A model-theoretic look at exponential fields

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Abstract

A model-theoretic look at exponential fields

An exponential function is a homomorphism from the additive group of a field to its multiplicative group. The most important examples are the real and complex exponentials, and these are naturally studied analytically.

However, one can also study the algebra of exponential fields and their logical theory. It turns out that the natural ways to do this take one outside the usual finitary classical logic of model theory and into positive/coherent logic, geometric logic, or other infinitary logics, or to the more algebraic and abstract setting of accessible categories.

I will describe some of this story, focussing on the more algebraic aspects of existentially closed exponential fields.

This is joint work with Levon Haykazyan.

Outline

1. Traditional model theory
2. Existentially closed models, positive and coherent logic
3. EC exponential fields

Outline

What is model theory, traditionally?

Example

The theory of $\mathbb{R}$-vector spaces has well-known axioms.
We can formalise these in the first-order language $\langle +, 0, (\lambda \cdot)_{\lambda \in \mathbb{R}} \rangle$.

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Example (Study the fields $\mathbb{C}$, $\mathbb{R}$, or $\mathbb{Q}$)

Take language $\langle +, -, \cdot, 0, 1 \rangle$ of rings.
Write axioms of fields.
What other axioms are needed to get a complete theory?
What are the other models?
What sets are definable?

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What if we add the exponential function to the language?

What is model theory, traditionally? An answer

Model theory naturally takes a theory (usually in first-order finitary classical logic), often described in a second-order way, for example as the theory of a structure like the real field.
We ask:

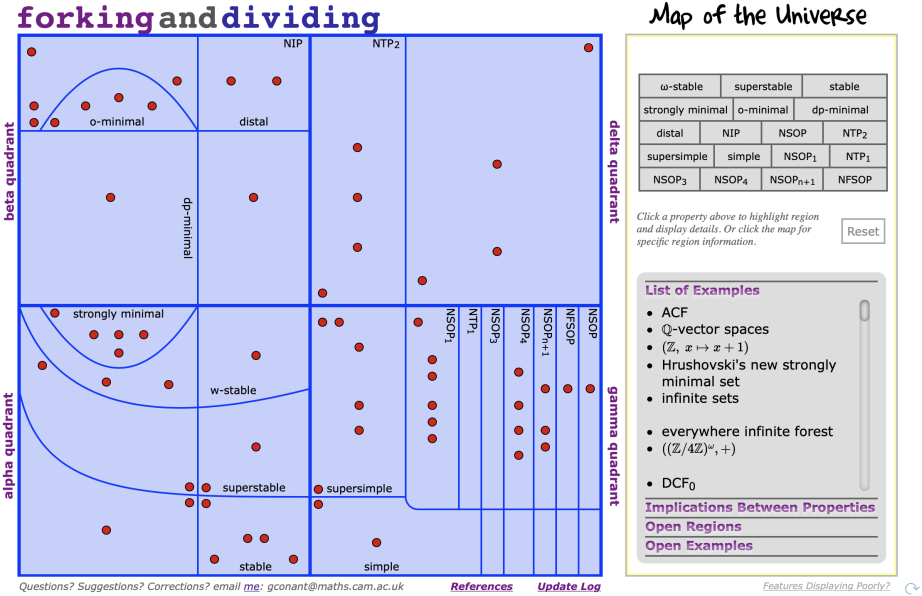
- Can we write a complete axiomatization of the theory?
- What are the models?
- What are the elementary embeddings of models?
- What are the definable sets – is there quantifier elimination, perhaps after expanding the language slightly?
- What are the types?

Stability theory

Then one can ask how combinatorially complicated the theory is:
stable / unstable, simple / non-simple, . . .

The stability hierarchy for complete first-order theories

From Gabriel Conant's forking and dividing website:



The Complex and Real fields

Example (The complex field)

- Axioms of algebraically closed fields
- All embeddings between ACFs are elementary
- Full quantifier elimination: definable sets are boolean combinations of zero sets of polynomials.
  Geometrically these are algebraic varieties.
- Types correspond to generic points on varieties.

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- Axioms of real-closed fields
- All embeddings between RCFs are elementary
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- Types in one variable related to Dedekind cuts and infinitesimals

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We get the theory of the class of existentially closed models of the theory of fields / ordered fields.

