Tangent ∞-categories and Goodwillie calculus

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Based on:

- *Tangent ∞-categories and Goodwillie calculus*, arxiv:2101.07819, joint with Kristine Bauer, Matthew Burke
- *Dual tangent structures for ∞-toposes*, arxiv:2101.08805

Inspired by:

- Kristine Bauer, Brenda Johnson, Christina Osborne, Emily Riehl and Amelia Tebbe, Directional derivatives and higher order chain rules for abelian functor calculus, *Topology Appl.* 235 (2018)

Two meanings of the phrase "tangent category"

(1) (nLab: tangent category on $\mathcal{C}$): the category of abelian group objects of the slice categories over $\mathcal{C}$ (aka Beck modules)

[Or the homotopical version: the ∞-category of parameterized spectra over $\mathcal{C}$.]

(2) (nLab: tangent bundle category $\mathbb{X}$): provides for each object $C \in \mathbb{X}$ a tangent bundle TC, a projection morphism $p : TC \to C$, a zero section $0 : C \to TC$, etc... such that ...

[E.g. $\mathbb{X} = \mathbb{M}\mathrm{fld}$, category of smooth manifolds and smooth maps.]

Goal of this project: connect these two meanings by making (1) an example of (2) with $\mathbb{X}$ equal to some ∞-category of ∞-categories.

Warning: We use the terminology tangent category for meaning (2), the reverse of the nLab choice!

Tangent categories

The notion of tangent category was developed by:

- Jiří Rosický, Abstract tangent functors, *Diagrammes* 12 (1984);
- Robin Cockett and Geoff Cruttwell, Differential structure, tangent structure and SDG, *Applied Categorical Structures* 22 (2014);

in order to axiomatize (some) properties of the tangent bundle functor

$$T : \mathrm{Mfld} \to \mathrm{Mfld}$$

A tangent structure on a category $\mathbb{X}$ consists of an endofunctor $T : \mathbb{X} \to \mathbb{X}$ together with natural transformations

- *projection* $p : TM \to M$
- *zero section* $0 : M \to TM$
- *addition* $+ : TM \times_M TM \to TM$
- *symmetry* $c : T(TM) \to T(TM)$
- *vertical lift* $\ell : TM \to T(TM)$

Tangent categories (continued)

such that

1. **lots** of diagrams commute
2. the vertical lift axiom: for each $M \in \mathbb{X}$, there is a pullback square

$$\begin{array}{ccc} TM \times_M TM & \longrightarrow & T(TM) \\ \big\downarrow & & \big\downarrow {\scriptstyle T(p)} \\ M & \xrightarrow{\;\;0\;\;} & TM \end{array}$$

In $\mathbb{M}\mathrm{fld}$ this axiom says that

$$T(T_x M) \cong T_x M \times T_x M.$$

smoothly in $x \in M$.

Tangent categories (examples)

1. $\mathbb{M}\mathrm{fld}$: smooth manifolds with ordinary tangent bundle
2. microlinear objects in a model of Synthetic Differential Geometry
3. differential λ-calculus [Ehrhard-Regnier]
4. $\mathbb{C}\mathbb{R}\mathrm{ing}$: commutative rings, $TR := R[x]/(x^2)$
5. $\mathbb{C}\mathbb{R}\mathrm{ing}^{op}$: $UR := \mathrm{Sym}(\Omega_R)$
6. $\mathbb{S}\mathrm{ch}$: schemes with Zariski tangent bundle

In order to generalize tangent structures to ∞-categories, we use a reformulation due to

- Poon Leung, Classifying tangent structures using Weil algebras, *Theory and Applications of Categories* 32(9), (2017).

Weil-algebras

Definition

Let $\mathbb{W}\mathrm{eil}$ be the monoidal category of Weil-algebras with:

- objects: augmented commutative semi-rings of the form

$$\mathbb{N}[x_1, \ldots, x_n]/(x_i x_j \mid i \sim j)$$

- morphisms: augmented semi-ring homomorphisms;
- monoidal structure: tensor product $\otimes$, unit $\mathbb{N}$.

