∞-groupoids in Lextensive Categories

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Theorem (Quillen, 1967)

The category of simplicial sets sSet *carries a proper cartesian model structure (the Kan–Quillen model structure) where*

- *weak equivalences are the weak homotopy equivalences,*
- *fibrations are the Kan fibrations,*
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Is it possible to internalize this theorem in categories more general than the category of sets? That is, for which categories $\mathcal{E}$ can we construct a model structure on the category of simplicial objects s$\mathcal{E}$ that specializes to the Kan–Quillen model structure when $\mathcal{E} = \mathrm{Set}$?

Theorem (constructive logic, CZF)

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- S. Henry, *A constructive account of the Kan-Quillen model structure and of Kan's* Ex^∞ *functor*, `https://arxiv.org/abs/1905.06160`
- N. Gambino, C. Sattler, K. Szumiło, *The constructive Kan-Quillen model structure: two new proofs*, `https://arxiv.org/abs/1907.05394`

Theorem

If $\mathcal{E}$ is a countably lextensive category, then the category of simplicial objects $\mathrm{s}\mathcal{E}$ carries a proper cartesian model structure (the effective model structure) where

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Definition

A category $\mathcal{E}$ is *countably lextensive* if

- it has finite limits,
- it has van Kampen countable coproducts.

Definition

Let $\mathcal{E}$ have finite limits. The colimit $Y_\star$ of a diagram $Y\colon D \to \mathcal{E}$ is

- *universal* if for every morphism $X_\star \to Y_\star$, $X_\star \cong \operatorname{colim}_d Y_d \times_{Y_\star} X_\star$.
- *effective* if for every cartesian transformation $X \to Y$, the colimit $X_\star$ of X exists and $X_d \cong Y_d \times_{Y_\star} X_\star$ for all $d \in D$.
- *van Kampen* if it is both universal and effective.

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Remark

If $\mathcal{E}$ has finite limits and countable coproducts, it is countably lextensive if and only if all countable coproducts are universal and disjoint.

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Definition

A coproduct $X_\star = \coprod_d X_d$ is *disjoint* if $X_d \times_{X_\star} X_{d'}$ is initial for all $d \neq d'$.

Examples

- All presheaf categories are completely lextensive.

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- The category of countable sets is countably lextensive.

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- The category of affine schemes is finitely lextensive.
  The category of schemes is completely lextensive.
- The category of countable sets is countably lextensive.
- The free κ-coproduct completion of a finitely complete category $\mathcal{E}$ is κ-lextensive.

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Proof.

- $(A \sqcup C) \sqcup_A X \cong X \sqcup C$
- $\mathrm{colim}(A_0 \to A_0 \sqcup A_1 \to A_0 \sqcup A_1 \sqcup A_2 \to \dots) \cong \coprod_i A_i$ $\square$

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Remark

If a functor $\mathcal{D} \to \mathcal{E}$ preserves coproducts, then the induced functor $\mathrm{s}\mathcal{D} \to \mathrm{s}\mathcal{E}$ preserves pushouts along levelwise complemented inclusions and colimits of sequences of levelwise complemented inclusions.

If S is a countable set, let $\underline{S} = \coprod_{s \in S} 1$ in $\mathcal{E}$. $S \mapsto \underline{S}$ preserves countable coproducts. We define sets of boundary inclusions and horn inclusions:

$$I = \{\underline{\partial\Delta[m]} \to \underline{\Delta[m]} \mid m \geq 0\}$$
$$J = \{\underline{\Lambda^i[m]} \to \underline{\Delta[m]} \mid m \geq i \geq 0, m > 0\}$$

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For $X, Y \in \mathrm{s}\mathcal{E}$, we define the Hom-presheaf $\mathrm{Hom}_{\mathrm{Psh}\,\mathcal{E}}(X, Y) \in \mathrm{Psh}\,\mathcal{E}$:

$$E \mapsto \mathrm{Hom}_{\mathrm{Set}}(X \times E, Y).$$

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If this presheaf is representable, then the representing object is denoted by $\mathrm{Hom}_{\mathcal{E}}(X, Y)$, e.g., $\mathrm{Hom}_{\mathcal{E}}(\underline{\Delta[m]}, Y) = Y_m$.