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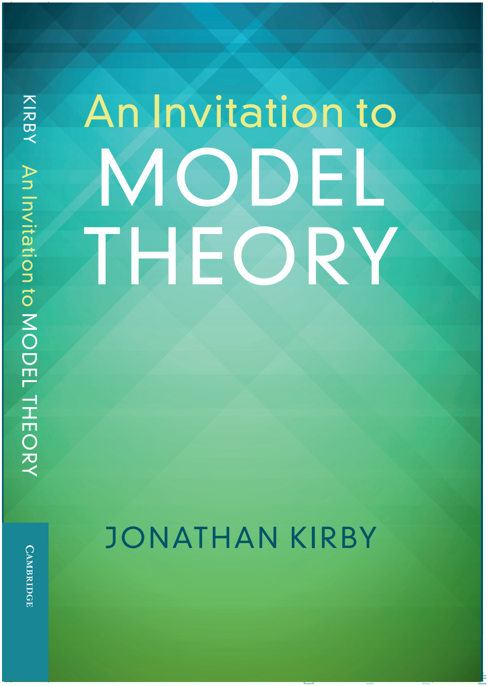

Exponential fields 1

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \mathbb{G}_a(F) \to \mathbb{G}_m(F)$.

Examples

$\mathbb{R}_{\exp}$, with kernel of exponential map $\{0\}$, and $\mathbb{C}_{\exp}$, with kernel $2\pi i\mathbb{Z}$.

Theorem (Wilkie, 1996)

The theory of $\mathbb{R}_{\exp}$ is model-complete (quantifier elimination up to existential formulas), and o-minimal. So it has an $\forall\exists$-axiomatization.

Theorem (Macintyre, Wilkie, 1996)

If Schanuel's conjecture of transcendental number theory is true then the theory of $\mathbb{R}_{\exp}$ has a recursive axiomatization, so is decidable.

Exponential fields 2

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \mathbb{G}_a(F) \to \mathbb{G}_m(F)$.

Observation

In $\mathbb{C}_{\exp}$, the ring of integers $\mathbb{Z}$ is definable via

$$z \in \mathbb{Z} \text{ iff } \mathbb{C}_{\exp} \models \forall x[\exp(x) = 1 \to \exp(zx) = 1].$$

Consequently, by Gödel incompleteness, the complete first-order theory of $\mathbb{C}_{\exp}$ is not decidable.
Nor is it stable, simple, or in any way tame.

Zilber (2005) had a project to study $\mathbb{C}_{\exp}$ algebraically, assuming whatever number-theoretic / geometric conjectures he needed.
He built an exponential field $\mathbb{B}_{\exp}$ which is conjecturally isomorphic to $\mathbb{C}_{\exp}$.
He gave a list of axioms in the logic $L_{\omega_1,\omega}(Q)$ which has exactly one model of each infinite cardinality, where $\mathbb{B}_{\exp}$ is the model of cardinality continuum.

Outline

Existentially closed (EC) models

Recall: A theory T is inductive if the union of a chain of models is a model,
iff $\mathrm{Mod}(T)$ is closed under colimits of directed systems
iff T is axiomatised by $\forall\exists$-sentences.

Definition

A model $M \models T$ is existentially closed (EC) if for all quantifier-free formulas $\phi(\bar{x}, \bar{y})$ and all $\bar{a}$ in M, if there is an extension $M \subseteq B$ such that $B \models T$ and $B \models \exists \bar{x} \phi(\bar{x}, \bar{a})$ then $M \models \exists \bar{x} \phi(\bar{x}, \bar{a})$.

Lemma

If T is inductive and $A \models T$ then there is $A \subseteq M$ with $M \models T$ and M is EC.

Proof: Build M as the union of a chain of extensions of A. □

Examples

Alg closed fields, real-closed ordered fields, differentially closed differential fields, random graphs, existentially closed groups, ...

Comparing first-order T and $\mathcal{EC}(T')$

There are good analogues between the category $\mathrm{Mod}(T)$ of models and elementary embeddings for a complete first-order T and the category $\mathcal{EC}(T)$ of EC models and embeddings.