E.g.

- $W = \mathbb{N}[x]/(x^2)$
- $W^2 = \mathbb{N}[x, y]/(x^2, xy, y^2)$
- $W \otimes W = \mathbb{N}[x, y]/(x^2, y^2)$
- $W^{n_1} \otimes \cdots \otimes W^{n_r}$

Tangent categories (revisited)

Theorem (Leung)

A tangent structure on category $\mathbb{X}$ is a (strong) monoidal functor

$$T^{\bullet} : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}(\mathbb{X})^{\circ}$$

that preserves certain pullbacks in $\mathbb{W}\mathrm{eil}$.

I.e. for each Weil-algebra A, we are given a functor $T^A : \mathbb{X} \to \mathbb{X}$:

- $T^{\mathbb{N}} \cong I$, the identity functor on $\mathbb{X}$
- $T^W = T$ is the tangent bundle functor
- $T^{W^{\otimes 2}} \cong T^2, \quad T^{W^2} \cong T \times_I T$

with natural transformations $p, 0, +, c, \ell$ given by applying T to the following morphisms in $\mathbb{W}\mathrm{eil}$:

- $p : W \to \mathbb{N}; x \mapsto 0, \quad 0 : \mathbb{N} \to W, \quad + : W^2 \to W; x, y \mapsto x$
- $c : W^{\otimes 2} \to W^{\otimes 2}; x \mapsto y, y \mapsto x, \quad \ell : W \to W^{\otimes}; x \mapsto xy$

Tangent ∞-categories

Definition (Bauer-Burke-C.)

A tangent structure on an ∞-category $\mathbb{X}$ is a (strong) monoidal functor

$$T^\bullet : \mathbb{W}\mathrm{eil}^{\otimes} \to \mathrm{End}(\mathbb{X})^{\circ}$$

that preserves those same pullbacks in $\mathbb{W}\mathrm{eil}$.

E.g. $\mathbb{DM}\mathrm{fld}$: derived manifolds (Spivak, Carchedi-Steffens)

There are notions of tangent functor and tangent natural transformation. We can form the $(\infty, 2)$-category of tangent ∞-categories

$$\mathbf{Tan}(\mathbf{Cat}_\infty).$$

More generally, there is a notion of tangent object in any $(\infty, 2)$-category $\mathbf{C}$ and an $(\infty, 2)$-category of tangent objects in $\mathbf{C}$:

$$\mathbf{Tan}(\mathbf{C}).$$

The Goodwillie tangent structure

An ∞-category is differentiable if it admits finite limits and sequential colimits, which commute.

There is a (very large) ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ with:

- objects: the (large) differentiable ∞-categories;
- morphisms: functors that preserve sequential colimits.

Let $\mathsf{Top}_{*,\mathsf{fin}}$ be the ∞-category of finite pointed CW-complexes.

Definition (Goodwillie tangent bundle on an ∞-category (Lurie))

For a (differentiable) ∞-category $\mathcal{C}$, set

$$T(\mathcal{C}) = \mathrm{Exc}(\mathsf{Top}_{*,\mathsf{fin}}, \mathcal{C})$$

the ∞-category of excisive functors $\mathsf{Top}_{*,\mathsf{fin}} \to \mathcal{C}$: those that map pushouts in $\mathsf{Top}_{*,\mathsf{fin}}$ to pullbacks in $\mathcal{C}$. For $F : \mathcal{C} \to \mathcal{D}$ we define

$$T(F) : T(\mathcal{C}) \to T(\mathcal{D}); \quad L \mapsto P_1(FL).$$

The Goodwillie tangent structure (continued)

Theorem (Bauer-Burke-C.)

