub(E) \to \mathcal{S}ub(F)$ defined as:

$$y \in \forall_\phi(A)(C) \iff \text{for all } f : D \to C \ \phi_D^{-1}(F(f)(y)) \subseteq A(D)$$

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In a presheaf topos like $\mathcal{S}et^{\mathcal{K}_{\geq\kappa^+,\leq\lambda}}$, the subobject above is easier to compute thanks to the Kripke-Joyal semantics: $M \Vdash \phi(\alpha)$ if and only if:

$$\begin{tikzcd} & {[[[\mathbf{x}, \phi]]]} \arrow[d, tail] \\ M \arrow[ur, dashed] \arrow[r, "\alpha"'] & {[[[\mathbf{x}, \top]]]} \end{tikzcd}$$

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It follows that $\phi = \top$ if $M \Vdash \phi(\alpha)$ for every $\alpha : M \to [[[\mathbf{x}, \top]]]$. Moreover, since the functor $ev : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \mathcal{S}et^{\mathcal{K}_{\geq\kappa^+, \leq\lambda}}$ is given by evaluation, we have $[[[\mathbf{x}, \phi]]] = ev_\phi$ for every κ-coherent ϕ, and thus, by Yoneda, $M \Vdash \phi(\alpha)$ if and only if $M \models \phi(\alpha)$.

Kripke-Joyal semantics

It follows that $\phi = \top$ if $M \Vdash \phi(\alpha)$ for every $\alpha : M \to [[[\mathbf{x}, \top]]]$.
Moreover, since the functor $ev : \mathcal{S}et[\mathbb{T}_{\kappa^+}]_{\kappa^+} \to \mathcal{S}et^{\mathcal{K}_{\geq\kappa^+,\leq\lambda}}$ is given by evaluation, we have $[[[\mathbf{x}, \phi]]] = ev_\phi$ for every κ-coherent ϕ, and thus, by Yoneda, $M \Vdash \phi(\alpha)$ if and only if $M \models \phi(\alpha)$.
If ϕ is not κ-coherent, the following properties are useful:

- $M \Vdash (\phi \to \psi)(\alpha)$ if and only if for every $f : M \to N$, $N \Vdash \phi(f(\alpha)) \implies N \Vdash \psi(f(\alpha))$
- $M \Vdash \neg\phi(\alpha)$ if and only if for every $f : M \to N$, $N \nVdash \phi(f(\alpha))$
- $M \Vdash \forall \mathbf{x}\phi(\mathbf{x})$ if and only if for every $f : M \to N$ and $\alpha \in N$, $M \Vdash \phi(\alpha)$

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Since this latter is two-valued, $[[\forall\mathbf{x}(\phi \to \psi)]]$ must be 1.

Thank you!