First-order T	$\mathcal{EC}(T')$ for inductive T'
Model M of T	M an EC model of T'
$M \preccurlyeq N$	$M \subseteq N$
Completion of T	JEP-refinement of T'
Substructure of a model, $A \models T_\forall$	$A \models T'$ or $A \models T'_\forall$ or an amalgamation base
$\mathrm{tp}(\bar{a}/A)$	$\mathrm{tp}_\exists(\bar{a}/A)$ provided A is an amalg base

- In stability theory, we look at T only up to Morleyisation
- Types are then quantifier-free types
- So the setting of the category of EC models of an inductive theory is more general than the setting of a complete first-order theory.

Positive logic – a generalisation of EC models

Definition

- *A positive formula is built using* $\perp$, $\top$, $\wedge$, $\vee$ *and* $\exists$.
  They have a normal form: $\exists \bar{y}[\phi(\bar{x}, \bar{y})]$, *where* ϕ *is positive and quantifier-free.*
- *An h-inductive sentence is a sentence of the form* $\forall \bar{x}[\theta(\bar{x}) \to \psi(\bar{x})]$ *where* θ *and* ϕ *are positive formulas.*

  h stands for homomorphism
- *An h-inductive theory is a set of h-inductive sentences.*

Remarks

- h-inductive theories are precisely those whose category of models is closed under colimits of directed systems of homomorphisms.
- Can specialise to embeddings by taking the symbol $\neq$ in the language, and formal negations of basic relations, with axioms like
  $$\forall xy[x = y \wedge x \neq y \to \perp] \quad \text{and} \quad \forall xy[\top \to x = y \vee x \neq y].$$
- This trick allows us to build as much negation as we choose into a theory.

Brief history of EC models and positive logic

The category of EC models

- Arises naturally from Abraham Robinson's approach to model theory.
- Shelah 1975: Defined and studied classes of Kind II and Kind III. Hrushovski later called the latter Robinson theories.
- Hodges books: Building Models by Games (1985), Model Theory (1993) contain the basic theory.
- Pillay (2000): Forking in the category of existentially closed structures

Positive logic / positive model theory

- Setups and basic properties: Poizat 2006, Ben Yaacov – Poizat 2007, Poizat – Yeshkeyev 2018
- Ben Yaacov (2003): Positive model theory and compact abstract theories
- Ben Yaacov (2003): Simplicity in compact abstract theories
- Haykazyan (2019): Spaces of Types in Positive Model Theory
- Kamsma (2020): Type space functors and interpretations in positive logic

Coherent logic and positive logic

Coherent logic = positive logic

Coherent logic and positive logic are the same, with a slightly different presentation. h-inductive sentences $\forall \bar{x}[\theta(\bar{x}) \to \psi(\bar{x})]$ are called coherent sequents, written $\theta(\bar{x}) \vdash_{\bar{x}} \psi(\bar{x})$.

Brief history

- Notion of Grothendieck topos (1960s)
- Connection with logic: Lawvere (1973)
- Connection with model theory: Makkai – Reyes (1976)
- Exposition: Mac Lane – Moerdijk (1994)
- Johnstone (2002), Sketches of an elephant, Part D

There is also Geometric logic: allows infinite disjunctions (but only finite conjunctions) within θ and ψ.
In geometric logic one can axiomatize the EC models of a coherent theory. (Hodges gives proof with different terminology.)

Coherent logic and positive logic 2

Much of model theory / stability theory can be generalised to positive logic

- Gödel completeness theorem — Deligne completeness theorem
- Compactness theorem still works
- Stone spaces of types — Spectral spaces of types (Haykazyan)
- Theory up to interpretation is “determined by type spaces” (Kamsma)
- Atomic models and omitting types (Haykazyan)
- Kim-Pillay theorem for simple theories (Ben Yaacov, also Kamsma in more generality)
- Other stability hierarchy properties later in this talk

Can also use ideas from topos theory

- Classifying toposes
- Models in categories other than **Set**

Outline

Exponential fields

Definition

An exponential field (E-field) is a field F of characteristic zero, with a homomorphism $\exp : \mathbb{G}_a(F) \to \mathbb{G}_m(F)$.
If the field is algebraically closed we call it an EA-field.
The theory $T_{\text{E-field}}$ of E-fields in $L_{\text{E-ring}} = \langle +, \cdot, -, 0, 1, \exp \rangle$ is inductive.