There is a tangent structure on the ∞-category $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$ with tangent bundle functor $T : \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}} \to \mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, projection $p : T(\mathcal{C}) \to \mathcal{C}; L \mapsto L(*)$, zero section $0 : \mathcal{C} \to T(\mathcal{C}); X \mapsto \mathrm{const}_X$, ...

For a Weil-algebra $A = \mathbb{N}[x_1, \ldots, x_n]/(R)$ and differentiable ∞-category $\mathcal{C}$, the tangent structure is given by

$$T^A(\mathcal{C}) := \mathrm{Exc}^A(\mathrm{Top}^n_{*,\mathrm{fin}}, \mathcal{C}) \subseteq \mathrm{Fun}(\mathrm{Top}^n_{*,\mathrm{fin}}, \mathcal{C})$$

the subcategory of functors that are A-excisive, i.e. take certain pushout squares to pullbacks in $\mathcal{C}$, depending on the relations R.

For a morphism $\phi : A \to A'$ in $\mathbb{W}\mathrm{eil}$, we define a corresponding functor

$$\tilde{\phi} : \mathrm{Top}^{n'}_{*,\mathrm{fin}} \to \mathrm{Top}^n_{*,\mathrm{fin}}$$

and then $T^\phi : T^A(\mathcal{C}) \to T^{A'}(\mathcal{C})$ is given by $L \mapsto P_{A'}(L\tilde{\phi})$.

Universality of vertical lift

The vertical lift axiom for the Goodwillie tangent structure: for each differentiable ∞-category $\mathcal{C}$, there is a pullback (of ∞-categories) of the form

$$\begin{array}{ccc} T(\mathcal{C}) \times_{\mathcal{C}} T(\mathcal{C}) & \longrightarrow & T(T(\mathcal{C})) \\ \downarrow & & \downarrow{\scriptstyle T(p)} \\ \mathcal{C} & \xrightarrow{\quad 0 \quad} & T(\mathcal{C}). \end{array}$$

This claim amounts to a splitting result for functors $\mathrm{Top}_{*,\mathrm{fin}} \times \mathrm{Top}_{*,\mathrm{fin}} \to \mathcal{C}$ that are excisive in each variable separately, and reduced in *one* variable, which is proved using Goodwillie's classification of homogeneous functors.

Differential objects in a tangent ∞-category

In any tangent ∞-category $\mathbb{X}$, a differential object is $C \in \mathbb{X}$ with a (suitable) splitting

$$TC \simeq C \times C.$$

The tangent *spaces* $T_x M$ (when they exist) are precisely the differential objects.

- In $\mathbb{M}\mathrm{fld}$, the differential objects are the Euclidean spaces $\mathbb{R}^n$.
- In $\mathbb{S}\mathrm{ch}$, the differential objects include the free affine schemes $\mathbb{A}^n$.
- In $\mathbb{C}\mathrm{at}_\infty^{\mathrm{diff}}$, the differentiable objects are... the stable ∞-categories, because tangent spaces are given by

$$T_X\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C}_{/X})$$

for $X \in \mathcal{C}$.

Jets in a tangent ∞-category

$F : C \to D$: morphism in a tangent ∞-category

The *n*-jet of F at $x : * \to C$ is (the equivalence class of) the morphism

$$T^n_x(F) : T^n_x(C) \hookrightarrow T^n(C) \xrightarrow{T^n(F)} T^n(D).$$

We say F, G agree to order n at x if $T^n_x(F) \simeq T^n_x(G)$.

- In $\mathbb{M}\mathrm{fld}$, $f, g : M \to N$ agree to order n at $x \in M$ if and only if their Taylor expansions (in local coordinates at x) agree up to degree n polynomials.
- In $\mathbb{C}\mathrm{at}^{\mathrm{diff}}_\infty$, $F, G : \mathcal{C} \to \mathcal{D}$ agree to order n at $X \in \mathcal{C}$ (via a natural transformation $\alpha : F \to G$) if and only if α induces an equivalence

  $$P^X_n(F) \xrightarrow{\sim} P^X_n(G)$$

  between Goodwillie's n-excisive approximations at X.