Definition

A morphism $i\colon A \to B$ in $s\mathcal{E}$ has the $\mathrm{Psh}\,\mathcal{E}$*-enriched left lifting property* with respect to $p\colon X \to Y$ if

$$\mathrm{Hom}_{\mathrm{Psh}\,\mathcal{E}}(B, X) \to \mathrm{Prob}_{\mathrm{Psh}\,\mathcal{E}}(i, p)$$

has a section, where

$$\mathrm{Prob}_{\mathrm{Psh}\,\mathcal{E}}(i, p) = \mathrm{Hom}_{\mathrm{Psh}\,\mathcal{E}}(A, X) \times_{\mathrm{Hom}_{\mathrm{Psh}\,\mathcal{E}}(A,Y)} \mathrm{Hom}_{\mathrm{Psh}\,\mathcal{E}}(B, Y).$$

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This refers to a weak factorization system on $\mathcal{E}$ of (complemented inclusions, split epimorphisms.).

Definition

A $\mathrm{Psh}\,\mathcal{E}$*-enriched weak factorization system* on $s\mathcal{E}$ is a pair $(\mathscr{L}, \mathscr{R})$ of classes of morphisms such that

- a morphism is in $\mathscr{L}$ if and only if it has the enriched left lifting property with respect to all morphisms of $\mathscr{R}$,
- a morphism is in $\mathscr{R}$ if and only if it has the enriched right lifting property with respect to all morphisms of $\mathscr{L}$,
- every morphism factors as a morphism of $\mathscr{L}$ followed by a morphism of $\mathscr{R}$.

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Theorem

If I is a countable set of levelwise complemented inclusions between finite objects of $\mathrm{s}\mathcal{E}$, then there is a $\mathrm{Psh}\,\mathcal{E}$-enriched weak factorization system, where

- *$\mathscr{R}$ are the I-fibrations, i.e., morphisms with the enriched right lifting property with respect to I,*
- *$\mathscr{L}$ are the I-cofibrations, i.e., morphisms with the enriched left lifting property with respect to $\mathscr{L}$.*

Definition

- A *(Kan) fibration* is a J-fibration.
- A *trivial (Kan) fibration* is an I-fibration.
- A *cofibration* is an I-cofibration.
- A *trivial cofibration* is a J-cofibration.

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Lemma

- *A morphism $A \to B$ is a cofibration if and only if it is a Reedy complemented inclusion, i.e., $A_k \sqcup_{L_k A} L_k B \to B_k$.*
- *A morphism $X \to Y$ is a trivial fibration if and only if it is a Reedy split epimorphism, i.e., $X_k \to M_k X \times_{M_k Y} Y_k$.*

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Definition

A morphism $X \to Y$ is a *pointwise weak equivalence* if

$$\mathrm{Hom}_{\mathrm{sSet}}(E, X) \to \mathrm{Hom}_{\mathrm{sSet}}(E, Y)$$

is a weak homotopy equivalence in sSet for all $E \in \mathcal{E}$.

Proposition

A fibration between fibrant objects is trivial if and only if it is a pointwise weak equivalence.

Proof.

This holds pointwise, i.e., on applying of $\mathrm{Hom}_{\mathsf{sSet}}(E, -)$ for all $E \in \mathcal{E}$. $\square$

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A morphism $X \to Y$ is a *weak homotopy equivalence* if its canonical fibrant replacement $\widehat{X} \to \widehat{Y}$ (produced by the enriched small object argument) is a pointwise weak equivalence.

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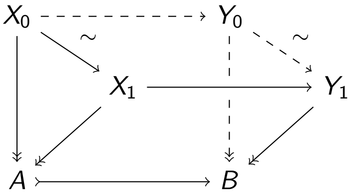
To complete the proof of the main theorem we need to generalize the proposition to fibrations between all objects.

Proposition (Equivalence extension property)

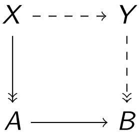
Given the diagram of morphisms between cofibrant objects



where the lower square is a pullback and $X_0 \to X_1$ is a homotopy equivalence over A, there is Y_0 and dashed morphisms such that the back square is a pullback and $Y_0 \to Y_1$ is a homotopy equivalence over B.

Proposition (Fibration extension property)

If X, A and B are cofibrant, $X \to A$ is a fibration and $A \to B$ a trivial cofibration, then there is a pullback square



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$$\begin{array}{ccc} X & \dashrightarrow & Y \\ \downarrow & & \vdots \\ A & \longrightarrow & B \end{array}$$

where $Y \to B$ is a fibration and $X \to Y$ is a trivial cofibration.