Examples

$\mathbb{R}_{\exp}$, with kernel $\{0\}$, and $\mathbb{C}_{\exp}$, with kernel $2\pi i\mathbb{Z}$.

Lemma

If F is an EC E-field then F is alg closed and $|\ker(F)| = |F|$.

Proof idea.

For $a \in F^{\times}$, we can build an extension B s.t. $B \models \exists x[e^x = 1 \wedge e^{x^2} = a]$. □

So $\mathbb{R}_{\exp}$ and $\mathbb{C}_{\exp}$ are not existentially closed.

Axiomatisation of EC E-fields

Definition

Let $V \subseteq \mathbb{G}_a(F)^n \times \mathbb{G}_m(F)^n$ be a subvariety, irreducible over F. Let $(\bar{x}, \bar{y}) \in V$, generic over F. Then V is additively free if $\bar{x}$ satisfies no equation of the form $\sum_{i=1}^{n} m_i x_i = a$ where $a \in F$ and $m_i \in \mathbb{Z}$, not all zero.

Theorem

An E-field F is existentially closed if and only if for every additively free V there is $\bar{a} \in F$ such that $(\bar{a}, \exp(\bar{a})) \in V$.

Lemma

If F is an EC E-field, and $a \in F$,

$$a \in \mathbb{Z} \text{ iff } F \models \forall x[\exp(x) = 1 \to \exp(ax) = 1].$$

Corollary

The class of EC E-fields is not first-order axiomatisable.

ECEF is $NSOP_1$ and TP_2

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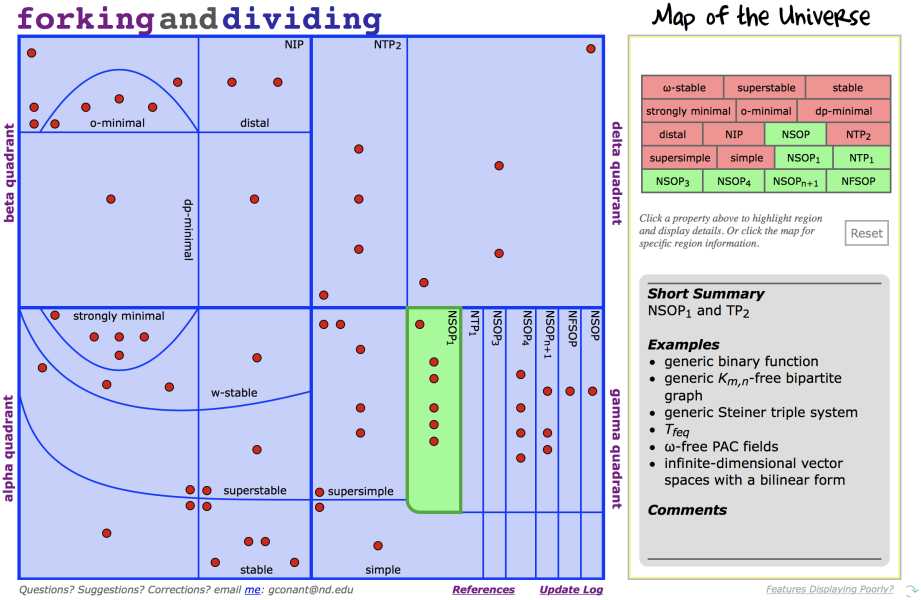

Thank you for your attention!

Amalgamation bases in general

Definition

An amalgamation base *for* $\mathrm{Emb}(T)$ *is* $A \models T$ *such that given* $B_1, B_2 \models T$ *and embeddings* $f_1 : A \to B_1$ *and* $f_2 : A \to B_2$ *there is* $C \models T$ *and embeddings* $g_1 : B_1 \to C$ *and* $g_2 : B_2 \to C$ *such that* $g_1 f_1 = g_2 f_2$.

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Fact

Every EC model is a (disjoint) amalgamation base for $\mathrm{Emb}(T)$.