Future directions

Other concepts developed for abstract tangent categories:

- Robin Cockett and Geoff Cruttwell, Differential bundles and fibrations for tangent categories, *Cah. Topol. Géom. Différ. Catég.* 59 (2018).
- Robin Cockett and Geoff Cruttwell, Connections in tangent categories, *Theory and Applications of Categories* 32 (2017).
- Geoff Cruttwell and Rory Lucyshyn-Wright, A simplicial foundation for differential and sector forms in tangent categories, *J. Homotopy Relat. Struct.* 13 (2018).
- Richard Blute, Geoff Cruttwell,and Rory Lucyshyn-Wright, Affine geometric spaces in tangent categories, *Theory and Applications of Categories* 34 (2019).

What do these concepts correspond to in $\mathbb{C}\mathrm{at}_{\infty}^{\mathrm{diff}}$?

∞-toposes

An ∞-topos is an ∞-category that is an accessible left-exact localization of a presheaf ∞-category $\mathcal{P}(\mathsf{C})$ for a small ∞-category C. E.g. $\mathcal{S}h(X)$ for a topological space X.

A geometric morphism $G : \mathcal{X} \to \mathcal{Y}$ between ∞-toposes is a functor that has a left-exact left adjoint $F : \mathcal{Y} \to \mathcal{X}$.

There are (very large) ∞-categories:

- $\mathbb{T}\mathrm{opos}_\infty$ of ∞-toposes and geometric morphisms
- $\mathbb{L}\mathrm{ogos}_\infty \simeq \mathbb{T}\mathrm{opos}_\infty^{op}$ of ∞-toposes and the left adjoints of geometric morphisms (i.e. the functors that preserve colimits and finite limits)

These correspond to two dual ways to think of ∞-toposes (geometric and algebraic), see

- Mathieu Anel and André Joyal, Topo-logie, available at http://mathieu.anel.free.fr/

Tangent structures on ∞-toposes

There are dual tangent structures on these ∞-categories:

Theorem (C.)

1. $\mathbb{L}\text{ogos}_\infty$ *has tangent structure* $T : \mathbb{L}\text{ogos}_\infty \to \mathbb{L}\text{ogos}_\infty$ *given by the restriction of the Goodwillie tangent structure to* $\mathbb{L}\text{ogos}_\infty \subseteq \mathbb{C}\text{at}_\infty^{\text{diff}}$.
2. $\mathbb{T}\text{opos}_\infty$ *has tangent structure* $U : \mathbb{T}\text{opos}_\infty \to \mathbb{T}\text{opos}_\infty$ *given by the right adjoint of the functor* T^{op}. *We call this the geometric tangent structure.*

Questions:

- What is $U(\mathcal{S}h(X))$ for a topological space X?
- What are the differentiable objects (i.e. tangent spaces) in the geometric tangent structure?
- What are the n-jets in the geometric tangent structure?

The geometric tangent structure on $\mathbb{T}\mathrm{opos}_\infty$

Any ∞-topos has an ∞-category of points

$$Pt(\mathcal{X}) := \mathrm{Fun}^R(\mathcal{S}, \mathcal{X}) \simeq \mathrm{Fun}^L(\mathcal{X}, \mathcal{S})$$

where $\mathcal{S}$ is the ∞-topos of spaces, the terminal object in $\mathbb{T}\mathrm{opos}_\infty$.

Theorem (C.)

There is an equivalence of tangent ∞-categories

$$Pt : \mathbb{I}\mathrm{nj}\mathbb{T}\mathrm{opos}_\infty \xrightarrow{\sim} \mathbb{C}\mathrm{at}_\infty^{\mathrm{pr,ca}}$$

between the geometric tangent structure on the injective ∞-toposes, and the Goodwillie tangent structure on presentable, compactly-assembled ∞-categories.