Corollary

A fibration between cofibrant objects is trivial if and only if it is a weak equivalence.

If G is a group and $\mathcal{E} = G$-Set, then the resulting model structure on G-sSet coincides with the genuine equivariant model structure.

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Theorem (Generalized Elmendorf's Theorem)

If $\mathcal{E}$ is a locally connected completely lextensive category, then $\mathrm{Ho}_\infty\, \mathrm{s}\mathcal{E}$ is equivalent to the $\mathrm{Ho}_\infty\, \mathrm{sPsh}(\mathcal{E}^{con})$ where $\mathcal{E}^{con}$ is the category of connected objects of $\mathcal{E}$.

Proof sketch.

- We consider the category of *small* simplicial preseaves $\mathrm{sPsh}(\mathcal{E}^{\mathrm{con}})$.
- This category carries a projective model structure with *class cofibrantly generated* weak factorization systems by results of Chorny and Dwyer.
- These weak factorization systems correspond to those of $\mathrm{s}\mathcal{E}$ under the restricted Yoneda embedding $\mathrm{s}\mathcal{E} \to \mathrm{sPsh}(\mathcal{E}^{\mathrm{con}})$. □

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Proposition

- *If $\mathcal{E}$ is κ-lextensive, then $\mathrm{Ho}_\infty \mathrm{s}\mathcal{E}$ satisfies κ-descent, i.e., κ-small colimits in $\mathrm{Ho}_\infty \mathrm{s}\mathcal{E}$ are van Kampen.*
- *If $\mathcal{E}$ is locally cartesian closed, then so is $\mathrm{Ho}_\infty \mathrm{s}\mathcal{E}$.*

Theorem

If $\mathcal{E}$ is either countably complete or countably lextensive, then $\mathrm{Ho}_\infty(s\mathcal{E}_{fib})$ *is equivalent to the full subcategory of simplicial presheaves over $\mathcal{E}$ that are homotopy colimits (geometric realizations) of Kan complexes in $\mathcal{E}$.*

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Proof sketch.

- There is a *fibration category* $\mathrm{s}_+\mathcal{E}_{\mathrm{fib}}$ of *semisimplicial objects* in $\mathcal{E}$ and an equivalence of fibration categories $\mathrm{s}\mathcal{E}_{\mathrm{fib}} \to \mathrm{s}_+\mathcal{E}_{\mathrm{fib}}$ provided that $\mathcal{E}$ is countably complete or countably lextensive.
- The category $\mathrm{Fam}\,\mathcal{E}$ (the coproduct completion of $\mathrm{s}\mathcal{E}$) is completely lextensive.
- In the diagram

$$\begin{array}{ccc} \mathrm{s}\mathcal{E}_{\mathrm{fib}} & \longrightarrow & \mathrm{s}_+\mathcal{E}_{\mathrm{fib}} \\ \downarrow & & \downarrow \\ (\mathrm{sFam}\,\mathcal{E})_{\mathrm{fib}} & \longrightarrow & (\mathrm{s}_+\mathrm{Fam}\,\mathcal{E})_{\mathrm{fib}} \end{array}$$

  the right functor is fully faithful on Ho_∞ and hence so is the left one.
- Apply Elmendorf's Theorem ($(\mathrm{Fam}\,\mathcal{E})^{\mathrm{con}} = \mathcal{E}$). □

Corollary

Under the same assumptions, the full subcategory of set-truncated objects in $\mathrm{Ho}(s\mathcal{E}_{fib})$ *is equivalent to the ex/lex completion of* $\mathcal{E}$.

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Conjecture

Under the same assumptions, the ex/lex completion of the ∞*-category* $\mathcal{E}$ *is equivalent to the full subcategory of* $\mathrm{Ho}_\infty(\mathsf{s}\mathcal{E}_{\mathsf{fib}})$ *on objects that are* n*-truncated for some* n.

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Conjecture

For general finitely complete $\mathcal{E}$, *the ex/lex completion of the* ∞*-category* $\mathcal{E}$ *is equivalent to the full subcategory of* $\mathrm{Ho}_\infty(\mathrm{s}_+\mathcal{E}_{\mathrm{fib}})$ *on objects that are* n*-truncated for some* n.