Corollary

We can work with monster models in $\mathcal{EC}(T)$*, just as in the first-order case.*

Types: analogue of quantifier-elimination

$\mathrm{tp}_{\exists}^{M}(\bar{a})$ is determined by the quantifier-free type of any amalgamation base A containing $\bar{a}$.

Amalgamation and types

Theorem

The amalgamation bases for $\mathrm{Emb}(T_{\text{E-field}})$ *are precisely the EA-fields. Furthermore they are disjoint amalgamation bases.*

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Corollary

Let M *be an EC E-field,* $\bar{a} \in M$*, and* A *the smallest EA-subfield of* M *containing* $\bar{a}$*. Then* $\mathrm{tp}_{\exists}(\bar{a})$ *is determined by the isomorphism type of* A *(with parameters for* $\bar{a}$*).*
For any $\bar{a}$ *there are* $2^{\aleph_0+|\bar{a}|}$ *such types.*

Corollary

A JEP-refinement of $\mathrm{Emb}(T_{\text{E-field}})$ *is determined by the EA-closure of* $\emptyset$*.*

TP_2 definition

Definition (T complete first-order theory)

$\phi(\bar{x}, \bar{y})$ has TP_2 if there are parameters $(\bar{a}_{i,j})_{i,j<\omega}$ such that:

(i) for all $\sigma \in \omega^\omega$ the set $\{\phi(\bar{x}, \bar{a}_{i,\sigma(i)}) : i < \omega\}$ is consistent,

(ii) for every $i, j, k < \omega$, if $j \neq k$ then $\phi(\bar{x}, \bar{a}_{i,j}) \wedge \phi(\bar{x}, \bar{a}_{i,k})$ is inconsistent.

Definition (T an inductive theory with the JEP)

An existential formula $\phi(\bar{x}, \bar{y})$ has TP_2 with respect to $\mathcal{EC}(T)$ if there is an amalgamation base $A \models T$, an existential formula $\psi(\bar{y}_1, \bar{y}_2)$ and parameters $(\bar{a}_{i,j})_{i,j<\omega}$ from A such that:

(i) for all $\sigma \in \omega^\omega$ the set $\{\phi(\bar{x}, \bar{a}_{i,\sigma(i)}) : i < \omega\}$ is consistent,

(ii) $T \vdash \neg\exists \bar{x}\bar{y}_1\bar{y}_2[\psi(\bar{y}_1, \bar{y}_2) \wedge \phi(\bar{x}, \bar{y}_1) \wedge \phi(\bar{x}, \bar{y}_2)]$,

(iii) for every $i, j, k < \omega$, if $j \neq k$ then $A \models \psi(\bar{a}_{i,j}, \bar{a}_{i,k})$.

T / $\mathcal{EC}(T)$ is NTP_2 if no formula / existential formula has TP_2.

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(iii) for every $i, j, k < \omega$, if $j \neq k$ then $A \models \psi(\bar{a}_{i,j}, \bar{a}_{i,k})$.

Proposition

Let T be a JEP-refinement of $T_{\text{E-field}}$. Then $\phi(x, yz) := \exp(y \cdot x) = z$ has TP_2 with respect to $\mathcal{EC}(T)$, witnessed by $\psi(y_1 z_1, y_2 z_2) := y_1 = y_2 \wedge z_1 \neq z_2$.

Proof idea.

ψ witnesses inconsistency because $\exp$ is a function.
Choose $b_i \in F$ such that $1, b_0, b_1, \ldots$ are $\mathbb{Q}$-linearly independent.
Choose $c_j \in F$ to be distinct and nonzero. Let $\bar{a}_{i,j} = b_i c_j$.
For any finite set of the $\bar{a}_{i,\sigma(i)}$, use additive freeness to get an extension of F to show consistency. Then apply compactness. $\square$

NSOP_1

SOP_1 is a tree property, similar to TP_2. Introduced by Džamonja and Shelah, 2004, in the full first-order setting.

It is an open question whether it is equal to TP_1.

Definition (T an inductive theory with the JEP)

An existential formula $\phi(\bar{x}, \bar{y})$ has SOP_1 with respect to $\mathcal{EC}(T)$ if there is an amalgamation base $A \models T$, an existential formula $\psi(\bar{y}_1, \bar{y}_2)$, and parameters $(\bar{a}_\eta : \eta \in 2^{<\omega})$ from A such that:

(i) For every branch $\sigma \in 2^\omega$, the set $\{\phi(\bar{x}, \bar{a}_{\sigma|_n}) : n < \omega\}$ is consistent,

(ii) $T \vdash \neg\exists \bar{x}\bar{y}_1\bar{y}_2[\psi(\bar{y}_1, \bar{y}_2) \wedge \phi(\bar{x}, \bar{y}_1) \wedge \phi(\bar{x}, \bar{y}_2)]$,

(iii) For every $\eta, \nu \in 2^{<\omega}$, if $\eta^\frown 0 \preceq \nu$, then $A \models \psi(\bar{a}_{\eta^\frown 1}, \bar{a}_\nu)$.

Definition

If no existential formula has SOP_1, we say that $\mathcal{EC}(T)$ is NSOP_1.

It is not feasible to prove directly that no formula has SOP_1.

Chernikov-Ramsey theorem

Chernikov and Ramsey proved a version for NSOP_1-theories of the Kim-Pillay theorem (for simple theories).

Theorem (Chernikov-Ramsey 2016, adapted for $\mathcal{EC}(T)$)

Let $\mathfrak{M}$ be a monster model for T. Assume that there is an $\mathrm{Aut}(\mathfrak{M})$ invariant independence relation $\mathop{\smile\!\!\!\!\!\!|}$ which satisfies the following properties for any small EC model M and any small tuples $\bar{a}$, $\bar{b}$ from $\mathfrak{M}$:

(i) Strong finite character: *if $\bar{a} \not\mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$, then there is an existential formula $\phi(\bar{x}, \bar{b}, \bar{m}) \in \mathrm{tp}_\exists(\bar{a}/\bar{b}M)$ such that for any $\bar{a}'$ realising ϕ, the relation $\bar{a}' \not\mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$ holds;*

(ii) Existence over models: *$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M M$ for any tuple $\bar{a} \in \mathfrak{M}$;*

(iii) Monotonicity: *$\bar{a}\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}\bar{b}'$ implies $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$;*

(iv) Symmetry: *$\bar{a} \mathop{\smile\!\!\!\!\!\!|}_M \bar{b}$ implies $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_M \bar{a}$;*

(v) Independent 3-amalgamation: *If $\bar{c}_1 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_2$, $\bar{b}_1 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_1$, $\bar{b}_2 \mathop{\smile\!\!\!\!\!\!|}_M \bar{c}_2$ and $\bar{b}_1 \equiv_M \bar{b}_2$ then there exists $\bar{b}$ with $\bar{b} \equiv_{\bar{c}_1 M} \bar{b}_1$ and $\bar{b} \equiv_{\bar{c}_2 M} \bar{b}_2$.*

Then $\mathcal{EC}(T)$ is NSOP_1.

Independence for EC E-fields

Definition

Let F be an EA-field. Let $A, B, C \subseteq F$ be subsets. Define $\langle A \rangle^{\mathrm{EA}}$ to be the smallest EA-subfield of F containing A.
$\langle AC \rangle^{\mathrm{EA}}$ means $\langle A \cup C \rangle^{\mathrm{EA}}$. Then define $\mathop{\smile\!\!\!\!\!\!|}$ by

$$A \underset{C}{\mathop{\smile\!\!\!\!\!\!|}} B \qquad \text{if} \qquad \langle AC \rangle^{\mathrm{EA}} \overset{\mathrm{ACF}}{\underset{\langle C \rangle^{\mathrm{EA}}}{\mathop{\smile\!\!\!\!\!\!|}}} \langle BC \rangle^{\mathrm{EA}}.$$

This notion of independence is quite weak. For example, it does not look at the behaviour of logarithms of elements of A, B, or C. Nor does it look at how A and B might otherwise be related in $\langle ABC \rangle^{\mathrm{EA}}$.
Stronger independence notions, suitable for exponentially closed fields, were explored in the thesis of Robert Henderson.

Theorem

This $\mathop{\smile\!\!\!\!\!\!|}$ satisfies the conditions of the Chernikov-Ramsey theorem. So all JEP-refinements of $\mathcal{EC}(T_{\text{E-field}})$ are NSOP_1.